Appendix A

Composed Weibull distribution

In Chapter 3 the Composed Weibull distribution is described. In this appendix expressions are derived for the significant wave height $H_{1/3}$ and the ratio of a wave height with a certain probability of exceedance to the significant wave height.

A.1 Mean of the highest 1/N-part

As mentioned in Chapter 3, $H_{1/N}$ is defined by:

$$H_{1/N} = \frac{\int_{H_N}^{\infty} H\, f(H)\, dH}{\int_{H_N}^{\infty} f(H)\, dH} = \frac{\int_{H_N}^{\infty} H\, f(H)\, dH}{\frac{1}{N}\int_{0}^{\infty} f(H)\, dH} = N \int_{H_N}^{\infty} H\, f(H)\, dH \qquad \text{(A.1)}$$

In this definition H_N is the wave height with an exceedance probability of $1/N$ (N>1). In coastal engineering practice the wave height with an exceedance probability of $1/N$ is often denoted by $H_{1/N\%}$. However, in this appendix H_N is used instead of $H_{1/N\%}$, in line with the definition of H_N. The determination of $H_{1/N}$ depends on the fact whether or not H_N exceeds H_{tr}.

Transitional wave height exceeds H_N

In Figure A.1 a wave height distribution with H_{tr}> H_N is shown.

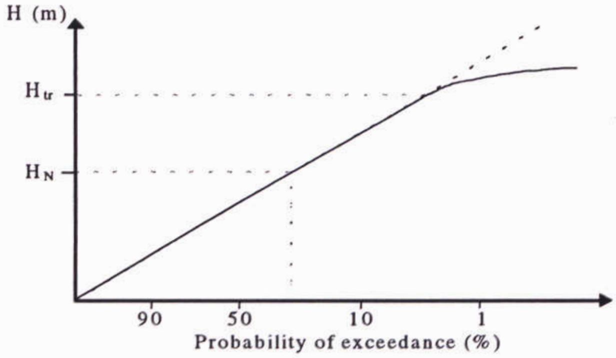

Figure A.1: Composed Weibull wave height distribution with H_{tr}>H_N.

In this case H_N is determined by:

$$Q_{\underline{H}_1}(H) \equiv \Pr_1\{\underline{H} \geq H\} = \frac{1}{N} = \exp\left[-\left(\frac{H_N}{H_1}\right)^{k_1}\right] \tag{A.2}$$

Yielding:

$$H_N = H_1[\ln(N)]^{\frac{1}{k_1}} \tag{A.3}$$

$H_{1/N}$ is evaluated via

$$H_{1/N} = N\int_{H_N}^{H_{tr}} H f_1(H)\, dH + N\int_{H_{tr}}^{\infty} H f_2(H)\, dH \tag{A.4}$$

The first integral can be rewritten into:

$$N\int_{H_N}^{H_{tr}} H f_1(H)\, dH = N\int_{H_N}^{\infty} H f_1(H)\, dH - N\int_{H_{tr}}^{\infty} H f_1(H)\, dH \tag{A.5}$$

yielding:

$$H_{1/N} = N\int_{H_N}^{\infty} H f_1(H)\, dH - N\int_{H_{tr}}^{\infty} H f_1(H)\, dH + N\int_{H_{tr}}^{\infty} H f_2(H)\, dH \tag{A.6}$$

Substitution of the probability density function (3.3) in (A.6) yields:

$$\begin{aligned} H_{1/N} = {} & N\int_{H_N}^{\infty} H \frac{k_1}{H_1^{k_1}}\, H^{k_1-1} \exp\left[-\left(\frac{H}{H_1}\right)^{k_1}\right] dH - N\int_{H_{tr}}^{\infty} H \frac{k_1}{H_1^{k_1}}\, H^{k_1-1} \exp\left[-\left(\frac{H}{H_1}\right)^{k_1}\right] dH \\ & + N\int_{H_{tr}}^{\infty} H \frac{k_2}{H_2^{k_2}}\, H^{k_2-1} \exp\left[-\left(\frac{H}{H_2}\right)^{k_2}\right] dH \end{aligned} \tag{A.7}$$

Assume

$$t = \left(\frac{H}{H_i}\right)^{k_i} \quad \Rightarrow \quad H = H_i\, t^{\frac{1}{k_i}} \quad \Rightarrow \quad dH = \frac{H_i}{k_i}\, t^{\frac{1}{k_i}-1}\, dt \qquad for \quad i = 1,2. \tag{A.8}$$

When this transformation is applied to (A.7) the following equation is obtained:

$$H_{1/N} = N H_1 \int_{\left(\frac{H_N}{H_1}\right)^{k_1}}^{\infty} t^{\frac{1}{k_1}} \exp[-t]dt - N H_1 \int_{\left(\frac{H_{tr}}{H_1}\right)^{k_1}}^{\infty} t^{\frac{1}{k_1}} \exp[-t]dt + N H_2 \int_{\left(\frac{H_{tr}}{H_2}\right)^{k_2}}^{\infty} t^{\frac{1}{k_2}} \exp[-t]dt \tag{A.9}$$

With the incomplete gamma functions described in Section A.4 equation (A.9) is rewritten into :

$$H_{1/N} = NH_1\left(\Gamma\left[\frac{1}{k_1}+1,\ \ln(N)\right]-\Gamma\left[\frac{1}{k_1}+1,\ \left(\frac{H_{tr}}{H_1}\right)^{k_1}\right]\right) + NH_2\Gamma\left[\frac{1}{k_2}+1,\ \left(\frac{H_{tr}}{H_2}\right)^{k_2}\right] \tag{A.10}$$

Transitional wave height exceeded by H_N

Figure A.2 shows a wave height distribution on a shallow foreshore for the case $H_{tr}<H_N$, which means that the transitional wave height is exceeded by the wave height with a certain probability of exceedance, H_N.

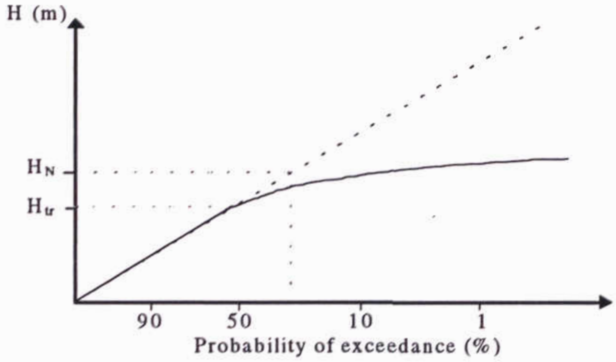

Figure A.2: Composed Weibull wave height distribution with $H_{tr}<H_N$.

When H_{tr} is exceeded by H_N, the determination of the $H_{1/N}$ is less complicated. H_N is determined by:

$$Q_{\underline{H}_2} \equiv \Pr_2\{\underline{H} \geq H\} = \frac{1}{N} = \exp\left[-\left(\frac{H_N}{H_2}\right)^{k_2}\right] \tag{A.11}$$

yielding:

$$H_N = H_2[\ln(N)]^{\frac{1}{k_2}} \tag{A.12}$$

$H_{1/N}$ is evaluated via

$$H_{1/N} = N \int_{H_N}^{\infty} H f_2(H) dH \tag{A.13}$$

When the probability density function $f_2(H)$ from the Composed Weibull distribution (3.3) is substituted, the following equation is obtained:

$$H_{1/N} = N \int_{H_N}^{\infty} H \frac{k_2}{H_2^{k_2}} H^{k_2-1} \exp\left[-\left(\frac{H}{H_2}\right)^{k_2}\right] dH \tag{A.14}$$

which can be transformed with (A.8) into:

$$H_{1/N} = N H_2 \int_{\left(\frac{H_N}{H_2}\right)^{k_2}}^{\infty} t^{\frac{1}{k_2}} \exp[-t] dt \tag{A.15}$$

With the incomplete gamma functions (Section A.4) equation (A.15) is rewritten into:

$$H_{1/N} = N H_2 \Gamma\left[\frac{1}{k_2}+1, \left(\frac{H_N}{H_2}\right)^{k_2}\right] \tag{A.16}$$

With (A.12) the following equation is obtained:

$$H_{1/N} = N H_2 \Gamma\left[\frac{1}{k_2}+1, \ln(N)\right] \tag{A.17}$$

In coastal engineering design practice one often is interested in the ratio of an extreme wave height to the significant wave height, $H_{2\%}/H_{1/3}$, $H_{1\%}/H_{1/3}$ or $H_{0.1\%}/H_{1/3}$. The H_N to $H_{1/3}$ ratio for the Composed Weibull distribution is derived in the following section.

A.2 Extreme wave height to significant wave height ratio

In the evaluation of the H_N to $H_{1/3}$ ratio three different situations must be distinguished:

1. $H_N < H_{tr}$
2. $H_3 < H_{tr} < H_N$
3. $H_{tr} < H_3$

with $H_3 < H_N$.

$H_N < H_{tr}$

In relatively deep water the transitional wave height is exceeding both H_3 and H_N, as is shown in Figure A.3. The H_N to $H_{1/3}$ ratio is obtained by evaluating the first part of the composed Weibull distribution $F_1(H)$. The wave height exceeded by 1/N-part of the waves heights is determined by (A.3) and the significant wave height is determined by equation (A.10).

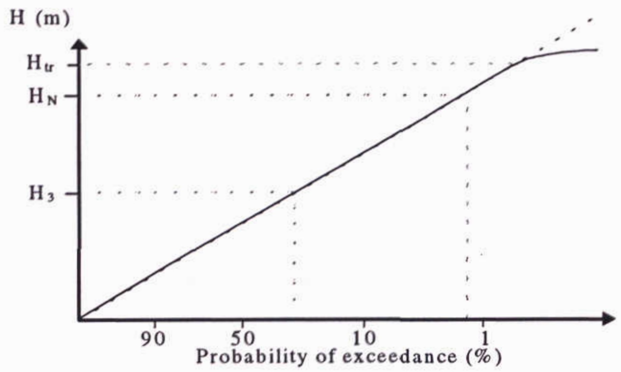

Figure A.3: Composed Weibull wave height distribution with $H_N<H_{tr}$.

The H_N to $H_{1/3}$ ratio becomes

$$\frac{H_N}{H_{1/3}} = \frac{H_1\left[\ln(N)\right]^{\frac{1}{k_1}}}{3H_1\left(\Gamma\left[\frac{1}{k_1}+1,\ln(N)\right]-\Gamma\left[\frac{1}{k_1}+1,\left(\frac{H_{tr}}{H_1}\right)^{k_1}\right]\right) + 3H_2\Gamma\left[\frac{1}{k_2}+1,\left(\frac{H_{tr}}{H_2}\right)^{k_2}\right]} \qquad \text{(A.18)}$$

$H_3<H_{tr}<H_N$

When the waves propagate into shallower water, the wave height at which the waves deviate from the Rayleigh distribution decreases.

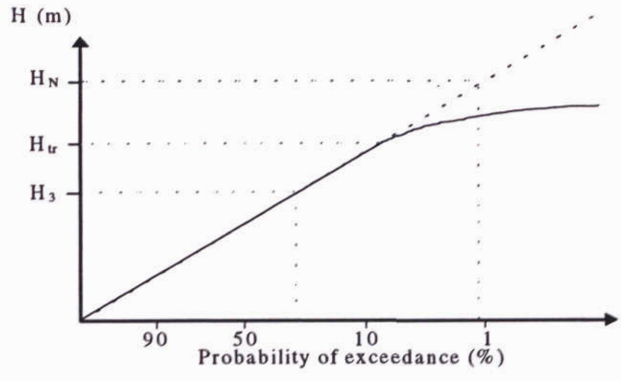

Figure A.4: Composed Weibull wave height distribution with $H_3<H_{tr}<H_N$.

As shown in Figure A.4, the transitional wave height is exceeded by the wave height with a probability of exceedance of $1/N$. Therefore H_N is determined by the second part of the composed wave height distribution via equation (A.12). The wave height with an exceedance probability of 1/3, H_3, is now obtained via

$$H_3 = H_1\left[\ln(3)\right]^{\frac{1}{k_1}} \qquad \text{(A.19)}$$

when H_3 is smaller than the transitional wave height. Equation (A.10) provides the mean of the highest 1/3-part of the wave height distribution, $H_{1/3}$. Thus, for $H_3 < H_{tr} < H_N$ the following equation determines the H_N to $H_{1/3}$ ratio:

$$\frac{H_N}{H_{1/3}} = \frac{H_2\left[\ln(N)\right]^{\frac{1}{k_2}}}{3H_1\left(\Gamma\left[\frac{1}{k_1}+1,\ \ln(3)\right]-\Gamma\left[\frac{1}{k_1}+1,\ \left(\frac{H_{tr}}{H_1}\right)^{k_1}\right]\right) + 3H_2\Gamma\left[\frac{1}{k_2}+1,\ \left(\frac{H_{tr}}{H_2}\right)^{k_2}\right]} \tag{A.20}$$

$H_{tr} < H_3$

In Figure A.5 the wave height, with an exceedance probability of 1/3, H_3, exceeds the transitional wave height.

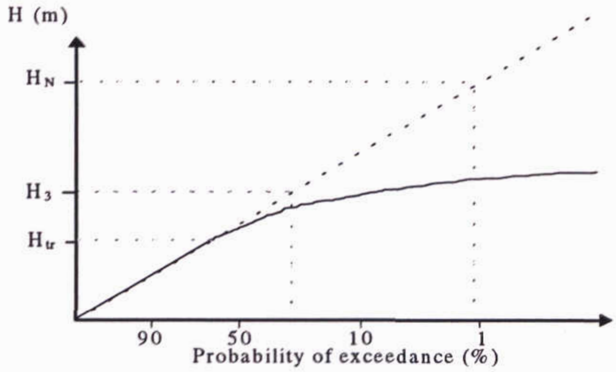

Figure 3.5: Composed Weibull height distribution with $H_{tr} < H_3$.

All wave heights exceeding H_{tr} obey the second part of the wave height distribution, $F_2(H)$, H_N is determined by (A.12) and the significant wave height by

$$H_{1/3} = 3\,H_2\,\Gamma\left[\frac{1}{k_2}+1,\ \ln(3)\right] \tag{A.21}$$

Therefore the following H_N to $H_{1/3}$ ratio is found:

$$\frac{H_N}{H_{1/3}} = \frac{\left[\ln(N)\right]^{\frac{1}{k_2}}}{3\Gamma\left[\frac{1}{k_2}+1,\ \ln(3)\right]} \tag{A.22}$$

A.3 Evaluation of the $H_{1\%}$ to $H_{1/3}$ ratio

In this section a specific ratio of a wave height with a certain probability of exceedance to the significant wave height is evaluated as an example. Since in coastal engineering practice $H_{1/N\%}$ is often used to represent a wave height with a certain probability of exceedance, in this section the same notation is used.

$H_{1\%}$ to $H_{1/3}$ ratio in deep water

In this section the $H_{1\%}$ to $H_{1/3}$ ratio is given for deep and extremely shallow water. In deep water, where all the wave heights obey the Rayleigh distribution $H_{tr} \to \infty$. This means that (A.18) reduces to

$$\lim_{H_{tr}\to\infty}\left(\frac{H_N}{H_{1/3}}\right) = \frac{[\ln(N)]^{\frac{1}{k_1}}}{3\Gamma\left[\frac{1}{k_1}+1,\ \ln(3)\right]} \tag{A.23}$$

In Section 3.3.2 assumptions are proposed for the exponents k_1 and k_2, i.e. k_1=2 and k_2=3.5. If k_1=2 and N=100 are substituted in equation (A.23), the following $H_{1\%}$ to $H_{1/3}$ ratio is obtained:

$$\frac{H_{1\%}}{H_{1/3}} = \frac{[\ln(100)]^{\frac{1}{2}}}{3\Gamma\left[\frac{1}{2}+1,\ \ln(3)\right]} \approx 1.52 \tag{A.24}$$

This is in line with the $H_{1\%}$ to $H_{1/3}$ ratio of the Rayleigh distribution, since from (2.4):

$$Q_{\underline{H}}(H) \equiv \Pr\{\underline{H} > H\} = \exp\left[-2\left(\frac{H}{H_{1/3}}\right)^2\right] \tag{A.25}$$

the same $H_{1\%}$ to $H_{1/3}$ ratio is obtained:

$$\frac{H_{1\%}}{H_{1/3}} = \sqrt{\tfrac{1}{2}\ln\left(Q_H^{-1}\right)} = \sqrt{\tfrac{1}{2}\ln(100)} \approx 1.52 \tag{A.26}$$

$H_{1\%}$ to $H_{1/3}$ ratio in extremely shallow water

In extremely shallow water, where $H_{1\%}$ and H_3 exceed the transitional wave height, equation (A.22) was found to represent the $H_{1\%}$ to $H_{1/3}$ ratio. When k_2= 3.5 is substituted in (A.22) the following constant $H_{1\%}$ to $H_{1/3}$ ratio is obtained:

$$\frac{H_{1\%}}{H_{1/3}} = \frac{[\ln(100)]^{\frac{1}{3.5}}}{3\Gamma\left[\frac{1}{3.5}+1,\ \ln(3)\right]} \approx 1.28 \tag{A.27}$$

This means that, when $H_{tr} < H_3$, the H_N to $H_{1/3}$ ratio fully depends on the shape of the second part of the composed wave height distribution.

A.4 Incomplete gamma functions

The gamma function is defined by:

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp[-t]\,dt \qquad 0 < t < \infty \tag{A.28}$$

This gamma function is a generalization of the factorial function. The gamma function has the following properties.

$$\Gamma(a+1) \;=\; a\Gamma(a) \tag{A.29}$$

$$\Gamma(a+1) \;=\; a! \qquad \textit{for} \quad a=1,2,\ldots,n. \tag{A.30}$$

The incomplete gamma function is defined by (Abramowitz and stegun 1965):

$$P(a,x) \;=\; \frac{\gamma(a,x)}{\Gamma(a)} \;=\; \frac{1}{\Gamma(a)}\int_0^x t^{a-1}\exp[-t]\,dt \qquad (a>0) \tag{A.31}$$

in which Γ(a) is the complete gamma function. Therefore $\gamma(a,x)$ is defined by:

$$\gamma(a,x) \;=\; \int_0^x t^{a-1}\exp[-t]\,dt \tag{A.32}$$

The complement of *P(a,x)* is defined by:

$$Q(a,x) \;=\; 1-P(a,x) \;=\; \frac{\Gamma(a,x)}{\Gamma(a)} \;=\; \frac{1}{\Gamma(a)}\int_x^\infty t^{a-1}\exp[-t]\,dt \tag{A.33}$$

yielding

$$\Gamma(a,x) \;=\; \int_x^\infty t^{a-1}\exp[-t]\,dt \tag{A.34}$$

Appendix C

***Table 7.1**: Characteristic nondimensional wave heights $\tilde{H}_x$ as function of $\tilde{H}_{tr}$.*

$\tilde{H}_{tr}$	$\tilde{H}_1$	$\tilde{H}_2$	$\tilde{H}_{1/3}$	$\tilde{H}_{1/10}$	$\tilde{H}_{2\%}$	$\tilde{H}_{1\%}$	$\tilde{H}_{0.1\%}$
0.05	9.949	1.029	1.249	1.438	1.520	1.592	1.788
0.10	5.916	1.029	1.249	1.438	1.520	1.592	1.788
0.15	4.365	1.029	1.249	1.438	1.520	1.592	1.788
0.20	3.518	1.029	1.249	1.438	1.520	1.592	1.788
0.25	2.976	1.029	1.249	1.438	1.520	1.593	1.788
0.30	2.595	1.029	1.249	1.438	1.520	1.593	1.788
0.35	2.312	1.030	1.249	1.438	1.520	1.593	1.788
0.40	2.092	1.030	1.250	1.438	1.520	1.593	1.789
0.45	1.916	1.030	1.250	1.438	1.521	1.593	1.789
0.50	1.772	1.030	1.250	1.439	1.521	1.594	1.790
0.55	1.651	1.031	1.251	1.440	1.522	1.595	1.791
0.60	1.549	1.032	1.252	1.441	1.523	1.596	1.792
0.65	1.462	1.033	1.253	1.442	1.525	1.598	1.794
0.70	1.387	1.035	1.255	1.445	1.528	1.600	1.797
0.75	1.322	1.037	1.258	1.448	1.531	1.604	1.801
0.80	1.265	1.040	1.262	1.452	1.535	1.608	1.806
0.85	1.216	1.043	1.266	1.457	1.540	1.614	1.812
0.90	1.174	1.048	1.271	1.463	1.547	1.621	1.820
0.95	1.137	1.053	1.278	1.470	1.555	1.629	1.829
1.00	1.106	1.059	1.285	1.479	1.564	1.638	1.840
1.05	1.079	1.066	1.294	1.489	1.575	1.650	1.852
1.10	1.056	1.075	1.304	1.501	1.587	1.663	1.867
1.15	1.037	1.084	1.314	1.514	1.601	1.677	1.883
1.20	1.021	1.094	1.321	1.528	1.616	1.693	1.901
1.25	1.008	1.105	1.327	1.544	1.632	1.710	1.920
1.30	0.998	1.117	1.331	1.561	1.650	1.729	1.941
1.35	0.989	1.130	1.335	1.578	1.669	1.748	1.963
1.40	0.983	1.144	1.338	1.597	1.689	1.769	1.987
1.45	0.978	1.158	1.341	1.617	1.709	1.791	2.011
1.50	0.974	1.172	1.344	1.637	1.731	1.813	2.036
1.55	0.972	1.187	1.347	1.654	1.753	1.836	2.062
1.60	0.970	1.202	1.350	1.669	1.775	1.860	2.088
1.65	0.970	1.218	1.354	1.682	1.798	1.884	2.115
1.70	0.969	1.233	1.357	1.694	1.821	1.908	2.142
1.75	0.970	1.249	1.361	1.704	1.844	1.932	2.170
1.80	0.971	1.265	1.364	1.714	1.868	1.957	2.197
1.85	0.972	1.281	1.368	1.722	1.891	1.981	2.225
1.90	0.974	1.297	1.372	1.730	1.915	2.006	2.252
1.95	0.975	1.312	1.375	1.737	1.926	2.030	2.280
2.00	0.977	1.328	1.379	1.744	1.929	2.055	2.307
2.05	0.979	1.344	1.382	1.750	1.933	2.079	2.334
2.10	0.981	1.359	1.386	1.756	1.936	2.103	2.361
2.15	0.983	1.374	1.389	1.761	1.940	2.105	2.388
2.20	0.985	1.390	1.392	1.766	1.944	2.109	2.414
2.25	0.986	1.404	1.395	1.770	1.947	2.113	2.440

$\tilde{H}_{tr}$	$\tilde{H}_1$	$\tilde{H}_2$	$\tilde{H}_{1/3}$	$\tilde{H}_{1/10}$	$\tilde{H}_{2\%}$	$\tilde{H}_{1\%}$	$\tilde{H}_{0.1\%}$
2.30	0.988	1.419	1.397	1.774	1.951	2.116	2.465
2.35	0.989	1.433	1.400	1.778	1.954	2.120	2.490
2.40	0.991	1.448	1.402	1.781	1.957	2.123	2.515
2.45	0.992	1.462	1.404	1.784	1.960	2.126	2.539
2.50	0.993	1.475	1.406	1.786	1.962	2.129	2.563
2.55	0.994	1.489	1.407	1.789	1.965	2.132	2.586
2.60	0.995	1.502	1.409	1.791	1.967	2.134	2.609
2.65	0.996	1.515	1.410	1.792	1.969	2.136	2.616
2.70	0.997	1.528	1.411	1.794	1.970	2.137	2.618
2.75	0.997	1.540	1.412	1.795	1.971	2.139	2.620
2.80	0.998	1.553	1.413	1.796	1.973	2.140	2.621
2.85	0.998	1.565	1.413	1.797	1.974	2.141	2.623
2.90	0.999	1.577	1.414	1.797	1.974	2.142	2.624
2.95	0.999	1.589	1.414	1.798	1.975	2.143	2.625
3.00	0.999	1.601	1.414	1.798	1.976	2.144	2.625
3.05	0.999	1.612	1.415	1.799	1.976	2.144	2.626
3.10	0.999	1.623	1.415	1.799	1.977	2.144	2.626
3.15	1.000	1.635	1.415	1.799	1.977	2.145	2.627
3.20	1.000	1.646	1.415	1.799	1.977	2.145	2.627
3.25	1.000	1.657	1.415	1.799	1.977	2.145	2.627
3.30	1.000	1.668	1.415	1.800	1.977	2.145	2.628
3.35	1.000	1.679	1.416	1.800	1.978	2.146	2.628
3.40	1.000	1.689	1.416	1.800	1.978	2.146	2.628
3.45	1.000	1.700	1.416	1.800	1.978	2.146	2.628
3.50	1.000	1.711	1.416	1.800	1.978	2.146	2.628

Notation

Roman letters:

A	:	parameter based on the exponent in the Glukhovskiy distribution (-)
d	:	water depth (m).
d_t	:	water depth at the toe of the sea-defence work (m).
$\tilde{d}$	:	ratio of mean wave height to water depth, H_m/d (-).
d^*	:	ratio of root mean square wave height to water depth, H_{rms}/d (-).
H	:	wave height (m).
H_i	:	scale wave height of the Composed Weibull distribution (m).
$H_{1/3}$	:	significant wave height defined as the mean of the highest 1/3-part of the wave heights in a wave field (m).
$H_{1/3,0}$	:	incident significant wave height (m).
$H_{1/3,t}$	:	significant wave height at the toe of the sea-defence work (m).
H_m	:	mean wave height (m).
H_{m0}	:	spectral significant wave height, $H_{m0} = 4\sqrt{m_0}$ (m).
H_N	:	the wave height with an exceedance probability of $1/N$ (m).
$H_{1/N}$	:	mean of the highest $1/N$-part of the wave heights in a wave field (m).
H_{rms}	:	root mean square wave height (m).
H_t	:	transition wave height of the split Weibull distribution (Van Vledder 1993) (m).
H_{tr}	:	transitional wave height of the Composed Weibull distribution (m).
$\tilde{H}_x$	:	nondimensionalized characteristic wave height, $\tilde{H}_x = H_x/H_{rms}$ (-).
k_i	:	exponent of the Composed Weibull distribution (-).
L_f	:	foreshore length (m).
L_{op}	:	deep water wavelength based on the peak period T_p (m).
$L_{0,2}$	:	local wave length based on the average zero-crossing period $T_{0,2}$ (m).
m_n	:	n^{th} moment of the frequency spectrum (m^2).
m_0	:	variance of the water surface elevation, i.e. the total wave energy (m^2).
$T_{0,2}$	:	average zero-crossing period, $T_{0,2} = \sqrt{m_0/m_2}$ (s).

Greek letters:

α	:	slope of the foreshore (-).
α_{H1}	:	empirical coefficient of the scale wave height forecasting function (-).
α_{tr}	:	empirical coefficient of the transitional wave height forecasting function (-).
β	:	coefficient of the Modified Glukhovskiy distribution (Klopman 1996) (-).
β_{tr}	:	empirical coefficient of the transitional wave height forecasting function (-).
ε_{rms}	:	root mean square error (-)
γ_{tr}	:	empirical breaker coefficient of the transitional wave height (-).
κ	:	exponent of the Glukhovskiy distribution (-).
κ^*	:	exponent of the Modified Glukhovskiy distribution (Klopman 1996) (-).
σ	:	scale wave height of the split Weibull distribution (Van Vledder 1993) (m).
Ψ	:	degree of saturation, $\Psi = \sqrt{m_0}/d$ (-).