Departamento de Matemática

Multivariate Analysis

Master in Eng. and Data Science & Master in Mathematics and Applications

1^{st} Test - Part I
Duration: 45min

1^{st} Semester – 2020/2021
04/02/2021 – 14:30

Please justify conveniently your answers

If the first letter of your second name is between "A" and "L" solve **Group I - Version A**, otherwise solve **Group I - Version B**.

Any wrong choice of Group I Version will not be classified.

Group I - Version A **10.0 points**

1. Assume that Z_1 and Z_2 are independent random variables with standard normal distribution. Write $\boldsymbol{X}$ has a function of Z_1 and Z_2, such that $\boldsymbol{X} \sim \mathcal{N}_2\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}\right)$ (2.0)

2. Suppose $\boldsymbol{X}$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu} = (2,1)^t$ and covariance matrix:

$$\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

   (a) Find $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in I\!R^2$, such that $\boldsymbol{\alpha}^t\boldsymbol{X}$ and $\boldsymbol{\beta}^t\boldsymbol{X}$ are independent random variables. Justify your answer. (2.0)

   (b) Find the regions of $\boldsymbol{X}$ centered on $\boldsymbol{\mu}$ which cover the area of the true parameter with probability 0.90. (2.0)

3. Let $\boldsymbol{X} = (X_1, X_2)^t$ and $\boldsymbol{Y} = (Y_1, Y_2)^t$ be a random vectors such that

$$\boldsymbol{X} \sim \mathcal{N}_2\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right),$$

$$(\boldsymbol{Y}|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{N}_2\left(\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad \text{and} \quad \boldsymbol{\Sigma}_{22} = \begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix}.$$

   (a) Determine $\mathrm{E}(\boldsymbol{Y})$. (1.5)

   (b) Determine the distribution of $\boldsymbol{X}+\boldsymbol{Y}$. (2.5)

Reminder: If $\boldsymbol{X} = (\boldsymbol{X}_1^t, \boldsymbol{X}_2^t)^t \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $E(\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$ and $Var(\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$.

If the first letter of your second name is between "A" and "L" solve **Group I - Version A**, otherwise solve **Group I - Version B**.

Any wrong choice of Group I Version will not be classified.

Group I - Version B **10.0 points**

1. Assume that Z_1 and Z_2 are independent random variables with standard normal distribution. Write $\boldsymbol{X}$ has a function of Z_1 and Z_2, such that $\boldsymbol{X} \sim \mathcal{N}_2\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\right)$ (2.0)

2. Suppose $\boldsymbol{X}$ has a bivariate normal distribution with mean vector $\boldsymbol{\mu} = (2,1)^t$ and covariance matrix:

$$\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix},$$

   (a) Find $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \mathbb{R}^2$, such that $\boldsymbol{\alpha}^t\boldsymbol{X}$ and $\boldsymbol{\beta}^t\boldsymbol{X}$ are independent random variables. Justify your answer. (2.0)

   (b) Find the regions of $\boldsymbol{X}$ centered on $\boldsymbol{\mu}$ which cover the area of the true parameter with probability 0.95. (2.0)

3. Let $\boldsymbol{X} = (X_1, X_2)^t$ and $\boldsymbol{Y} = (Y_1, Y_2)^t$ be a random vectors such that

$$\boldsymbol{X} \sim \mathcal{N}_2\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right),$$

$$(\boldsymbol{Y}|\boldsymbol{X} = \boldsymbol{x}) \sim \mathcal{N}_2\left(\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad \text{and} \quad \boldsymbol{\Sigma}_{22} = \begin{pmatrix} 3 & 3 \\ 3 & 7 \end{pmatrix}.$$

   (a) Determine $\mathrm{E}(\boldsymbol{Y})$. (1.5)

   (b) Determine the distribution of $\boldsymbol{X} - \boldsymbol{Y}$. (2.5)

Reminder: If $\boldsymbol{X} = (\boldsymbol{X}_1^t, \boldsymbol{X}_2^t)^t \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ then $E(\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\boldsymbol{x}_2 - \boldsymbol{\mu}_2)$ and $Var(\boldsymbol{X}_1|\boldsymbol{X}_2 = \boldsymbol{x}_2) = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$.