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The Rational Number Sub-Constructs as a Foundation for Problem Solving

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Abstract

One of the many roles of two year community colleges in the United States is to bridge the gap between secondary school and college for students who graduate from high school with weak mathematics skills that prevent them from enrolling in college level mathematics courses. At community colleges remedial or developmental mathematics courses review the pre-algebra and/or algebra skills required for college level mathematics. Fractions are often cited as the most difficult topic for students due to their abstract nature (Wilensky, 1991). This study with adult pre-algebra students is based upon a teaching research experiment in which the Kieren's fraction sub-constructs of part-whole, ratio, operator, quotient, measure and the fractional equivalence were used as foundational concept knowledge during problem solving. In the first quantitative part of this study, students' proficiency with Kieren's rational number sub-constructs are used as independent variables in a multiple linear regression model to

predict or explain students' competency in formal problem solving. This part of the study supplies hypothetical or statistical suggested pathways for students learning and transition from fraction concepts to proportional reasoning. Then in the second qualitative part of this study, transcripts from classroom lectures during the teaching research experiment are reviewed in order to understand how students used these rational number sub-constructs during problem solving with ratio, quotient, proportion, and percent.

Keywords: Adult remedial mathematics, fractions, sub-construct, ratio, operator, quotient, measure, informal and formal proportional reasoning

Introduction

Proportional reasoning is often cited as a critical component in the transition from informal to formal mathematical thought. In the pre-algebra curriculum proportion typically come after fraction and ratio, however many educators believe it should be introduced earlier and the connections between these topics should be emphasized (Streefland, 1984). The claim that instruction in proportions should be based upon and connected to students' understanding of fractions puts more emphasis on this important concept. Fractions represent a difficult concept for many students. Almost every instructor has heard a student proclaim, "I hate fractions." In an effort to clarify the relationships between various fraction concepts the Kieren (1976) model of fraction and the extension of this model by Behr, Lesh, Post and Silver (1983) was studied using quantitative analyses by Charalambous & Pitta-Pantazi (2007), with children, and Baker, Czarnocha, Dias, Doyle and Prabhu (2009) with adults. The Behr et al. (1985) extension of Kieren's work was used as a theoretical foundation to study the relationship between procedural and conceptual knowledge for adult students reviewing fraction concepts in Baker, Czarnocha, Dias, Doyle, Kennis and Prabhu (2012).

The first objective of this study is to test an underlying hypothesis inherent in the Behr et al. (1983) extension that the rational number sub-constructs provide a foundation for problem solving in the realm of proportions. This is done by using student proficiency with these constructs and fractional equivalence as independent variables in an analysis of variation (ANOVA) linear regression model to predict student competency with problem solving.

The second objective involves analyzing classroom transcripts during the teaching research project in order to determine how these rational number concepts are used during student informal reasoning with ratio, proportion and percent problems.

Literature Review

Problem-Solving

Cognitive theorists suggest that all learning takes place in a problem solving or goal directed environment. A problem solver acquires methods and strategies to obtain a goal in one of three manners. The first is through direct instruction, the second is by discovery and the third is using analogy to previous solutions. Learning and increased proficiency in a domain is characterized by the ability to recognize chunks or patterns of elements which repeat over problems of a similar structure (Anderson, 1995). These chunks or patterns can be identified with problem solving schema which are triggered whenever "an individual tries to comprehend, understand, organize or make sense of a new situation" (Steele & Johanning, 2004, p. 67) The ability of a

student to recognize elementary schema that relates to previous situations is viewed by math educators as the first (recognition) stage in the development of problem solving (Cifareli, 1998).

Direct instruction in problem solving in a mathematics classroom frequently takes place through modeling correct problem-solving behavior. Then students are given problems with similar structure to strengthen their skills at recognition and the use of analogy. That is educators employ repetition, recognition and generalization often by adapting problem solving sequences with increasing difficulty and generalization (Steele & Johanning, 2004). Unfortunately, weak problem solvers tend to employ strategies dominated by superficial aspects of a problem and in a classroom situation their ability to recognize a pattern and transfer knowledge is heavily influenced by what cognitive psychologist refer to as "temporal proximity," that is whatever type of problem they are solving in class is what they expect to use (Anderson, 1995). Another frequently observed trait is referred to by Lamon (2007) as "non-conservation of operation" this behavior is characterized by the choice of an operation that is easy to perform given the numerical values presented without consideration of problem structure. A student who replies that "when 3 lbs. are divided into 9 packages the result is 3," would be exhibiting such problem solving behavior.

The inability of many students to assimilate information about the problem structure into their choice of operation(s) makes an over reliance on modeling correct problem solving behavior ineffective. The insight that these students need to directly engage in the process has lead to reforms that emphasize student discovery during problem solving. For cognitive psychologist the discovery or formation of new methods and techniques for problem solving are built upon a "rich conceptual knowledge base" (Byrnes & Wasik, 1991, p. 778).

Concept development and problem solving are frequently treated as separate branches of mathematics. However, several educational researchers suggest a dynamic interaction between them (Steele & Johanning, 2007; Lesh, R., Landau, M. & Hamilton, E., 1983). Tracy Goodson-Epsy (1998) uses both the stages of problem solving introduced by Cifareli based upon the ability to recognize and mentally represent solution strategies to a given problem and the stages of concept development based upon the work of Piaget. She concludes that students in the lower stages of problem solving, "recognition and re-presentation, typically held weak conceptions of variable and equality" (p.244).

The Kieren Model and Behr et al. Extension

Kieren proposed that the concept of a fraction can be viewed as the composition of five related but distinct sub-constructs, the primary sub-construct of part-whole knowledge and the four secondary sub-constructs of ratio, operator, quotient and measure. An extension of this model to corresponding fraction operations, equivalence and problem solving was developed by Behr et al. (1983).

In Figure 1, the primary sub-construct of part-whole and the row of the secondary sub-constructs: ratio, operator, quotient and measure can be viewed as conceptual knowledge. The bottom row, added by Behr et al. (1983), includes, problem solving which is the focus of this study, as well as the procedural knowledge of multiplication and addition, which were the focus of an earlier related study Baker et al. (2012).

In Figure 1 neither procedural knowledge nor fractional equivalence is given a role in promoting problem solving. Educational researchers consider fractional equivalence and

equivalence schemes as "basic constructive mechanisms for rational number knowledge-building" (Pitkethly & Hunting, 1996, p.8). The results of Baker et al. (2009) corroborate the idea that fractional equivalence is considered as conceptual knowledge and its role in determining student competency with problem solving is analyzed in this study.

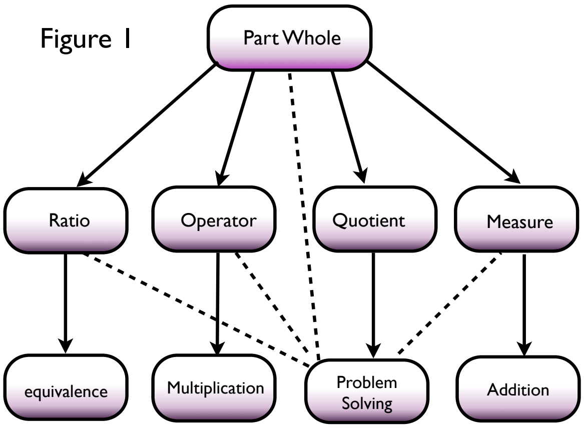

Figure 1 Model of Behr et al., 1983, p.100

The arrows in Figure 1 from all four sub-constructs pointing to problem solving represent an underlying hypothesis that knowledge of these concepts lead to competency with problem solving. Lamon (2007) uses the Kieren sub-constructs as a foundation to promote proportional reasoning and thus agrees that solving proportion and related problems should be based upon these rational number concepts, in particular, she notes that students develop rational number sense through encounters with different representations of rational numbers.

In the first quantitative component of this study, equivalence and the other rational number sub-constructs are used to investigate the hypothesis that competency with these sub-constructs promotes proficiency with problem solving based upon ratio, rates and proportion. In the second qualitative component student use of these rational number concepts during the transition from informal to formal proportional reasoning in the math classroom is analyzed.

Proportional Reasoning

Proportional reasoning has been described as a foundation or core of algebra and higher mathematics (Berk, Taber, Gorowara & Poetzl, 2009; Lo & Watanabe, 1997). Despite the importance of proportional reasoning in subsequent math courses, educators point out that, many college students fail to manifest effective formal proportional reasoning (Adi & Pulos, 1980). Lamon (2007) affirms that the lack of ability to reason proportionally is widespread when she notes, "a sense of urgency about the consistent failure of students and adults to reason proportionally… my own estimate is that more than 90% of adults do not reason proportionally…" (p.637)

Informal Proportional Reasoning

Fischbein (1999) noted that there is no commonly accepted definition for intuitive knowledge or informal reasoning. However, informal reasoning is frequently used in mathematics education

to refer to problem solving strategies demonstrated by children before formal instruction in mathematics. Carpenter (1986) found that children who used informal strategies were fairly successful at solving word problems. His characterization of children's strategies as informal is reminiscent of Vygotksy's (1997) notion of "spontaneous concepts" that children develop before instruction as opposed to the "scientific concepts" characterized by a hierarchy of connections which is the structure they learn during formal instruction.

Many educators share the view that formal instruction in (proportional) reasoning should be based upon informal reasoning in real life situations and this has lead them to lament the lack of this connection in formal schooling, "…too often, we ignore the child's experience with ratio and proportions outside of formal mathematics lessons and teach children algorithms, which utilize techniques that are alien to them…." (Singh, 2000, p.291)

In this study informal reasoning strategies were presented during math instruction, therefore a characterization of informal reasoning based upon processes and elementary schema is more appropriate than one based upon spontaneous or pre instructional thought.

Transition from informal to formal Proportional Reasoning

Intuitive reasoning has been studied within the domain of proportions (Fernandez, Llinares, Modestou, Gagatsis, 2010) in particular during the transition from informal to formal proportional reasoning (Karplus, Pulos, & Stage, 1983). Nahors (2003) relates educational studies of children's schema with rational number concepts to the work of educators who have mapped out the transition from informal to formal proportional reasoning and serves as an excellent framework to define and analyze the intuitive reasoning exemplified in the classroom transcripts.

The example used by Fischbein (1999) to illustrate informal proportional reasoning is, "if one liter of juice costs 5 shekels then how much does 3 liters of juice cost?" (p. 15) Nahors (2003) outlines the steps an individual might use to solve this proportion problem at different levels of conceptual development. These steps include observing the two referents (liters and shekels), the rate or equivalence between them, and the understanding this equivalence is invariant under multiplication. At the initial level an individual begins an additive process of counting or iterating by the given composite referent quantities. In this case 1 liter to 5 shekels, 2 liters to 10 shekels…. Using the schema terminology of Steffe and Olive (1988) Nahors refers to this reasoning as a "coordinate unit-coordinating scheme." (p.137) In a second level of development an individual understands that the new amount of 3 liters is three times the original 1 liter and then multiplies the cost times 3. Nahors refers to the process involved in this approach as "iterable composite units coordinating scheme." (p.138) Nahors considers this a slightly more sophisticated and powerful version of the coordinate unit-coordinating scheme due to its multiplicative nature.

The third level is an intermediate step in proportional reasoning and is often described by educational researchers as the unit rate approach (Karplus, Pulos & Stage, 1983; Nahors, 2003). In a proportion problem, it involves first finding the unit rate between the given referents and then a multiplicative based iteration strategy as described in the iterable composite units coordinating scheme to solve the proportion. This level of concept development is considered by educators to begin formal proportional reasoning.

In the qualitative part of this study, the analysis of classroom transcripts is based upon the work of Nahor. The objective is identify the processes and elementary schemes students use when applying rational number concepts during informal reasoning with ratio, quotient, proportions and percent problems and the difficulties they experience.

The Sub-constructs of Rational Number Sense

The definitions of the fraction sub-constructs are taken as in Charalambous & Pitta-Panzini (2007). The part-whole sub-construct interprets the symbol notation *p/q* to represent the partitioning of a whole entity into q equal shares and then taking *p* out of the *q* shares. The part-whole sub-construct is used as a foundation for developing rational number sense in the mathematics curricula. However the part-whole sub-construct is limited in that it does not readily illustrate the concept of an improper fraction. The measure sub-construct is frequently evaluated through placement of a fraction on the number line. Measure involves an application of the part-whole concept by determining the placement of *p/q* on an interval with a designated unit. The unit is partitioned into *q* equal parts and the resulting sub-unit *1/n* is iterated *p* times.

Through this process the measure sub-construct extends the part-whole concept to include improper fractions. The quotient sub-construct interprets *p/q* as the amount obtained when *p* quantities are divided into *q* equal shares. The quotient sub-constructs supports a dual interpretation of *p/q* as the number of equal shares obtained when a quantity *p* is divided into *q* equal sized shares. The ratio sub-construct interpretation of *p/q* involves a comparison between two quantities *p* and *q* and thus it extends the part-whole interpretation to include part-part.

Operator is synonymous with the process of taking a fraction of some quantity, thus the operator sub-construct interpretation of *p/q* involves multiplication by p and division by *q*. The operator concept is associated with the input-output box in which the output is a fractional amount of the input quantity. The exercises used to evaluate the part-whole and ratio sub-constructs are mostly pictorial, measure is evaluated through the number line, operator through the input-output box and quotient through problem situations often involving sharing a pizza. Exercises used to evaluate the equivalence sub-construct are based primarily upon translation between part-whole pictorial representations i.e. identifying the fraction associated with a picture containing 2 out of 5 objects shaded and then shading the appropriate number of boxes out of 15 objects that corresponds to the equivalent fraction.

Also included are solving missing value problems that can be solved through scalar multiplication i.e. the second level of intuitive reasoning an example would be, find x in the proportion $2/5 = x/20$. The exercises used to evaluate the rational number sub-constructs are included in the appendix and are essentially identical to those of Charalambous & Pitta-Pantazi (2007). The results of factor analysis and reliability tests on the exercise sets used to evaluate these sub-constructs are given in this appendix as well.

Research Questions

Research question 1: To what extent do the Kieren's rational number sub-constructs predict or explain students' competency with formal problem solving based upon proportional reasoning?

Research question 2: How do students use Kieren's rational number concepts when reasoning informally during proportion and percent problem solving? Specifically what

schemes are observed during student use of this reasoning and what difficulties do students experience?

Setting

The quantitative data in this study came from the same source as Baker et al. (2012) and like this article involves student proficiency with fraction concepts. However, unlike the earlier article is also includes data from these students with proportional reasoning. Thus, in both articles the data was collected over several semesters from 334 adult students enrolled in pre-algebra courses taught by six professors of Mathematics at Hostos Community College (HCC) and Bronx Community College (BCC) both urban community colleges in the City University of New York (CUNY) system.

The teaching research experiment ([1]) from which this data was collected was designed on an educational approach in which the rational number sub-constructs served as a basis to develop competency with problem solving involving ratio, rates, proportion and percent. The classroom sessions were focused on problem solving with an emphasis on guidance and encouragement rather than direct instruction. In this sense the common methodology of the instructors could be described as constructivist instruction i.e. based upon discovery learning. Classroom transcripts of several of these professors during this teaching research project are analyzed for student reasoning with the rational number concepts during informal proportional reasoning.

The assessment of the original teaching research project contained a control group (n=34), and experimental group (n=46), using a pre-test and post-test that focused on problem solving with ratio, rates, proportions and percent. The same professors taught sections of each group. There was no significant difference between the mean scores of the pre-test between the groups but the experimental group significantly outperformed the control group on the post-test at the $p < 0.001$ level.

As noted in Baker et al. (2012), "the student body at these community colleges is predominately female (70%-80%) and minority (85%-95%) and is the mathematically weakest group of students applying to the CUNY system. These students have failed both the algebra and pre-algebra placement exams in mathematics and are not eligible to take college level mathematics course until they pass these courses. At these community colleges the pre-algebra course lasts fifteen weeks, it covers real numbers such as decimals and fractions, proportions, percent and an introduction to algebra."

Methodology

The exercises sets for these sub-constructs were adapted from those used by Charalambous & Pitta-Pantazi (2007) and except for problem solving are identical to those used in Baker et al. (2012). Problem solving was evaluated through application problems involving ratio, rates and proportions that were taken from the adult curriculum. Principal factor analysis and reliability tests were conducted on the exercise sets (Cramer, Post & dellMas, 2002) in order to determine the components within each set and the reliability of the set of exercises. All problem sets and the results of these analyses are listed in the appendix.

[1] This study was partially funded by the grant. Problem Solving in Remedial Mathematics: A jump start to reform, CUNY College Collaborative Incentive Research Grant Program (2010), Czarnocha, Prabhu, Dias & Baker.

Results and Discussion

Quantitative Analysis

The quantitative analysis of correlations between variables used in this study is identical to that used in Baker et al. (2012) and is based upon the assumption that the mean scores of two variables are significantly different (T-test) and there is a positive and significant correlation between them. As noted in Baker et al. (2012) in such a situation, "the underlying premise is that students' knowledge of the easier concept will precede and be used to acquire knowledge of the more difficult concept. Thus knowledge of easier concept X will imply knowledge of more difficult Y this will be written as, X $\Rightarrow$Y. Furthermore, the square of the correlation coefficient r^2 indicates the percent variation of Y explained by X. This will be written as X$\Rightarrow$Y, (r^2%)." (p.47) For example, if X$\Rightarrow$Y, (40%) then, given a class of students proficient in X one can expect 40% to be competent with Y. The first research question involves quantitative analysis of student competency with the rational number sub-constructs, fractional equivalence, and problem solving. The means and correlations between these sub-constructs are listed.

Student Performance: Mean and Standard Deviation

A two sided T-test confirms that the mean score of part-whole is significantly easier than the other sub-constructs. In a second tier are equivalence and ratio. The third tier is operator and quotient, then measure and finally problem solving or proportional reasoning.

Table 1 Mean scores and standard deviations on sub-constructs (n=334)

Sub-construct	$\bar{x}$*	SD
1) Part-whole	0.74	0.18
2) Equivalence	0.68	0.28
3) Ratio	0.67	0.24
4) Operator	0.62	0.27
5) Quotient	0.55	0.25
6) Measure	0.49	0.28
7) Problem Solving	0.41	0.29

Correlations between Sub-constructs

The correlations in Table II confirm Lamon's (2007) statement that the rational number sub-constructs are very connected to one another and suggest the Behr et al. (1983) hypothesis that fraction concepts leads to problem solving is valid. In order to determine the extent to which the rational number sub-constructs predict proportional reasoning we employ multiple linear regression with Kieren's rational number sub-constructs and equivalence used as independent variables and students' competency with formal problem solving based on proportional reasoning as the dependent variable.

Baker et al. (2012) worked with an underlying assumption for a linear regression or analysis of variance (ANOVA) model that, "each independent variable correlates significantly

with the dependent variable and there is a significant difference between the mean score of each independent and dependent variable"(p.47). Each of these assumptions has been demonstrated in either Table I or II. As noted in Baker et al. (2012), "In an ANOVA the F-value indicates the strength of the relationship between the independent variables and dependent variable and the p-value determines whether the model is significant. When the p-value of the model indicates it is significant the relevant question becomes what is the interaction between the independent variables as they predict or explain the dependent variable."(p.47) As explained in Baker et al. (2012) the interaction between the independent and dependent variables is quantified by first the significance or p-value of each variable and second the beta value of each independent variable which determines how much of the dependent variable it explains in the given ANOVA model.

Table 2 Correlations between Sub-constructs & Problem Solving

	PW	EQ	Ratio	Op	Qu	Mea	PS
Part-whole -PW	1.00	0.64	0.53	0.40	0.38	0.46	0.35
Equivalence-EQ		1.00	0.53	0.54	0.38	0.41	0.50
Ratio			1.00	0.54	0.46	0.51	0.51
Operator-Op				1.00	0.46	0.37	0.55
Quotient-Qu					1.00	0.47	0.49
Measure-Mea						1.00	0.46
Problem Solving-PS							1.00

Correlations R-values listed are all significant at 0.01 level (2-sided), n=334

Formal Proportional Reasoning

A multiple regression analysis with the rational number sense sub-constructs and equivalence as independent variables to predict formal problem solving was conducted. The results, using 5 independent variables and 334 students, $F(5,334)=44.5$, $p < 0.001$ with adjusted R-square value of 0.44, reveal a highly significant model which explains 44% of the variance in formal proportion problems.

Table 3 Beta and significance values: Problem Solving

Predictor Variable	Beta	p-value
1) ratio	0.13	$p<0.026$
2) operator	0.27	$p<0.001$
3) quotient	0.18	$p=0.001$
4) measure	0.14	$p=0.001$
5)equivalence	0.17	$p=0.002$

In response to the first research question, the foundational factor of part-whole is not significant in predicting problem solving. However, all the other rational number sub-constructs and equivalence are significant. The means in Table II and the beta values and p values in Table III indicate that the more difficult concepts of measure, quotient and operator are more influential in explaining student competency with problem solving than the easier concepts of ratio, equivalence and part whole.

This result, combined with the high correlations exhibited in Table II suggest that the rational number sub-constructs development in a hierarchical manner with the more difficult concepts built upon the foundation of earlier concepts in much the way Vygotsky (1997) would describe scientific concepts and schema theorist like Sfard (1991) would describe structured schema. Furthermore, the more developed an individual's hierarchy of concepts the more competent they will be with problem solving. In this case 44% of student competency with proportional reasoning can be explained by their proficiency with the rational number sub-constructs. This statistical analysis validates educators who stress a connection between concept development and problem solving.

In the qualitative part of this study classroom transcripts are reviewed to understand how the rational number sub-constructs are used when engaged in informal reasoning during problem solving with ratio, rate, quotient, proportion and percent.

Qualitative Data Review of Transcript

A student asks to review the following problem given on an earlier test. Two students participated in the dialogue (GT & SP).

> 6.2.1 A taxi charges \$6.50 for the first quarter mile and \$0.50 for each additional mile. What is the charge for 1 ¾ mile?
> The teacher (T) calls upon a struggling yet determined student who had the problem correct on the test (GT).
>
> GT: I made a line with quarters 0 ¼ ½ ¾ 1 ¼ ½ ¾ 2
> T: (The teacher draws out the number line) |---|---|---|---|---|---|--|--|--|→
> T: What next?
> GT: First is \$6.50, then \$7.00 she counts slowly and are carefully until she reaches 1¾ and proclaims the answer \$9.50.
> T: Very good, did anyone do it differently?
> Silence
> T: Do we have to count out each \$0.50 (Directs this question to the class—a second student SP answers)
> SP: No, we know there are 6 additional quarters so we can multiply 6(\$.50) = \$3.00 and add this to the \$6.50 to get the answer.

This is one problem (other than percent) that students independently constructed and used the number line consistently to solve. GT understood the initial rate 1 quarter mile to \$6.50, distinguished this from the subsequent rate 1 quarter mile to \$0.50 and she successfully represented this rate on the number line. SP also distinguished between the rates 1 quarter mile to \$6.50 and 1 quarter mile to \$0.50 and she also represented these rates on the number line. However, SP iterated the quarter miles 6 times and then coordinated this with the appropriate dollar amount through multiplication. The reasoning of GT is an example of a "coordinate unit coordinating scheme" (Nahors, 2003) while the multiplicative reasoning of SP is a good example of an "iterable composite units coordinating scheme." Example 6.2.1 shows that students can and will return to informal strategies effectively and independently when a problem situation is amenable (Nahors, 2003).

The teacher (T) presents the following quotient problem during a lecture on fractions after multiplication and addition of fractions had been discussed. Students who participated include (JM, YM, GT & JA).

6.2.2 Each package of meat is to contain $\frac{2}{3}$ lb. There are 8 lb. of meat to be made into packages. How many packages will there be?

T: What do we do?
JM: we multiply: $\frac{8}{1}\times\frac{2}{3}$
T: Class, do we all agree? Can anyone explain why?
JM: We divide. (JM changes his mind after the instructor-teacher questions his reasoning)
T: Okay, why? (Asks the entire class to see if someone can supply an answer as to why division—a second student answers)
YM: Because we are taking the $\frac{2}{3}$ lb as a part from the whole.
T: But why is this division? I mean this could indicate subtraction or even perhaps some other operation why division?
YM: Because we are dividing the 8lb. into parts
T: Good, we are partitioning or dividing the 8lb into parts each $\frac{2}{3}$ lb. So how do we divide?
JM: $\frac{2}{3}\div\frac{8}{1}$
T: Okay let's work this out: $\frac{2}{3}\div\frac{8}{1}=\frac{2}{3}\times\frac{1}{8}=\frac{2}{24}=\frac{1}{12}$ so what is the answer? (Asks entire class. It was not clear who called out the following two responses)
Class: The answer is 12
T: Okay but why did we get $\frac{1}{12}$?
Class: It's basically the same answer
T: Well it's not exactly the same and it can be confusing if you forget to reduce the $\frac{2}{24}$ to $\frac{1}{12}$ you would probably get this wrong on a test. Okay, guys we did something wrong can anyone tell me what is wrong?
JM: We should have divided $\frac{8}{1}\div\frac{2}{3}$
T: Yes, we divide and the 8 goes first because it is the 8lb. that is being partitioned or divided up into groups. (Teacher works out the problem)
T: I want to look at this same problem using a number line. Where does the fraction $\frac{2}{3}$ go on the number line, between what two whole numbers?
GT: between 2 and 3.
T: No anyone else?
YM: Between 0 and 1.
T: good (draws a number line) Let's count 1 package is ⅔lb. counting over two more thirds we have (points to 1⅓) 2 packages then counting over two more thirds we have 2 lbs. is 3 packages.

```
0   ⅓  ⅔  1  ⅓       2 lbs.
 |---|---|---|---|---|---|
         1       2        3 packages
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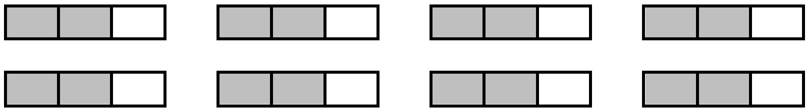
T: Class if 2 lbs. is 3 packages then 8 lbs. requires how many?
YM: 12
GT: Why, I don't see this?
T: Okay, in proportion form: $\frac{2lb}{3pac}=\frac{8lb}{X}$ what is the value of X? How do we find X?
GT: Okay I get it.
T: Class does everyone get this? How do we solve this problem? (Teacher works out the solution to this proportion)
T: Did anyone do this problem differently?

JA: I drew out 8 figures each 1 lb. and took ⅔ lb.

T: Wait let me draw this on the board (draws 8 rectangular pieces representing 1 lb. each and partitioned them into two-thirds for each package)



T : Okay what did you do next?
JA: There are 8 packages of ⅔ lb. each so I take these. Then there are 8 parts of ⅓ remaining. She counts these up in pairs (1,2,3,4) for a total of 12.
GT: I don't get it. What is she counting?
T: After taking the ⅔lb. there is ⅓ lb. remaining (pointing to the un-shaded part) correct?
GT: Yes I see it.
T: We now count in pairs we do this because we need two of these parts to make a ⅔lb. package Correct?
GT: Okay
T: So we count by pairs, how many pairs are there?
GT: Okay now I see it, there are 4 pairs.
T: So we add these 4 to the original 8 to get 12.

The student JM was guessing which operation and when he understood it was division he then had trouble setting up the division correctly, confusing the dividend with the divisor. This is an example of non-conservation of operation. After the 2/3 lb. to 1 package rate was placed on the number line, the students followed the additive iteration of this equivalence as the teacher counted up to, 2 lbs. to 3 packages. Then after understanding that 2/3 lbs per package is equivalent to 2 lbs per 3 packages the students followed the informal scalar multiplication to arrive at the solution.

In this example the teacher used the number line or measure concept to coordinate the referents in much the same way as GT and SP had in the previous example 6.2.1. As indicated by the incorrect response of the student GT as to where the fraction 2/3 goes on the number line and the relatively low mean score of measure (0.49) in Table I the use of a number line-measure concept is difficult for students when a fractional referent in this case 2/3 lb. is involved. Thus, the measure concept, while very useful for students during informal reasoning can be an area of difficultly when fractions are involved. This provides a partial answer to research question 2.

The quotative method used independently by the student JA employs the part-whole concept to take 2/3 of each 1 lb. iterating the result to 8 lbs. while coordinating this process with the remainder 1/3 lb. (un-shaded areas) and the referent number of packages. Thus it represents a slightly more sophisticated version of the "coordinate unit coordinating scheme." (Nahors, 2003) In this way it directly addresses the issue of non-conservation of division which is clearly a big problem for students as JM's response indicated.

The following example was presented during an introduction to percent lecture. The previous example was how to take a percent of a given number. It had been discussed using both informal iteration on the number line and formal reasoning with proportion. Typically after exposure to formal techniques students prefer to translate into an equation using the structural identification of the phase '% of' as multiplication or they set up a proportion using the phrase '% of' to identify the base-whole, and the phrase "is" to identify the amount-part. The teacher uses the number line to clarify the amount and the base and expected the class to use a proportion to solve this problem. Students involved in discussion are (AH, EZ & JA).

6.2.3 30% of some number is 900 find the number?
T: Any ideas?
AH: We can put this on the number line like the previous one.
T: Okay, draws the following line (The line was decomposed into 10% as familiar reference points, however these were not labeled)

```
 0%           30%     50%                  100%
  |------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
 0
```

T: Where does the 900 go? Is with the 100% or with the 30%?
EZ: It goes with the 30%
T:Why?
EZ: Because, we have 30% of some number is 900, so 30% is 900 and X goes with the 100%.
T: Okay, (places 900 on the number line as shown)

```
 0%           30%     50%                  100%
  |------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
 0            900                            X
```

AH: Why does the 900 go with the 30%?
T: 30% of some number is 900. Thus, we take a 30% of an unknown number, this means we take a part of this number and get 900. The number is bigger than the 900 we take 30% of a bigger number that we don't know. When we don't know a number we label it X so this unknown X is 100%. Do you understand?
AH: I think so.
T: How do we find 100% which is the X? (The expectation was that students would suggest a proportion as in the problem done before this one)
JA: We can find 10% which is 300.
(Teacher was surprised although in previous examples the number line had been used to give meaning to setting up a percent proportion and the informal iteration strategy had been used the decomposition with percent problems had not been shown in class).
T: (Marks the equivalence 10% is 300 on the appropriate scale point) Does everyone see how she got this? (The class nods agreement, it is clear to them)

```
0%   10%     30%     50%                   100%
  |------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
 0    300     900                            X
```

T: Okay, what should we do next?
JA: 100% is 3000. (The speed in which she answers indicates the use of scalar multiplication of 10 on the equivalence of 10% is 300

The student EZ successfully identified the amount and the base with the assistance of the number line and yet could not immediately set up a proportion when the teacher asked. Perhaps this is because the base was unknown. Despite having been exposed to formal proportions JA continue to effectively use decomposition into an appropriate rate (10% is 300) represent this on the number line and then used scalar multiplication to solve this percent problem. The sequence of first decomposition of 30%—900 to the equivalent 10%—300 followed by scalar multiplication by JA contains the essence of the unit rate approach and thus documents a student transitioning between informal and formal reasoning independent of the teacher's guidance.

In general, representation of percent through measure is easier for students than for fraction and the instructors frequently observed students independently using the measure sub-construct as for reasoning proportionally when solving percent problems during this project.

The following example was done near the end of a percent lecture after students had been exposed to formal proportion and percent equations. It mixes the fraction operator concept with percent. The student involved in the class dialogue was (RG, JM & EZ).

6.2.4 Find 60% of $\frac{2}{3}$ of 600.

T: How should we do this? (Silence) Okay how about we find $\frac{2}{3}$ of 600 first.(Silence)

T: If you had to describe the fraction $\frac{2}{3}$ how would you do this? (Silence) Suppose you wanted to explain $\frac{2}{3}$ to a child who did not know what it meant how would you do this?

RG: I would draw it. (Teacher draws a circle.)

RG: Then make a peace sign.

T: (After drawing the circle divides it into 3 parts with a peace sign).
What would you do next? How do you represent the 2?

RG: Take 2 of these parts.

(Teacher shades in two of the three parts.)

T: How would you relate this picture of $\frac{2}{3}$ to the 600?

RG: The 600 is the total.

T: So how much is the $\frac{2}{3}$?

RG: It is 400.

T: Class how did he get the 400?

JM: Each part is 200 so two of them are 400.

T: Good, does everyone see this? Can anyone tell me how to do this mathematically or formally in a faster way without pictures?

JM: $\frac{2}{3} \div 600$

T: $\frac{2}{3} \div \frac{600}{1} = \frac{2}{3} \times \frac{1}{600} = \frac{1}{900}$ is this correct?

JM: No

T: When taking a fraction of something the word 'of' indicates multiplication.

(Writes out $\frac{2}{3} \times 600/1$ and works it out to obtain 400)

T: What is the next step?

JM: We find 60% of 400

T: How do we do this?

JM: We set up a proportion $\frac{60}{100} = \frac{n}{400}$

T: good. (Solves the proportion) Did anyone else do it differently? What operation is indicated by the word "of"?

EZ: Multiplication

T: Good (Writes out 60% × 400 => 0.60 × 400 and solves)

The responses of JM are an example of non conservation of operation. JM had forgotten how to take a fraction of a quantity and even after the teacher demonstrated an informal solution to the problem he appeared to be guessing which operation corresponded to this informal reasoning process. The teacher employed the part-whole sub-construct to represent 2/3 as a visual picture. The students RG & JM were able to relate the 600 to the total and decompose this to find the unit rate or equivalence of 200 to a 1/3 part before taking twice this as the answer. These processes make up an elementary schema that corresponds to the operator concept. The operator concept is closely related to adult students' part-whole knowledge and proficiency with multiplication, Baker et al. (2012).

In answer to research question 2, the reasoning schemes used effectively in these examples were basically of two types, one that corresponds to the operator concept supported by part-whole and the other involving iteration and decomposition into equivalent rates supported by the measure concept. The informal strategy based upon the elementary schemes of iteration and decomposition were those presented by Nahors (2003). The students (GT & SP) used such iterative reasoning supported by measure in 6.2.1 to solve the taxi problem. The teacher also used iterative reasoning supported by the number line during the quotient problem (6.2.2) when 2/3 lb. per package was iterated to 2 lb. per 3 packages.

The elementary schema associated with the operator concept of taking a fraction or percent of a quantity involves the processes of identify and distinguishing between the base-whole and partial-amount. The visual for this schema was represented by a circle-fraction or number line—percent and the student related problem information to the corresponding visual concept. This elementary schema supported by part-whole pictures was used by the teacher to help students understand the process of taking 2/3 of 600 (6.2.4). Example 6.2.3 demonstrates the coordination of these schemes. First, EZ used the operator/amount-base schema to represent 30% to 900 on the number line. Then JA applied decomposition to an equivalent rate of 10% to 300 and finally she iterated to find the solution.

These examples demonstrate how concepts in visual form stimulate student engagement in problem solving through the process of relating problem information to a relevant picture whether a circle, rectangular bar or number line. These concepts also help shape and formulate the reasoning process. For example, RG and JM intuitively know to divide the 600 by three and double the result after relating it the whole circle in 6.2.4. In like manner JA intuitively decomposed the 30% to 900 to its equivalent 10% to 300 and iterated to solve after EZ had correctly represented it on the number line.

Conclusion

In the first part of this research study it was demonstrated that the rational number sub-constructs explain about 44% of a students' problem solving based upon proportional reasoning. This validates the Behr et al. (1983) extension of the Kieren sub-constructs to competency with problem solving and verifies Lamon's (2007) assertion that the rational number sub-constructs provide a foundation for proportional reasoning. The generalization of this result connects two areas of mathematical research, concept development and problem solving and suggests the interaction should be of significant interest to mathematics education.

In the second part of this study transcript of classroom lectures in which the rational number sub-constructs serve as a foundation for informal reasoning are analyzed for evidence of how these concepts were used. An analysis of student reasoning shows the concepts of part-whole-circles and number line-measure when represented in visual form acted as a catalyst for students reasoning. That is students became engaged in the reasoning process when they related problem information to these pictures. These picture-concepts also supported informal reasoning as students formulated strategies and applied processes based upon how problem information was represented.

Students used an elementary operator schema that involved identifying the partial-amount and base-whole with problem information and representing this on a diagram. They also used these picture-concepts especially measure to support the processes of iteration and decomposition. These results support educational research on the benefits of concept-pictures or visuals during problem solving (Goodson-Epsy, 1998; Steele & Johanning, 2004; Caddle & Brizuela, 2011).

The application of the processes of iteration and decomposition to rates on the number line-measure concept supported student transition to formal proportional reasoning. However, the measure concept was difficult for many students to grasp especially when a fractional quantity was involved. One exception was the use of whole number percent. The adult students independently applied their part-whole knowledge to represent given percent amount and base information on the number line and then applied the processes of iteration and decomposition.

Most adult textbooks and curriculum are based upon the same sequence of topics as that presented for children. Thus, percent is introduced after proportion and rates which are introduced after decimals and fractions. An effort to arrange the topics for adults according to the informal reasoning observed in this study would suggest the integration of whole number percent earlier in the adult curriculum with fraction, rate and proportion.

Reflection upon Learning Theories

In mathematics education there are separate branches for and corresponding models of concept development and problem solving. In the APOS (action-process-object-schema) model, concept development begins with an individual's actions upon existing concepts and then with reflection upon these actions they become internalized processes that eventually lead to new concepts-objects and ultimately schema (Czarnocha, Dubinsky, Prabu & Viadokovic, 1999). On the other hand, the development of a problem solving schema for Cifareli (1998) focuses on recognition of strategies from problems with similar structure and reflection upon the processes involved.

The informal reasoning processes described in this study are dictated by the problem structure and reflection upon such structure is an essential component in the development of problem solving (Cifareli, 1998). On the other hand, these processes can be viewed as actions upon conceptual knowledge of part-whole and measure and in the APOS model reflection upon such processes leads to concept development. This suggests that such reflection is a point of commonality between these models of learning. Although an analysis of how these models are connected is beyond the scope of this study, cognitive theories that situate learning, including concept development within the framework of problem solving remind us that such a link exists.

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Appendix

Principal factor analysis and reliability tests were conducted on the exercise sets (Cramer, Post & dellMas, 2002). The Kaiser-Meyer-Olkin measure of sampling adequacy was 0.6 or more for all sets of exercises, thus indicates the factor analysis was appropriate. The exercises sets are listed by the factor-component they fall in. The Cronbach's alpha value of sampling reliability was 0.6 or more for all sets of exercises except quotient thus, all exercises sets are considered reliable except quotient.

Ratio

Component 1
1) In a History class there are 2 male to every 3 female students, use fraction notation to write the ratio of male to female students in the class.(*)
2) In a History class there are 2 male to every 3 female students, use fraction notation to write the ratio of female to male students in the History class. (*)
3) Use fraction notation to write the ratio of female to total students in the History class. (*)

Component 2
4) Write the ratio 4 to 36 in simplest terms. (*)(†)
5) Write the ratio 48 to 16 in simplest terms. (*)(†)

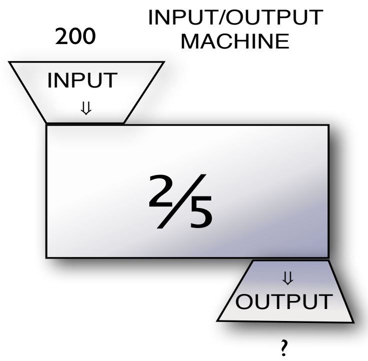
Component 3
Juan and María are making lemonade. Given the following recipes whose lemonade is going to be sweeter?
6) Juan uses 2 spoons of sugar for every 5 glasses of lemonade
María uses 1 spoon of sugar for every 7 glasses of lemonade (*)
7) Juan uses 2 spoons of sugar for every 5 glasses of lemonade
María uses 4 spoon of sugar for every 8 glasses of lemonade (*)

Component 4
Jose jogs each morning before work. Determine which of the following days he jogged at a faster rate. Please answer a) Monday b) Tuesday or c) Not able to determine from information given.
8) On Tuesday he jogged a longer distance than he did on Monday. On both days he jogged exactly the same amount of time. (*)
9) On Tuesday he jogged a shorter distance than he did on Monday. On both days he jogged exactly the same amount of time. (*)
The Kaiser-Meyer-Olkin measure of sampling adequacy for these 9 questions was 0.64.
These 4 components explained 76% of the variation of the exercise set.
The Cronbach's alpha value for this exercise set was 0.67 and thus it is reliable.
(*) commonality with other exercises in this set is more than 0.5
(†) Added by present authors and not found in Charalambous and Pitta-Pantazi (2007).

Operator

Component 1
There were two exercises that evaluated the operator concept through functional input-output boxes.

1) The following diagram represents a machine that outputs $\frac{2}{5}$ of the input number. If the input number is 200 then what is the output number? (*)



2) An input-output machine has outputs that is $\frac{1}{5}$ of the input. If the input number is 480 then that is the output number? (*)
3) Find half of $1\frac{1}{2}$ hours (*)
4) Find $\frac{4}{5}$ of $\frac{7}{8}$ of 40,000 (†)
5) Find $\frac{3}{5}$ of $\frac{5}{8}$ of 4000 (†)

Component 2
6) Taking $\frac{2}{5}$ of a number is the same as dividing the number by 5 and multiplying this result by 2, True/False? (*)
7) If we divide a number by six and multiply by twenty-four this the same as multiplying by the fraction $\frac{1}{4}$ True/False? (*)

Component 3
8) A recipe calls for 1 ½ cup of flour. Which of the following expresses the amount of flour required for $\frac{1}{3}$ of this recipe? (*)(†)
a) $\frac{3}{2} \div \frac{1}{3}$ b) $\frac{1}{2} \div \frac{1}{3}$ c) $\frac{3}{2} \times \frac{1}{3}$ d) $1\frac{1}{2} - \frac{1}{3}$ e) not given

9) Find $\frac{3}{4}$ of $\frac{1}{5}$ (*)
The Kaiser-Meyer-Okin measure of sampling adequacy was 0.69 with 58% of the variation explained by these three components.
The Chronbach's alpha was 0.69 and thus operator is a reliable set of exercises.
(*) Commonality was at least 0.5
(†) Added by present authors and not found in Charalambous and Pitta-Pantazi (2007).

Measure
Component 1
Locate the following numbers on this number line:
#1) $\frac{1}{6}$ #2) $\frac{4}{3}$ #3) $\frac{5}{6}$

0 $\frac{1}{3}$ 1
←-----|----|----|----|----|----|---------------→

Component 2
#4) Locate the number one "1" on the number line below:

0 $\frac{4}{9}$
←---|------------|----------------------------→

#5) Locate the number one "1" on the number line below:

0 $2\frac{1}{4}$

←---|---------------------------|-------------→

The Kaiser-Meyer-Okin measure of sampling adequacy was 0.63 with 72% of the variation explained by these two components.
Chronbach's alpha 0.69 thus measure is a reliable set of exercises.
Commonality was at least 0.5 or above for all exercises.
All exercises were used in Charalambous and Pitta-Pantazi (2007).

Quotient
Component 1
1) Three pizzas are shared equally among four students what fraction of a pizza will each receive? (*)
2) It takes ¾ kg of apples to make one pie. How many pies can be made using 20 kg? (*)

Component 2
3) A beach $\frac{3}{7}$ miles long is divided into 6 equal parts. How long is each part? (*)
4) Two pizzas were shared equally among a group of students. If each student received $\frac{2}{5}$ of a pizza then how many students were there?
5) If 3 pizzas are shared evenly among seven girls while 1 pizza is shared evenly among three boys. Who gets more pizza, a girl or boy? (*)
(*) Commonality was 0.5 or above
All exercises were used in Charalambous and Pitta-Pantazi (2007).
The Kaiser-Meyer-Okin measure of sampling adequacy was 0.621 and 52% of the variation was explained by these two components.
The Chronbach's alpha was 0.49 thus quotient was not a reliable set of exercises.

Equivalence
Component 1
1) A 2x3 rectangular array of equal squares is given with 1 square shaded. Next to this are 24 un-shaded identical objects the student is asked to shade in the appropriate number of objects to represent an equivalent fraction.
2) Four identical objects are presented with 1 circled. Next to this is a 2x8 rectangular array of equal squares the student is asked to shade in the appropriate number of objects to represent an equivalent fraction.
3) A 2x3 rectangular array of equal squares is given with 4 squares shaded. Next to this are 24 un-shaded identical objects the student is asked to shade in the appropriate number of objects to represent an equivalent fraction.
4) Four identical objects are given with 3 circled. Next to this is a 4x4 rectangular array of equal squares the student is asked to shade in the appropriate number of objects to represent an equivalent fraction.
5) 16 un-shaded identical objects are presented. Next to this is a 2x2 rectangular array of equal squares the student is asked to shade in the appropriate number of objects to represent an equivalent fraction.
6) A 2x16 rectangular array of equal squares is given with 4 squares shaded. Next to this are 6 un-shaded identical objects the student is asked to shade in the appropriate number of objects to represent an equivalent fraction

Component 2
7) $\frac{2}{3} = \frac{?}{12}$ 8) $\frac{25}{40} = \frac{5}{?}$ 9) $\frac{7}{9} = \frac{42}{?}$

All exercises were used in Charalambous and Pitta-Pantazi (2007).

The Commonality was 0.5 or above for all exercises.
The Kaiser-Meyer-Okin measure of sampling adequacy was 0.86 and 67% of the variation was explained by these two components.
The Chronbach's alpha was 0.85 thus equivalence was a reliable set of exercises.

Part-Whole
Component 1
1a) Given a picture of four triangles and five circles; the question is what fraction of the objects are triangles? (*)
1b) Given two triangles; what fraction of the total triangles do these represent? (*)

Component 2
2) Given a picture of a circle with 2 out of 5 equal parts shaded; the question is what fraction of the circle is shaded? (*)
3) Given figure composed of seven squares, three of which are shaded; the question is what fraction of the squares are shaded? (*)
4) Given five equivalent objects three of which are circled; the question is what fraction of the objects are circled? (*)

Component 3
5) Given a rectangle array composed of six equal squares one of which is shaded; the question is what fraction of the squares are shaded? (*)
6) Given four identical objects one of which are circled; the question is what fraction of the objects are circled? (*)
7) Given a figure composed four equivalent objects three of which are circled: the question is what fraction of the objects is circled? (*)

Component 4
8) The fraction $\frac{2}{3}$ corresponds to taking a chocolate bar, dividing it into three equal parts and taking two of these parts. T/F? (*)
9) The fraction $\frac{2}{3}$ corresponds to taking a set of objects dividing it into three equal parts and taking two of them. T/F? (*)

Component 5
10) Given a 3x2 rectangle array composed of six equal rectangles four of which is shaded; the question is, does the shaded region corresponds to the fraction $\frac{2}{3}$? (*)
11) Does the shaded part of this rectangle correspond to the fraction ⅔? (*)

12) Given a 2x6 array of circles 8 of which are shaded; the question is does the shaded objects represent the fraction ⅔? (*)
13) Given a 1x5 rectangular array of equal squares with 2 shaded; the question is does the shaded part of the rectangle correspond to the faction ⅔?
All exercises were used in Charalambous and Pitta-Pantazi (2007).
(*) The commonality value was at least 0.5
The Kaiser-Meyer-Okin measure of sampling adequacy was 0.72 and 60% of the variation was explained by these three components.
The Chronbach's alpha value was 0.7 thus part-whole is a very reliable set of exercises.

Formal Proportional Reasoning
Component 1
1) Hank drove 500 miles in $8\frac{1}{3}$ miles, what was his average speed or rate in miles per hour? (*)

2) If ¾ cup of coleslaw contains 120 calories. How many calories are there in $\frac{2}{5}$ cup? (*)
3) If the ratio of a:b is 2:3 and b is 4200 then find the value of a. (*)
4) If $\frac{2}{5}$ of 4000 is equal to $\frac{1}{4}$ of some number then find the number. (*)
5) If 0.5 ml of medicine are mixed with 2 ml of water to form a solution then what is the ratio of drug to water in simplest terms?(*)
Component 2
6) Which of the following fractions is closest to 1? (*)
a) $\frac{2}{3}$ b) $\frac{3}{4}$ c) $\frac{4}{5}$ d) $\frac{5}{6}$
7) Circle the smallest fraction: (*)
a) $\frac{2}{11}$ b) $\frac{3}{13}$ c) $\frac{4}{23}$ d) $\frac{5}{6}$
All exercises were taken from end of year departmental exit exam.
(*) The commonality value was at least 0.5
The Kaiser-Meyer-Okin measure of sampling adequacy was 0.75 and 51.3% of the variation was explained by these two components.
The Chronbach's alpha value was 0.70 thus the formal proportional reasoning exercises formed a very reliable set of exercises.