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Multivariate Analysis
1st Exam 7 jan 2013

Group I

1. $(X,Y)^t \sim N_2(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$, where $\sigma_{11}=\sigma_{22}$,

$Z_1 = X+Y$
$Z_2 = X-Y$ show that $Z_1 \perp\!\!\!\perp Z_2$

Two things have to be proved:

(i) $(Z_1, Z_2) \sim N_2(\underset{\sim}{\mu}_Z, \underset{\sim}{\Sigma}_Z)$

(ii) $\text{cov}(Z_1, Z_2) = 0$

Dem (i):

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} X+Y \\ X-Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \underset{\sim}{A} \begin{bmatrix} X \\ Y \end{bmatrix}$$

since $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N_2\left(\underset{\sim}{\mu}, \underset{\sim}{\Sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ & \sigma_{11} \end{bmatrix}\right)$

then for any $\underset{\sim}{A}$ (2x2) of constant then

$$\underset{\sim}{A}\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\underset{\sim}{A}\underset{\sim}{\mu}, \underset{\sim}{A}\underset{\sim}{\Sigma}\underset{\sim}{A}^t)$$

Dem (ii):

$$\underset{\sim}{\Sigma}_Z = \text{Var}\left(\underset{\sim}{A}\begin{pmatrix} X \\ Y \end{pmatrix}\right) = \underset{\sim}{A}\,\underset{\sim}{\Sigma}\,\underset{\sim}{A}^t = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{11} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{11}+\sigma_{12} & -\sigma_{12}+\sigma_{11} \\ \sigma_{11}+\sigma_{12} & \sigma_{12}-\sigma_{11} \end{bmatrix} = \begin{bmatrix} 2\sigma_{11}+2\sigma_{12} & 0 \\ 0 & 2\sigma_{11}+2\sigma_{12} \end{bmatrix}$$

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Then $\text{cov}(Z_1, Z_2) = 0$

Thus, $Z_1 \perp\!\!\!\perp Z_2$ i.e. $X+Y \perp\!\!\!\perp X-Y$.

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2. $\underset{\sim}{X}_1 \perp\!\!\!\perp \underset{\sim}{X}_2$, $\underset{\sim}{X}_i \sim \underset{\sim}{N}(\underset{\sim}{\mu}_i, \underset{=}{\Sigma})$

$(\underset{\sim}{X}_{11}, \dots, \underset{\sim}{X}_{1n_1}) \perp\!\!\!\perp (\underset{\sim}{X}_{21} \dots \underset{\sim}{X}_{2n_2})$

(a) Prove that:

$$\frac{n_1 n_2}{n_1+n_2} (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))^t \underset{=}{\Sigma}^{-1} (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)) \sim \chi^2_{(p)}$$

Since $\underset{\sim}{X}_{1i} \sim N_p(\underset{\sim}{\mu}_1, \underset{=}{\Sigma})$ indep from

$\underset{\sim}{X}_{2j} \sim N_p(\underset{\sim}{\mu}_2, \underset{=}{\Sigma})$

Then:

(i) $\bar{X}_i \sim N_p(\mu_i, \frac{\underset{=}{\Sigma}}{n_i})$ because $\bar{X}_i$ is a linear combination of indep. random variables with multivariate normal dist.

(ii) $\bar{X}_1 - \bar{X}_2 \sim N_p(E(\bar{X}_1 - \bar{X}_2), Var(\bar{X}_1 - \bar{X}_2))$

since this is the difference of two indep random variables with multivariate normal dist.

(iii) $E(\bar{X}_1 - \bar{X}_2) = E(\bar{X}_1) - E(\bar{X}_2) = \mu_1 - \mu_2$

$$Var(\bar{X}_1 - \bar{X}_2) \underset{\bar{X}_1 \perp\!\!\!\perp \bar{X}_2}{\uparrow =} Var(\bar{X}_1) + Var(\bar{X}_2) =$$

$$= \frac{1}{n_1}\underset{=}{\Sigma} + \frac{1}{n_2}\underset{=}{\Sigma} =$$

$$= \frac{n_2+n_1}{n_1 n_2}\underset{=}{\Sigma}$$

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Thus,

$$(\bar{X}_1 - \bar{X}_2 - E(\bar{X}_1 - \bar{X}_2))^t\, Var(\bar{X}_1 - \bar{X}_2)^{-1} (\bar{X}_1 - \bar{X}_2 - E(\bar{X}_1 - \bar{X}_2)) \sim \chi^2_{(p)}$$

$(\Leftrightarrow)$

$$(\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))^t \frac{n_1 n_2}{n_1+n_2} \underset{=}{\Sigma}^{-1} (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))^t \sim \chi^2_{(p)}$$

(b) $n_1 = 20$, $n_2 = 30$

$\bar{x}_1 = (23.5,\ 30.3,\ 40.4)^t$

$\bar{x}_2 = (25.5,\ 31.7,\ 42.3)^t$

$\bar{X}_i \sim N_3(\mu_i, \underset{=}{\Sigma})$ $i=1,2$ with

$$\underset{=}{\Sigma}^{-1} = \begin{bmatrix} 0.1750 & -0.1250 & 0.0750 \\ & 0.2083 & -0.1250 \\ & & 0.1750 \end{bmatrix}$$

Hypothesis:

$H_0: \mu_1 = \mu_2$

$H_1: \mu_1 \neq \mu_2$

Pivotal Quantity:

$$T = (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))^t \frac{n_1 n_2}{n_1+n_2} \underset{=}{\Sigma}^{-1} (\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2))^t \sim \chi^2_{(p)}$$

Test Statistics:

If H_0 is true , $T_0 = T | H_0$ is true

$$T_0 = (\bar{X}_1 - \bar{X}_2)^t \frac{n_1 n_2}{n_1+n_2} \Sigma^{-1} (\bar{X}_1 - \bar{X}_2) \sim \chi^2_{(3)}$$

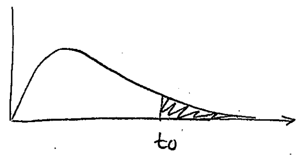
Observed value of test statistic:

$$t_0 = (23.5-25.5 \quad 30.3-31.7 \quad 40.4-42.3) \frac{20\times 30}{50} \Sigma^{-1} \begin{bmatrix} 23.5-25.5 \\ 30.3-31.7 \\ 40.4-42.3 \end{bmatrix}$$

$$= 12\ (0.945018)$$

$$= 11.3402$$

p-value:



$$p = P(T_0 \geq t_0 \mid H_0 \text{ is true}) =$$
$$= 1 - F_{\chi^2_{(3)}}(11.3402)$$
$$= 1 - 0.9899784$$
$$= 0.0100\,2159$$

Using the tables:

$$F_{\chi^2_{(3)}}(11.34) = 0.99$$

thus, p-value $= 1 - F_{\chi^2_{(3)}}(11.34) = 0.01$ //

Decision Rule:

If $\alpha > 0.01$ ⇒ Reject H_0
otherwise ⇒ do not Reject H_0

Decision:

For 5% and 10% do not Rej H_0.

Group II

$U_1 \perp\!\!\!\perp U_2$, $U_i \sim N(0,1)$

$\underset{\sim}{X} = (U_1, U_2, U_1+U_2, U_1-U_2)$

1. $$\underset{\sim}{X} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$

where

$Var(U_1) = Var(U_2) = 1$

Thus,

$$\underset{\sim}{\Sigma} = Cov(\underset{\sim}{X}) = \underset{\sim}{A}\ Var(\underset{\sim}{U})\ \underset{\sim}{A}^t$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix}$$

$$Cor(\underset{\sim}{X}) = \underset{4\times 2}{\underset{\sim}{D}^{-1/2}}\ \underset{4\times 4}{\Sigma}\ \underset{4\times 1}{\underset{\sim}{D}^{-1/2}}$$

where $\underset{\sim}{D}^{-1/2} = diag\left(\frac{1}{\sqrt{1}}, \frac{1}{\sqrt{1}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^t$

$$\underset{\sim}{R} = \operatorname{cor}(\underset{\sim}{X}) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 2/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & \frac{2}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \frac{1}{\sqrt{2}} & \\ & & & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \end{bmatrix}$$

2. $\lambda_1 = \lambda_2 = 2$ and $\operatorname{tr}(\underset{\sim}{R}) = 4$ so
$\lambda_3 = \lambda_4 = 0$

Thus, $\dfrac{\lambda_1+\lambda_2}{\operatorname{tr}(\underset{\sim}{R})} = 1$, i.e. the first two PC's explain 100% of the total variability.

this is expected, since $\underset{\sim}{X} = \underset{\sim}{g}(U_1, U_2)$.

3. Eingenvectors of $\underset{\sim}{R}$ are:

$\gamma_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1, 0\right)^t$ and $\gamma_2 = \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0, 0\right)^t$

$$(\underset{\sim}{R} - 2\underset{\sim}{I})\,\underset{\sim}{x} = \underset{\sim}{0} \iff \begin{bmatrix} -1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & -1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \underset{\sim}{0}$$

$$\iff \begin{cases} -x + \frac{z}{\sqrt{2}} + \frac{w}{\sqrt{2}} = 0 \\ -y + \frac{z}{\sqrt{2}} - \frac{w}{\sqrt{2}} = 0 \\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} - z = 0 \\ \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - w = 0 \end{cases} \iff \begin{cases} x = \frac{z+w}{\sqrt{2}} \\ y = \frac{z-w}{\sqrt{2}} \\ - \\ - \end{cases}$$

$$\underset{\sim}{e}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \\ 0 \end{bmatrix} \qquad \underset{\sim}{e}_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 1 \end{bmatrix}$$

$$\|\underset{\sim}{e}_1\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 1} = \sqrt{2} \qquad \|\underset{\sim}{e}_2\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 1} = \sqrt{2}$$

Thus,

$$\underset{\sim}{\gamma}_1 = \frac{1}{\sqrt{2}}\underset{\sim}{e}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \quad ; \quad \underset{\sim}{\gamma}_2 = \frac{1}{\sqrt{2}}\underset{\sim}{e}_2 \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

e $\underset{\sim}{\gamma}_1^t \underset{\sim}{\gamma}_2 = \frac{1}{2}\left(\frac{1}{4} - \frac{1}{4} + 0 + 0\right) = 0$, thus the vectors are orthogonal.

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4. $$PC_1 = \frac{X_1^* + X_2^*}{2} + \frac{1}{\sqrt{2}} X_3^* = \frac{X_1 + X_2}{2} + \frac{1}{\sqrt{2}}\left(\frac{X_3}{\sqrt{2}}\right)$$

$$= \frac{U_1 + U_2}{2} + \frac{U_1 + U_2}{2}$$

$$= U_1 + U_2 = X_3 \text{, where } X_i^* = \frac{X_i - E(X_i)}{\sqrt{var(X_i)}},$$

where $X_1^* = U_1$; $X_2^* = U_2$; $X_3^* = \frac{U_1+U_2}{\sqrt{2}}$; $X_4^* = \frac{U_1 - U_2}{\sqrt{2}}$

Thus, the first PC is equal to the sum of U_1 and U_2 and equal to X_3. So is porportional to the sample mean of U_1 and U_2. High (low) values of U_1 and U_2 leads to high (low) values of PC_1.

$$CP_2 = \frac{X_1^* - X_2^*}{2} + \frac{1}{\sqrt{2}} X_4^* = \frac{X_1 - X_2}{2} + \frac{1}{\sqrt{2}} \frac{X_4}{\sqrt{2}}$$

$$= \frac{U_1 - U_2}{2} + \frac{1}{2}(U_1 - U_2)$$

$$= U_1 - U_2 = X_4$$

Thus, the second PC is equal to X_3, i.e. is a contrast between U_1 and U_2.

High (low) values of U_1 and low (high) values of U_2 leads to high (low) values of PC_2.

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Group III

X_1 - Taste

X_2 - Goodbuy for money

X_3 - Flavor

X_4 - Suitable for snack

X_5 - Provids lots of energy

1. Communalities: (original solution)

$$h_i^2 = \sum_{j=1}^{2} l_{ij}^2 = \begin{cases} 0.56^2 + 0.82^2 = 0.986 \\ 0.78^2 + 0.52^2 = 0.8788 \\ 0.65^2 + 0.75^2 = 0.985 \\ 0.94^2 + 0.10^2 = 0.8936 \\ 0.80^2 + 0.54^2 = 0.9316 \end{cases}$$

Specific variances:

$$\psi_{ii} = 1 - h_i^2 = \begin{cases} 1 - 0.986 = 0.014 \\ 1 - 0.8788 = 0.1212 \\ 1 - 0.985 = 0.015 \\ 1 - 0.8936 = 0.1064 \\ 1 - 0.9316 = 0.0684 \end{cases}$$

Since all communalities are near $1 = Var(X_i)$ and all specific variances are near zero, so all of them are well represented by two common factors.

2. $\hat{\underset{\sim}{R}} = \Lambda \Lambda^t + \Psi$

$$= \begin{bmatrix} 0.56 & 0.82 \\ 0.78 & -0.52 \\ 0.65 & 0.75 \\ 0.94 & -0.10 \\ 0.80 & -0.54 \end{bmatrix} \begin{bmatrix} 0.56 & 0.78 & 0.65 & 0.94 & 0.80 \\ 0.82 & -0.52 & 0.75 & -0.10 & -0.54 \end{bmatrix}$$

$$+ \begin{bmatrix} 0.014 & & & & \\ & 0.1212 & & & \\ & & 0.015 & & \\ & & & 0.1064 & \\ & & & & 0.0684 \end{bmatrix}$$

$$= \begin{bmatrix} 1.0000 & 0.0104 & 0.979 & 0.444 & 0.0052 \\ & 1.0000 & 0.117 & 0.7852 & 0.9048 \\ & & 1 & 0.5360 & 0.1150 \\ & & & 1 & 0.8060 \\ & & & & 1 \end{bmatrix}$$

3. The interpretation is easier based on the rotated solution since the loadings associated with each observed random variable show clear pattern on the two common factors.

$Z_1 = 0.02 f_1^* + 0.99 f_2^* + e_1$

$Z_2 = 0.94 f_1^* - 0.01 f_2^* + e_2$

$Z_3 = 0.13 f_1^* + 0.98 f_2^* + e_3$

$Z_4 = 0.84 f_1^* + 0.43 f_2^* + e_4$

$Z_5 = 0.97 f_1^* - 0.02 f_2^* + e_5$

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where $Z_i = \frac{X_i - \hat{\mu}_i}{\hat{\sigma}_i}$

Thus f_1^* has higher weights on X_2, X_4, X_5 and f_2^* has higher weights on X_1, X_3

So, f_2^* is essencial a measure of Taste and flavour of the new product

and

f_1^* is a measure of less sensorial characteristics of the new product like

- economic aspects - Good buy for money
- easier to eat - suitable for snack
- energetic characteristics - Provides lots of energy

4. We know that:

$$\hat{\Lambda} = \begin{bmatrix} 0.56 & 0.82 \\ 0.78 & -0.52 \\ 0.65 & 0.75 \\ 0.94 & -0.10 \\ 0.80 & -0.54 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1}\, \gamma_1 & \sqrt{\lambda_2}\, \gamma_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \end{bmatrix}$$

and $0.574 = \frac{\lambda_1}{tr(\underset{\sim}{R})} = \frac{\lambda_1}{5}$

$0.935 = \frac{\lambda_1 + \lambda_2}{5}$

or

Thus, $\lambda_1 = \overbrace{5 \times 0.574 = 2.870}$ $\simeq \sum_{i=1}^{5} \alpha_{i1}^2 = \overbrace{2.8681}$

$\lambda_2 = \underbrace{5 \times 0.935 - \lambda_1}_{1.805} \simeq \underbrace{\sum_{i=1}^{5} \alpha_{i2}^2}_{1.8069}$

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$$\gamma_1 = \frac{1}{\sqrt{\lambda_1}} \underset{\sim}{\alpha}_1 = \frac{1}{\sqrt{2.8681}} \begin{bmatrix}0.56\\0.78\\0.65\\0.94\\0.80\end{bmatrix} = \begin{bmatrix}0.331\\0.461\\0.384\\0.555\\0.472\end{bmatrix}$$

$$\gamma_2 = \frac{1}{\sqrt{\lambda_2}} \alpha_2 = \frac{1}{\sqrt{1.8069}} \begin{bmatrix}0.82\\-0.52\\0.75\\-0.10\\-0.54\end{bmatrix} = \begin{bmatrix}0.610\\-0.387\\0.558\\-0.074\\-0.402\end{bmatrix}$$

$$CP_1 = 0.331\,Z_1 + 0.461\,Z_2 + 0.384\,Z_3 + 0.555\,Z_4 + 0.472\,Z_5$$

$$CP_2 = 0.610\,Z_1 - 0.387\,Z_2 + 0.558\,Z_3 - 0.074\,Z_4 - 0.402\,Z_5$$

where $Z_i = \frac{X_i - \bar{x}_i}{s_i}$ are the standardized version of the variables:

X_1 - Taste
X_2 - Good buy for money
X_3 - flavor
X_4 - Suitable for snack
X_5 - Provides lots of energy

CP_1 - All weights (γ_{1j}) are similar so it can be understood as a weighting average of the 5 attributes under study. Moreover, product rate as excelent (extremely bad) in all attributes will have high (low) values on this PC.

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CP_2 - can be understood as a contrast between

Z_1 - taste
Z_3 - flavor

and

Z_4 - Good buy for money
Z_5 - Provides lots of energy

being a contrast between how the food tastes and additional characteristics of the product (like economic and energetic value)

Thus, high (low) values of CP_2 represents products with high (low) values of food taste and low (high) values for the economic and energetic attributes.

5. When comparing the first common factors verses first PC's we can say that they have very different interpretation and for the goal of this analysis FA seems more interesting, since it gives an explanation how the common consumer rates the products, and that can be useful to develop target group for publicity, etc.

Group IV

$P(Y=0)=0.6 \quad , \quad P(Y=1)=0.4$

$(X|Y=j) \sim \text{Binomial}(10, q_j) \qquad q_j = \begin{cases} 0.3 , & j=0 \\ 0.5 , & j=1 \end{cases}$

1. $TPM = P(Y=0)\, P(\text{classif } X \text{ in } \{Y=1\} \mid Y=0\}$

$+ P(Y=1)\, P(\text{classif } X \text{ in } \{Y=0\} \mid Y=1\}$

$$= 0.6 \sum_{x\in R_1} P(X=x|Y=0) + 0.4 \sum_{x\in R_2} P(X=x|Y=1)$$

$$= 0.6 \sum_{x\in R_1} P(X=x|Y=0) + 0.4 \left(1 - \sum_{x\in R_1} P(X=x|Y=1)\right)$$

$$= 0.4 + \sum_{x\in R_1} \left[0.6\, P(X=x|Y=0) - 0.4\, P(X=x|Y=1)\right]$$

R_1 is such that: min TPM is equivalent of having:

$x \in R_1 : \quad 0.6\, P(X=x|Y=0) - 0.4\, P(X=x|Y=1) \leq 0$

$$(\Leftrightarrow) \quad \frac{P(X=x|Y=0)}{P(X=x|Y=1)} \leq \frac{0.4}{0.6}$$

$$(\Leftrightarrow) \quad \frac{\cancel{\binom{10}{x}}\, 0.3^x (1-0.3)^{10-x}}{\cancel{\binom{10}{x}}\, 0.5^x (1-0.5)^{10-x}} \leq \frac{2}{3}$$

$$(\Leftrightarrow) \quad \left(\frac{3}{5}\right)^x \left(\frac{7}{5}\right)^{10-x} \leq \frac{2}{3}$$

$$(\Leftrightarrow) \quad \left(\frac{3}{5}\right)^x \left(\frac{7}{5}\right)^{10} \left(\frac{5}{7}\right)^x \leq \frac{2}{3}$$

$$(\Leftrightarrow) \quad \left(\frac{3}{7}\right)^x \leq \frac{2}{3}\left(\frac{5}{7}\right)^{10}$$

$$(\Leftrightarrow) \quad x \log\left(\frac{3}{7}\right) \leq \log\left(\frac{2}{3}\right) + 10 \log\left(\frac{5}{7}\right)$$

$$(\Leftrightarrow) \quad x \log\left(\frac{7}{3}\right) \geq \log\left(\frac{3}{2}\right) + 10 \log\left(\frac{7}{5}\right)$$

$$(\Leftrightarrow) \quad x \geq \frac{\log(3/2) + 10\log(7/5)}{\log\left(\frac{7}{3}\right)} \simeq 4.450$$

<u>Optimal classif Rule:</u>

$\begin{cases} \text{If } x_0 \geq 4.450 , & \text{classif } x_0 \text{ in } \{Y=1\} \\ \text{otherwise } (x_0 \leq 4) , & \text{classif } x_0 \text{ in } \{Y=0\} \end{cases}$

i.e. $\begin{cases} \text{If } x_0 \in \{5,6,\ldots,10\} , & \text{classify } x_0 \text{ in } \{Y=1\} \\ \text{If } x_0 \in \{1,2,3,4\} , & \text{classify } x_0 \text{ in } \{Y=0\} \end{cases}$

2. $P(\text{classif } X \text{ in } \{Y=1\} \mid Y=0) =$

$= P(X \geq 5 \mid Y=0) = 1 - P(X \leq 4 \mid Y=0)$

$= 1 - F_{Bin(10,0.3)}(4) = 0.1503$

$P(\text{classif } X \text{ in } \{Y=0\} \mid Y=1) =$

$= P(X \leq 4 \mid Y=1) = F_{Bin(10,0.5)}(4) = 0.3770$

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$TPM = 0.6 \times 0.1503 + 0.4 \times 0.3770 = 0.2409$

3. If $x_0 = 5 \geq 4.450$, assign x_0 to $\{Y=1\}$.