WAVE DISPERSION EQUATION

Linear Wave Theory

$$\frac{L}{L_0} = \tanh\left(\frac{2\pi h}{L}\right) = \tanh\left(\frac{k_0 h}{\left(\frac{L}{L_0}\right)}\right) \qquad L_0 = \frac{gT^2}{2\pi} \text{ and } k_0 = \frac{2\pi}{L_0} \qquad [0]$$

or $\frac{L}{L_0} = \phi(k_0 h)$

The APS/GEP solutions

Below there is a list of direct solutions of the Wave Dispersion Equation, an equation that has to be solved in typical coastal engineering calculations. They were obtained with APS-Automatic Problem Solver (www.gepsoft.com), a software program that applies GEP[1]-Gene Expression Programming to evolve computer programs (www.gene-expression-programming.com).

Approximate direct solutions

$$\frac{L}{L_0} \approx \tanh\left(\frac{k_0 h}{(\tanh(k_0 h))^{1/4} \cdot \sqrt{\tanh\left(\sqrt{\sinh(k_0 h)}\right)}}\right) \qquad [1]$$

(Maximum error: 0,04%)

$$\frac{L}{L_0} \approx \tanh\left(\frac{k_0 h}{\left(\tanh\left(\frac{k_0 h}{\left(\tanh\left(\frac{k_0 h}{\sinh\left(\tanh\left(\sqrt{k_0 h}\right)\right)}\right)\right)}\right)\right)}\right) \qquad [2]$$

(Maximum error: 0,05%)

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[1] *"GEP is a new evolutionary algorithm that evolves computer programs. The individuals of gene expression programming are encoded in linear chromosomes which are expressed or translated into expression trees (branched entities). Thus, in GEP, the genotype (the linear chromosomes) and the phenotype (the expression trees) are different entities (both structurally and functionally) that, nevertheless, work together forming an indivisible whole. As in nature, the linear chromosomes consist of the genetic material that is passed on with modification to the next generation. Therefore, in GEP, all the genetic modifications take place in the chromosomes, and only the chromosomes are transmitted in the process of reproduction. After reproduction the new chromosomes are expressed forming the body or expression trees (ETs). The ETs are themselves computer programs evolved to solve a particular problem and are selected according to their capabilities in solving the problem at hand. With time, populations of such computer programs discover new traits and become better adapted to a particular selection environment (for instance, a set of experimental results) and, hopefully, a good solution evolves. Due to the genotype/phenotype representation and to the multigenic organization of GEP chromosomes, this new algorithm surpasses the old GP system in 100-10,000 times"*. Source: www.gene-expression-programming.com.

$$\frac{L}{L_0} \approx \tanh\left(\frac{k_0 h}{\tanh\left(\dfrac{k_0 h}{\sinh\left(\tanh\left(\sqrt{k_0 h}\right)\right)}\right)}\right) \qquad [3]$$

(Maximum error: 0,14%)

$$\frac{L}{L_0} \approx \left(\tanh\left(k_0 h\right)\right)^{1/4} \cdot \sqrt{\tanh\left(\sqrt{\sinh\left(k_0 h\right)}\right)} \qquad [4]$$

(Maximum error: 0,20%)

$$\frac{L}{L_0} \approx \tanh\left(\left(\frac{6}{5}\right)^{k_0 h} \cdot \sqrt{k_0 h}\right) \qquad [5]$$

(Maximum error: 0,27%)

$$\frac{L}{L_0} \approx \tanh\left(\frac{k_0 h}{\tanh\left(\sinh\left(\sqrt{k_0 h}\right)\right)}\right) \qquad [6]$$

(Maximum error: 0,40%)

$$\frac{L}{L_0} \approx \sqrt{\tanh\left(\left(k_0 h\right) \cdot \sin\left(\cosh\left(\tanh\left(\ln\left(\tanh\left(\sinh\left(\sqrt{k_0 h}\right)\right)\right)\right)\right)\right)\right)} \qquad [7]$$

(Maximum error: 0,44%)

$$\frac{L}{L_0} \approx \sqrt{\tanh\left(\sqrt{k_0 h} \cdot \tanh\left(\sqrt{k_0 h}\right)\right)} \qquad [8]$$

(Maximum error: 0,89%)

$$\frac{L}{L_0} \approx \tanh\left(\sinh\left(\sqrt{k_0 h}\right)\right) \qquad [9]$$

(Maximum error:1,12%)

$$\frac{L}{L_0} \approx \left(\tanh\left(\sqrt{k_0 h}\right)\right)^{\left(1/\cosh\left(k_0 h\right)\right)} \qquad [10]$$

(Maximum error: 1,26%)

$$\frac{L}{L_0} \approx \sqrt{\tanh(k_0h)} \cdot \tanh\left(k_0h + \frac{1}{\sqrt{k_0h}}\right) \qquad [11]$$

(Maximum error: 1,33%)

$$\frac{L}{L_0} \approx (\tanh(k_0h))^{((k_0h+4)/8)} \qquad [12]$$

(Maximum error: 1,40%)

$$\frac{L}{L_0} \approx \sqrt{\tanh\left(\frac{\tanh(\tanh(k_0h))}{\tanh(1/(k_0h))}\right)} \qquad [13]$$

(Maximum error: 1,49%)

$$\frac{L}{L_0} \approx \left(1 + \frac{1}{(k_0h)^2}\right)^{-1/4} \qquad [14]$$

(Maximum error: 3,28%)

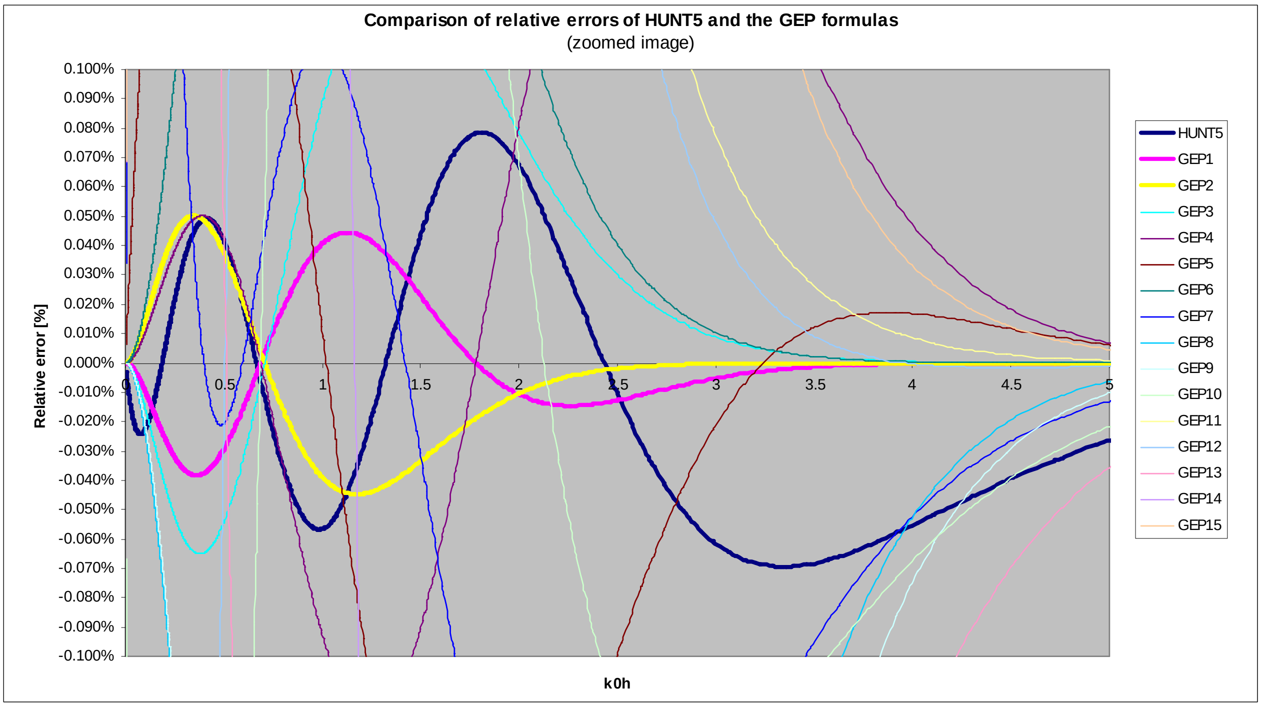
$$\frac{L}{L_0} \approx \sqrt{\tanh(k_0h)} \qquad [15]$$

(Maximum error: 5,24%)

Notice that some of these formulas were obtained with APS/GEP but they can also be derived applying the following iterative equation to some of the simpler GEP equations:

$$\left(\frac{L}{L_0}\right)_{n+1} = \tanh\left(\frac{k_0h}{\left(\frac{L}{L_0}\right)_n}\right)$$

Error comparison



Conclusions

Formulas [1] and [2] have almost symmetrical relative errors for k_0h below 1.5, and very small errors for the higher k_0h values, so they can be combined into a single formula in order to produce an extremely accurate direct solution to the Wave Dispersion Equation, with maximum error 0.012% for all k_0h values (below 0.006% for $k_0h \leq 1.5$), which is significantly more accurate than Hunt's 5th order formula. The corresponding final formula is then:

$$\frac{L}{L_0} \approx \frac{1}{2}\tanh\left(\frac{k_0h}{\left(\tanh(k_0h)\right)^{1/4}\cdot\sqrt{\tanh\left(\sqrt{\sinh(k_0h)}\right)}}\right)+\frac{1}{2}\tanh\left(\frac{k_0h}{\tanh\left(\frac{k_0h}{\tanh\left(\frac{k_0h}{\sinh\left(\tanh\left(\sqrt{k_0h}\right)\right)}\right)}\right)}\right) \qquad [16]$$

(Maximum error: 0,012%)

Just like in the case above, other GEP formulas can also be combined, so that their relative errors are compensated and better accuracy is achieved, although with smaller degrees of success. For example, the combination of formulas [3] and [5] produce the following formula, with maximum error 0,11% for all k_0h, which is almost as good as Hunt's 5th order formula (maximum error of 0.08% for all k_0h):

$$\frac{L}{L_0} \approx \frac{1}{2}\tanh\left(\left(\frac{6}{5}\right)^{k_0h}\cdot\sqrt{k_0h}\right)+\frac{1}{2}\tanh\left(\frac{k_0h}{\tanh\left(\frac{k_0h}{\sinh\left(\tanh\left(\sqrt{k_0h}\right)\right)}\right)}\right) \qquad [17]$$

(Maximum error: 0,11%)

(These are preliminary results that need further investigation and validation).