University of Lisbon - Instituto Superior Técnico

Masters in Data Science and Engineering

Computational Methods in Finance

Submitted by: Ricardo Fraga Simões, nº 93674

To: Prof. Dr. João Janela

Lisbon, 10/07/2022

General Comments

This report presents the results obtained for the Computational Methods in Finance 2nd project. Discussion about the numerical results was included, and in some cases, also the linkage with the mathematical concepts that were used. Relevant images were also included to facilitate the results' interpretation.

Alongside this report, a Python notebook was also delivered with comments that help the reader to better understand the code. The notebooks were made using Google Collab, and the reader can obtain **exactly all** the same results here presented with these notebooks. It is advised to import these notebooks into Google Collab, since they were developed there.

All results were calculated on a computer with an Intel Core i3-7100U CPU, 2.4 GHz with 4096Mb of RAM.

European Vs. American Options

European and American Options are two very similar financial products acquired by traders and investors, for several purposes. They are characterized as a contract between two identities, where they agree to buy or sell an asset at a predetermined (strike) price in the future. These can be a call or put option, depending on the value the purchaser forecasts; call options allow the holder to buy the asset (bullish sentiment), while put options permit to sell (bearish sentiment).

Nevertheless, these two products have a particular difference. While the European options can only be exercised on the expiration date, the American options can be exercised at any time before the expiration date. Naturally, since they can be exercised when the price moves in favor of the trader, this comes with a premium that they have to pay. So to make a profit with American options, traders and investors need the asset to move far enough from the strike price.

There are other differences. For example, while most stocks and ETFs have American options, equity Indices have European options. Other differences reside in the settlement price and the day the option stops trading, but these are less relevant. Appropriate literature will be referenced regarding these differences, but the most important one is the first one mentioned above.

Analytical Solution for European Options

The solution to the famous Black-Scholes (BS) equation gives us the formula to calculate the value of a call or put option. However, deriving the solution is not a trivial task. For the sake of simplicity (and since the proof is well-known and can be found in different places) we won't present the deduction, but we will leave useful Literature in case the reader wants to see the proof. We now present BS equation and the boundary conditions for a call and put option, and the corresponding analytical solution in the case of European options. The BS equation is

$$\frac{\partial V(S,t)}{\partial t} + rS\frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} - rV(S,t) = 0,$$

where V is the price of the option, S is the stock price, t is the time, r is the risk-free interest rate and σ is the volatility of the stock.

Now, let $d_1 = \frac{\log\left(\frac{S}{K}\right)+\mathrm{T}\left(\mathrm{r}+\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$, $d_2 = \frac{\log\left(\frac{S}{K}\right)+\mathrm{T}\left(\mathrm{r}-\frac{\sigma^2}{2}\right)}{\sigma\sqrt{T}}$, and $N(x)$ the cumulative normal distribution with 0-mean and unit variance.

For a call Option:

$$B.C.: \begin{cases} V(S=0,t)=0 \\ V(S,t)=S-Ke^{-r(T-t)}, \quad if\ S\to\infty \\ V(S,t=T)=\max(S-K,0) \end{cases}$$

$$Analytical\ Solution:$$
$$V(S,t)=S*N(\mathrm{d}_1)-K*e^{-r(T-t)}*N(\mathrm{d}_2)$$

For a put Option:

$$B.C.: \begin{cases} V(S=0,t)=Ke^{-r(T-t)} \\ V(S,t)=0, \quad if\ S\to\infty \\ V(S,t=T)=\max(K-S,0) \end{cases}$$

$$Analytical\ Solution:$$
$$V(S,t)=K*e^{-r(T-t)}*N(-\mathrm{d}_2)-S*N(-d_1)$$

The boundary conditions are also defined as function of the strike price K and of the expiration time T.

In Table 1, we define the set of parameters for the call and put options that we will use in our simulations.

Table 1 – Parameters of the Option

Strike Price, $\boldsymbol{K}$	Risk-Free rate, $\boldsymbol{r}$	Volatility, $\boldsymbol{\sigma}$	Expiration Time, $\boldsymbol{T}$
100	5%	20%	1

For instance, plugging in the known parameters, we can obtain the value of the option at t = 0 (see table 2) for different values of $\boldsymbol{S}$. Let's consider $S\ \epsilon\{\mathbf{80},\mathbf{90},\ldots,\mathbf{130}\}$.

Table 2 – Values of the call option for different S values

Call	
Price, S	**Value, $\boldsymbol{V(S,0)}$**
80	1.85942
90	5.09122
100	10.45058
110	17.66295
120	26.16904
130	35.44027

Put	
Price, S	**Value, $\boldsymbol{V(S,0)}$**
80	16.98236
90	10.21416
100	5.57353
110	2.7859
120	1.29199
130	0.56321

Forward Euler Explicit Scheme for the BS equation

We will proceed to approximate the BS equation by finite differences, using the Forward Euler explicit scheme:

$$\frac{V_{i,j}-V_{i-1,j}}{\Delta t}+r(j\Delta s)\frac{V_{i,j+1}-V_{i,j-1}}{2\Delta s}+\frac{1}{2}\sigma^2(j\Delta s)^2\frac{V_{i,j+1}-2V_{i,j}+V_{i,j-1}}{\Delta s^2}-rV_{i,j}=0$$

Now we can re-write the equation as,

$$V_{i-1,j} = \frac{1}{2}\Delta t(\sigma^2 j^2 - rj)V_{i,j-1} + [1 - \Delta t(\sigma^2 j^2 + r)]V_{i,J} + \left[\frac{\Delta t}{2}(\sigma^2 j^2 + rj)\right]V_{i,j+1}$$

Starting at $T = 1$, we can solve a system of equations backward (for the previous time-step), which will allow us to calculate the value V at all time steps, but in particular for $t = 0$. As exemplified in Fig. 1, since we know the value at all the points identified with a green circle, using the previous point it is easy to find the value at all the interior points of the grid.

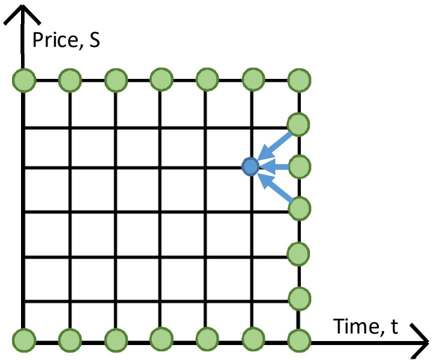

Figure 1 - Explicit Scheme representation

We decided to price the European call option with an accuracy of three decimal places at t = 0 and at the spot price S = 100. The corresponding analytical solution is equal to 10.45058. The results seen in Table 3 show that we achieve that by using as grid refinement $\Delta t = 0.00005$ and $\Delta s = 0.25$. Therefore, we will use the previous refinement $\Delta t = 0.00001$ and $\Delta s = 0.5$ in the next simulations, since it is much more time efficient in terms of CPU time.

Table 3 – European Call option with pricing accuracy of three decimal places – Explicit scheme. Values at t = 0 and at the spot price S = 100.

Δt	Δs	Value	CPU Time
0.00001	1	10.44812	2min 14s
0.00001	0.5	10.44998	3min 55s
0.000005	0.25	10.45043	15min 42s

In Table 4, we present the results for American and European call and put options, at t = 0 and at the spot price S = 100, using $\Delta t = 0.00001$ and $\Delta s = 0.5$. For the European ones, we will also present the corresponding analytical solution.

Table 4 - Explicit Scheme results for European and American call and put options. Values at t = 0 and at the spot price S = 100.

Price	Analytical Solution	Numerical Solution	CPU Time
European Call	10.45058	10.44998	3 min 55s
European Put	5.57353	5.57291	3 min 54s
American Call	*NA*	10.18992	*45min 54s*
American Put	*NA*	5.79045	*45min 23s*

The accuracies are almost perfect for the explicit scheme with this refinement. The figures below (Figs. 2-5) present 2D and 3D plots of the numerical solutions for European and American Call and Put options.

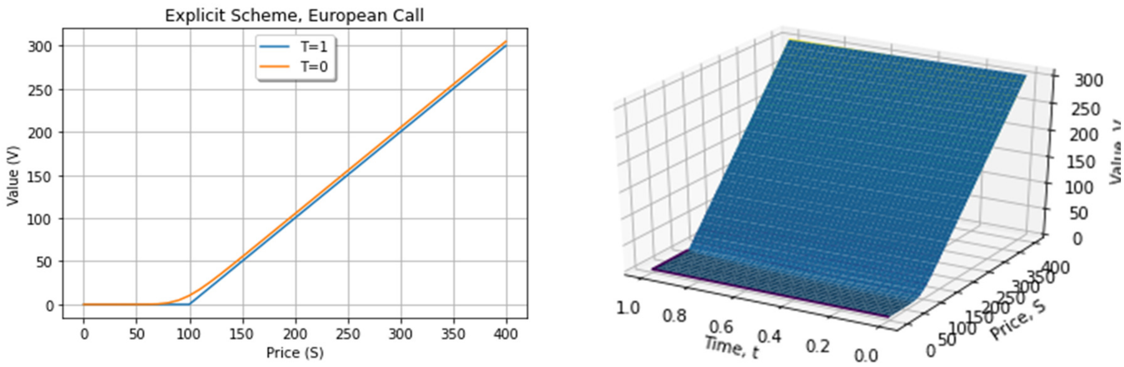

Figure 2 – Explicit Scheme results for European Calls

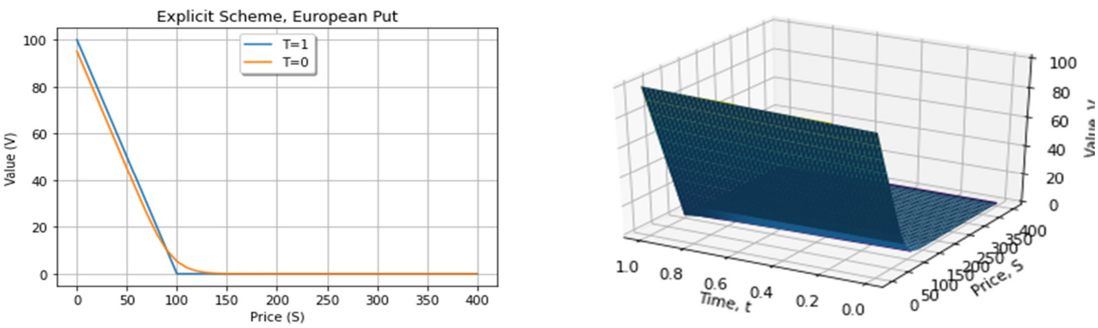

Figure 3 - Explicit Scheme results for European Put

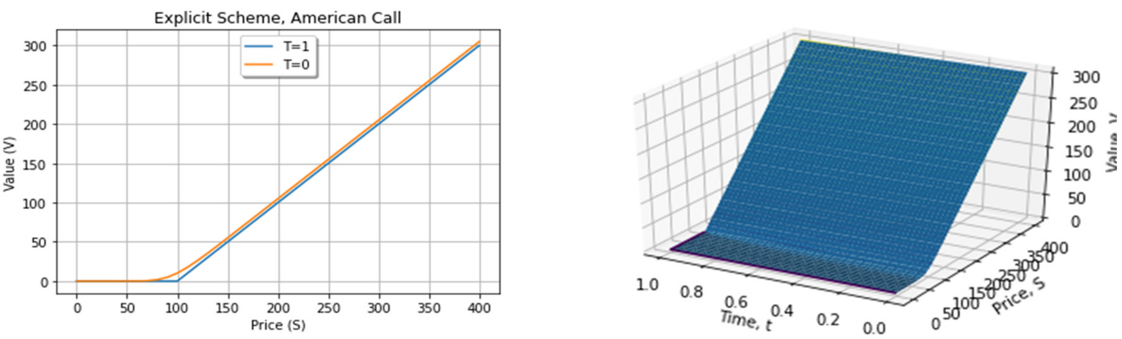

Figure 4 - Explicit Scheme results for American Calls

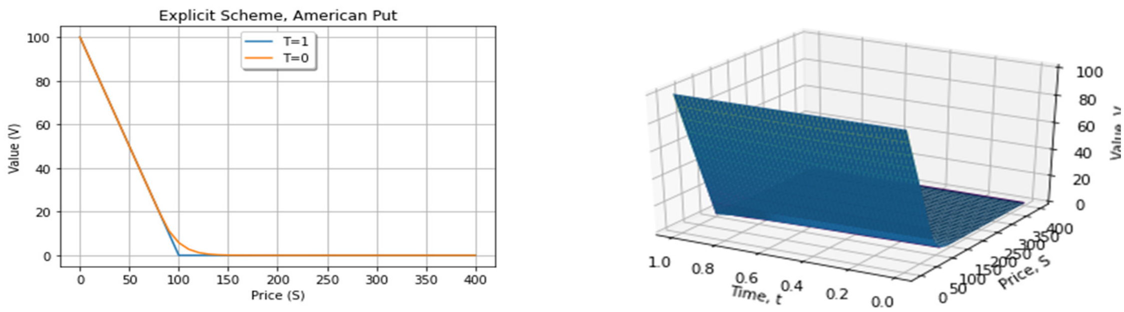

Figure 5 - Explicit Scheme results for American Puts

Crank Nicolson Scheme for BS equation

In case we use the Crank Nicolson method to solve the BS equation, then the corresponding implicit finite difference scheme is,

$$\left[-\frac{\Delta t}{4}(\sigma^2 j^2 - rj)\right]V_{i-1,j-1} + \left[1 - \left(-\frac{\Delta t}{2}(\sigma^2 j^2 + r)\right)\right]V_{i-1,j} - \left[\frac{\Delta t}{4}(\sigma^2 j^2 + rj)\right]V_{i-1,j+1} =$$

$$= \left[\frac{\Delta t}{4}(\sigma^2 j^2 - rj)\right]V_{i,j-1} + \left[1 + \left(-\frac{\Delta t}{2}(\sigma^2 j^2 + r)\right)\right]V_{i,j+} \left[\frac{\Delta t}{4}(\sigma^2 j^2 + rj)\right]V_{i,j+1} \Leftrightarrow$$

$$\Leftrightarrow a_j V_{i-1,j-1} + (1 - b_j)V_{i-1,j} - c_j V_{i-1,j+1} = a_j V_{i,j-1} + (1 + b_j)V_{i,j} + c_j V_{i,j+1}$$

This will lead to a linear system of equations,

$$AV^{i-1} = BV^i$$

where,

$$A = \begin{bmatrix} 1 - b_1 & -c_1 & 0 & \cdots & 0 \\ -a_2 & 1 - b_2 & -c_2 & \dots & 0 \\ 0 & -a_3 & 1 - b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -c_{n-2} \\ 0 & 0 & \cdots & -a_{n-1} & 1 - b_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 1 + b_1 & c_1 & 0 & \cdots & 0 \\ a_2 & 1 + b_2 & c_2 & \dots & 0 \\ 0 & a_3 & 1 + b_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & c_{n-2} \\ 0 & 0 & \cdots & a_{n-1} & 1 + b_{n-1} \end{bmatrix}$$

and V^{i-1} and $V^i$are, respectively, the vectors with the values of the options' values at the current time-step we are determining and at the previous time-step already determined. The rationale is the same as in the previous explicit scheme but now we have two additional terms – the right and left values from the current time-step, marked as yellow (see the image in Fig. 6 which illustrates this fact).

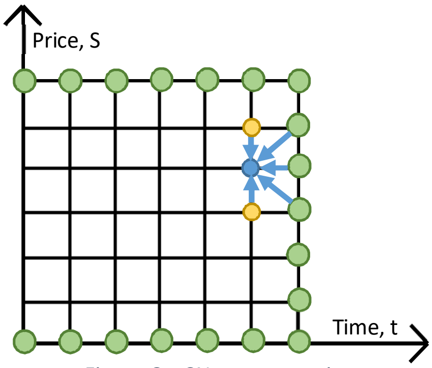

Figure 6 – CN representation

We decided again to price the European call option with an accuracy of three decimal places. The results are compared with the analytical solution we presented above (10.45058) at t = 0 and at the spot price S = 100.

The results seen in Table 5 show that we achieve that accuracy by using as grid refinement $\Delta t = 0.01$ and $\Delta s = 0.25$. Therefore, we will use the previous refinement $\Delta t = 0.01$ and $\Delta s = 0.5$ in the next simulations, since it is much more time efficient in terms of CPU time.

Table 5 – European Call option with pricing accuracy of three decimal places – Implicit CN scheme. Values at t = 0 and at the spot price S = 100.

Δt	Δs	Value	CPU Time
0.01	1	10.44814	1.25s
0.01	0.5	10.44999	4.63s
0.01	0.25	10.44958	24.7s

We will now present the results for American and European call and put options. For the European ones, we will also present the corresponding analytical solution.

Table 6 – CN scheme results for European and American call and put options. Values at t = 0 and at the spot price S = 100.

Price	Analytical Solution	Numerical Solution	CPU Time
European Call	10.45058	10.44999	4.63s
European Put	5.57353	5.57293	4.79s
American Call	*NA*	10.44999	*5.87s*
American Put	*NA*	6.08424	*5.73s*

In general, the numerical solutions obtained with the CN scheme are very good because they are very close to the analytical solution and they are obtained with a time refinement coarser than the one used in the explicit scheme. Consequently, in terms of CPU time, it seems that the CN scheme is preferable because it is faster since it allows to use larger time increments. The equivalent slice plot and global 3D solution plot for the CN scheme can be found below, in Figs. 7-10.

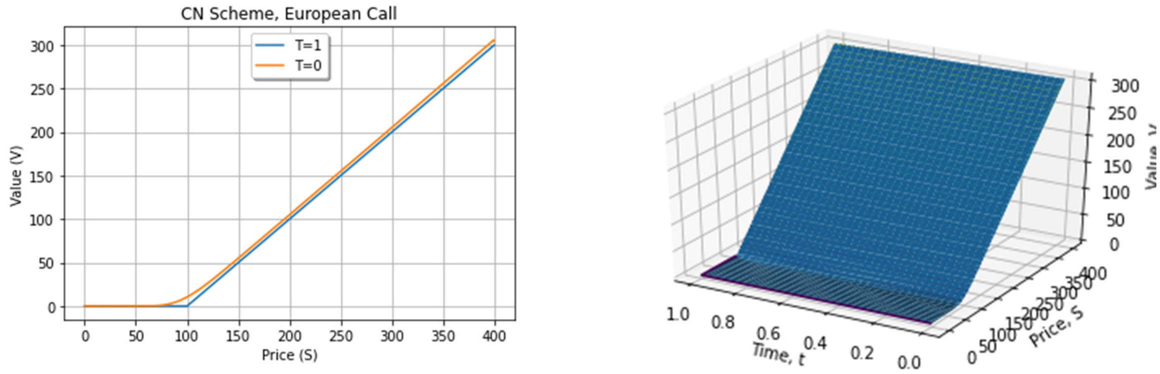

Figure 7 - CN Scheme results for European calls

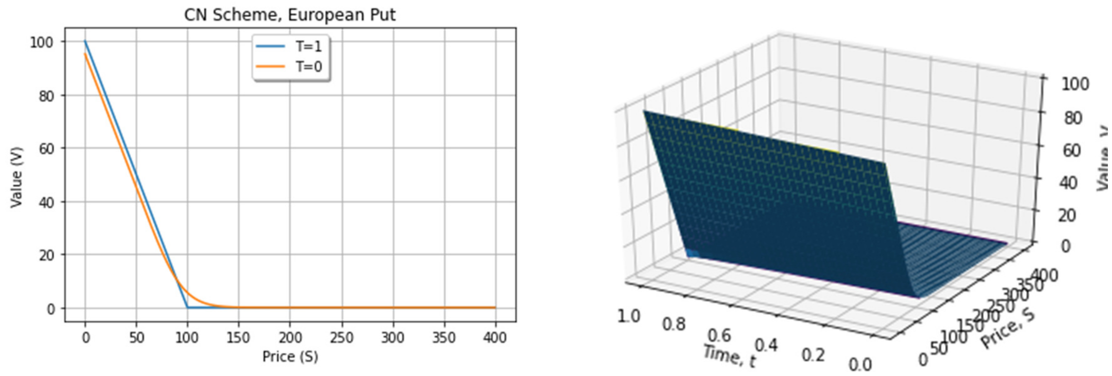

Figure 8 - CN scheme for European Puts

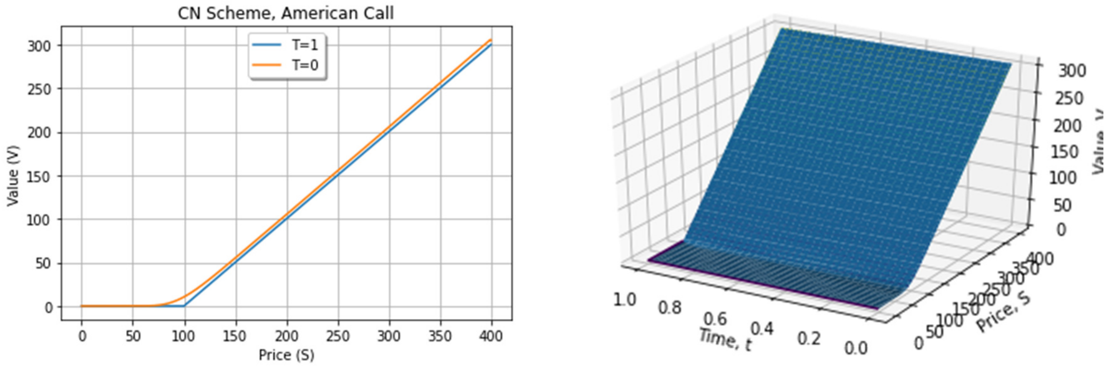

Figure 9 - CN scheme for American Calls

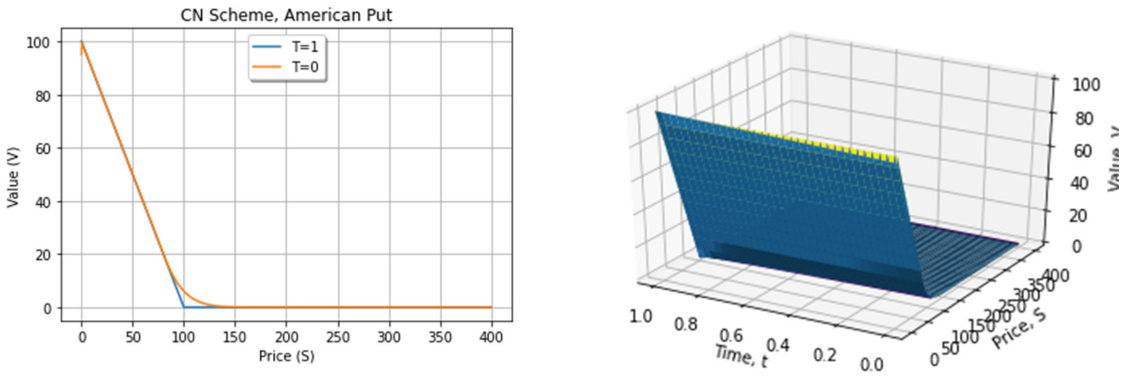

Figure 10 - CN scheme for American Puts

Binomial Method

The binomial method is another methodology used for valuing options. The fundamental ideas of Binomial methods are the discrete movement of time ($\Delta t, 2\Delta t$, ... up to $M\Delta t = T$), the possibility that the current price S only has two possible price movements between two consecutive time-steps (one up uS and one down, δS), a probability inherent to these price movements (p for up movement and $(1-p)$ for down movement), and the assumption of a risk-neutral world. A more detailed description of the methodology can be found in the Literature provided.

There are different ways of picking the parameters u, δ and p. In fact, we only need to assert one of the variables. This is due to the fact that the Binomial method assumes that discrete and continuous random walk have equal properties (mean value and variance). We decided two use two different and approaches, as presented below.

Picking $p = \frac{1}{2}$ implies:

$$u = e^{r\Delta t}\left(1 + \sqrt{e^{\sigma^2 \Delta t} - 1}\right)$$

$$\delta = e^{r\Delta t}(1 - \sqrt{e^{\sigma^2 \Delta t} - 1})$$

Picking $u = \frac{1}{\delta}$ implies:

$$A = \frac{1}{2}(e^{-r\Delta t} + e^{(r+\sigma^2)\Delta t})$$

$$u = A + \sqrt{A^2 - 1}$$

$$\delta = A - \sqrt{A^2 - 1}$$

$$p = \frac{e^{r\delta t} - \delta}{u - \delta}$$

Table 7 – Binomial Method results for European and American call and put options. Values at t *= 0 and at the spot price S = 100.*

$\boldsymbol{u = 1/\delta}$			
Price	Analyt. Solution	Num. Solution	CPU Time
Eu. Call ($\boldsymbol{\Delta t = 0.01}$)	10.45058	10.43545	0.00113s
Eu. Call ($\boldsymbol{\Delta t = 0.0001}$)	10.45058	10.45043	93s
Eu. Put ($\boldsymbol{\Delta t = 0.01}$)	5.57353	5.55839	0.00164s
Eu. Put ($\boldsymbol{\Delta t = 0.0001}$)	5.57353	5.57337	85s
Am. Call ($\boldsymbol{\Delta t = 0.01}$)	NA	10.43545	*0.00239s*
Am. Call ($\boldsymbol{\Delta t = 0.0001}$)	NA	10.45043	*183s*
Am. Put ($\boldsymbol{\Delta t = 0.01}$)	NA	6.08719	*0.00200s*
Am. Put ($\boldsymbol{\Delta t = 0.0001}$)	NA	6.09034	*179s*

$\boldsymbol{p = 1/2}$			
Price	Analyt. Solution	Num. Solution	CPU Time
Eu. Call ($\boldsymbol{\Delta t = 0.01}$)	10.45058	10.46178	0.00139s
Eu. Call ($\boldsymbol{\Delta t = 0.0001}$)	10.45058	10.45079	91s
Eu. Put ($\boldsymbol{\Delta t = 0.01}$)	5.57353	5.58473	0.00366s
Eu. Put ($\boldsymbol{\Delta t = 0.0001}$)	5.57353	5.57373	90s
Am. Call ($\boldsymbol{\Delta t = 0.01}$)	NA	10.46178	0.00211s
Am. Call ($\boldsymbol{\Delta t = 0.0001}$)	NA	10.45079	180s
Am. Put ($\boldsymbol{\Delta t = 0.01}$)	NA	6.10177	0.00243s
Am. Put ($\boldsymbol{\Delta t = 0.0001}$)	NA	6.09053	175s

The results in Table 7 show that for a smaller $\boldsymbol{\Delta t}$, equal to 0.0001, the numerical solutions are very close to the existing analytical ones, so this refinement seems appropriate for options pricing. We can also see that when we choose $u = 1/\delta$ the numerical solutions are smaller than the analytical ones, and the opposite occurs when we choose $p = 1/2$. Nevertheless, one approach does not seem to be superior to the other.

The image in Fig. 11 shows a slice plot of the Binomial method's solution when using $u = 1/\delta$, for European Call options, at $t = 0.5$. Here, we can see that the three different approaches (explicit scheme, CN scheme and binomial method) lead to very similar results not just at t = 0 but also at other time instants. For the sake of simplicity, we decided to only present the plot for this instant ($t = 0.5$) and just for European Call options but exploring different time instants and different assets, it can be seen that all three methodologies lead to very similar results.

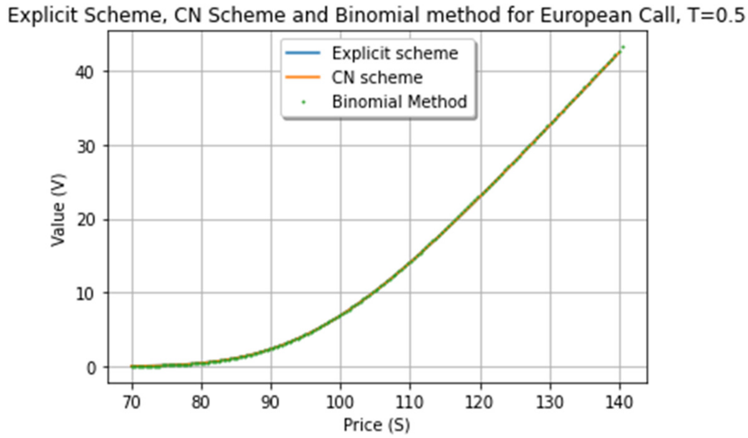

Figure 11 - Explicit and CN Schemes compared with Binomial method for European Call options at t=0.5

Therefore, our decision about the method that seems to be the most suitable one resides on computational complexity (comparing CPU times). The numerical results of the Binomial method show that this method is faster than the explicit scheme for the BS equation, but a little slower than the CN scheme, so the CN scheme seems to be the best among these three.

Bibliography

- Strikwerda, J., 2004. Finite difference schemes and partial differential equations. 2nd ed. Philadelphia: Society for Industrial and Applied Mathematics.

- Reti, D., 2021. Option pricing using the Black-Scholes model, without the formula. [online] Medium. Available at: <https://towardsdatascience.com/option-pricing-using-the-black-scholes-model-without-the-formula-e5c002771e2f> [Accessed 5 May 2022].

- Fernandes, M., 2009. Finite Differences Schemes for Pricing of European and American Options. [online] Lisbon. Available at: <https://docplayer.net/15563956-Finite-differences-schemes-for-pricing-of-european-and-american-options.html> [Accessed 5 May 2022].

- Uddin, M., Ahmed, M. and Bhowmilk, S., 2014. A Note on Numerical Solution of a Linear Black-Scholes Model. GANIT: Journal of Bangladesh Mathematical Society, 33, pp.103-115.