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WS 2022/2023

Exercise Sheet 1
Generalized Linear Models

Discussion of the tutorial exercises on October 24 and 27, 2021

Problem 1 (*) Prove the following property of the bivariate normal distribution.

Let $\boldsymbol{X} \sim N_2(\boldsymbol{\mu}, \Sigma)$. Then with $A \in \mathbb{R}^{2\times 2}$ and $\boldsymbol{b} \in \mathbb{R}^2$, it holds

$$\boldsymbol{Y} := A\boldsymbol{X} + \boldsymbol{b} \sim N_2(A\boldsymbol{\mu} + \boldsymbol{b}, A\Sigma A^T).$$

Hint: Use the moment-generating function of $\boldsymbol{X}$ which is of the form

$$M_{\boldsymbol{X}}(\boldsymbol{t}) = \mathbb{E}\left[\exp(\boldsymbol{t}^T\boldsymbol{X})\right] = \exp\left(\boldsymbol{t}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{t}^T\Sigma\boldsymbol{t}\right)$$

for $\boldsymbol{t} \in \mathbb{R}^2$.

Problem 2 (*) Load the data set `USCRIME` which is provided on the *moodle* course side. It contains a crime rate per 1,000,000 population in 47 states of USA. Additionally, 13 covariates are available which may have an influence on the crime rate `R`. Investigate this assumption in a linear model.

Remark: For a) and b) we exclude the covariables `LF` and `NW` since they need a special transformation.

Proceed the following steps:

a) Perform an explorative data analysis with the help of pairs plots. What are three important findings?

b) Fit a linear model to the data using the **R** function `lm()`. Which covariates are significant at the $\alpha = 0.05$ level? Give an estimate of the error variance σ^2.

c) Transform the covariates `LF` and `NW` using the least-squares-method as follows. For `LF`, use a polynomial f of degree 2. For `NW`, use a polynomial g of degree three. Then, the least-squares-method for `LF` is of the form

$$\min_{\beta_0,\beta_1,\beta_2} ||R - \left(\beta_0 + \beta_1 LF + \beta_2 LF^2\right)||^2.$$

Use the results to transform `LF.tr`=f(`LF`) and `NW.tr`=g(`NW`).

Plot the `LF` vs. `R` scatterplot and the `NW` vs. `R` scatterplot including the respective calculated polynomial. Furthermore, plot the scatterplots of the transformed covariates together with `R`.

Hint: You can use the **R** function `optim` with `method="BFGS"` and 0 for the initial values to find the minimizing parameters of the least-squares problem.

Problem 3 (Additional) Let

$$\mathbb{R}^n \ni \mathbf{Y} = X_1\boldsymbol{\beta}_1^{(1)} + \boldsymbol{\epsilon}$$

with $X_1 \in \mathbb{R}^{n\times p}$, $\boldsymbol{\beta}_1^{(1)} \in \mathbb{R}^p$. Assume that X_1 has full rank and that $\boldsymbol{\epsilon} \sim N_n(\mathbf{0}, \sigma^2\mathbf{I}_n)$. Further let $\widehat{\boldsymbol{\beta}}_1^{(1)}$ denote the least squares estimator of $\boldsymbol{\beta}_1^{(1)}$. Consider the second linear model

$$\mathbb{R}^n \ni \mathbf{Y} = X_1\boldsymbol{\beta}_1 + X_2\boldsymbol{\beta}_2 + \boldsymbol{\epsilon}$$

with $\boldsymbol{\beta}_1 \in \mathbb{R}^p$, $X_2 \in \mathbb{R}^{n\times q}$, $\boldsymbol{\beta}_2 \in \mathbb{R}^q$. Assume that $X = (X_1 \vdots X_2)$ is of full rank as well. Let $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}_1^\top, \widehat{\boldsymbol{\beta}}_2^\top)^\top$ be the least-squares-estimator of $\boldsymbol{\beta} = \left(\boldsymbol{\beta}_1^\top, \boldsymbol{\beta}_2^\top\right)^\top$. When does $\widehat{\boldsymbol{\beta}}_1^{(1)} = \widehat{\boldsymbol{\beta}}_1$ hold?