Incremental Stability of Discrete-Time First-Order Projection Elements

Riccardo Bertollo

Sebastiaan van den Eijnden

W.P.M.H. (Maurice) Heemels

{r.bertollo, s.j.a.m.v.d.eijnden, w.p.m.h.heemels}@tue.nl

IFAC Conference on Analysis and Design of Hybrid Systems

Boulder, July 1-3, 2024

Incremental Stability of Discrete-Time First-Order Projection Elements

[clearly not a talk on learning]

Riccardo Bertollo

Sebastiaan van den Eijnden

W.P.M.H. (Maurice) Heemels

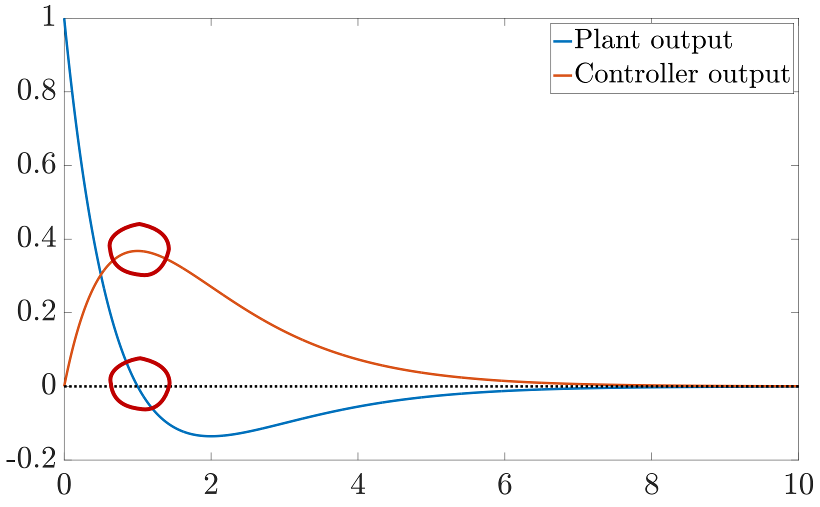
{r.bertollo, s.j.a.m.v.d.eijnden, w.p.m.h.heemels}@tue.nl

IFAC Conference on Analysis and Design of Hybrid Systems

Boulder, July 1-3, 2024

Motivation:
Improving performance of linear controllers

Integrator-based (linear) controllers are needed to reject constant disturbances and overcome friction, but they have some **intrinsic limitations**

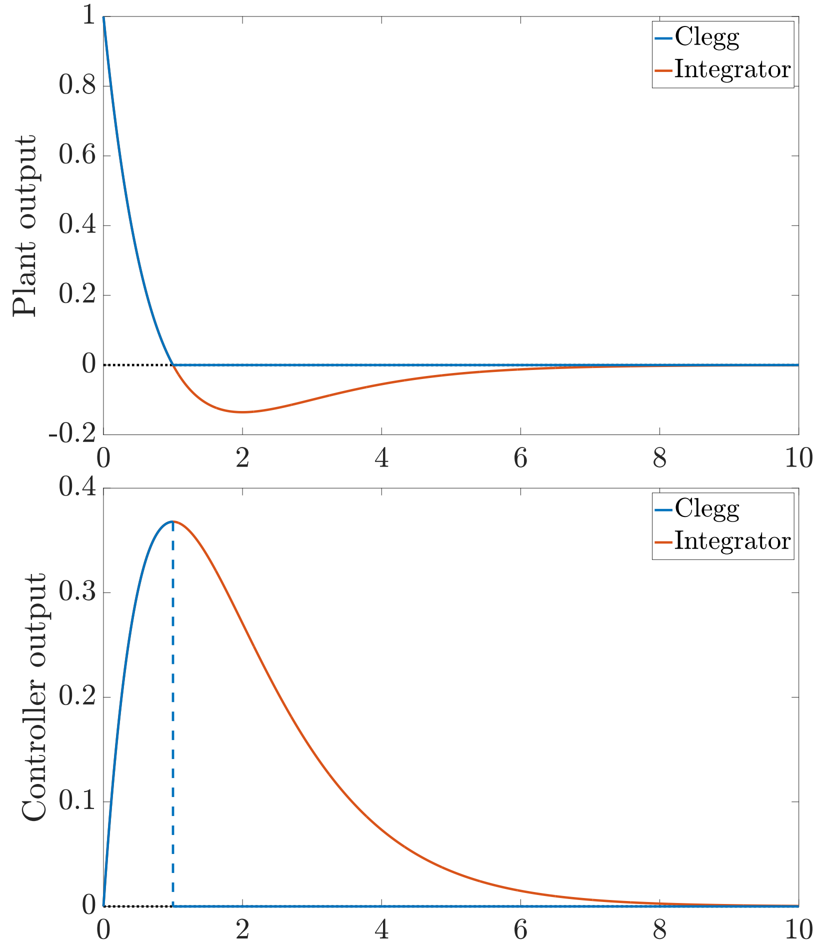


Motivation:
Improving performance of linear controllers

Integrator-based (linear) controllers are needed to reject constant disturbances and overcome friction, but they have some **intrinsic limitations**

[Clegg, 1958] proposed the addition of a resetting mechanism

$$\mathcal{H} := \begin{cases} \dot{u} = e, & ue \geq 0 \\ u^+ = 0, & ue < 0 \end{cases}$$



Motivation:
Improving performance of linear controllers

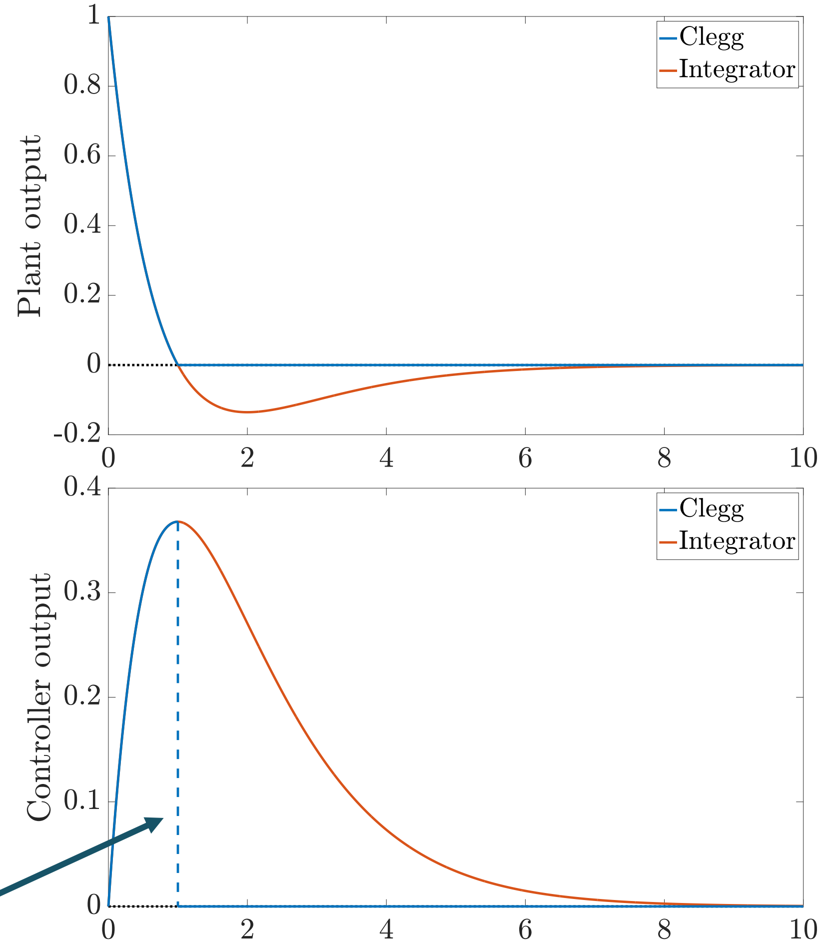
Integrator-based (linear) controllers are needed to reject constant disturbances and overcome friction, but they have some **intrinsic limitations**

[Clegg, 1958] proposed the addition of a resetting mechanism

$$\mathcal{H} := \begin{cases} \dot{u} = e, & ue \geq 0 \\ u^+ = 0, & ue < 0 \end{cases}$$

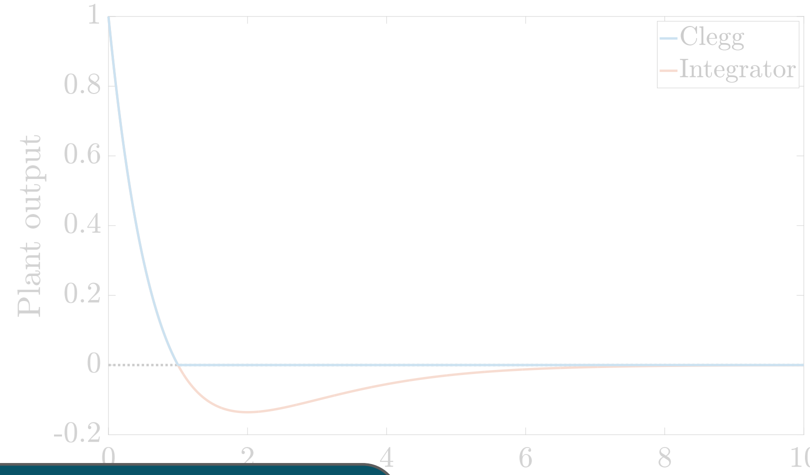
HOWEVER:

resets might be **too aggressive** to implement in practice



Motivation:
Improving performance of linear controllers

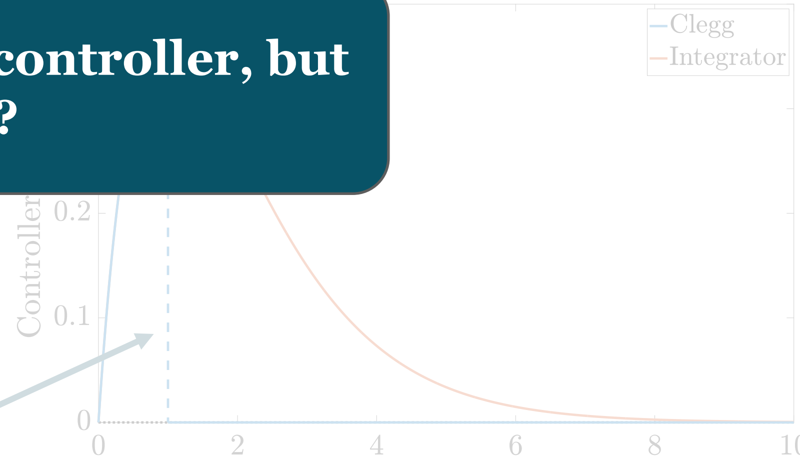
Integrator-based (linear) controllers are needed to reject constant disturbances and overcome friction, but they have some **intrinsic limitations**



[Clegg, 1958] p

Can we keep the good properties of the controller, but with continuous solutions?

$$\mathcal{H} := \begin{cases} \\ u^+ = 0, \qquad ue < 0 \end{cases}$$



HOWEVER:

resets might be **too aggressive** to implement in practice

A new approach: Projection-based control

IDEA:

The “good thing” about the Clegg integrator is that $eu \geq 0$

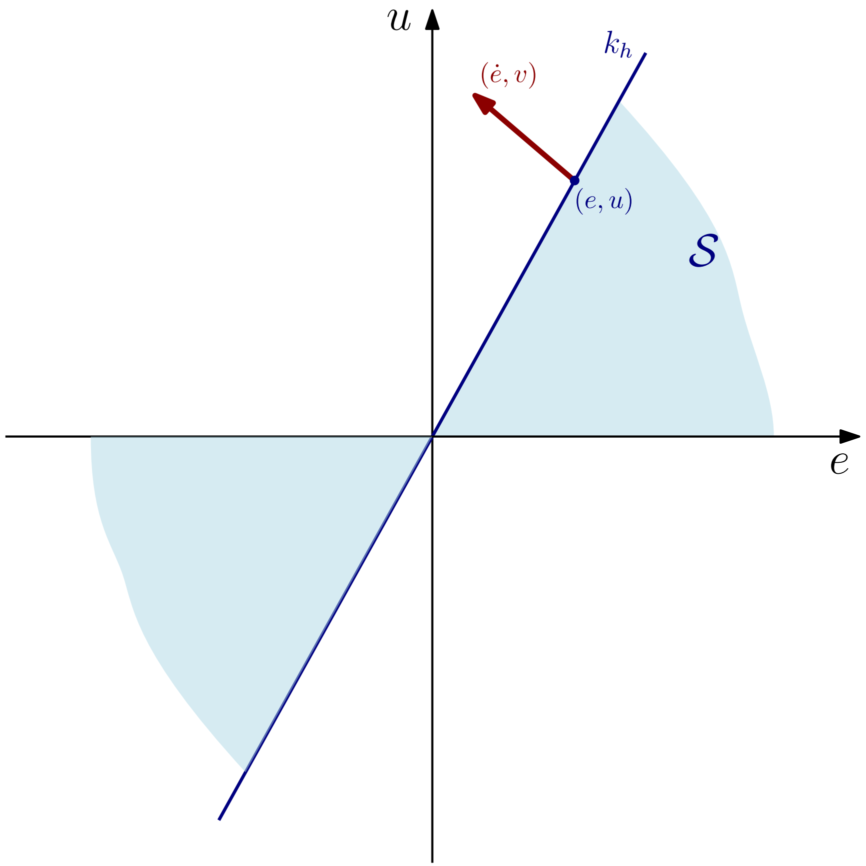
So, what if we **project the dynamics** instead of resetting?

A new approach: Projection-based control

IDEA:

The “good thing” about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?

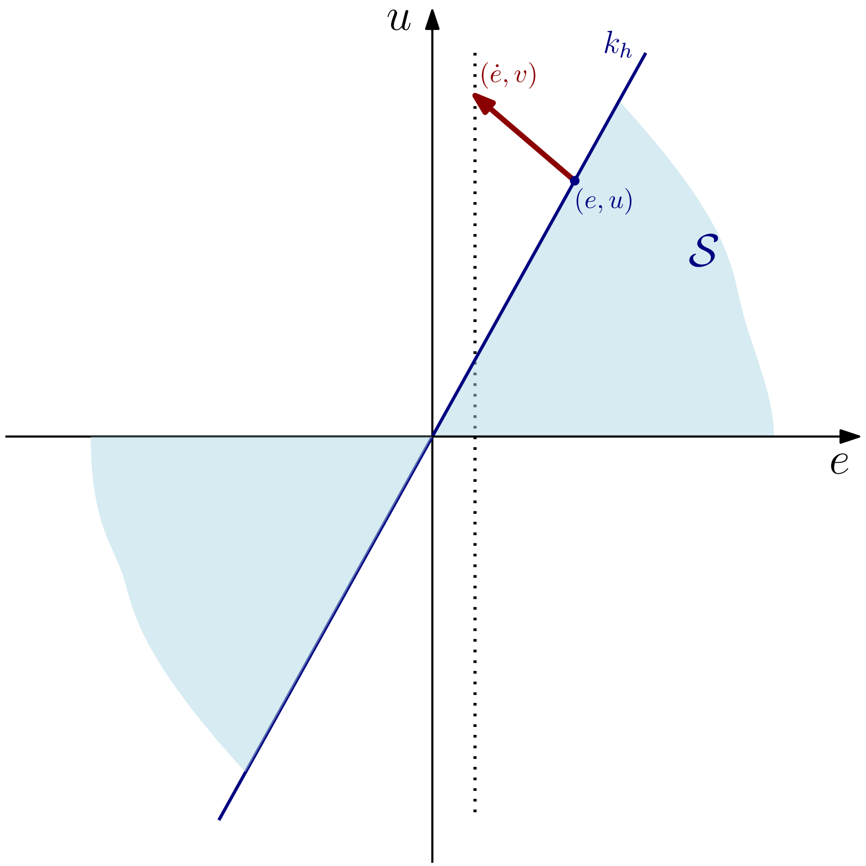


A new approach:
Projection-based control

IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?

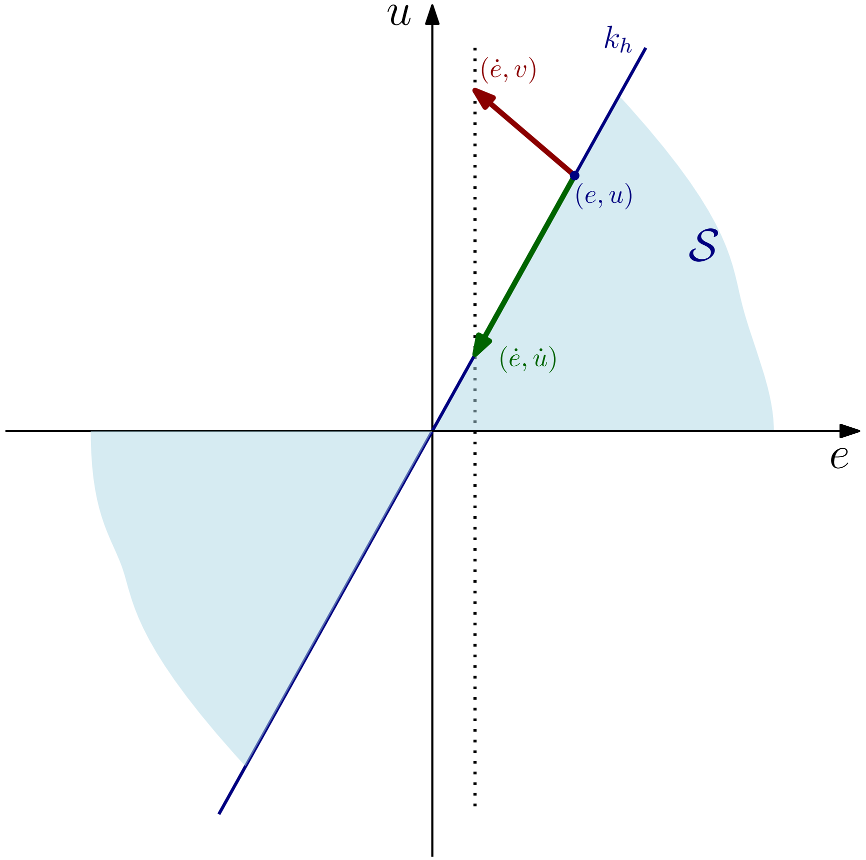


A new approach: Projection-based control

IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?



A new approach: Projection-based control

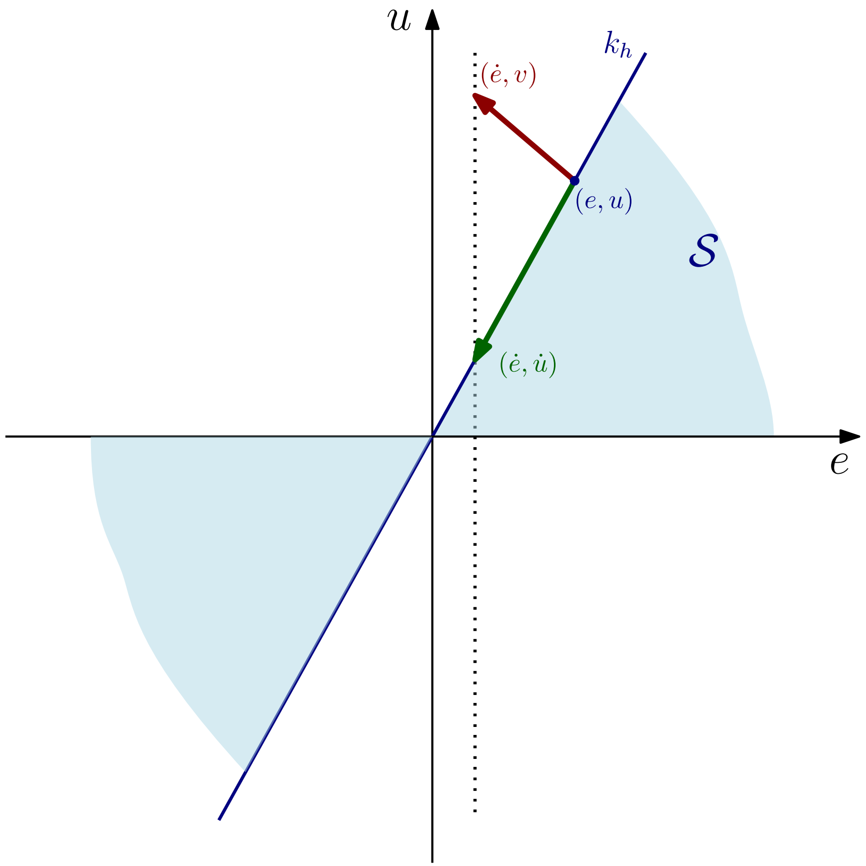
IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?

Hybrid Integrator-Gain Systems (HIGS) [Deenen et al., 2021]

$$\dot{u} = \begin{cases} \omega_h e, & (\omega_h e, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (\omega_h e, \dot{e}) \notin T_S(u) \end{cases}$$



A new approach: Projection-based control

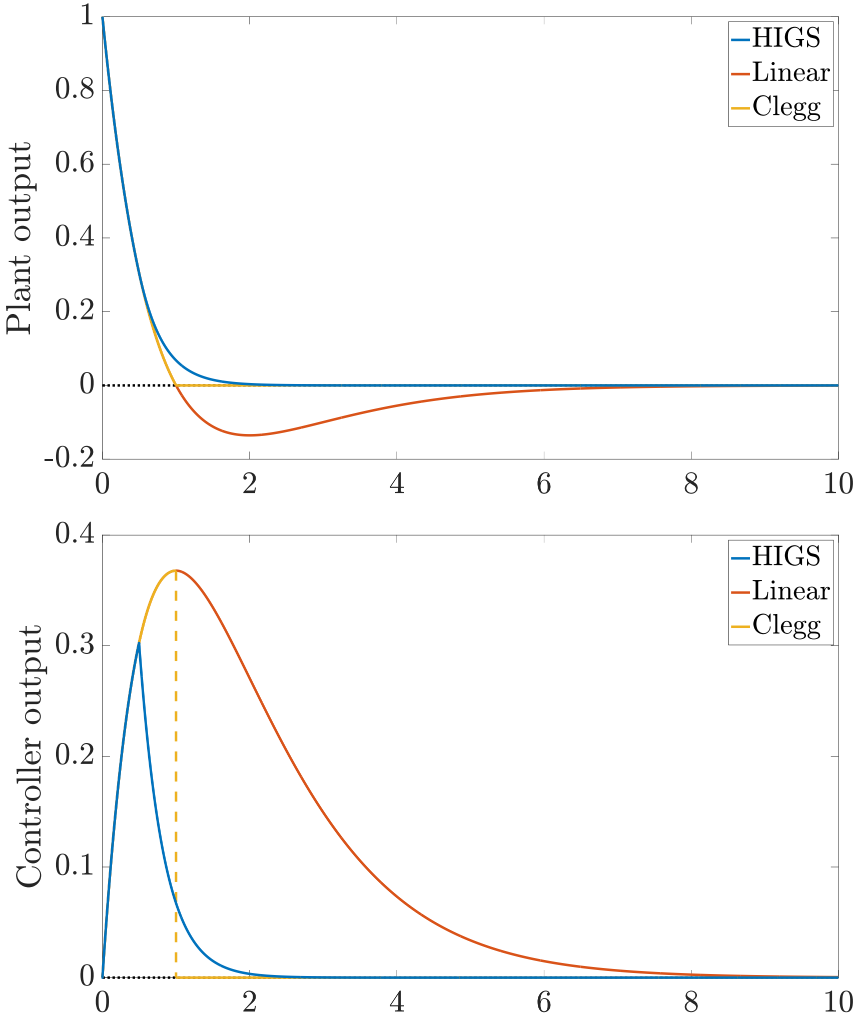
IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?

Hybrid Integrator-Gain Systems (HIGS) [Deenen et al., 2021]

$$\dot{u} = \begin{cases} \omega_h e, & (\omega_h e, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (\omega_h e, \dot{e}) \notin T_S(u) \end{cases}$$



A new approach: Projection-based control

IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

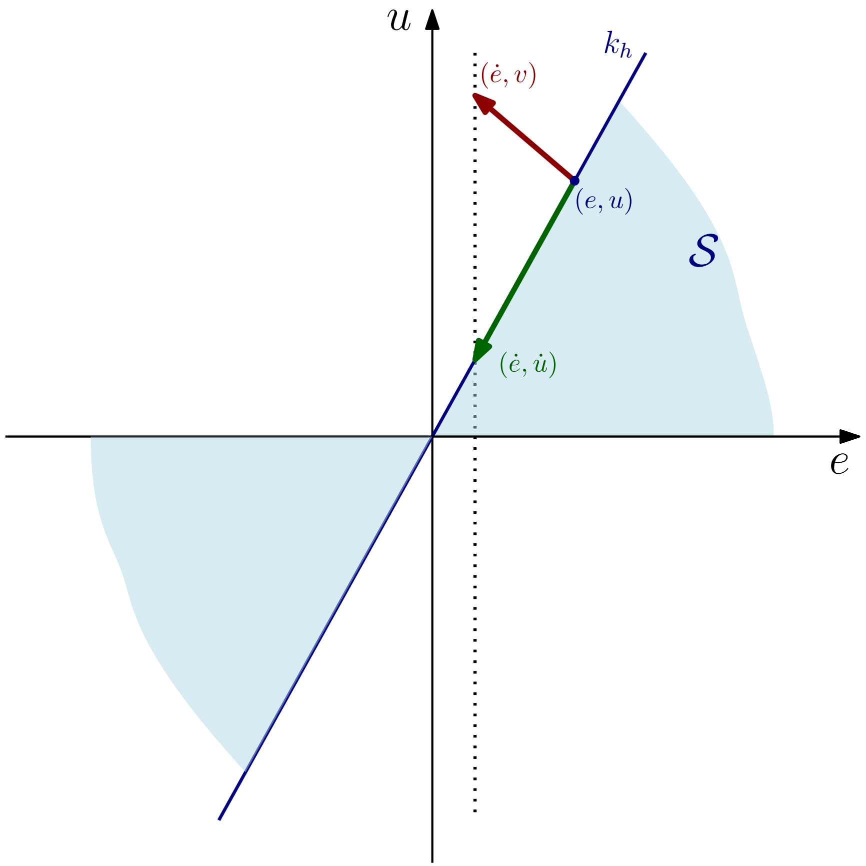
So, what if we **project the dynamics** instead of resetting?

Hybrid Integrator-Gain Systems (HIGS) [Deenen et al., 2021]

$$\dot{u} = \begin{cases} \omega_h e, & (\omega_h e, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (\omega_h e, \dot{e}) \notin T_S(u) \end{cases}$$

We can extend this to any underlying dynamics

$$\dot{u} = \begin{cases} v, & (v, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (v, \dot{e}) \notin T_S(u) \end{cases}$$



A new approach: Projection-based control

IDEA:

The "good thing" about the Clegg integrator is that $eu \geq 0$

So, what if we **project the dynamics** instead of resetting?

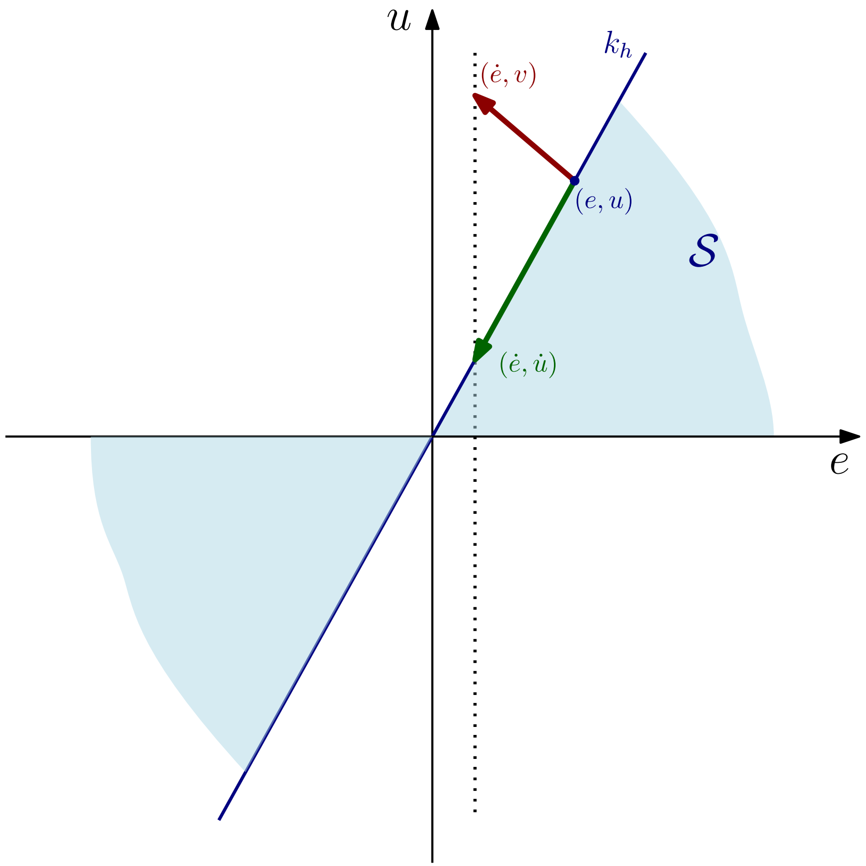
↓

Hybrid Integrator-Gain Systems (HIGS) [Deenen et al., 2021]

$$\dot{u} = \begin{cases} \omega_h e, & (\omega_h e, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (\omega_h e, \dot{e}) \notin T_S(u) \end{cases}$$

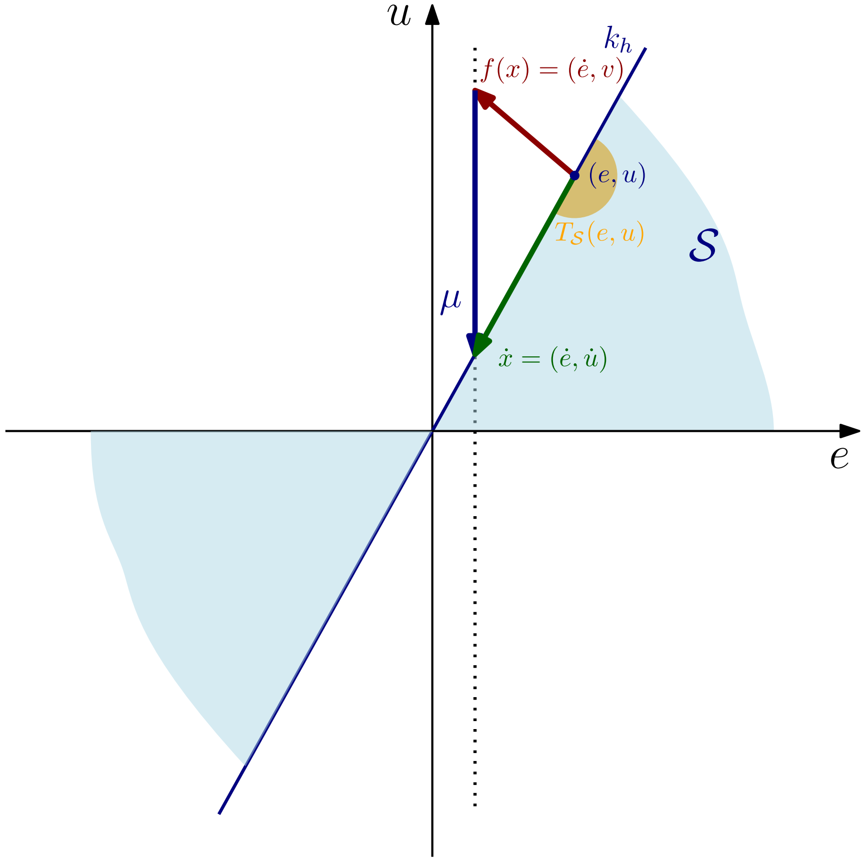
First-Order Projection Element (FOPE)

$$\dot{u} = \begin{cases} au + be, & (au + be, \dot{e}) \in T_S(u) \\ k_h \dot{e}, & (au + be, \dot{e}) \notin T_S(u) \end{cases}$$



Projection-based control: the ePDS framework

Since the error derivative cannot be modified, we can **project** only in **selected directions**



Projection-based control: the ePDS framework

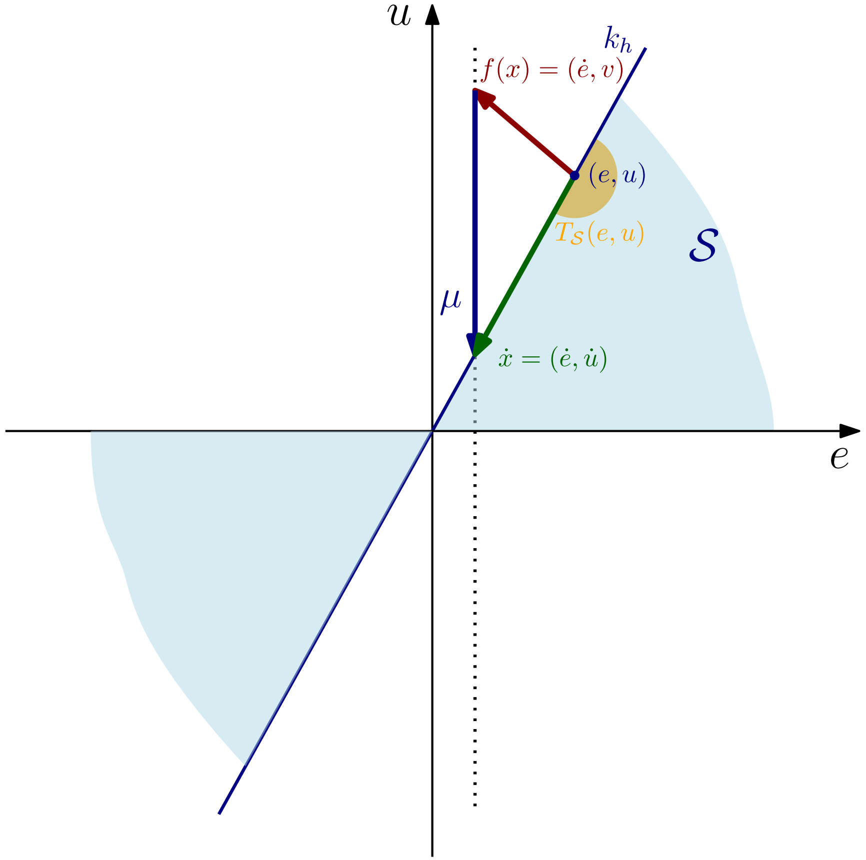
Since the error derivative cannot be modified, we can **project** only in **selected directions**

Extended Projected Dynamical Systems (ePDS)

$$\dot{x} = \prod_{\mathcal{S},E} f(x) = f(x) + \mu$$

where

$$\mu \coloneqq argmin\ ||w|| \quad s.t. \quad f(x) + w \in T_{\mathcal{S}}(x), \quad \boldsymbol{w} \in \boldsymbol{Im}(\boldsymbol{E})$$



Projection-based control: the ePDS framework

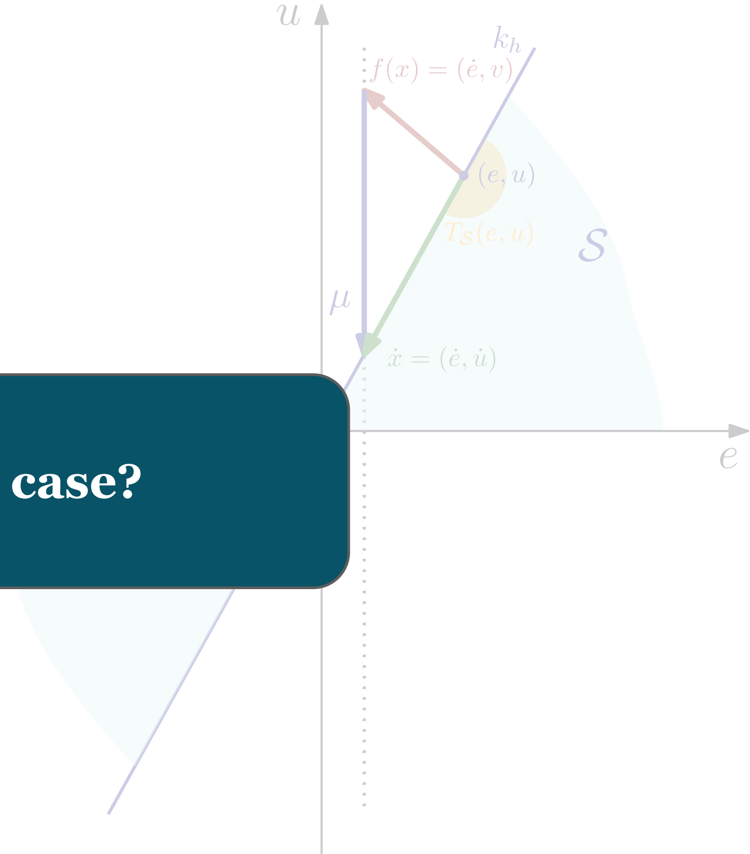
Since the error derivative cannot be modified, we can **project** only in **selected directions**

Extended Proj

$\dot{x} =$

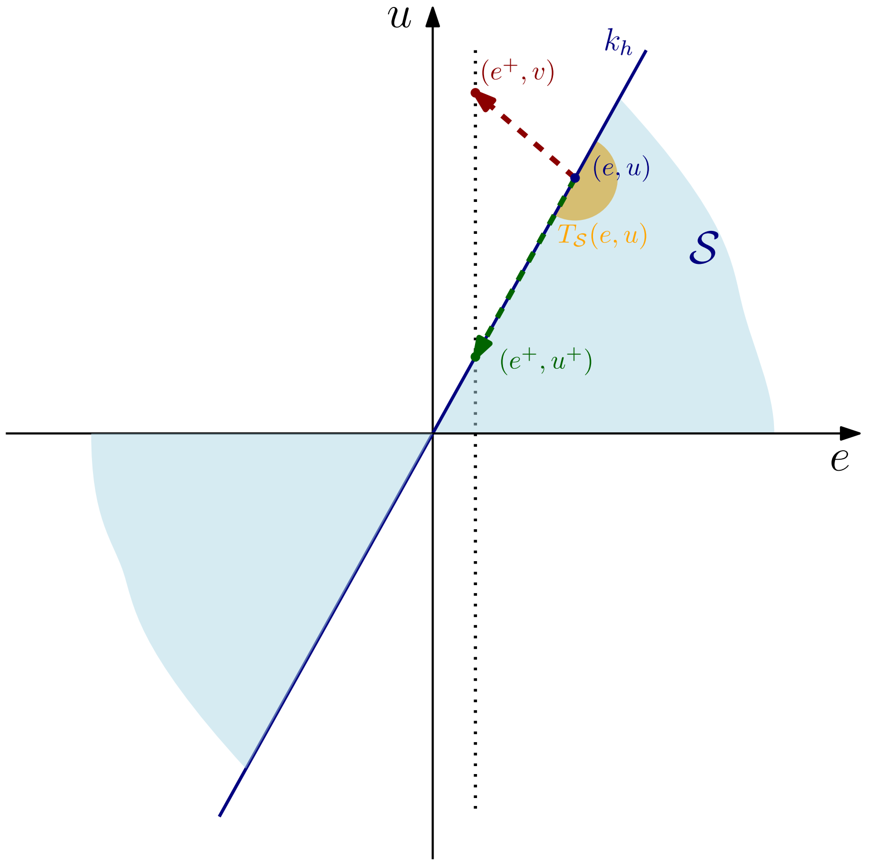
where

$$\mu \coloneqq argmin\,||w||$$
$$s.t. \quad f(x) + w \in T_S(x)$$
$$\boldsymbol{w \in Im(E)}$$



What happens in the discrete-time case?

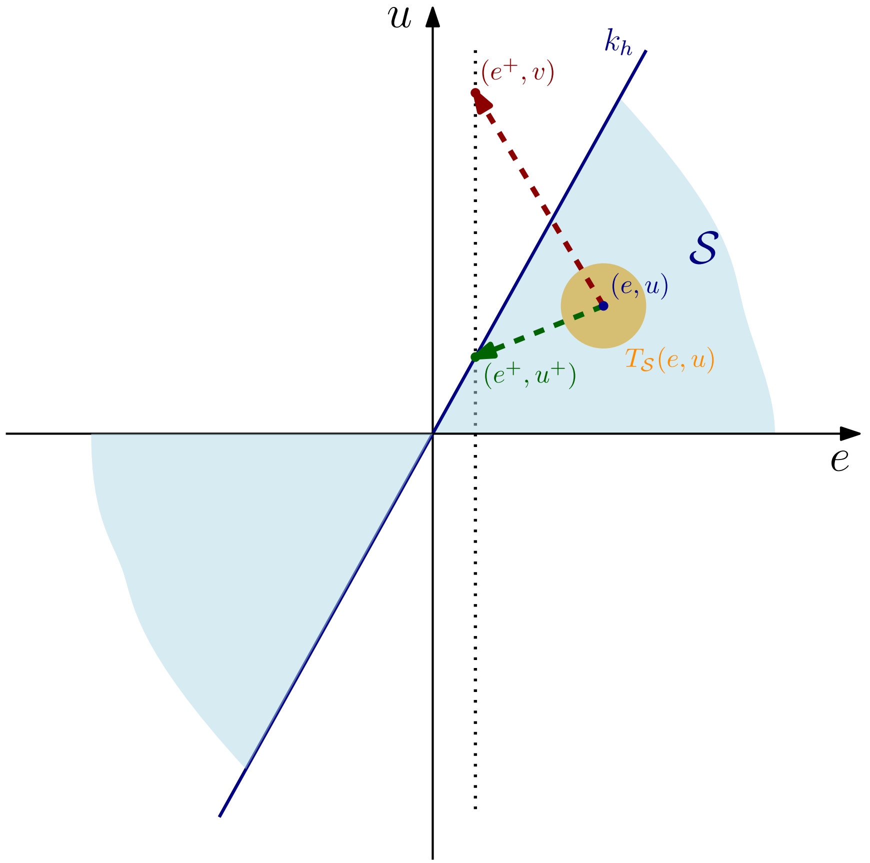
(DT) First-Order Projection Elements



(DT) First-Order Projection Elements

Possible issues:

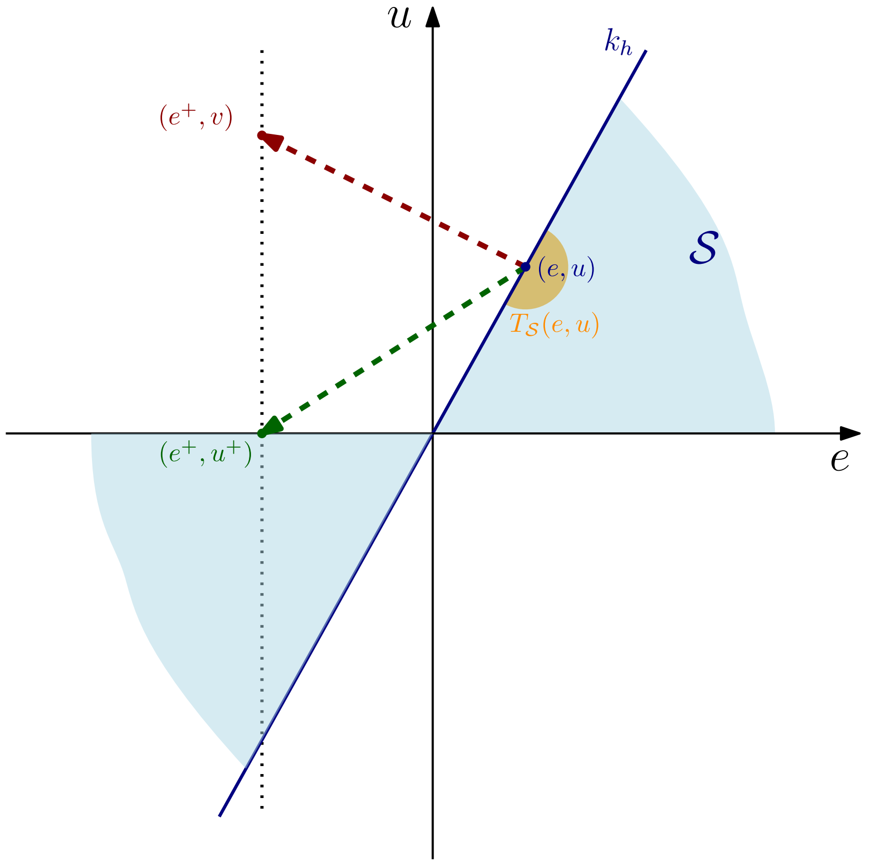
- **not necessarily** from the **boundary** of the set



(DT) First-Order Projection Elements

Possible issues:

- **not necessarily** from the **boundary** of the set
- e can **change sign**



(DT) First-Order Projection Elements

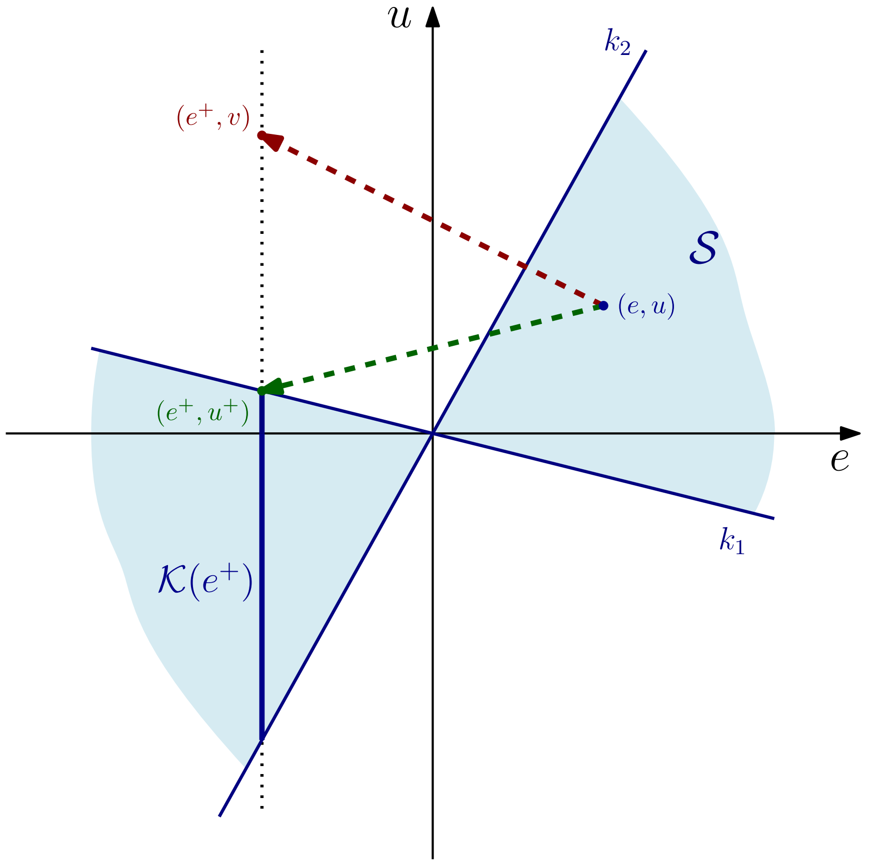
Possible issues:

- **not necessarily** from the **boundary** of the set
- e can **change sign**

First-Order Projection Element: projection on an **input-dependent set**

$$u^{+} = \Pi_{K(e^{+})}(v) = \boldsymbol{\Pi_{K(e^{+})}(au + be^{+})}$$

(where $|a| \leq 1, b > 0$)



(DT) First-Order Projection Elements

Possible issues:

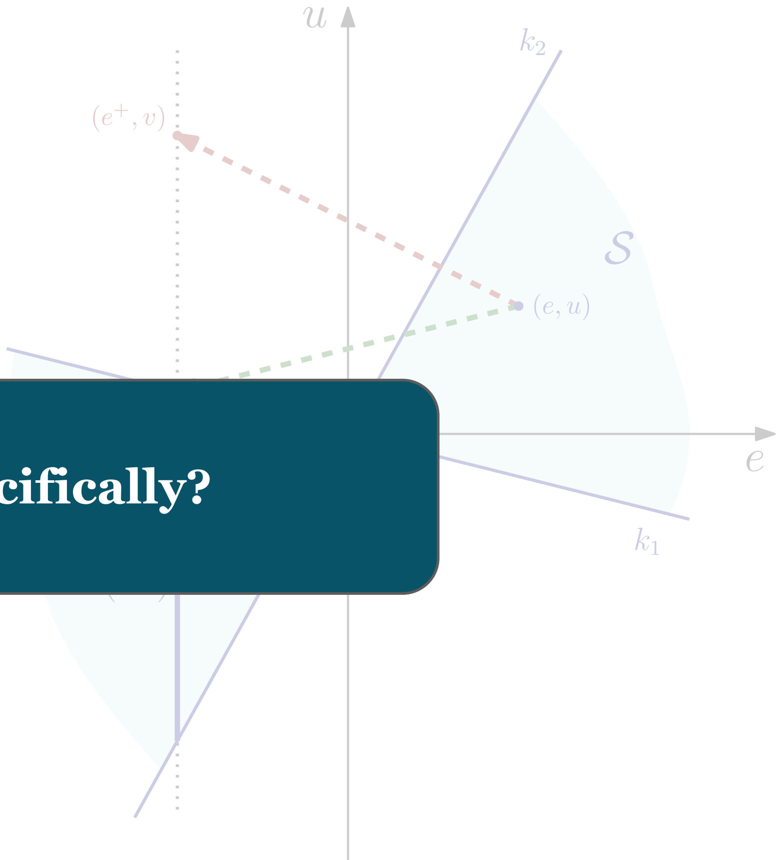
- **not necessarily** from the **boundary** of the set
  - e can **change sign**

What are we investigating specifically?

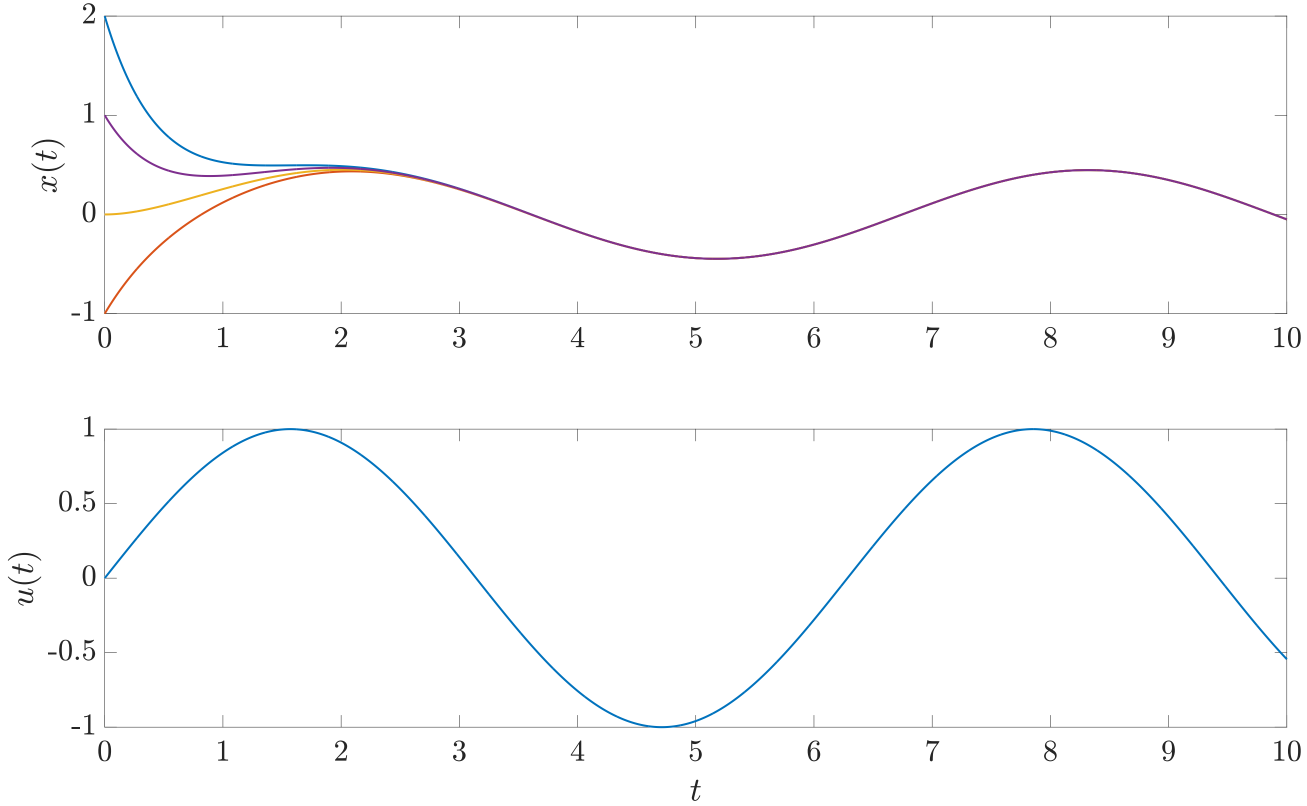
First-Order Projection Element: projection on an **input-dependent set**

$$\boldsymbol{u^+} = \Pi_{K(e^+)}(v) = \boldsymbol{\Pi_{K(e^+)}(au + be^+)}$$

(where $|a| \leq 1, b > 0$)



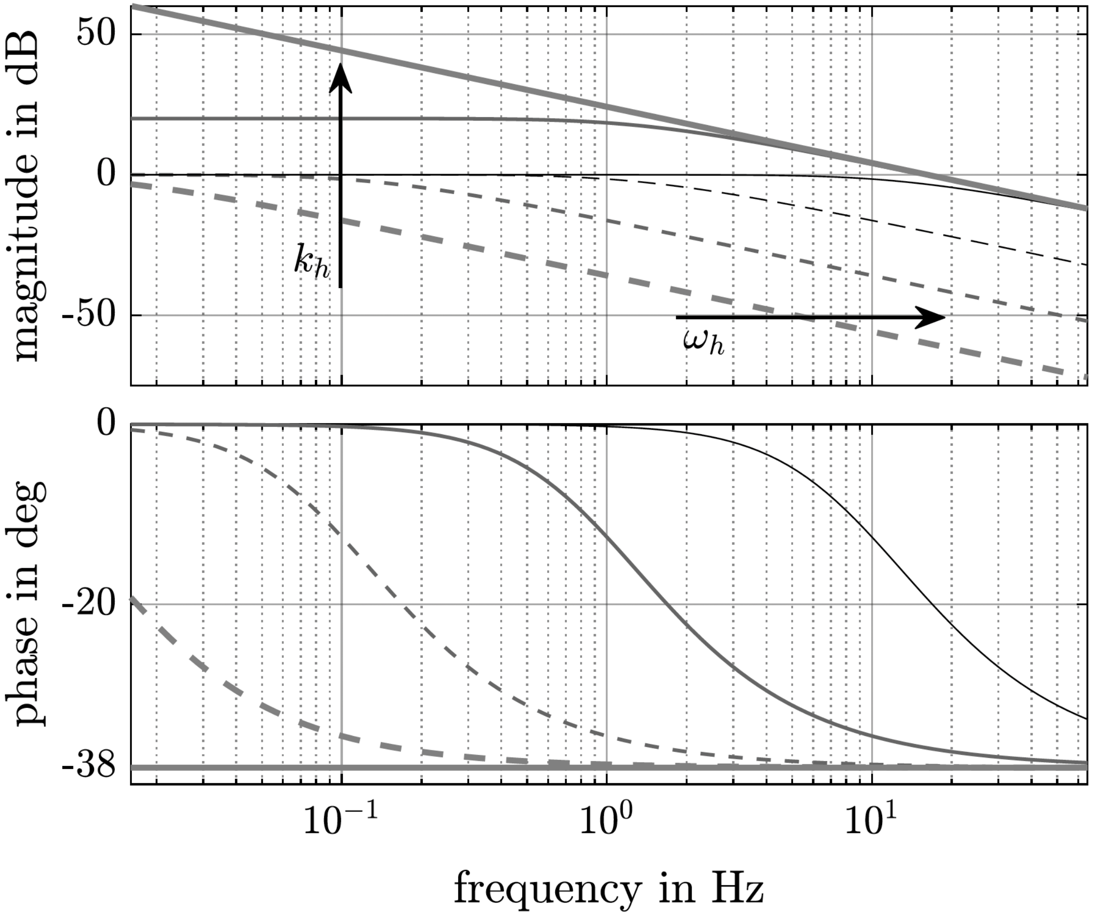
Research question:
Discrete-time Incremental Stability



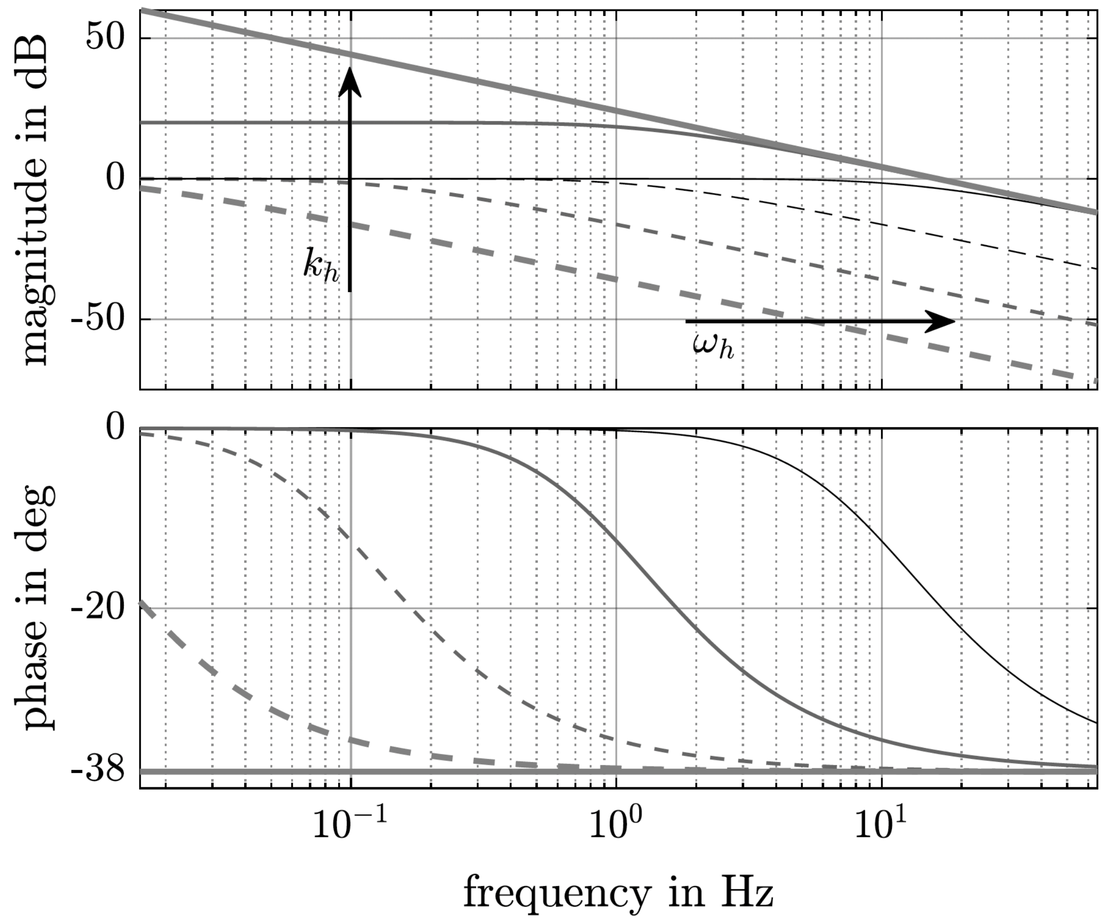
DEFINITION: Incremental (asymptotic/exponential) stability

If two copies of the system are subject to the **same input**, their **trajectories converge to each other** (asymptotically/exponentially)

Literature overview

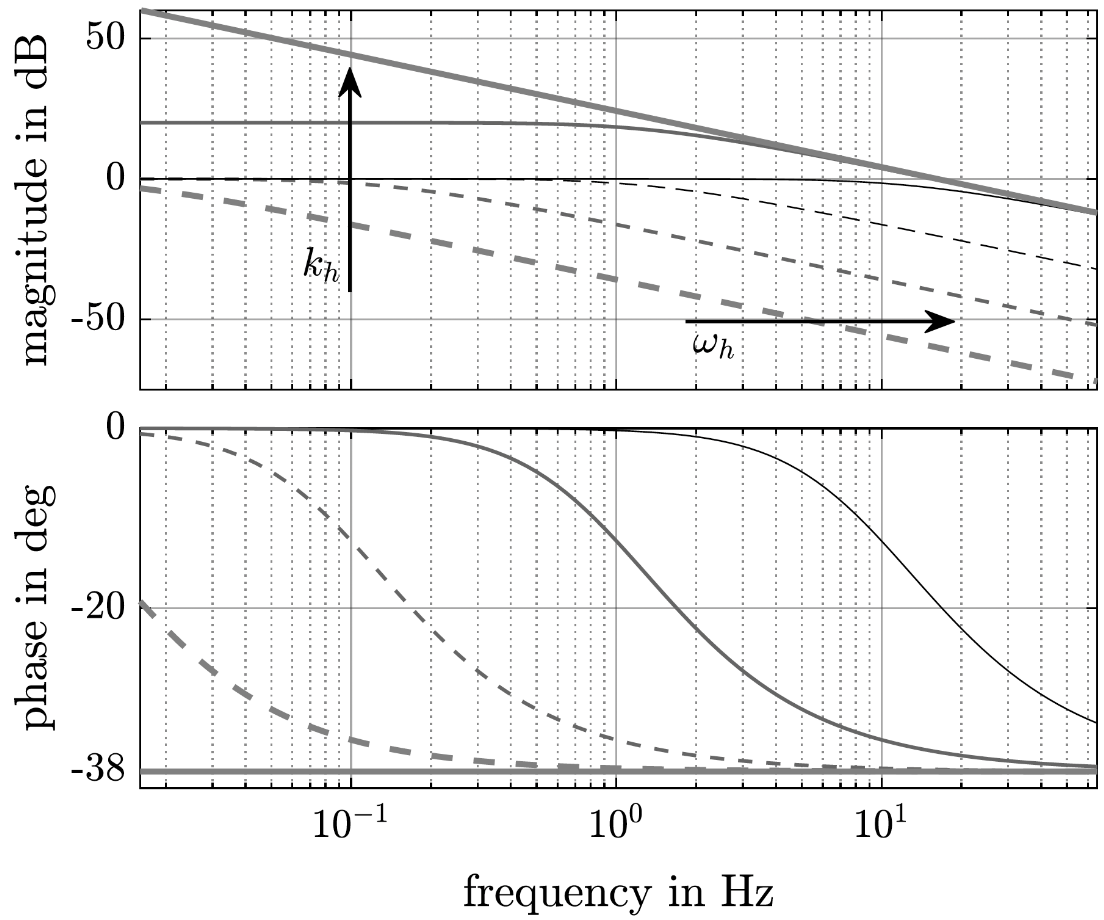


Literature overview



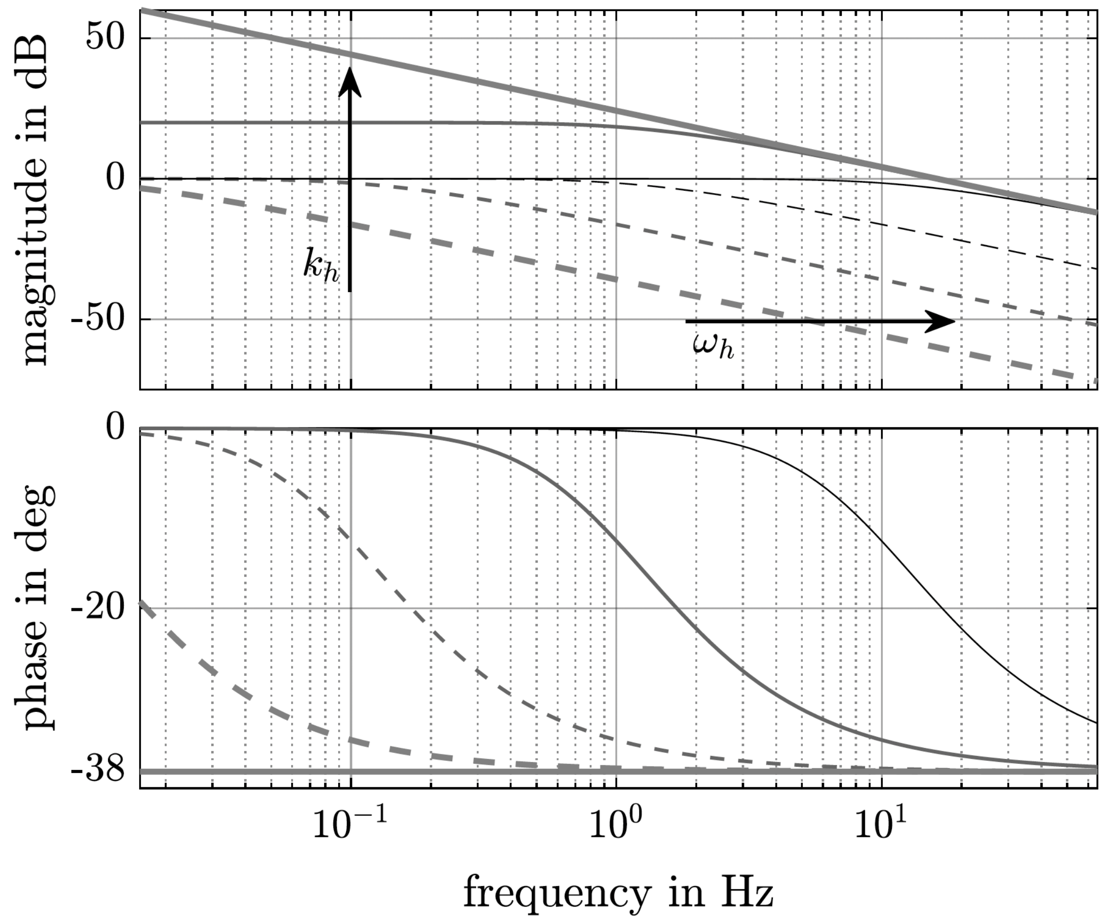
Research focus	Corresponding article(s)
CT **incremental stability**	Van den Eijnden et al., L-CSS, 2023

Literature overview



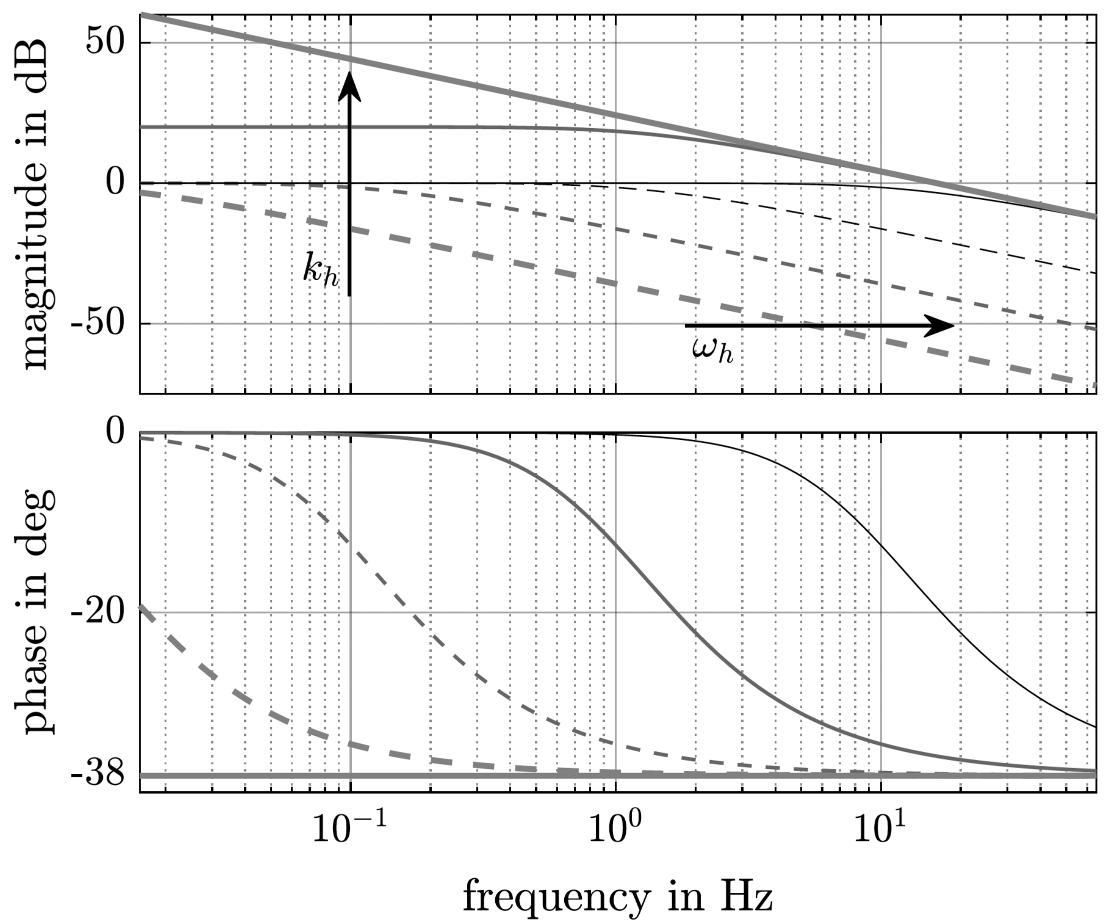
Research focus	Corresponding article(s)
CT **incremental stability**	Van den Eijnden et al., L-CSS, 2023
DT formulation and stability	Sharif et al., CDC, 2022 Sharif et al., Automatica, 2024

Literature overview



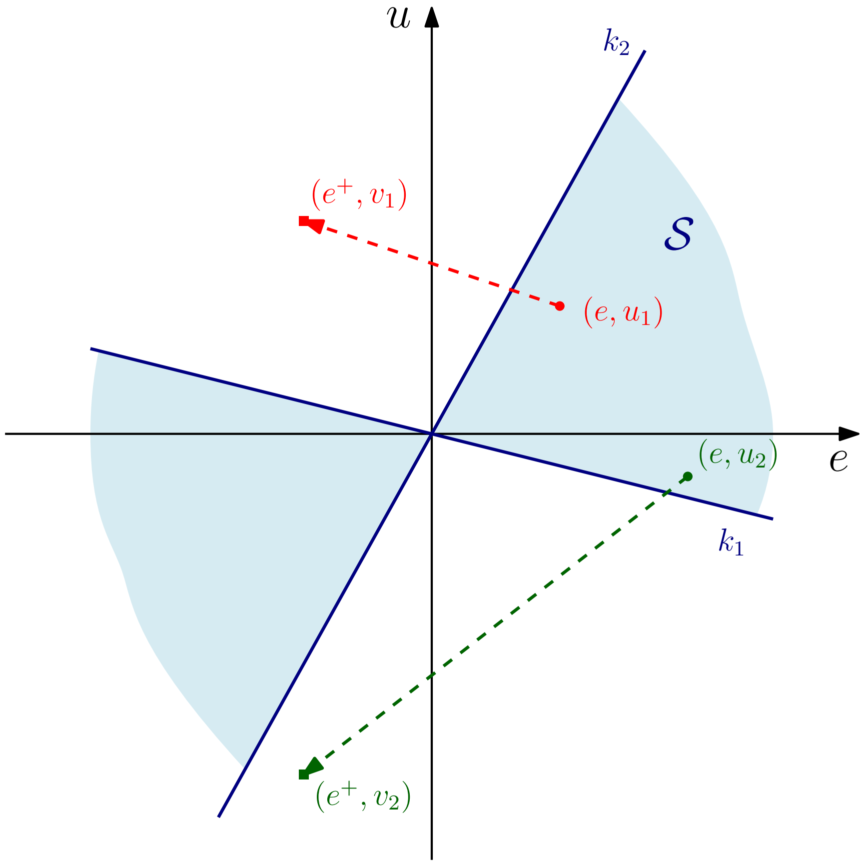
Research focus	Corresponding article(s)
CT **incremental stability**	Van den Eijnden et al., L-CSS, 2023
DT formulation and stability	Sharif et al., CDC, 2022 Sharif et al., Automatica, 2024
DT/SD convergence properties	Pavlov et al., TAC, 2012 Tren et al., TAC, 2019

Literature overview

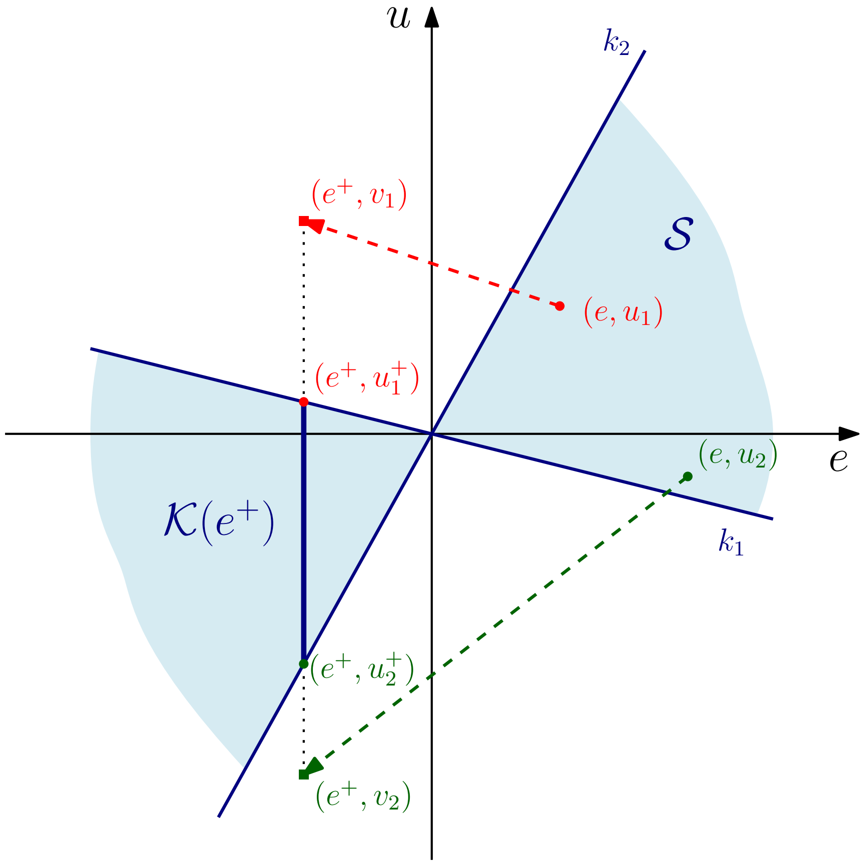


Research focus	Corresponding article(s)
CT **incremental stability**	Van den Eijnden et al., L-CSS, 2023
DT formulation and stability	Sharif et al., CDC, 2022 Sharif et al., Automatica, 2024
DT/SD convergence properties	Pavlov et al., TAC, 2012 Tren et al., TAC, 2019
DT/SD incremental stability and convergence	Bertollo et al., ADHS, 2024

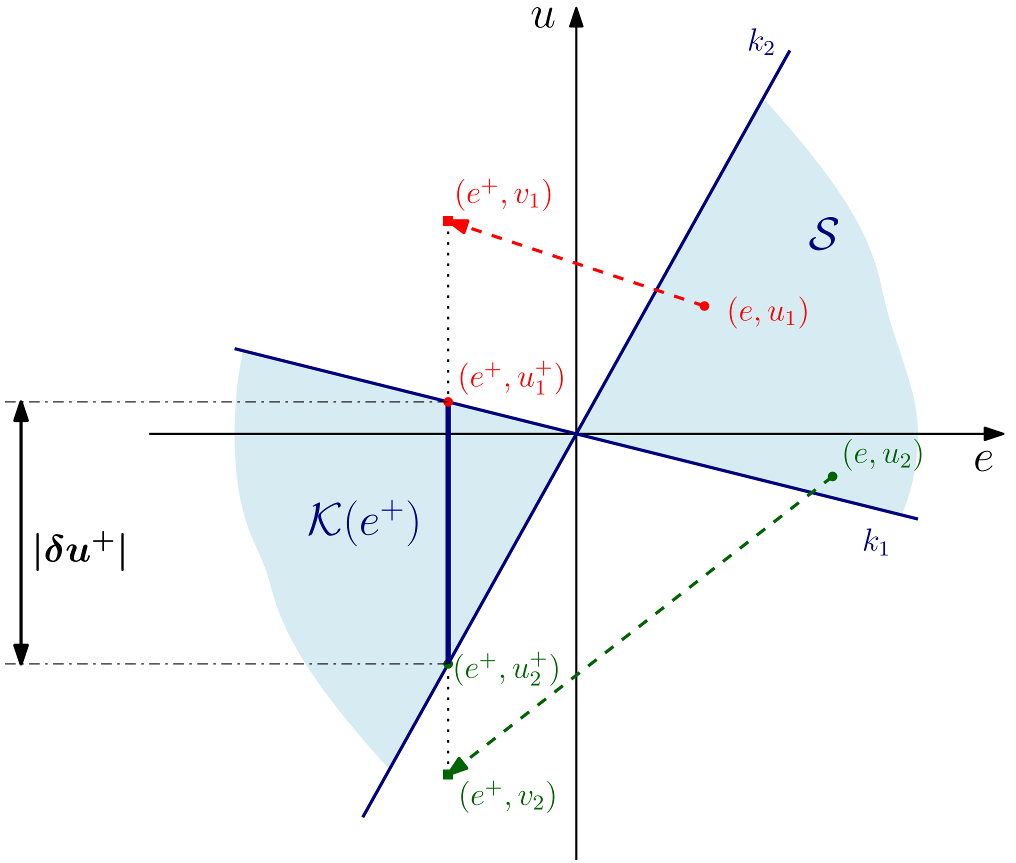
Open-loop Incremental Stability



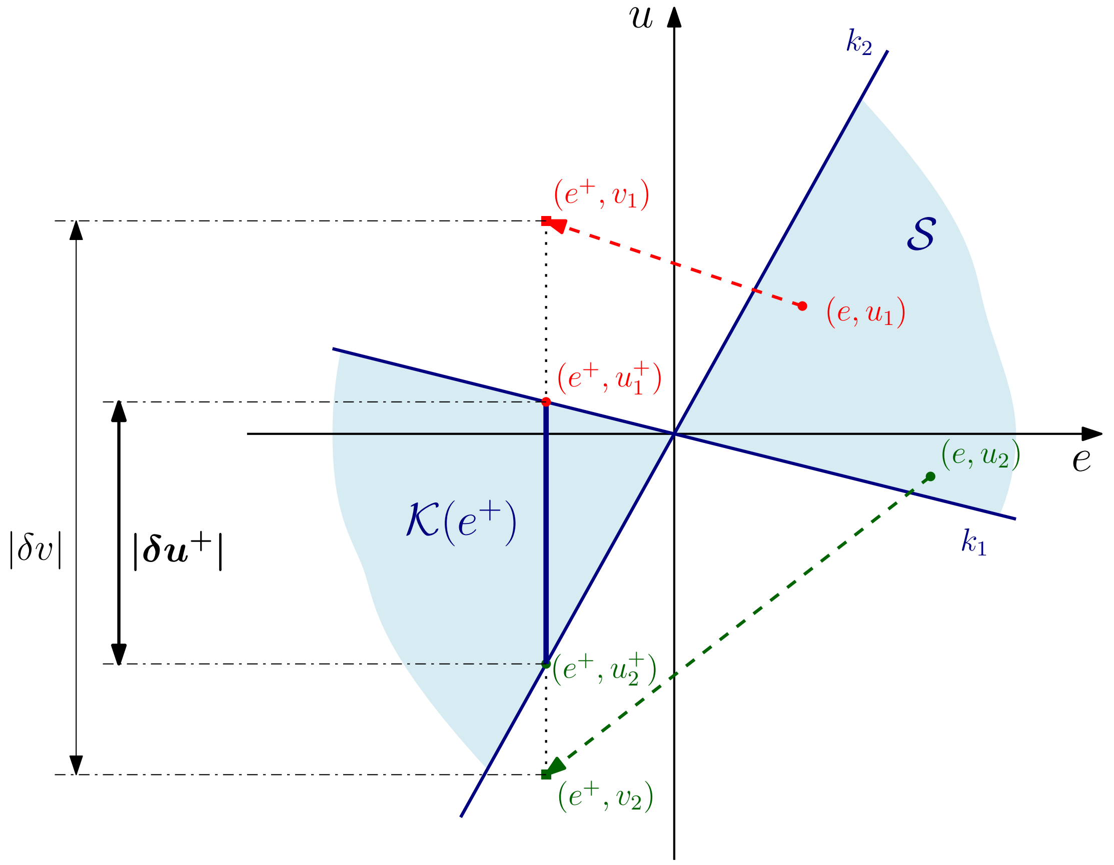
Open-loop Incremental Stability



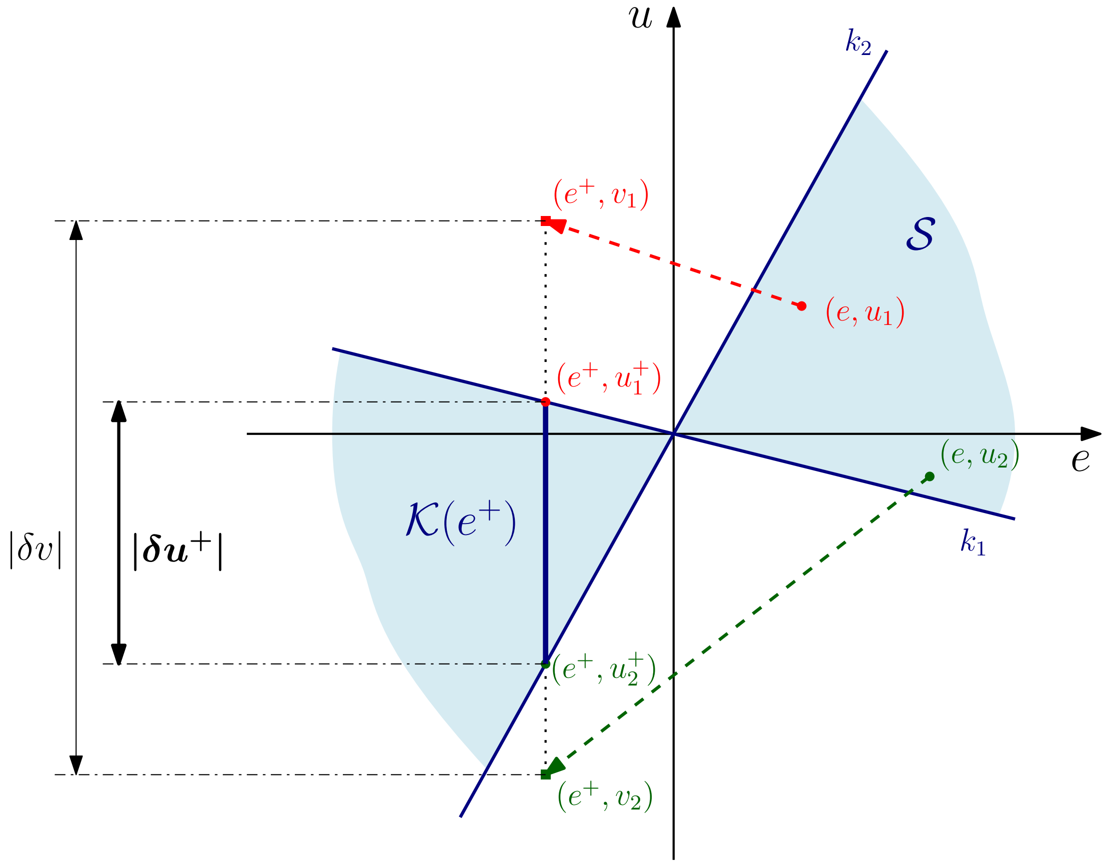
Open-loop Incremental Stability



Open-loop Incremental Stability



Open-loop Incremental Stability



Lemma 1:

The projection on the same set *K(e)* is **non-expansive**

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

$$|\delta u^{+}| \leq |\delta v|$$

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

$$|\delta u^+| \leq |\delta v|$$

$$\leq |au_1 + be^+ \; - au_2 \; - be^+|$$

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

$$|\delta u^+| \leq |\delta v|$$

$$\leq |au_1 + be^+ - au_2 - be^+|$$

$$= |a||\delta u|$$

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

$$
\begin{aligned}
|\delta u^{+}| &\leq |\delta v| \\
&\leq |au_1 + be^{+} - au_2 - be^{+}| \\
&= |a||\delta u|
\end{aligned}
$$

THEOREM 1:

If the underlying dynamics is GES (Globally Exponentially Stable), the FOPE is **incrementally exponentially stable**

Open-loop Incremental Stability: GES underlying dynamics

The case $|\boldsymbol{a}| < \mathbf{1}$ is **straightforward**:

$$
\begin{aligned}
|\delta u^+| &\leq |\delta v| \\
&\leq |au_1 + be^+ - au_2 - be^+| \\
&= |a||\delta u|
\end{aligned}
$$

If a=1, this is not enough!

THEOREM 1:

If the underlying dynamics is GES (Globally Exponentially Stable), the FOPE is **incrementally exponentially stable**

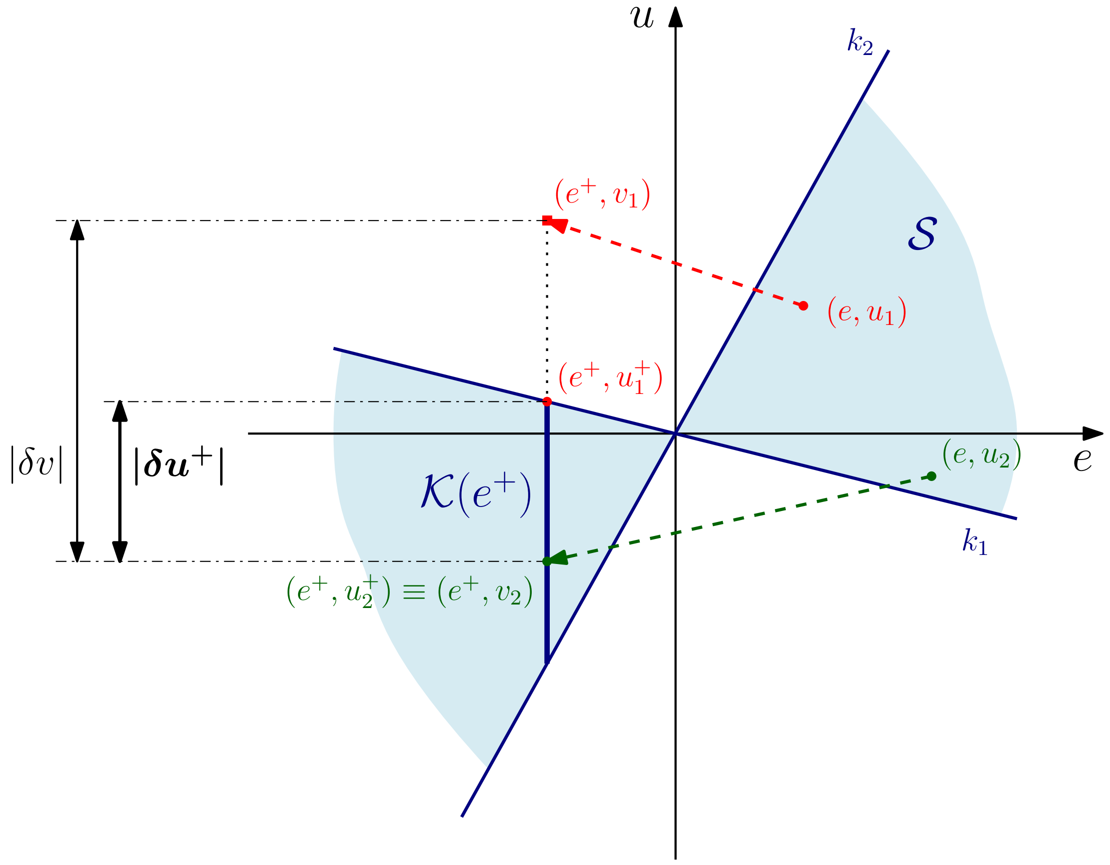
Open-loop Incremental Stability: Integrator case

- **Integrator** dynamics is **not incrementally stable**

Open-loop Incremental Stability: Integrator case

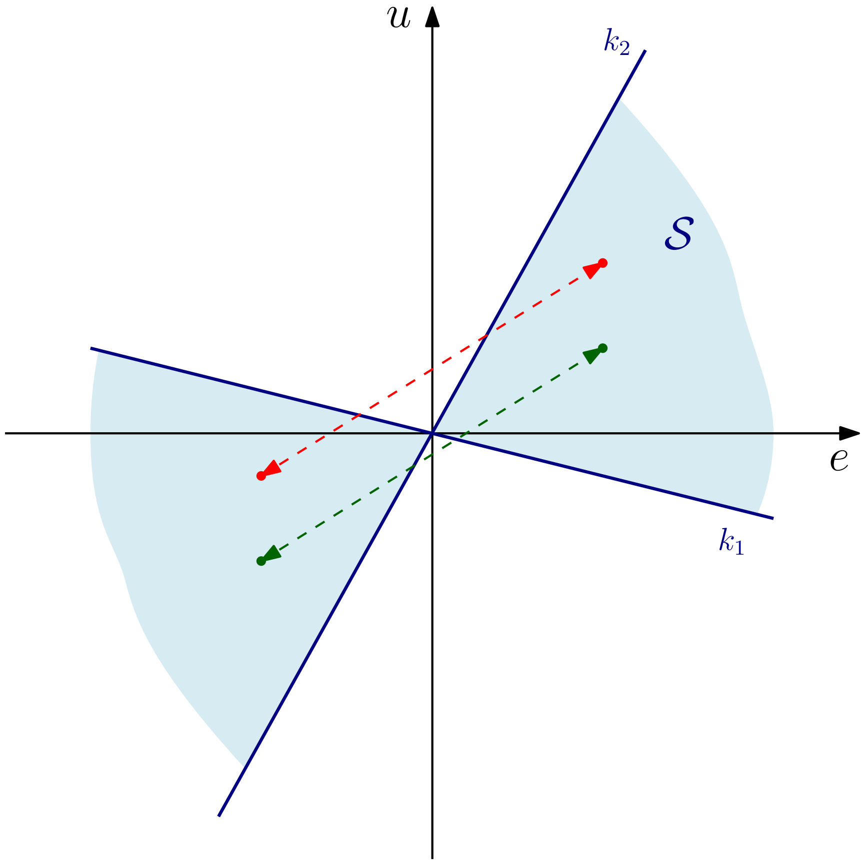
- **Integrator** dynamics is **not incrementally stable**

- **When the projection is active** for (at least) one of the two trajectories, the **inequality** $|\delta u^+| < |\delta v|$ is **strict**



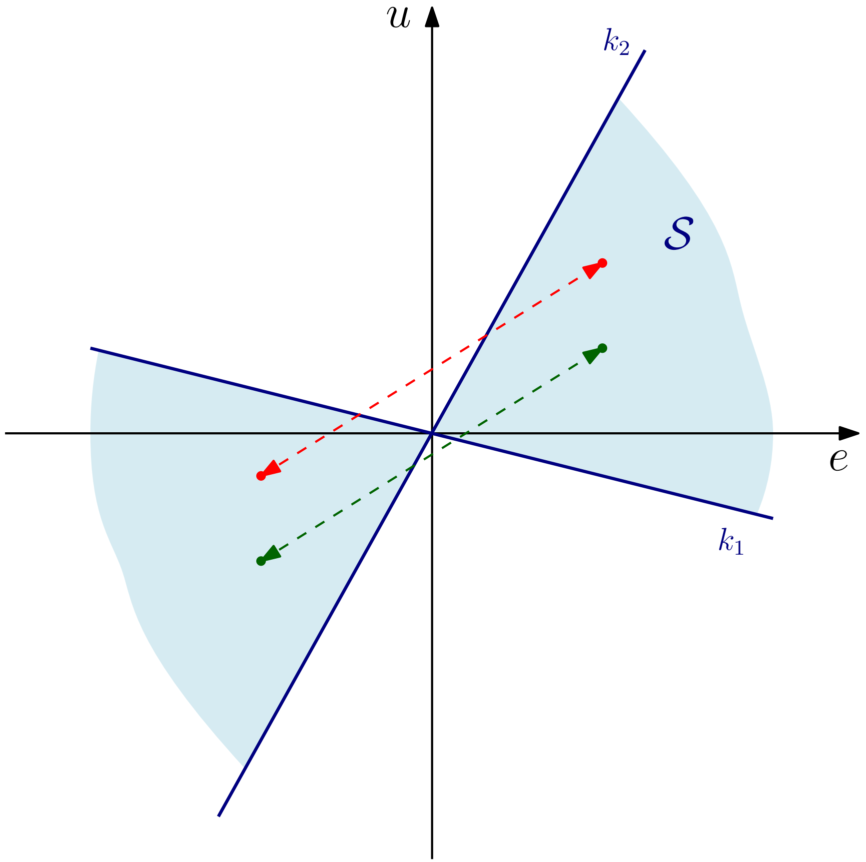
Open-loop Incremental Stability: Integrator case

- **Integrator** dynamics is **not incrementally stable**
- **When the projection is active** for (at least) one of the two trajectories, the **inequality** $|\delta u^+| < |\delta v|$ is **strict**
- **Without** additional **constraints**, it is possible that the projection is **never active**



Open-loop Incremental Stability: Integrator case

- **Integrator** dynamics is **not incrementally stable**
- **When the projection is active** for (at least) one of the two trajectories, the **inequality** $|\delta u^+| < |\delta v|$ is **strict**
- **Without** additional **constraints**, it is possible that the projection is **never active**



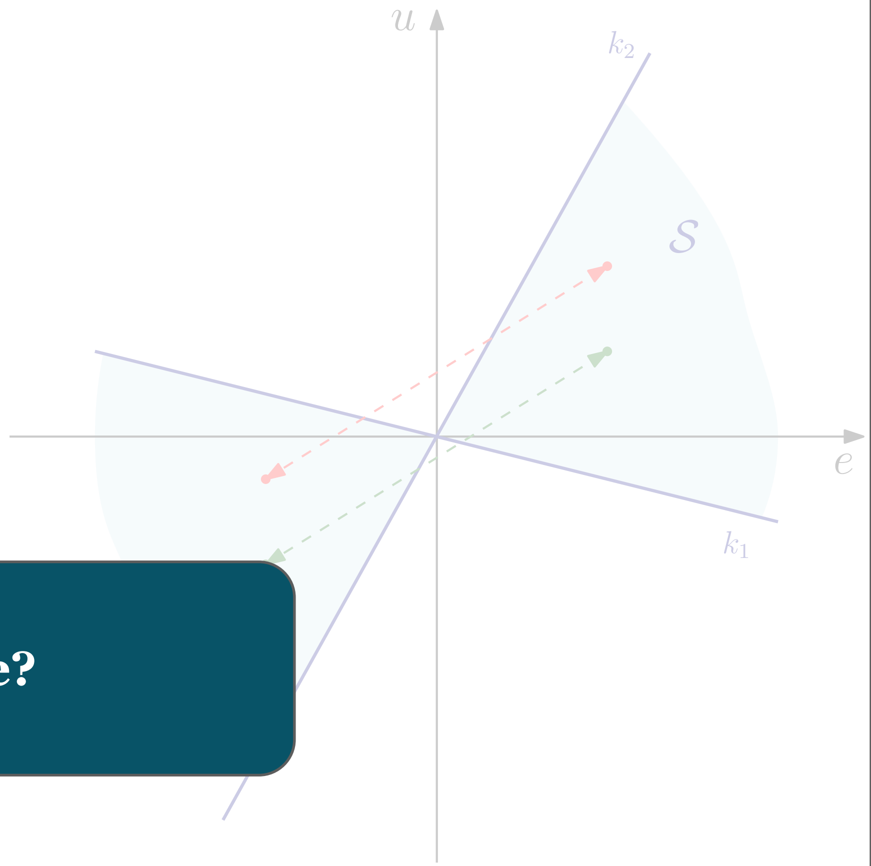
THEOREM 2:

If a=1, then the FOPE is **incrementally asymptotically stable if and only if** the parameter b>0 satisfies

$$b \notin (2k_1, 2k_2)$$

Open-loop Incremental Stability: Integrator case

- **Integrator** dynamics is **not incrementally stable**
- **When the projection is active** for (at least) one of the two trajectories, the **inequality** $|\delta u^+| < |\delta v|$ is **strict**
- **Without** additi… projection is **ne**…



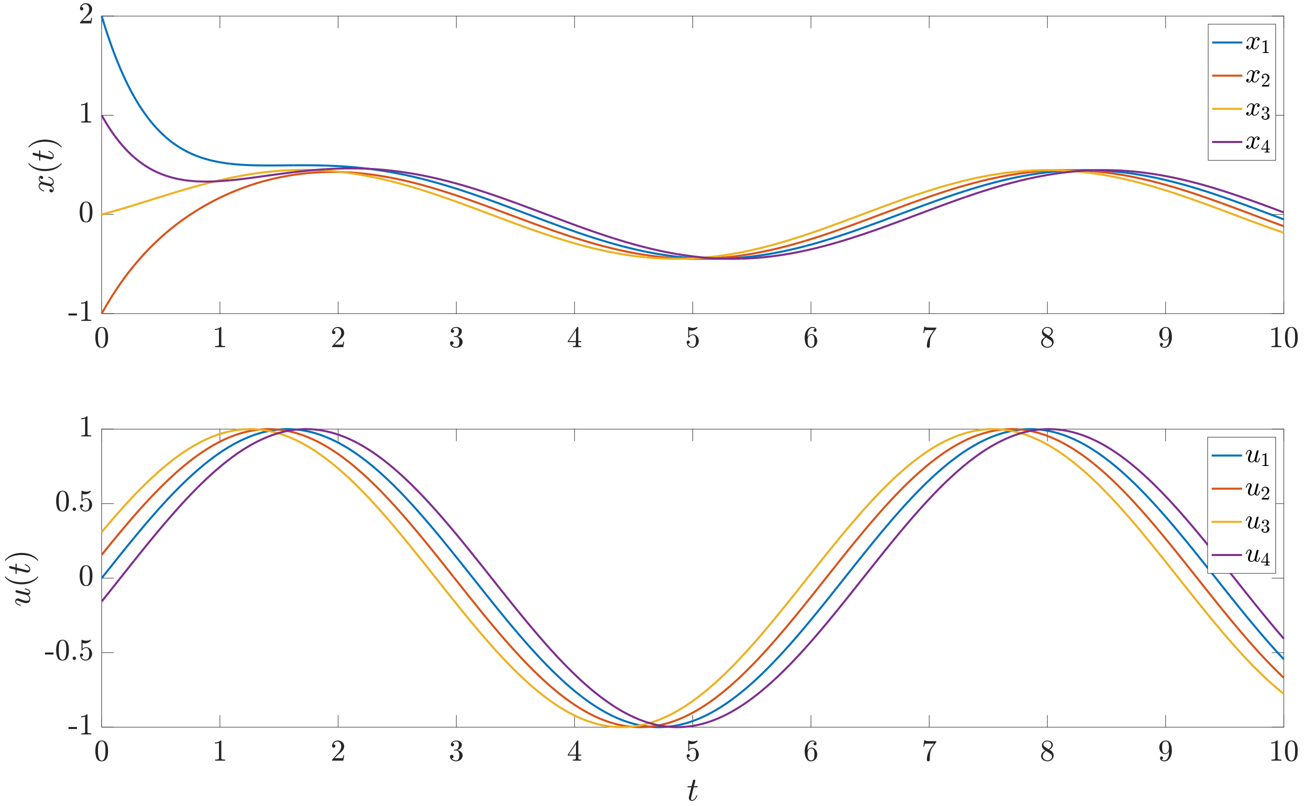
What if the inputs are not the same?

THEOREM 2:

If a=1, then the FOPE is **incrementally asymptotically stable if and only if** the parameter b>0 satisfies

$$b \notin (2k_1, 2k_2)$$

Research question: DT Incremental ISS



DEFINITION: Incremental input-to-state stability (ISS)

If two copies of the system are subject to **different inputs**, the **bound on the distance between** their **trajectories** converges to some positive value **related to the distance between** the **inputs**

Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

Open-loop Incremental ISS

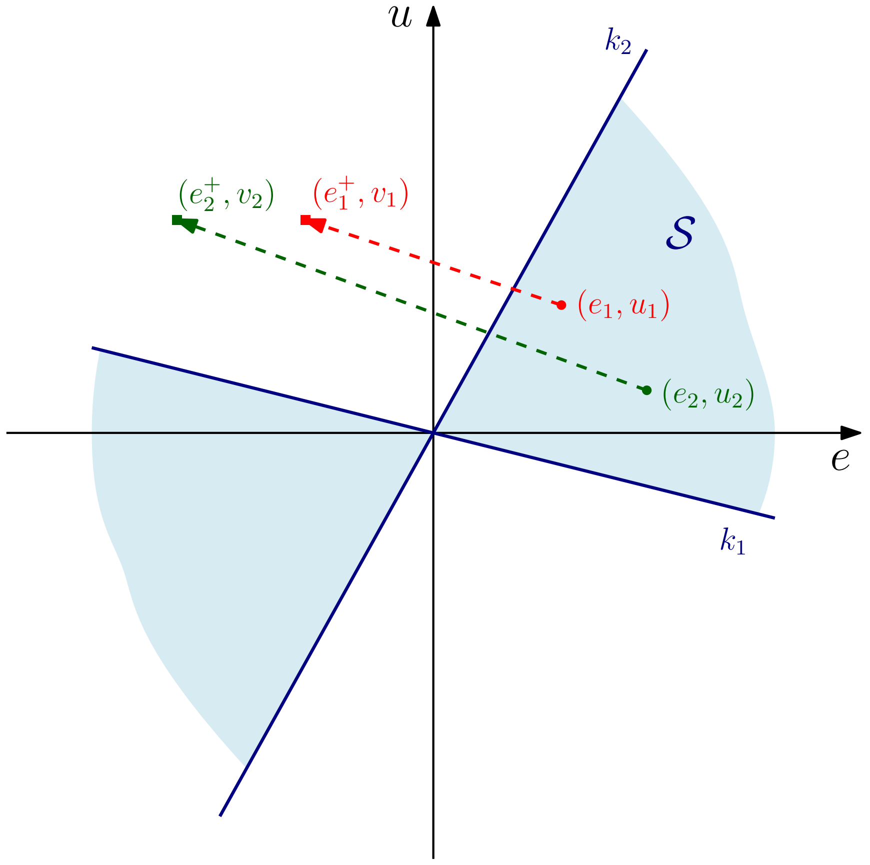
- What happens if two copies of a FOPE are subject to two **different error signals**?

- In general, this might **increase the distance** between u_1 and u_2

Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

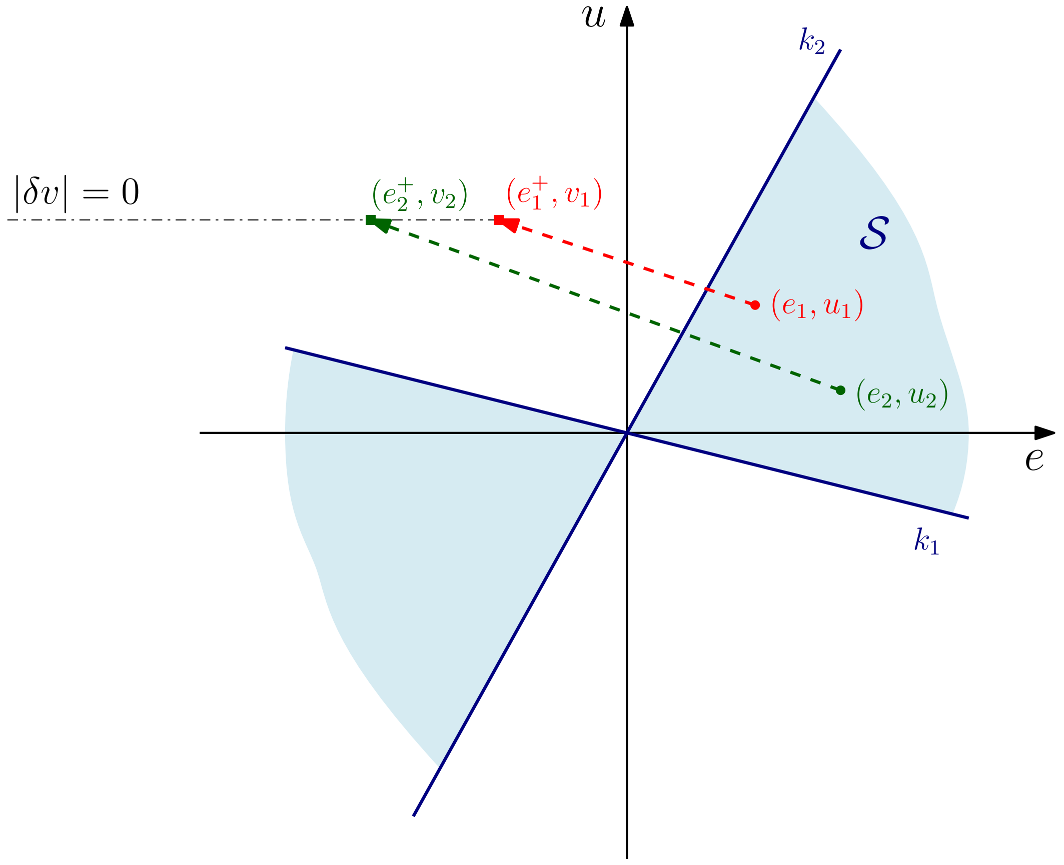
- In general, this might **increase the distance** between u_1 and u_2



Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

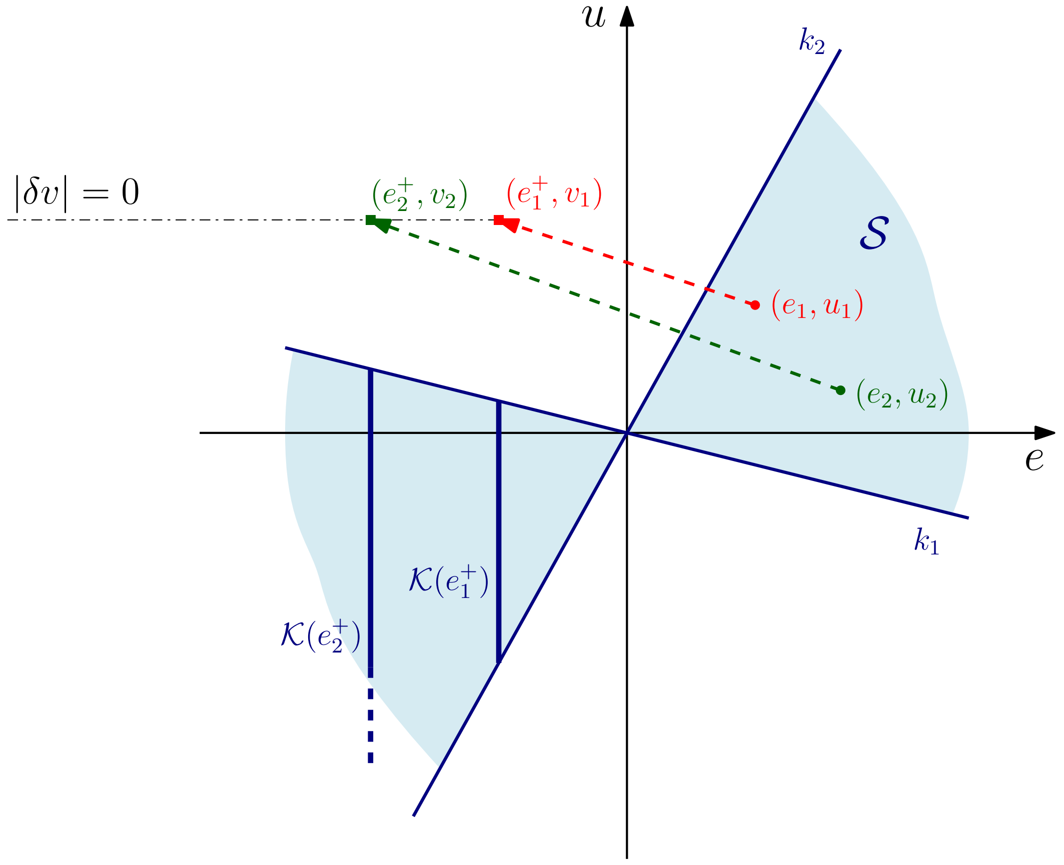
- In general, this might **increase the distance** between u_1 and u_2



Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

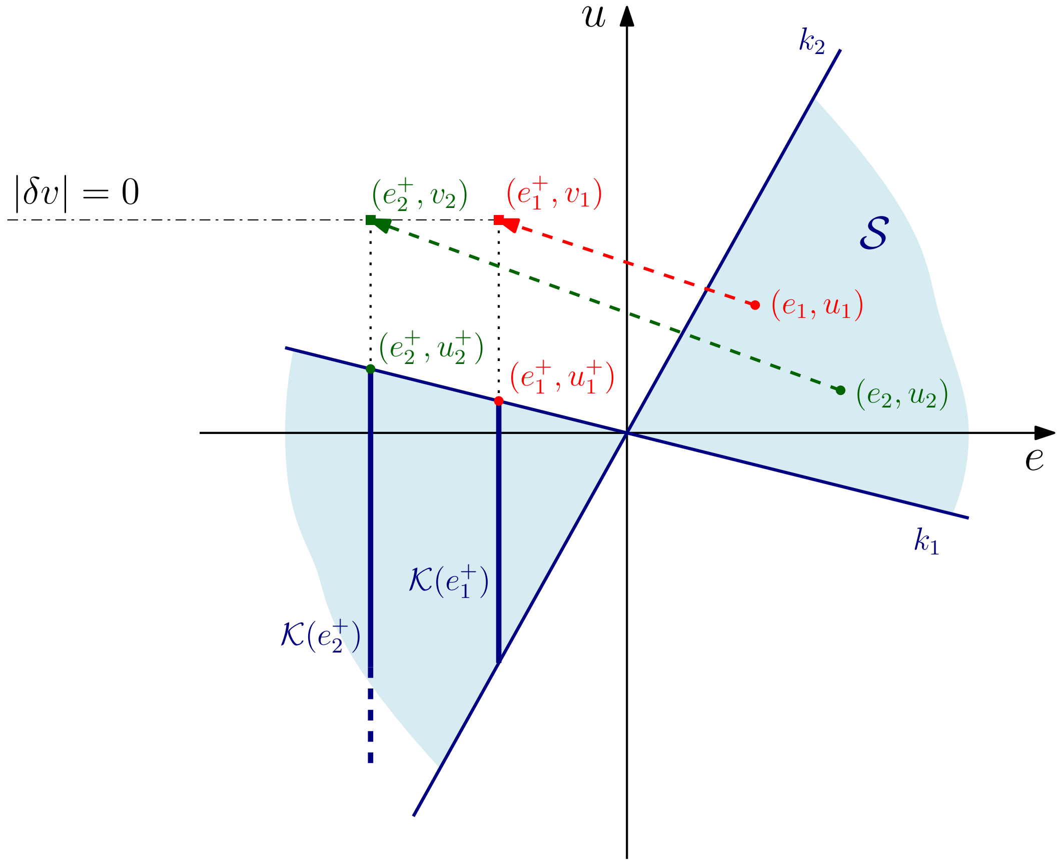
- In general, this might **increase the distance** between u_1 and u_2



Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

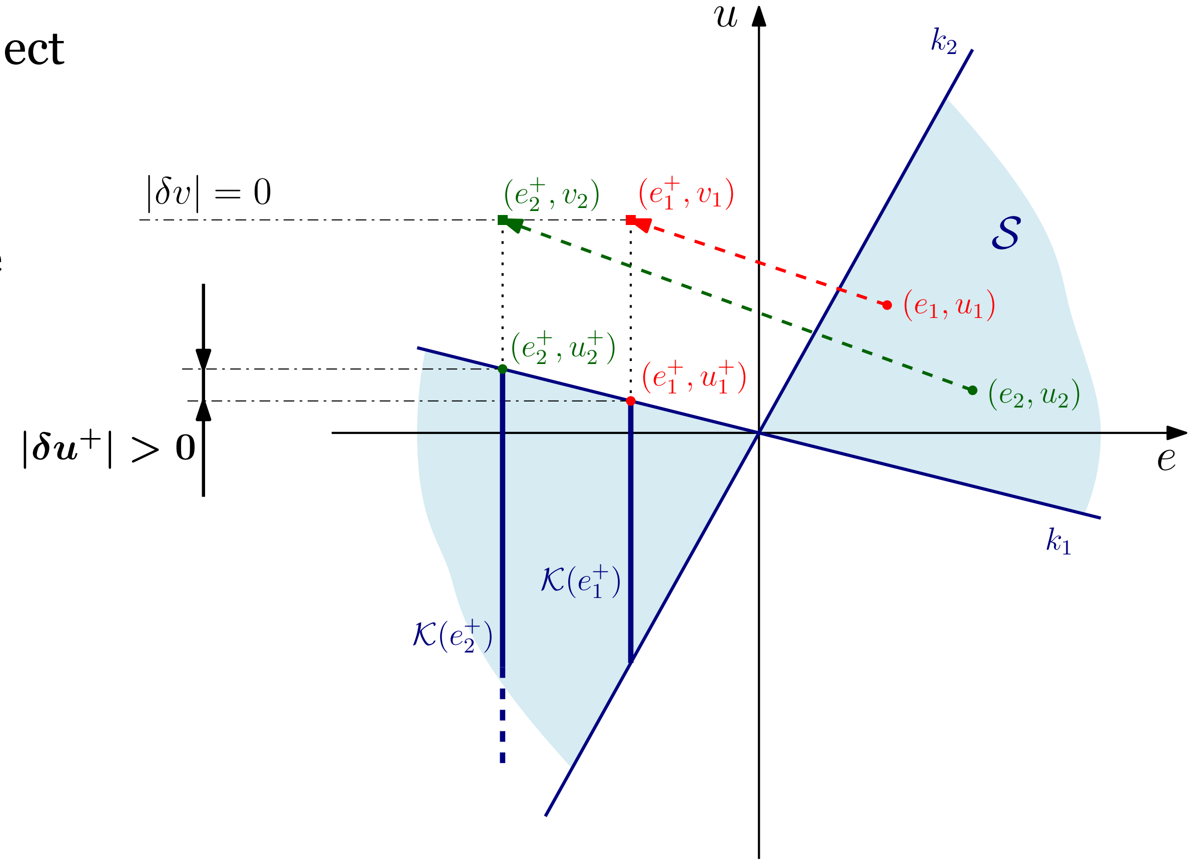
- In general, this might **increase the distance** between u_1 and u_2



Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

- In general, this might **increase the distance** between u_1 and u_2

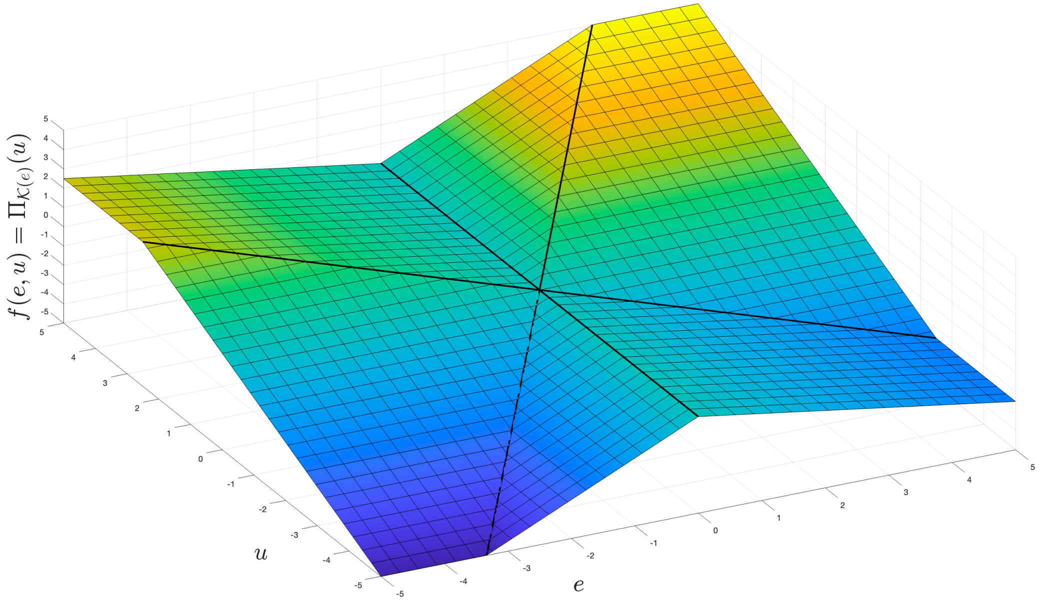


Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?

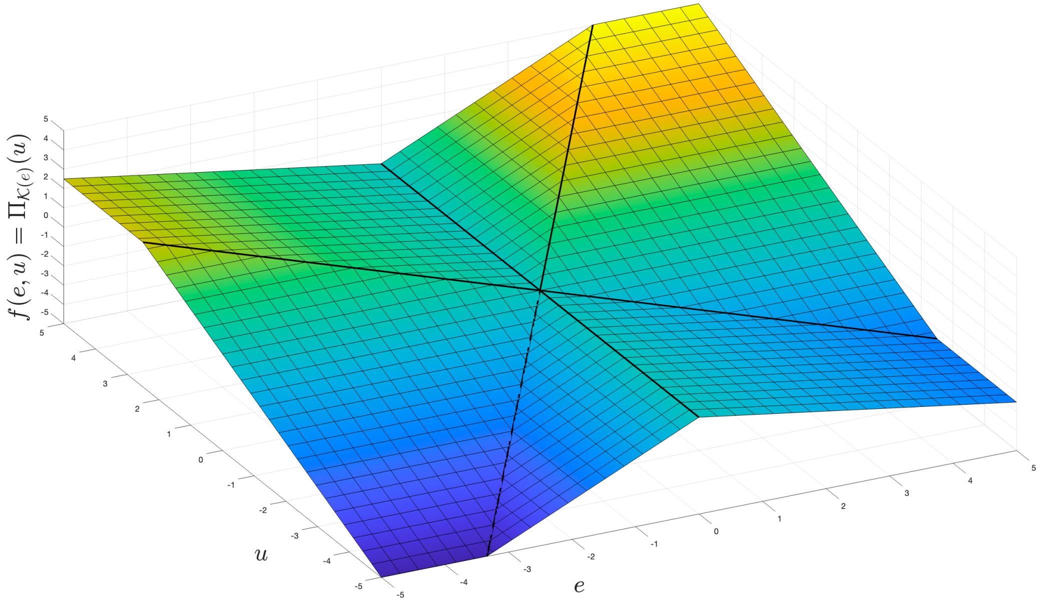
- In general, this might **increase the distance** between u_1 and u_2

- However, the projection (written as a 2D map) is **Lipschitz continuous**, with constant $\bar{k} = max\{|k_1|, |k_2|\}$



Open-loop Incremental ISS

- What happens if two copies of a FOPE are subject to two **different error signals**?
- In general, this might **increase the distance** between u_1 and u_2
- However, the projection (written as a 2D map) is **Lipschitz continuous**, with constant $\bar{k} = max\{|k_1|, |k_2|\}$



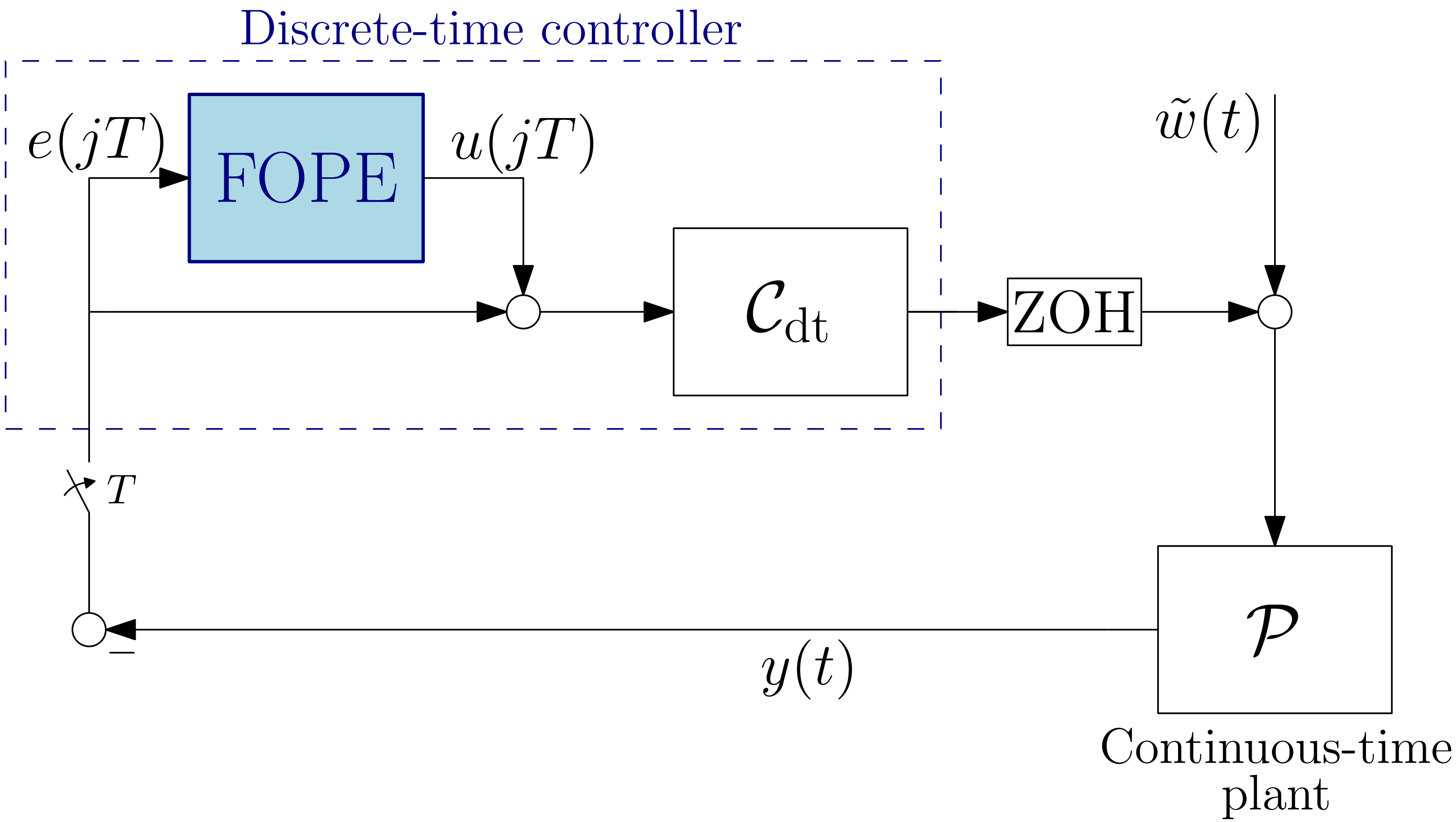
LEMMA 2:

If the underlying dynamics is GES (i.e., $|a|<1$), then

$$|\delta u^+| \leq |a||\delta u| + (b + \bar{k})|\delta e^+|$$

And the FOPE is **incrementally exponentially input-to-state stable**

Research question: Closed-loop Incremental ISS



Closed-loop incremental ISS: Incremental small gain theorem

We can use the following result

THEOREM[1]**:**

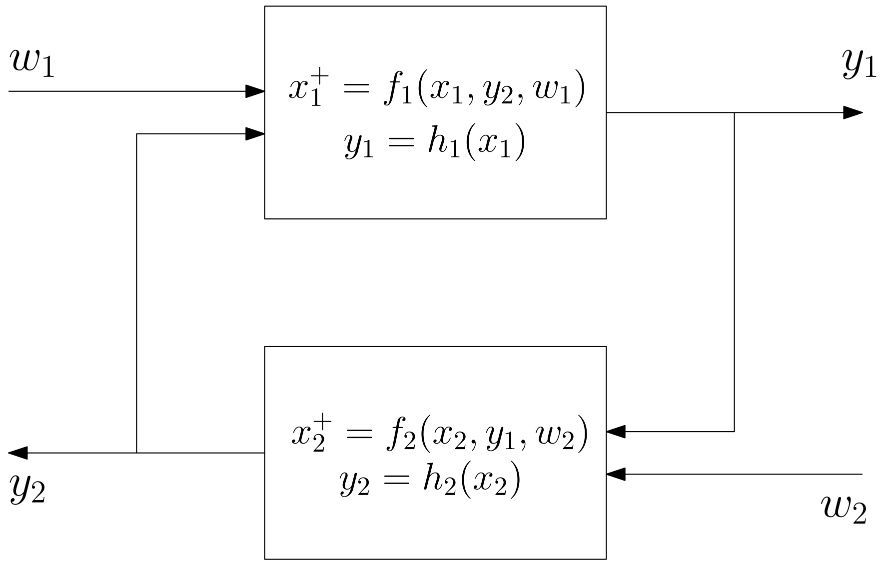
When two DT systems are interconnected in feedback as shown, if the following hold

$$|y_1(j)| \leq \beta_1(j, |x_1(0)|) + \boldsymbol{\gamma_{y,1}} ||\boldsymbol{y_2}||_\infty + \gamma_{w,1} ||w_1||_\infty$$

$$|y_2(j)| \leq \beta_2(j, |x_2(0)|) + \boldsymbol{\gamma_{y,2}} ||\boldsymbol{y_1}||_\infty + \gamma_{w,2} ||w_2||_\infty$$

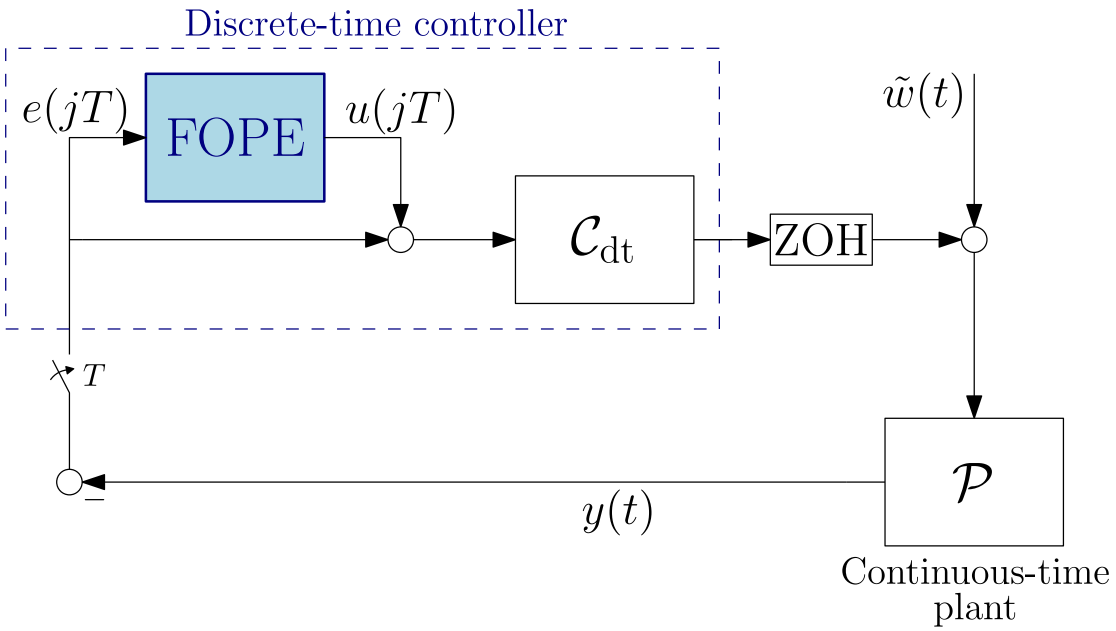
$$\boldsymbol{\gamma_{y,1} \cdot \gamma_{y,2} < 1}$$

then the **closed-loop dynamics is ISS**.



[1] Z.P. Jiang, A.R. Teel, L. Praly (1994). Small-gain theorem theorem for ISS sytems and applications. *Mathematics of Control, Signals and Systems*.

Consequences of incremental ISS: Incremental small gain theorem



We can easily write an **incremental version** and apply it to the **sampled-data interconnection** of the discrete-time FOPE and a continuous-time linear plant

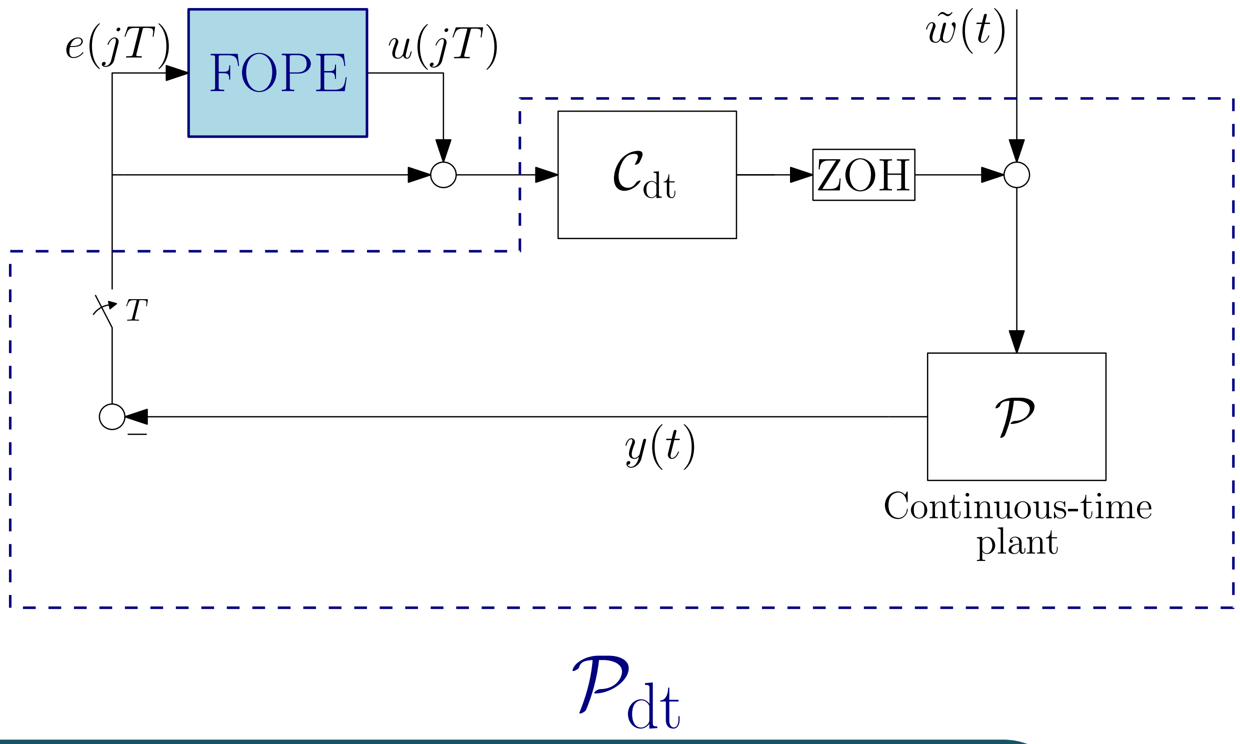
THEOREM 3:

If the parameters of the FOPE are selected so that

$$\frac{b+\bar{k}}{1-|a|} < \frac{1}{||P_{dt}||_{\ell_1}}$$

then the **sampled-data interconnection** of the FOPE with a linear plant is **incrementally ISS**.

Consequences of incremental ISS: Incremental small gain theorem



We can easily write an **incremental version** and apply it to the **sampled-data interconnection** of the discrete-time FOPE and a continuous-time linear plant

THEOREM 3:

If the parameters of the FOPE are selected so that

$$\frac{b+\bar{k}}{1-|a|} < \frac{1}{||P_{dt}||_{\ell_1}}$$

then the **sampled-data interconnection** of the FOPE with a linear plant is **incrementally ISS**.

Simulations

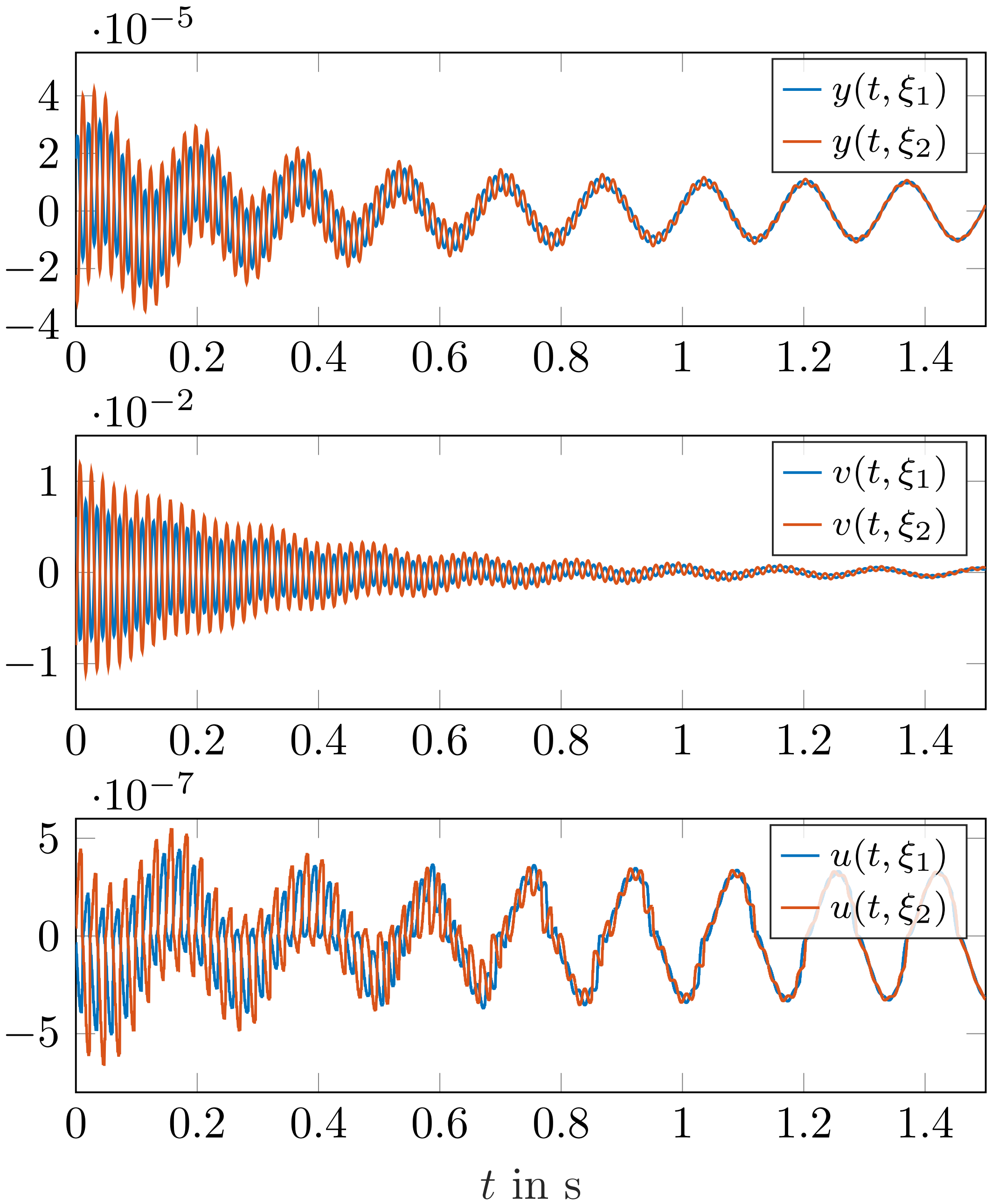
- Plant = **precision positioning system** (2DoF)
- Closed loop is perturbed by a sinusoidal disturbance with frequency 6π rad/s
- Simulations starting from different ICs

Simulations

- Plant = **precision positioning system** (2DoF)
- Closed loop is perturbed by a sinusoidal disturbance with frequency 6π rad/s
- Simulations starting from different ICs

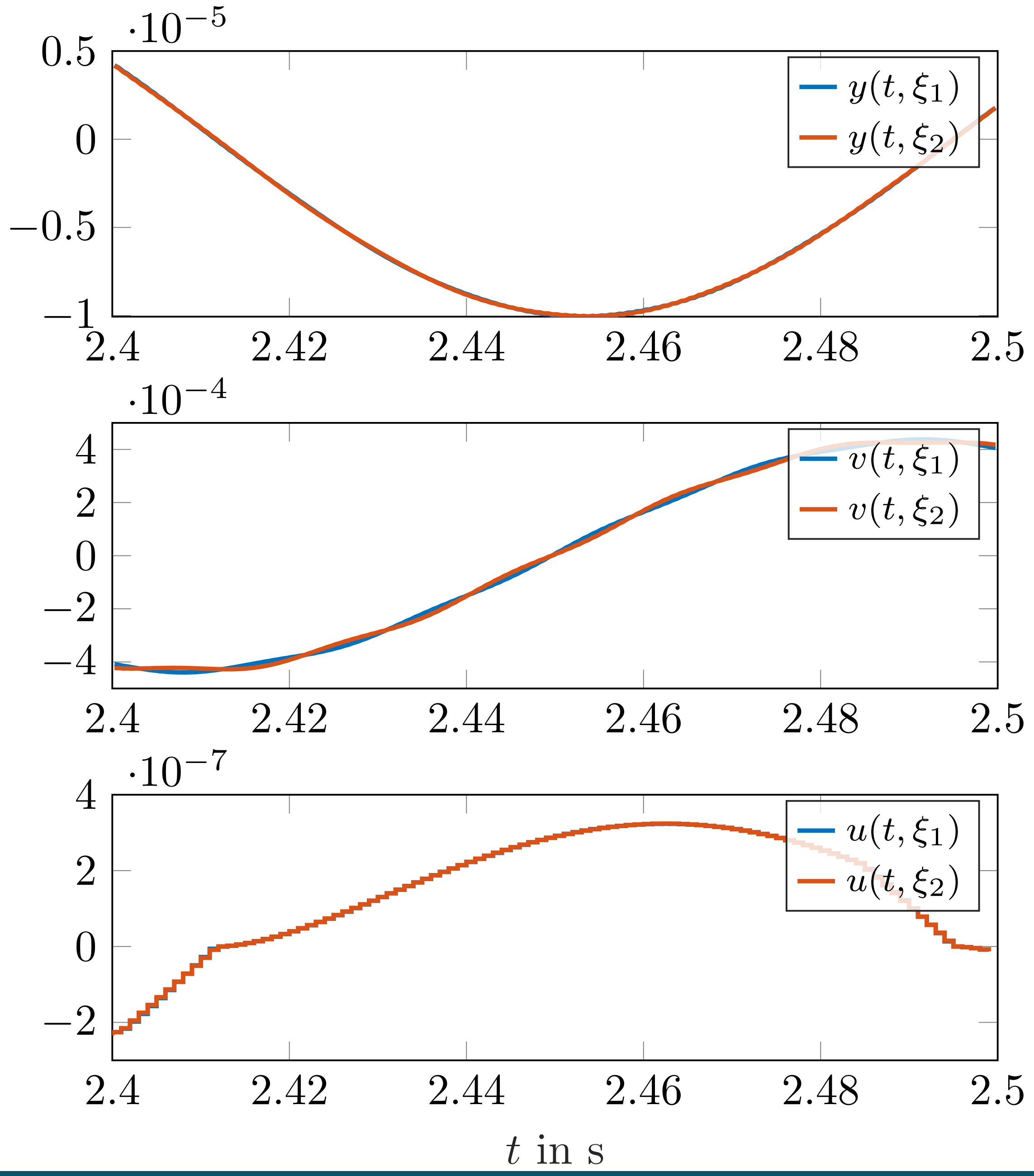
RESULTS:

- The states of the closed loop converge to the same trajectory
- The trajectories have the **same frequency as the input signal**



Simulations

Close-up on the previous plots, to highlight the sampled-data nature of the feedback loop



Conclusion

- We analyzed the dynamics of the **discrete-time FOPE**, a control element obtained by projecting a first order dynamics onto a sector
- We showed necessary and sufficient conditions for **incremental stability in open-loop** and sufficient conditions for **incremental ISS in closed loop**

Conclusion

- We analyzed the dynamics of the **discrete-time FOPE**, a control element obtained by projecting a first order dynamics onto a sector
- We showed necessary and sufficient conditions for **incremental stability in open-loop** and sufficient conditions for **incremental ISS in closed loop**

FUTURE WORK:

Conclusion

- We analyzed the dynamics of the **discrete-time FOPE**, a control element obtained by projecting a first order dynamics onto a sector
- We showed necessary and sufficient conditions for **incremental stability in open-loop** and sufficient conditions for **incremental ISS in closed loop**

FUTURE WORK:

- Formally prove **convergence properties** (all the ingredients are there, but we need to be careful about some details)

Conclusion

- We analyzed the dynamics of the **discrete-time FOPE**, a control element obtained by projecting a first order dynamics onto a sector
- We showed necessary and sufficient conditions for **incremental stability in open-loop** and sufficient conditions for **incremental ISS in closed loop**

FUTURE WORK:

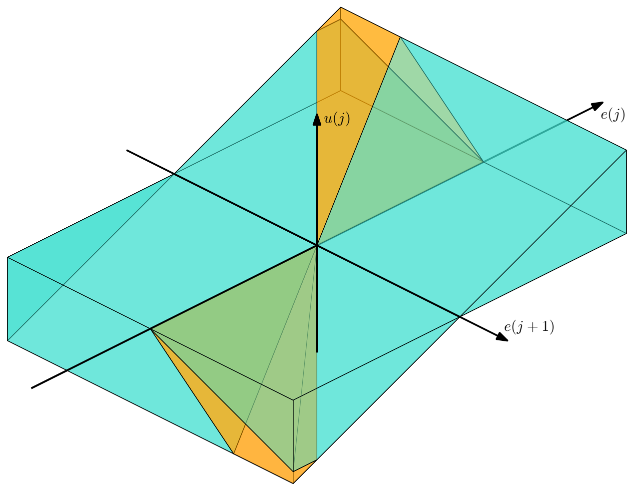
- Formally prove **convergence properties** (all the ingredients are there, but we need to be careful about some details)
- Prove **incremental ISS** of the closed loop through **less conservative conditions**

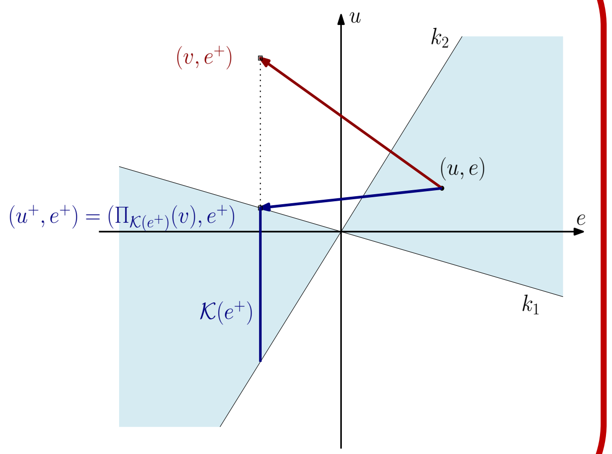
Conclusion

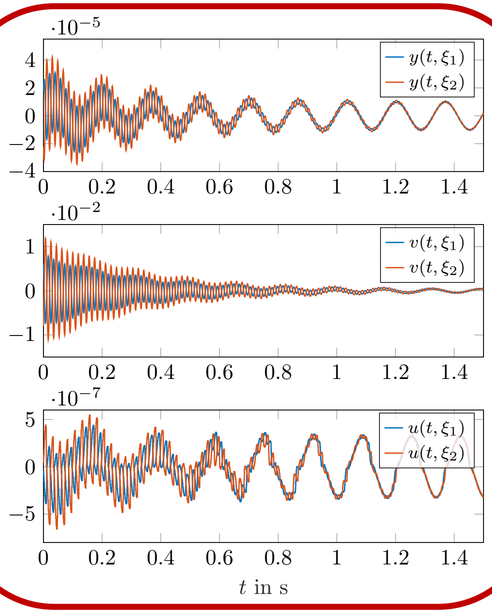
- We analyzed the dynamics of the **discrete-time FOPE**, a control element obtained by projecting a first order dynamics onto a sector
- We showed necessary and sufficient conditions for **incremental stability in open-loop** and sufficient conditions for **incremental ISS in closed loop**

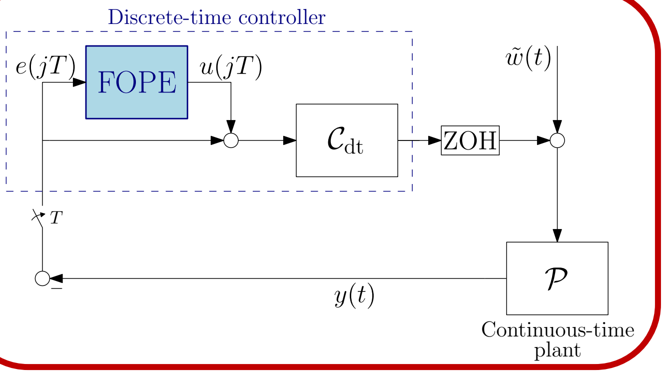
FUTURE WORK:

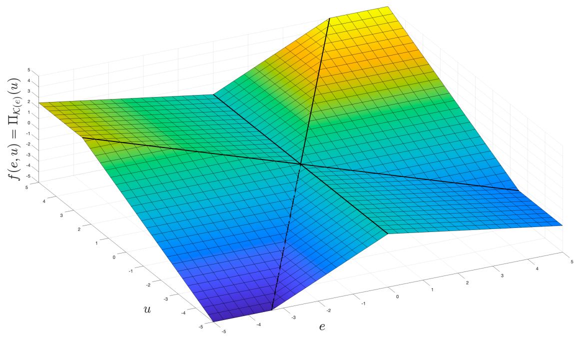
- Formally prove **convergence properties** (all the ingredients are there, but we need to be careful about some details)
- Prove **incremental ISS** of the closed loop through **less conservative conditions**
- The condition for incremental stability with *a=1* are quite conservative: is there a **better formulation?**











Riccardo Bertollo
r.bertollo@tue.nl