Name: SOLUTIONS

UM Student ID:

EEN 336 Fall 2014 Exam 2

Work through all the problems. **Clearly explain your steps and reasoning for full credit. I CANNOT READ YOUR MIND!!**

Problem 1: For each of the $X(z)$ defined below, do the following steps:

1. Find the poles and zeros
2. Determine all possible ROCs
3. For each ROC, find the corresponding $x(n)$
4. Determine whether each $x(n)$ is causal, and whether each $x(n)$ is BIBO stable.

(a)

$$X(z) = 1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

(b)

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

Problem 2: Suppose $f(n)$ is a signal with values $f(-1) = 1,\ f(0) = -1,\ f(1) = 3,\ f(2) = -1,\ f(3) = 2$ and zero otherwise, and $g(n)$ is a signal with values $g(0) = 1,\ g(1) = -2,\ g(3) = 3$ and zero otherwise. Determine $q(n) = f(n) * g(n)$, and plot the result $q(n)$.

Problem 3: a causal discrete-time LTI system with input $x(n)$ and output $y(n)$ is characterized by the following difference equation:

$$y(n) = \frac{1}{5}y(n-1) + x(n)$$

Suppose the input to this system is a particular $x(n)$ given by

$$x(n) = 2^n u(-n)$$

Determine the corresponding output $y(n)$.

$$\sum_{K=0}^{N} \gamma^{K} = \frac{1-\gamma^{N+1}}{1-\gamma}$$

$$\sum_{K=0}^{\infty} \gamma^{K} = \frac{1}{1-\gamma} \quad , \text{if } |\gamma| < 1$$

TABLE 3.1 SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m > 0$) or ∞ (if $m < 0$)
5. $a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
9. $[\cos\omega_0 n]u[n]$	$\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1,\\ 0, & \text{otherwise}\end{cases}$	$\dfrac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

TABLE 3.2 SOME z-TRANSFORM PROPERTIES

Property Number	Section Reference	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1	3.4.1	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
2	3.4.2	$x[n-n_0]$	$z^{-n_0}X(z)$	R_x, except for the possible addition or deletion of the origin or ∞
3	3.4.3	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
4	3.4.4	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x, except for the possible addition or deletion of the origin or ∞
5	3.4.5	$x^*[n]$	$X^*(z^*)$	R_x
6		$\mathcal{Re}\{x[n]\}$	$\frac{1}{2}[X(z)+X^*(z^*)]$	Contains R_x
7		$\mathcal{Im}\{x[n]\}$	$\frac{1}{2j}[X(z)-X^*(z^*)]$	Contains R_x
8	3.4.6	$x^*[-n]$	$X^*(1/z^*)$	$1/R_x$
9	3.4.7	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	Contains $R_{x_1}\cap R_{x_2}$
10	3.4.8	Initial-value theorem: $x[n]=0,\quad n<0$	$\lim\limits_{z\to\infty} X(z)=x[0]$	

Table 3.1 Some common z-transform pairs

	Sequence $x[n]$	z-Transform $X(z)$	ROC
1.	$\delta[n]$	1	All z
2.	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\|z\| > 1$
3.	$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4.	$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
5.	$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
6.	$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
7.	$(\cos\omega_0 n)u[n]$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
8.	$(\sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
9.	$(r^n \cos\omega_0 n)u[n]$	$\dfrac{1-(r\cos\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$
10.	$(r^n \sin\omega_0 n)u[n]$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

- The ROC *cannot* include any poles.
- The ROC is a connected (that is, a single contiguous) region.
- For finite duration sequences the ROC is the entire z-plane, with the possible exception of $z = 0$ or $z = \infty$.

Table 3.2 Some z-transform properties.

	Property	Sequence	Transform	ROC
		$x[n]$	$X(z)$	R_x
		$x_1[n]$	$X_1(z)$	R_{x_1}
		$x_2[n]$	$X_2(z)$	R_{x_2}
1.	Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
2.	Time shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z = 0$ or ∞
3.	Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|R_x$
4.	Differentation	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R_x
5.	Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
6.	Real-part	$\mathrm{Re}\{x[n]\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	At least R_x
7.	Imaginary part	$\mathrm{Im}\{x[n]\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	At least R_x
8.	Folding	$x[-n]$	$X(1/z)$	$1/R_x$
9.	Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1} \bigcap R_{x_2}$
10.	Initial-value theorem	$x[n] = 0$ for $n < 0$	$x[0] = \lim_{z\to\infty} X(z)$	

(1)

(a) $X(z) = 1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$

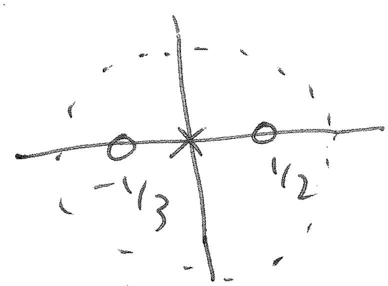
Mult. z^2/z^2

$$X(z) \stackrel{\swarrow}{=} \left(z^2 - \frac{1}{6}z - \frac{1}{6}\right)\frac{1}{z^2}$$

$$= \left(z - \frac{1}{2}\right)\left(z + \frac{1}{3}\right)\frac{1}{z^2}$$

zeros at $z = \frac{1}{2}$, $z = -\frac{1}{3}$

2 poles at $z = 0$



The only ROC : $|z| > 0$ (entire z plane except for $z = 0$)

with this $X(z)$,

(3) $\longrightarrow$ Then $x(n) = \delta(n) - \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$

(4) $\longrightarrow$ $x(n) = 0$ for $n < 0$, so the system/signal is causal.

$\longrightarrow$ The ROC includes $|z| = 1$, so the system is BIBO stable. ALSO $\sum_{n=-\infty}^{\infty} |x(n)| = 1 - \frac{1}{6} - \frac{1}{6} = 2/3 < \infty$.

(b)

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)}$$

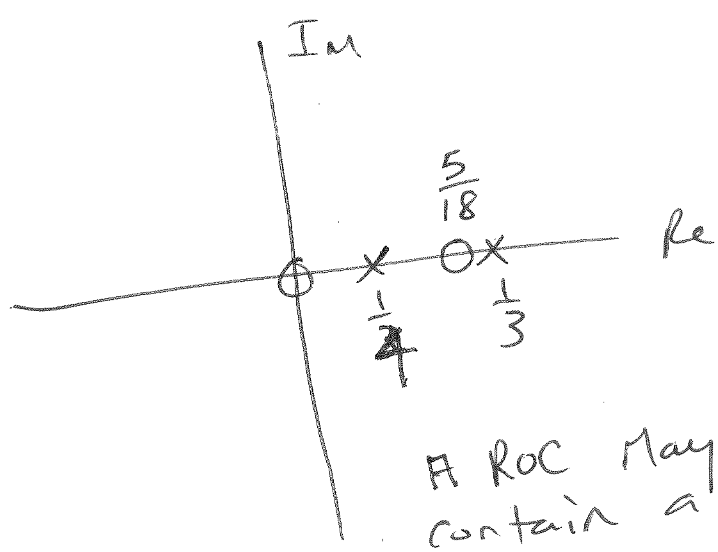
$\frac{z^2}{z^2}$

$$= \frac{3z^2 - \frac{5}{6}z}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)}$$

$$= \frac{3z\left(z - \frac{5}{18}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{3}\right)}$$

zeros: $z = 0$, $z = \frac{5}{18}$

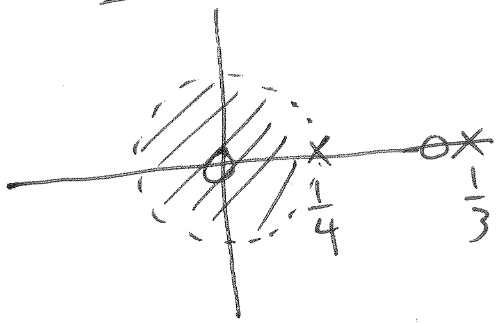
poles: $z = \frac{1}{4}$, $z = \frac{1}{3}$



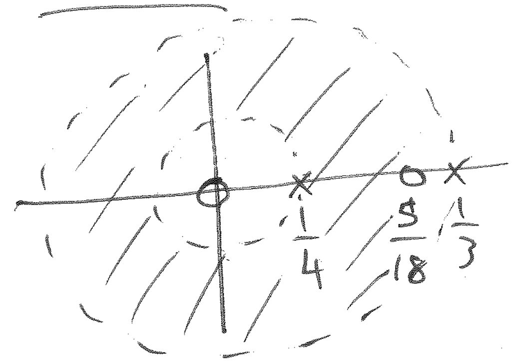
A ROC may not contain a pole.

There are 3 possible ROCs

Case 1: $|z| < \frac{1}{4}$

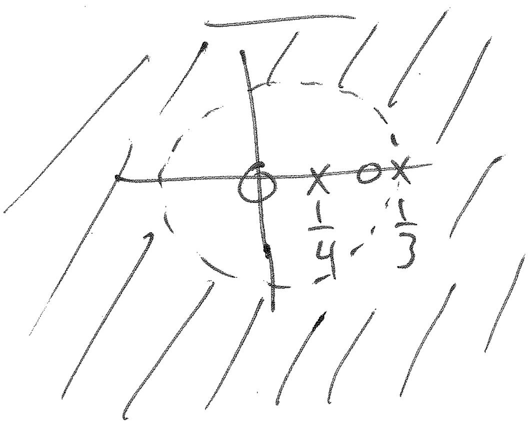


Case 2:



$\frac{1}{4} < |z| < \frac{1}{3}$

Case 3:



$|z| > \frac{1}{3}$

Let's examine each case.
But first, we expand $X(z)$ using partial fractions.

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 - \frac{1}{3}z^{-1})}$$

$$= \frac{A}{(1 - \frac{1}{4}z^{-1})} + \frac{B}{(1 - \frac{1}{3}z^{-1})}$$

$$A = (1 - \tfrac{1}{4}z^{-1})X(z)\Big|_{z^{-1}=4} = \frac{3 - \frac{5}{6}(4)}{(1 - \frac{1}{3}4)} = \frac{3 - \frac{10}{3}}{-\frac{1}{3}}$$

$$= -3(-\tfrac{1}{3}) = 1$$

$$B = (1 - \tfrac{1}{3}z^{-1})X(z)\Big|_{z^{-1}=3} = \frac{3 - \frac{5}{6}(3)}{(1 - \frac{1}{4}3)} = \frac{3 - \frac{5}{2}}{1 - \frac{3}{4}} = \frac{1/2}{\frac{1}{4}}$$

$$= 2$$

$$X(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} + \frac{2}{1 - \frac{1}{3}z^{-1}}$$

Now consider *each* *ROC* *case*.

Case 1

$|z| < \frac{1}{4} \Rightarrow$ Results from $\{|z| < \frac{1}{4}\} \cap \{|z| < \frac{1}{3}\}$

This consists of 2 left sided sequences since the ROC is inward.

Tables:

$$-a^n u(-n-1) \longleftrightarrow \frac{1}{1-az^{-1}}, \quad |z| < |a|$$

Then by inspection

$$x(n) = -\left(\frac{1}{4}\right)^n u(-n-1) - 2\left(\frac{1}{3}\right)^n u(-n-1)$$

→ This is NOT causal since $x(n) \neq 0$ for $n < 0$

ALSO, ~~the~~ the ROC is inward, and NOT the exterior of a circle

→ This is NOT BIBO stable since the ROC does NOT include the unit circle $|z| = 1$.

Case 2:

$\frac{1}{4} < |z| < \frac{1}{3} = \{|z| > \frac{1}{4}\} \cap \{|z| < \frac{1}{3}\}$

Right sided   Left sided.

This is a two-sided sequence. One pole is left sided, while the other is right-sided.

Then Table: $a^n u(n) \longleftrightarrow \frac{1}{1 - az^{-1}}, \ |z| > |a|$

we see

$$x(n) = \left(\frac{1}{4}\right)^n u(n) + -2\left(\frac{1}{3}\right)^n u(-n-1)$$

→ NOT causal: $x(n) \neq 0$ for $n < 0$ and the ROC is NOT the exterior of a circle

→ NOT BIBO stable since $|z| = 1$ is NOT in the ROC.

Case 3:

$$|z| > \frac{1}{3} = \{|z| > \frac{1}{4}\} \cap \{|z| > \frac{1}{3}\}$$

Then we have 2 right-sided sequence produced by the poles.

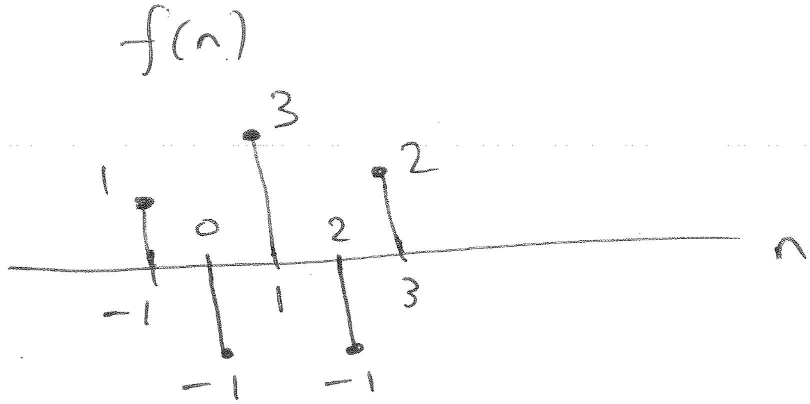
By inspection

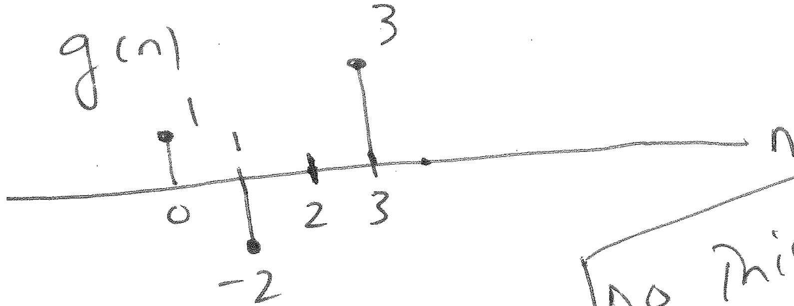
$$x(n) = \left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(n)$$

– causal? Yes, $x(n) = 0$ for $n < 0$.

Also, the ROC is the exterior of a circle.

→ Yes, BIBO stable since $|z| = 1$ is in the ROC.





!!

Do this in time or z-domain!

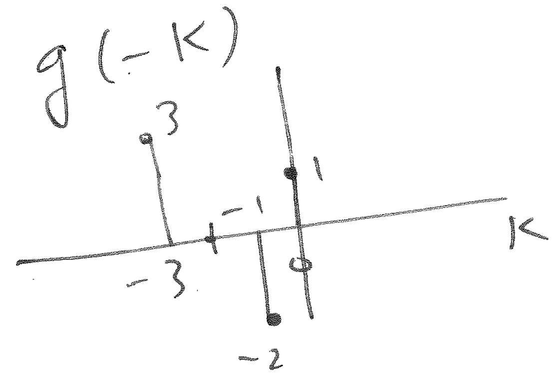
$$g(n) = f(n) * g(n)$$

TIME DOMAIN:

$$= \sum_{K=-\infty}^{\infty} f(K)\, g(n-K)$$

Since $f(K) = 0$ for $K < -2$ and $K > 4$

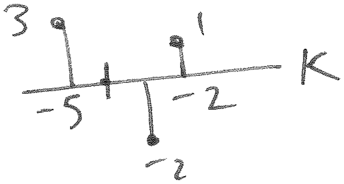
$$g(n) = \sum_{K=-1}^{3} f(K)\, g(n-K)$$



Now Do
The convolution

$n = -2$

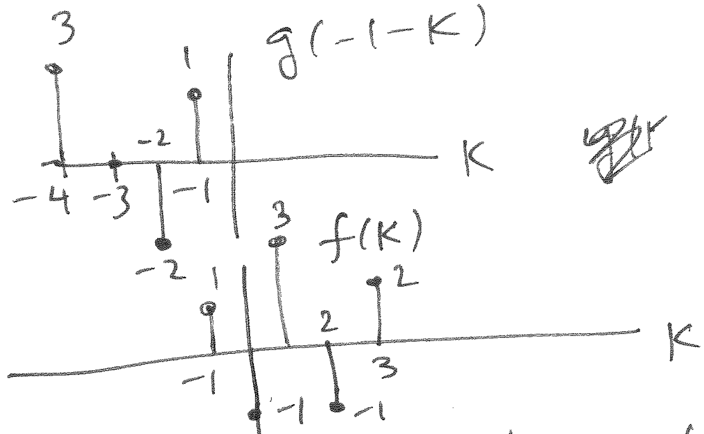
$g(-2-K)$



No overlap between $f(K)$, $g(-2-K)$

$y(n) = 0$ for $n \le -2$

$n = -1$



overlap at $K = -1$

visualize as ← only overlap

$$y(-1) = \sum_{K=-1}^{3} f(K)g(-1-K)$$

$$= f(-1)g(-1+1) + f(0)\overbrace{g(-1)}^{0} + f(1)\overbrace{g(-2)}^{0} + f(2)\overbrace{g(-3)}^{0} + f(3)\overbrace{g(-4)}^{0}$$

$$= f(-1)g(0)$$ → which is the only overlap

$g(-1) = 1$

$g(0)\ ? = \sum_{k=-1}^{3} f(k)\, g(0-k)$

visualize as

$g(-k)$							
3	0	-2	1				
		1	-1	3	-1	2	$f(k)$

$g(0) = (-2)(1) + (1)(-1)$

$= -2 + -1 = -3$

$g(1) = \sum_{k=-1}^{3} f(k)\, g(1-k)$; write this out or

$g(1-k)$						
3	0	-2	1			
	1	-1	3	-1	2	$f(k)$

$g(1) = (0)(1) + (-2)(-1) + (1)(3)$

$= 2 + 3 = 5$

$g(2) = \sum_{k=-1}^{3} f(k)\, g(2-k)$

$g(2-k)$					
3	0	-2	1		
1	-1	3	-1	2	$f(k)$

$y(2) = (3)(1) + (0)(-1) + (2)(3) + (1)(-1)$

$= 3 + 6 + -1 = \cancel{8}\ -4$

$$y(3) = \sum_{k=-1}^{3} f(k)g(3-k)$$

	3	0	-2	1
1	-1	3	-1	2

$$y(3) = (3)(-1) + (0)(3) + (-2)(-1) + (1)(2)$$
$$= -3 + 2 + 2 = 1$$

$$y(4) = \sum_{k=-1}^{3} f(k)g(4-k)$$

		3	0	-2	1
1	-1	3	-1	2	

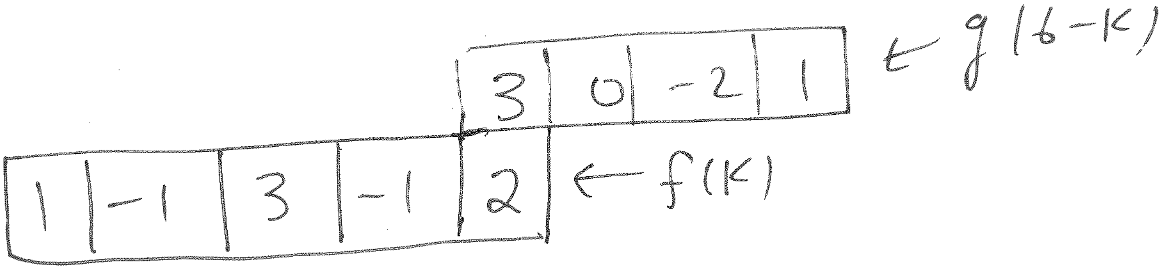
$$y(4) = (3)(3) + (0)(-1) + (-2)(2)$$
$$= 9 - 4 = 5$$

$$y(5) = \sum_{k=-1}^{3} f(k)g(5-k)$$

			3	0	-2	1
1	-1	3	-1	2		

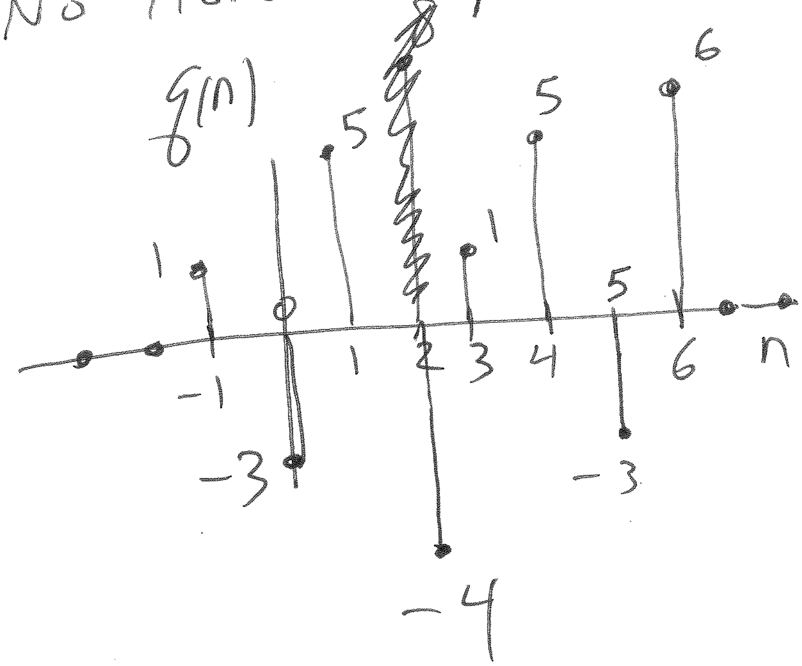
$$y(5) = (3)(-1) + (0)(2) = -3$$

$$g(6) = \sum_{K=-1}^{3} f(K)\, g(6-K)$$



$$g(6) = (3)(2) = 6$$

$g(7) = 0$, No more overlap



To summarize

$g(-1) = 1$

$g(0) = -3$

$g(1) = 5$

$g(2) = \cancel{8}\ -4$

$g(3) = 1$

$g(4) = 5$

$g(5) = -3$

$g(6) = 6$

$g(n) = 0$ otherwise

$$g(n) = \delta(n+1) - 3\delta(n) + 5\delta(n-1) + -4\,\delta(n-2) + \delta(n-3) + 5\,\delta(n-4) - 3\delta(n-5) + 6\delta(n-6)$$

OR

Z - DOMAIN

You could have done The following

$$F(z) = z + -1 + 3z^{-1} + -1z^{-2} + 2z^{-3}$$

$$G(z) = 1 - 2z^{-1} + 3z^{-3}$$

$$Q(z) = F(z)\,G(z)$$

Multiply out, group terms

$$\begin{array}{r} z + -1 + 3z^{-1} - 1z^{-2} + 2z^{-3} \\ 1 - 2z^{-1} + 3z^{-3} \\ \hline \end{array}$$

Add

$$\begin{array}{l} z + -1 + 3z^{-1} - z^{-2} + 2z^{-3} \\ \quad -2 + 2z^{-1} - 6z^{-2} + 2z^{-3} - 4z^{-4} \\ \quad\quad 3z^{-2} - 3z^{-3} + 9z^{-4} \\ \quad\quad\quad + \left(-3z^{-5} + 6z^{-6}\right) \\ \hline \end{array}$$

SAME AS Before.

$$Q(z) = z + -3 + 5z^{-1} + -4z^{-2} + z^{-3} + 5z^{-4} - 3z^{-5} + 6z^{-6}$$

$$\Rightarrow g(n) = \delta(n+1) - 3\delta(n) + 5\delta(n-1) - 4\delta(n-2) + (1)\delta(n-3) + 5\delta(n-4) - 3\delta(n-5) + 6\delta(n-6)$$

Causal LTI

(3) $x(n) \rightarrow \boxed{} \rightarrow y(n) = \frac{1}{5} y(n-1) + x(n)$

$x(n) = 2^n u(-n)$

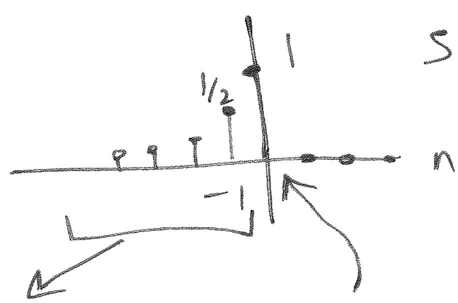
Then, Take z transforms

$Y(z) = \frac{1}{5} z^{-1} Y(z) + X(z)$ ROC: $|z| > \frac{1}{5}$

$Y(z)\left[1 - \frac{1}{5} z^{-1}\right] = X(z)$; $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{5} z^{-1}}$

with

$x(n) = 2^n u(-n)$



split into two segments. why ?

$x(n) = \left(\frac{1}{2}\right) 2^n u(-n-1) + (1)\,\delta(n)$

Tables:

$\delta(n) \longleftrightarrow 1$

$-a^n u(-n-1) \longleftrightarrow \frac{1}{1 - a z^{-1}}, \quad |z| < |a|$

Then

$$\left(\tfrac{1}{2}\right)2^{n}u(-n-1) \longleftrightarrow \frac{-1/2}{1-2z^{-1}}, \quad \underline{|z|<2} \ \text{ROC}$$

$$Y(z) = H(z)X(z)$$

$$\text{ROC } Y = \{\text{ROC } H\} \cap \{\text{ROC } X\}$$

$$Y(z) = H(z)X(z)$$

$$\Rightarrow \quad X(z) = \frac{-1/2}{1-2z^{-1}} + 1, \quad \text{ROC } |z|<2$$

ROC $|z| > \frac{1}{5}$

$$Y(z) = \left[\frac{1}{1-\frac{1}{5}z^{-1}}\right]\left[\frac{-1/2}{1-2z^{-1}} + 1\right]$$

$$= \underbrace{\frac{-1/2}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-2z^{-1}\right)}}_{\text{Do partial Fractions here.}} + \frac{1}{1-\frac{1}{5}z^{-1}}$$

$$\frac{-1/2}{\left(1-\frac{1}{5}z^{-1}\right)\left(1-2z^{-1}\right)} = \frac{A}{1-\frac{1}{5}z^{-1}} + \frac{B}{1-2z^{-1}}$$

$$A = \left.\frac{-1/2}{1-2z^{-1}}\right|_{z^{-1}=5} = \frac{-1/2}{1-2(5)} = \frac{-1/2}{1-10} = \frac{-1/2}{-9} = \frac{-1}{18}$$

$$B = \left.\frac{-1/2}{1-\frac{1}{5}z^{-1}}\right|_{z^{-1}=\frac{1}{2}} = \frac{-1/2}{1-\frac{1}{5}\left(\frac{1}{2}\right)} = \frac{-1/2}{1-\frac{1}{10}}$$

$$= \frac{-1/2}{9/10} = -\frac{1}{2}\,\frac{10}{9}$$

$$= -\frac{5}{9}$$

$|z| > \frac{1}{5}$ left sided

$|z| < 2$ Right sided

$|z| > \frac{1}{5}$

$$Y(z) = \frac{-\frac{1}{18}}{1-\frac{1}{5}z^{-1}} + \frac{-\frac{5}{9}}{1-2z^{-1}} + \frac{1}{1-\frac{1}{5}z^{-1}}$$

ROC: $\frac{1}{5} < |z| < 2$

$$y(n) = \frac{-1}{18}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{9}\,2^n u(-n-1) + \left(\frac{1}{5}\right)^n u(n)$$