MITx: Statistics, Computation & Applications

Statistics Refresher

Lecture 1: Observational Studies and Experiments

Mammography and breast cancer

- Breast cancer is one of the most common malignancies among women in the United States
- Mammography: screening women for breast cancer by X-rays

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- Becomes clear when comparing the death rates from all other causes
- Instead compare the whole treatment group against the whole control group

✶ **Intention-to-treat analysis**

Hypothesis testing

- Death rate from breast cancer in control group: 0.0020 ($= \frac{63}{31000}$)
- Death rate from breast cancer in treatment group: 0.0013 ($= \frac{39}{31000}$)

Is the difference in death rates between the treatment and control group sufficient to establish that mammography reduces the risk of death from breast cancer?

$\Rightarrow$ Perform a **hypothesis test**

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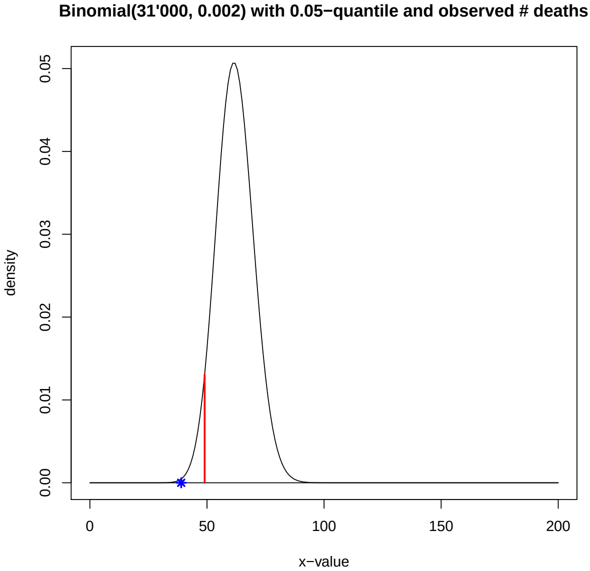
4. Determine a **significance level** (α), i.e. the probability of rejecting H_0 when H_0 is true: e.g. $\alpha = 0.05$

Binomial distribution



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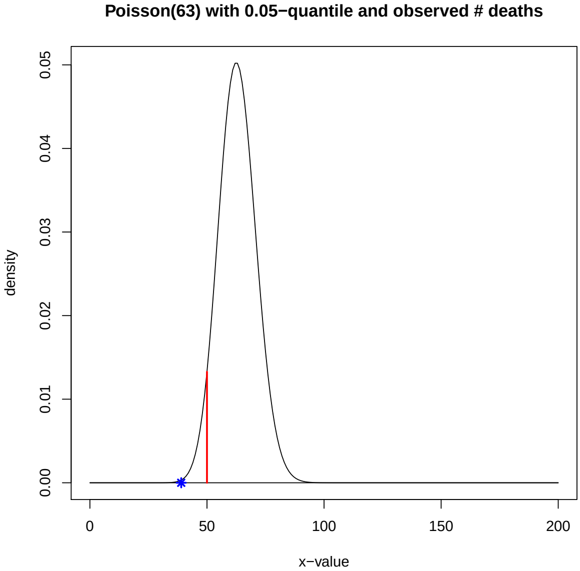

P-value

- Probability under H_0 to obtain the observed value or a more extreme value of the test statistic

  $\Rightarrow$ p-value is always between 0 and 1!

  For mammography study: p-value is 0.0012 under binomial model and 0.0008 under Poisson model

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- Note that there is a trade-off between the probability of making a type I error and the probability of making a type II error (Why?)
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Recap: Hypothesis testing

	breast cancer deaths (rate)	alive	total
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1. Model: $X_1, \ldots, X_{31'000} \sim \text{Bernoulli}(\pi)$ or $Y \sim \text{Poisson}(\lambda)$
2. Null hypothesis (H_0): $\pi = 0.002$ or $\lambda = 63$
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3. Test statistic T = Number of deaths under H_0
   $T \sim \text{binomial}(31'000, 0.002)$ or $T \sim \text{Poisson}(63)$
4. Significance level: $\alpha = 0.05$

Any important assumption that we should relax?

Alternative test: assume no knowledge of π_{control}

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- Test statistic T: number of deaths among the treated individuals
- Model: **Hypergeometric distribution**:

$$\mathbb{P}_{H_0}(T = 39) = \frac{\binom{31'000}{39}\binom{31'000}{63}}{\binom{62'000}{102}}$$

- p-value $= 0.011$

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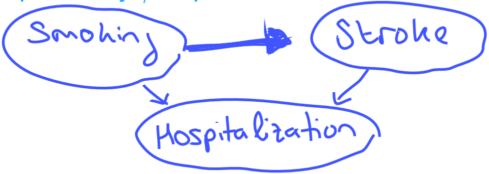

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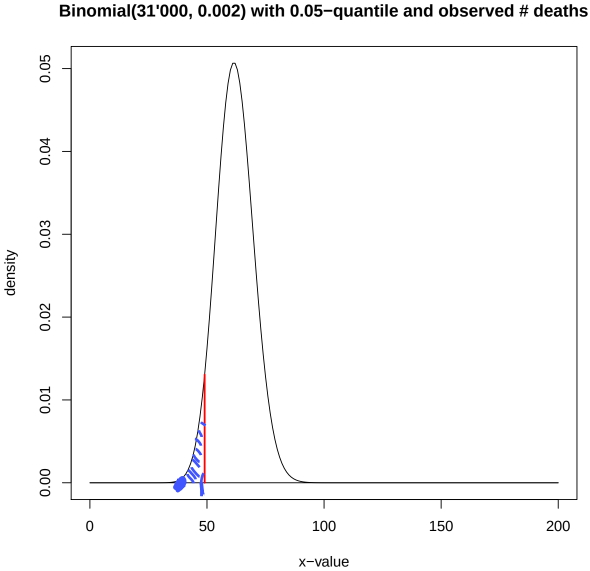

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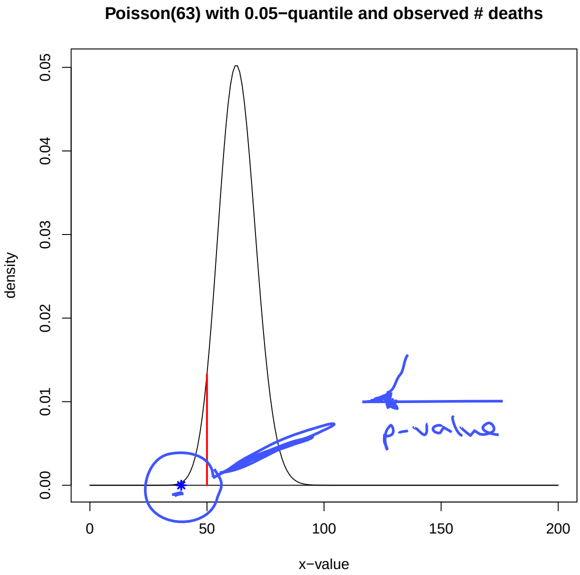

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- Model: **Hypergeometric distribution**:

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- p-value $= 0.011$ $= \sum_{i=0}^{39} \mathbb{P}_{H_0}(T = i)$

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