Gain-Scheduling Integral Controller Applied to a Quadrotor

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***Abstract*— The control of non-linear systems presents major challenges when compared to the control of linear systems. To downsize the complexity of non-linear systems, a common practice is to linearize the system around an equilibrium point and, subsequently, use known linear control techniques. Problems arise if the state of the system deviates greatly from the equilibrium point around which the system is linearized. To overcome this issue, gain-scheduling control uses different gains, calculated for different equilibrium points. This approach allows for an adaptation of the control parameters depending on the state of the system. Here, we compare a fixed-gain and a gain-scheduling controller by applying them to the problem of tracking a time-varying reference where the non-linear system in consideration is a quadrotor. We experimentally show that a fixed gain controller does not allow, in general, for the quadrotor to follow a time-varying reference. However, this problem is circumvented by applying a gain-scheduling controller.**

I. Introduction

Quadrotors are arguably the most studied and research type of vehicle in the control community. On one hand, quadrotors are used in a wide range of applications. As testimony to this there is the mainstream use of drones by the general public in the past years. On the other hand, the seemingly simple structure of a quadrotor hides its complex dynamics, which are of particular interest in control theory. The control of a quadrotor is, however, not an easy task. To start, it is an under-actuated system: it has six control outputs, namely the position and its three orientation angles, but only four independent control inputs. Moreover, and most importantly, a quadrotor is a non-linear system.

A common practice when trying to control a non-linear system is to linearize the system around a desired equilibrium point (also referred to as a stable state) and then appeal to the results and techniques of linear control designing. This strategy has proven to be successful in stabilizing a non-linear system around an equilibrium state, given that the initial state of the system is within a certain neighborhood of that equilibrium state. The stabilization control problem can be regarded as the tracking of a constant reference. Determining the neighborhood of the equilibrium point in which the controller is guaranteed to work is not trivial. As such, in general, a controller obtained by linearizing a non-linear system does not allow for the system to track a time-varying reference. Control theory offers a tool to circumvent this problem: gain scheduling. Gain scheduling is a technique that can extend the validity of the linearization approach to a range of operating points [1], thus allowing for the system to track a wider set of possibly time varying references.

In this short report we will study, by simulation, the behaviour of a quadrotor tracking a time-varying reference, both with a fixed gain feedback controller and a gain-scheduling. We will show that a linear fixed-gain integral controller can track a time-varying reference in the particular case when the yaw angle of the quadrotor is constant along the reference, that is, when the quadrotor is parallel to the ground and has a fixed orientation at all times. However, if the yaw angle changes with the reference, for example, being tangential to the trajectory, the fixed-gain controller does not allow for the tracking of the reference. In this last case, a gain-scheduling integral controller will be used to successfully track the reference signal. We will assume that the quadrotor state is always available in-line, so the designed control strategies will be of state feedback. In practice, we often do not have access to the full state of the system, in which case an observer may be needed. This is, however, out of the scope of this report.

This short report is organized as follows. Section II presents a summarized overview of the mathematical model of the quadrotor. Section III illustrates the linearization strategy. Section IV reviews the control theory for a fixed-gain controller. Section V introduces the gain-scheduling technique for integral controller design. Section VI describes the implementation of the quadrotor simulation and analyses the obtained results. Finally, section VII concludes the report.

II. Quadrotor Model

A quadrotor is a type of helicopter with four rotors where each rotor can generate a vertical force and a torque. Different combinations of forces and toques are responsible for changing the orientation (ϕ, θ, ψ), given by the roll-pitch-yaw angles [2], and position in space (x, y, z) of the quadrotor. The necessity of using different coordinate frames arises from the fact that dynamics and kinematics equations are applied to different frames. Five coordinate frames are used: the inertial frame is an earth fixed coordinate system with the z direction pointing into earth; the vehicle frame is identical to the inertial frame but has its origin in the center of mass of the quadrotor; the vehicle-1 frame is identical to the vehicle frame but rotated about the z axis by an yaw angle ψ; the vehicle-2 frame is identical to the vehicle-1 frame but rotated about the y axis by a pitch angle θ; finally, the body frame is identical to the vehicle-2 frame but rotated about about the x axis by a roll angle ϕ. In summary, the body frame is identical to the vehicle frame after the application of Euler angles (ψ, θ, ϕ) using the ZYX convention [3]. In figure 1 the inertial and the body frame are depicted, together with the axis where the roll, pitch and yaw angles are applied.

Using the defined coordinate frames we can obtain the state equations of the quadrotor using the equation of Coriolis, quadrotor kinematics, rigid body dynamics and the rotation matrices that describe the transformation between each frame. The full procedure is detailed in 1.2 to 3 of reference [4]. The variables to be used are the following:

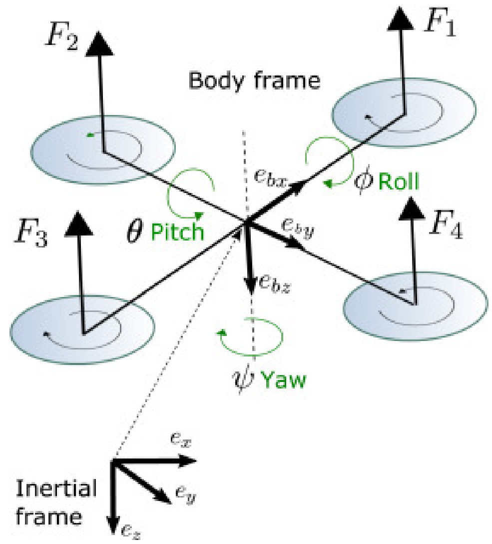

Fig. 1: Representations of 2 of the 5 coordinate frames together with the roll-pitch-yaw angles.

$$
\begin{aligned}
x &= \text{quadrotor's position in the inertial frame along } e_x\\
y &= \text{quadrotor's position in the inertial frame along } e_y\\
h &= \text{quadrotor's position in the inertial frame along } -e_z\\
u &= \text{quadrotor's velocity in the body frame along } e_{b_x}\\
v &= \text{quadrotor's velocity in the body frame along } e_{b_y}\\
w &= \text{quadrotor's velocity in the body frame along } e_{b_z}\\
\phi &= \text{roll angle defined in the vehicle-2 frame}\\
\theta &= \text{pitch angle defined in the vehicle-1 frame}\\
\psi &= \text{yaw angle defined in the vehicle frame}\\
p &= \text{the roll angle rate in the body frame}\\
q &= \text{the pitch angle rate in the body frame}\\
r &= \text{the yaw angle rate in the body frame}
\end{aligned}
\tag{1}
$$

The non linear equations of the system are the following:

$$
\begin{cases}
\dot{x} = (c\theta c\psi)u + (s\phi s\theta c\psi - c\phi s\psi)v + (c\phi s\theta c\psi + s\phi s\psi)w\\
\dot{y} = (c\theta c\psi)u + (s\phi s\theta c\psi + c\phi s\psi)v + (c\phi s\theta s\psi - s\phi c\psi)w\\
\dot{h} = (s\theta)u - (s\phi c\theta)v - (c\phi c\theta)w\\
\dot{u} = rv - qw - g\sin\theta\\
\dot{v} = pw - ru + g\cos\theta\sin\phi\\
\dot{w} = qu - pv + g\cos\theta\cos\phi - F/m\\
\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta\\
\dot{\theta} = q\cos\phi - r\sin\phi\\
\dot{\psi} = q\sin\phi/\cos\theta + r\cos\phi/\cos\theta\\
\dot{p} = \frac{J_y - J_z}{J_x}qr + \frac{\tau_\phi}{J_x}\\
\dot{q} = \frac{J_z - J_x}{J_y}pr + \frac{\tau_\theta}{J_y}\\
\dot{r} = \frac{J_x - J_y}{J_z}pq + \frac{\tau_\psi}{J_z}
\end{cases}
\tag{2}
$$

where c is for cos, s is for sin, F is the combined force of the four rotors and τ_ϕ, τ_θ and τ_ψ are the resulting rolling, pitching and yawing torques, obtained by different combinations of forces of the four rotors, as stated in [4].

III. Equilibrium Points and Linearization

With equations (1) and (2), we introduce the state-space representation of the quadrotor system. The state $\mathbf{x}$, the input $\mathbf{u}$ and the output $\mathbf{y}$ are given by

$$\mathbf{x} = (x, y, h, u, v, w, \phi, \theta, \psi, p, q, r) \tag{3}$$
$$\mathbf{u} = (F, \tau_\phi, \tau_\theta, \tau_\psi) \tag{4}$$
$$\mathbf{y} = (x, y, h, \psi) \tag{5}$$

We are assuming that all states are available in-line. The state-equation of the system is given by (2) and, together with the equation of the output, the system can be written in a more general form,

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \tag{6}$$
$$\mathbf{y} = h(\mathbf{x}). \tag{7}$$

The equilibrium points of the system can be obtained by setting $\dot{\mathbf{x}} = 0$,

$$0 = f(\mathbf{x_{ss}}, \mathbf{u_{ss}}), \tag{8}$$

where $\mathbf{x_{ss}}$ is the stable-state and $\mathbf{u_{ss}}$ the stable-input that maintains the equilibrium at $\mathbf{x_{ss}}$. For the case of the quadrotor, $\mathbf{x_{ss}}$ and $\mathbf{u_{ss}}$ are of the form

$$\mathbf{x_{ss}} = (x_{ss}, y_{ss}, h_{ss}, 0, 0, 0, 0, 0, \psi_{ss}, 0, 0, 0), \tag{9}$$
$$\mathbf{u_{ss}} = (mg, 0, 0, 0), \tag{10}$$

with (x_{ss}, y_{ss}, h_{ss}) the stable-state position, ψ_{ss} the stable-state yaw angle, m the quadrotor mass and mg the total thrust necessary to cancel the weight of the quadrotor. These stable state points are according to intuition: the quadrotor is in equilibrium if it is at a given position while parallel to the ground (only the yaw angle can be non-null), and the force produced by its rotors compensates the weight of the quadrotor. Also, notice that, given the form of the output variable $\mathbf{y}$ in (7), any reference $\mathbf{r}(t)$, possibly time-varying, will be a collection of equilibrium points.

Having the equilibrium points of the quadrotor, we now establish a linear model for its dynamic behaviour in a neighborhood of a point $(\mathbf{x_{ss}}, \mathbf{u_{ss}})$. For that, we consider the first-order approximation of the state-equation (6),

$$(\mathbf{x} \dot{-} \mathbf{x_{ss}}) = A(\mathbf{x} - \mathbf{x_{ss}}) + B(\mathbf{u} - \mathbf{u_{ss}}), \tag{11}$$

where

$$A = \frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{u})\Big|_{(\mathbf{x_{ss}}, \mathbf{u_{ss}})} \text{ and } B = \frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}, \mathbf{u})\Big|_{(\mathbf{x_{ss}}, \mathbf{u_{ss}})}. \tag{12}$$

Since the resulting A matrix depends on the stable-state yaw angle ψ_{ss} but not on the stable-state position (x_{ss}, y_{ss}, h_{ss}), we can say that $A \equiv A(\psi_{ss})$. This fact is important when designing a control strategy for the problem of tracking a time-varying trajectory. Linearization of the output equation around a point $\mathbf{x_{ss}}$ yields

$$(\mathbf{y}-\mathbf{y_{ss}}) = C(\mathbf{x}-\mathbf{x_{ss}}), \quad \mathbf{y_{ss}} = h(\mathbf{x_{ss}}),$$
$$C = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x})\Big|_{\mathbf{x_{ss}}}. \tag{13}$$

IV. Fixed Gain Integral Controller

A. Feedback Control

Feedback control is a technique that allows for the stabilization of a system around the origin. Given a linear system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, with output equation $\mathbf{y} = C\mathbf{x}$, the application of a state feedback control law $\mathbf{u} = -K\mathbf{x}$ transforms the state equation into $\dot{\mathbf{x}} = (A - BK)\mathbf{x}$. If we manage to find the gains K such that the eigenvalues of matrix $A - BK$ are placed in the negative complex semi-plane, we can stabilize the system around the origin.

Stabilizing a non-linear system is complicated and a less understood problem when compared to the stabilization of linear systems. For that reason, the simplest approach to the stabilization of non linear systems is the use of the neat results obtained for the linear case. Stabilizing a non linear system around an equilibrium point $(\mathbf{x_{ss}}, \mathbf{u_{ss}})$ is reduced to first, linearizing the system around the desired equilibrium point; then, shifting the origin to the equilibrium point, that is, proceeding to the change of variables $\mathbf{x}_\delta = \mathbf{x} - \mathbf{x_{ss}}$ and $\mathbf{u}_\delta = \mathbf{u} - \mathbf{u_{ss}}$, as illustrated in equations (11) and (12); and finally, compute the vector of gains K, as in the linear case. The resulting, closed-loop system is of the form

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}, -K(\mathbf{x} - \mathbf{x_{ss}}))\\ \mathbf{y} &= h(\mathbf{x}),\end{aligned}$$

where $\mathbf{u_{ss}} = K\mathbf{x_{ss}}$ is a feedforward term that speeds up the convergence.

B. Integral Control

We now turn our attention to a more general control problem, the tracking of a possibly time-varying reference signal $\mathbf{r}(t)$. The basic goal of this control problem is to design the control input $\mathbf{u}$ such that

$$\lim_{t\to\infty} \mathbf{y}(t) - \mathbf{r}(t) = 0. \tag{14}$$

It is well known in control theory that, in order to track a varying reference, there is the need to include integral action in the feedback control law. To introduce integral action, we integrate the regulation error, $\boldsymbol{e} = \mathbf{y} - \boldsymbol{r}$, such that

$$\dot{\boldsymbol{\sigma}} = \mathbf{e}. \tag{15}$$

We then augment the state equation in (6) to obtain

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \tag{16}$$
$$\dot{\boldsymbol{\sigma}} = h(\mathbf{x}) - \boldsymbol{r}. \tag{17}$$

We reduced our reference tracking problem to a stabilization problem around the equilibrium point of the augmented system, $(\mathbf{x_{ss}}, \boldsymbol{\sigma_{ss}})$. To that extend, we assume a linear state feedback control law of the form

$$\mathbf{u} = -\mathbf{K_1}\mathbf{x} - \mathbf{K_2}\boldsymbol{\sigma}. \tag{18}$$

The equilibrium points of the augmented system relate to the equilibrium points of the original system by

$$\mathbf{u_{ss}} = -\mathbf{K_1}\mathbf{x_{ss}} - \mathbf{K_2}\boldsymbol{\sigma_{ss}}. \tag{19}$$

This is a proportional-integral (PI) controller, has the first term can be regarded as a proportional term, and the second an integral term.

Linearizing equations (16) and (17) around the equilibrium point of the augmented system, $(\mathbf{x_{ss}}, \boldsymbol{\sigma_{ss}})$, and substituting the control law of equation (18) into the resulting, linearized, equations, we obtain the closed-loop state equation for the augmented system,

$$\dot{\boldsymbol{\xi}} = (\mathcal{A} - \mathcal{B}\mathcal{K})\boldsymbol{\xi}, \tag{20}$$

where

$$\boldsymbol{\xi} = \begin{bmatrix}\boldsymbol{x} - \boldsymbol{x_{ss}}\\ \boldsymbol{\sigma} - \boldsymbol{\sigma_{ss}}\end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix}A & 0\\ C & 0\end{bmatrix},$$
$$\mathcal{B} = \begin{bmatrix}B\\ 0\end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix}K_1 & K_2\end{bmatrix}, \tag{21}$$

and A, B and C are the matrices in (12) and (13). Once again, if $\mathcal{K}$ is designed such that the eigenvalues of matrix $(\mathcal{A} - \mathcal{B}\mathcal{K})$ are placed in the negative complex semi-plane, then $(\mathbf{x_{ss}}, \boldsymbol{\sigma_{ss}})$ is a stable equilibrium point of the closed-loop system, and, as a consequence, the system follows the reference $\mathbf{r}$. The controlled, closed-loop system then takes the form

$$\dot{\mathbf{x}} = f(\mathbf{x}, -K_1(\mathbf{x} - \mathbf{x_{ss}}) - K_2\boldsymbol{\sigma} + \mathbf{u_{ss}}), \tag{22}$$
$$\dot{\boldsymbol{\sigma}} = \mathbf{y} - \mathbf{r}. \tag{23}$$

In the particular case of the quadrotor, since the reference $\mathbf{r}(t)$ is always a collection of equilibrium points, we can set $\mathbf{x_{ss}}$ in (22) as $C^T\mathbf{r} = (x_r, y_r, h_r, 0, 0, 0, 0, 0, \psi_r, 0, 0, 0)$, as stated in [1].

We end this section by noting that, since we are linearizing the system around a specific equilibrium point, the controller is only guaranteed to work in a neighborhood of that equilibrium point. To tackle this issue, we will use the strategy of gain-scheduling, described in the next section.

V. Gain Scheduling Integral Controller

Gain scheduling is a technique that can extend the validity of the linearization to a range of operating points. In many situations, it is known how the dynamics of a system change with its operating points. It may also be possible to model the system such that these operating points are parameterized by one or more variables called the *scheduling variables*, α. In situations where we move from one equilibrium point to others, we may linearize the system at several equilibrium points and design a linear feedback controller for each point. After that we can implement the resulting family of linear controllers as a single controller whose parameters are changed by monitoring the scheduling variables α. The resulting controller can be obtained by interpolating the parameters of the several linear controllers obtained, and may be non-linear. Here, we consider a simpler approach, as stated in [5], where the gain vector of the resulting controller, $\mathcal{K}$, is a piecewise constant function of the scheduling variables α of the form

$$\mathcal{K}(\alpha) = \{\mathcal{K}_l \text{ if } \alpha \in \mathcal{D}_l\}, \tag{24}$$

where $\mathcal{K}_l(\alpha)$ is the gain vector of the controller obtained via linearization around the equilibrium point l, and $\mathcal{D}_l$ the corresponding parametrizing set.

For the particular case of the quadrotor, we start by recalling that the matrix A defined in (12) only depends on the yaw angle, ψ. As such, this will be our scheduling variable. Since ψ can range from $-\pi$ to π, we start by dividing this interval into n equal intervals, $\mathcal{D}_l = [l2\pi/n, (l+1)2\pi/n]$, $0 \leq l < n$. For each interval, we linearize the system as stated in section III, with $\psi_{ss} = l2\pi/n$, and obtain the linear state-feedback controller as in (18), yielding the vector of gains $\mathcal{K}_l = [K_1^l \ K_2^l]$.

Finally, we introduce a slight modification, called the *velocity algorithm* (see references [5] and [6]) to improve the controller performance. The control law now yields

$$\mathbf{u} = -K_1(\psi)(\mathbf{x} - C^T\mathbf{r}) + \boldsymbol{\eta}, \tag{25}$$

where $\boldsymbol{\eta}$ is such that

$$\dot{\boldsymbol{\eta}} = -K_2(\psi)(\mathbf{y} - \mathbf{r}). \tag{26}$$

The augmented state is now the concatenation of $\mathbf{x}$ and $\boldsymbol{\eta}$ instead of $\mathbf{x}$ and $\boldsymbol{\sigma}$. By commuting the integrator with the gain $-K_2(\psi)$, the control variable does not depend directly on the scheduling variable and the linearization of the controller does not introduce any additional dynamics [7].

The overall closed-loop system with the gain-scheduled controller is now

$$\dot{\mathbf{x}} = f(\mathbf{x}, -K_1(\psi)(\mathbf{x} - C^T\mathbf{r}) - K_2(\psi)\boldsymbol{\eta} + \mathbf{u_{ss}}), \tag{27}$$

and $\dot{\boldsymbol{\eta}}$ given by equation (26).

VI. Implementation and Results

In this section we discuss the implementation of the quadrotor simulator, as well as the developed controllers and the corresponding obtained results. The simulator was developed in MATLAB, and is available in [10].

A. Reference path

The main goal of the developed simulation is to control a quadrotor by having it pass through some user defined waypoints. A reference path is constructed by interpolating the waypoints with a cubic spline. The reference $\mathbf{r}$ is then a sequence of points of the form $(x_{ref}, y_{ref}, h_{ref}, \psi_{ref})$. In order to make the vizualization of the trajectory easier, the altitude h_{ref} will always have a fixed value of 5 meters. The x and y positions will depend on the chosen waypoints. These are introduced in the XY plane, and the x_{ref} and y_{ref} coordinates can have any value from -10 to 10 meters.

The reference parameters are defined as a function of time, t, such that $t \in [0,\ t_{max}]$. By adjusting the parameter t_{max}, the average velocity of the quadrotor during the trajectory is changed. Intuitively, as t_{max} increases, the easier it is to control the quadrotor, in exchange of a higher computation time.

For the reference yaw angle, ψ_{ref}, we will consider two cases. First, we will study the case where $\psi_{ref}(t) \equiv 0$, that is, the orientation of the quadrotor remains constant at all times. In the second case, we will consider that the yaw angle is tangential to the reference path, thus allowing for a more natural movement of the quadrotor. The reference for the yaw angle, ψ_{ref} is computed with the use of the x and y components of the reference:

$$\psi_{ref}(t) = arctan\left(\frac{\frac{d}{dt}y_{ref}(t)}{\frac{d}{dt}x_{ref}(t)}\right). \tag{28}$$

B. Fixed Gain Controller

1) Implementation: Starting with the use of a fixed gain integral controller, the K_1 and K_2 gains in equation (18) were calculated using the LQR formulation [8]. The cost matrices Q and R used were the following:

$$Q = diag(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1), \tag{29}$$
$$R = diag(0.1, 0.1, 0.1, 0.1). \tag{30}$$

The Q matrix was set fix and the R matrix was chosen to simultaneously have small tracking errors and to also have realistic values for the input control signal, $\mathbf{u}$.

The overall implemented system was the one in equations (22) and (23).

2) Results: We present the results obtained with the fixed gain controller for two different paths: a first one (figure 2), where the slope of the line tangent to the xy curve varies smoothly, and a second one (figure 4), where the slope of the line tangent to the xy curve varies more abruptly.

For each path, we performed two simulations, with different reference signals, as discussed in section VI-A. In one, the reference for the yaw angle is set to be always zero. In the other, the reference for the yaw angle is given by expression (28).

Starting with the first path, in the case where $\psi_{ref}(t) \equiv 0$, the obtained results are depicted in figure 2. Notice that the reference x and y positions are correctly tracked, apart from small deviations where the curve is more abrupt, near the end of the trajectory (at the right). The reference h position and yaw angle are followed by the quadrotor perfectly, that is, it keeps a constant altitude and orientation at all times.

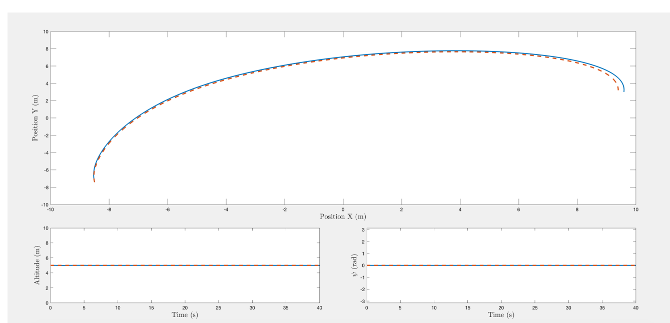

Fig. 2: Reference tracking with zero yaw angle of an user introduced path using a fixed gain controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 40 seconds.

We now consider the case of the tangential yaw angle, still for the first path. The simulation results are presented in figure 3. Notice that the tracking of the position and orientation fails completely after the 30 second mark.

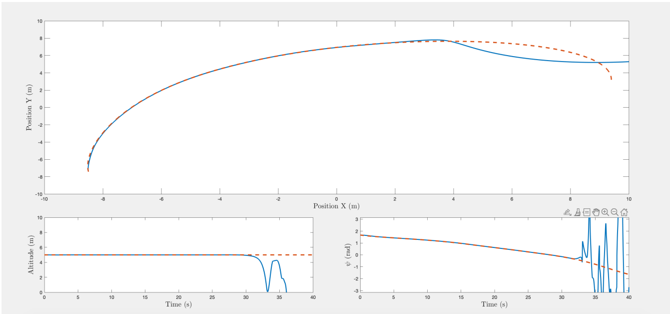

Fig. 3: Reference tracking with tangential yaw angle of an user introduced path using a fixed gain controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 40 seconds.

The repetition of the same procedure for the second path yields the results presented in figure 4, for the constant yaw angle and figure 5, for the tangential yaw angle.

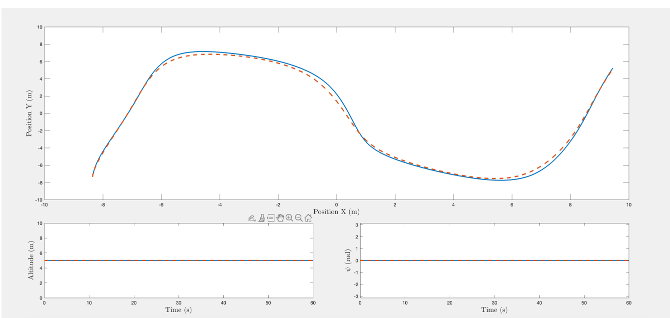

Fig. 4: Reference tracking with zero yaw angle of an user introduced path using a fixed gain controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 60 seconds.

It is clear, by figure 4, that the more abrupt curves result in a slight discrepancy between the reference and the actual path. However, the quadrotor is, in general, able to track the reference. In figure 5, just as in figure 3, it is evident that the controller is not able to guide the position or orientation of the quadrotor to the reference. This is because the controller is tracking a reference yaw angle that progressively departs from the angle about which the linearization was considered. This observation motivates the design of a gain-scheduling integral controller, using as scheduling variable the yaw angle. Notice that this is the only free parameter in matrix A, this is why the quadrotor with a fixed gain controller is able to track references that have time-varying positions but not references that have a time-varying yaw angle, in general.

C. Gain Scheduling Controller

1) Implementation: In section VI-B.2 we concluded that a fixed gain controller does not allow for the tracking of a reference with varying yaw angle. As such, we will now use the gain scheduling procedure presented in section V. To that extent, the gain scheduled controller was developed in the following way: every time that the yaw angle departs from the angle about which the linearization was considered by more than a given tolerance value, the gains are recalculated, with the current reference point as the new equilibrium point (this is possible since, as mentioned before, all the points constituting the reference are stable state points). At all times, the new gains are computed using the LQR formulation with the same Q and R matrices of equations (29) and (30). After some experiments, the tolerance value was set to $\pi/8$.

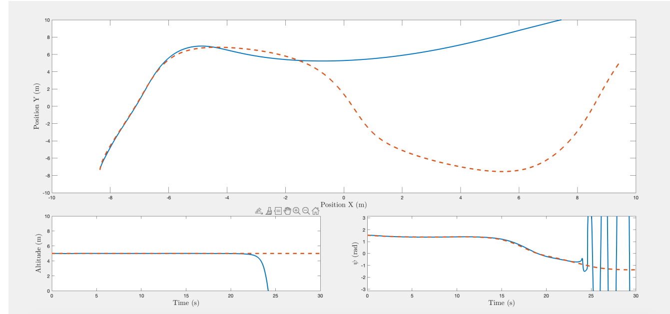

Fig. 5: Reference tracking with tangential yaw angle of an user introduced path using a fixed gain controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 60 seconds but only 30 seconds of the trajectory are shown.

2) Results: The results of the reference tracking with tangential yaw angle for both paths are depicted in figures 6 and 7. Regarding the x, y and h positions, the reference is tracked perfectly. The slight deviations from the reference in the abrupt curved that used to be present when using the fixed gain controller disappeared completely. Furthermore, the scheduled gain controller also allows for the quadrotor to track the tangential yaw angle reference perfectly, something that was not possible with the fixed gain controller.

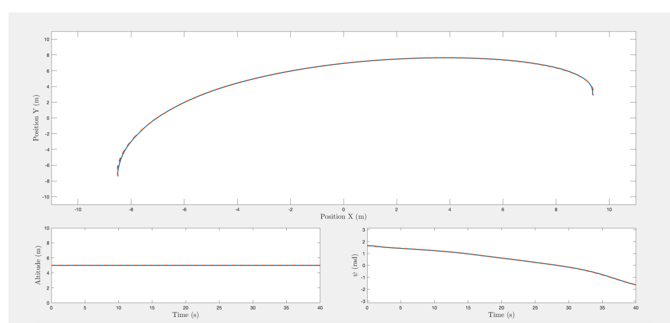

Fig. 6: Reference tracking with tangential yaw angle of an user introduced path using a gain scheduled controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 40 seconds.

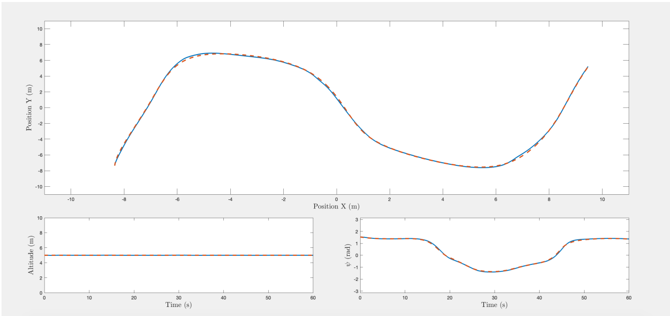

Fig. 7: Reference tracking with tangential yaw angle of an user introduced path using a gain scheduled controller. The red dashed lines correspond to the references that must be tracked and the continuous blue lines correspond to the actual sequence of positions of the quadrotor. The t_{max} parameter was set to 60 seconds.

VII. Conclusion

A fixed-gain controller and a gain-scheduled controller were applied to a quadrotor, a non-linear system. We showed that, in general, the quadrotor cannot track a time-varying reference when a fixed-gain controller is applied in the case where the yaw reference is time-varying. This occurs due to the fact that the linear, fixed-gain controller, obtained by linearizing the system about a given equilibrium point, is only guaranteed to work in a neighborhood of said equilibrium point. We also show that the linearization technique can be extended to several equilibrium points, which we referred to as operation points, thus extending the region of validity of the obtained controller and allowing for the quadrotor to track a time-varying reference.

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