Stability analysis of nonsmooth systems for control

Riccardo Bertollo

(Eindhoven University of Technology)

UCLouvain – January 21, 2025

TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

UCLouvain

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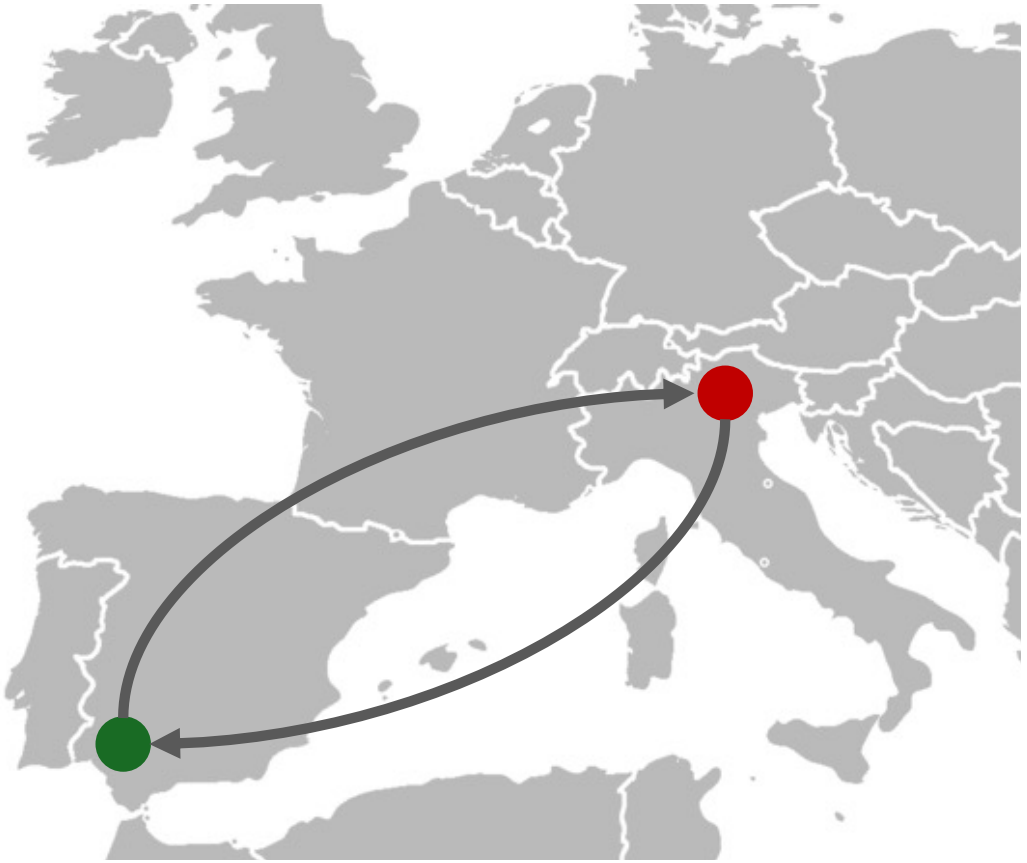

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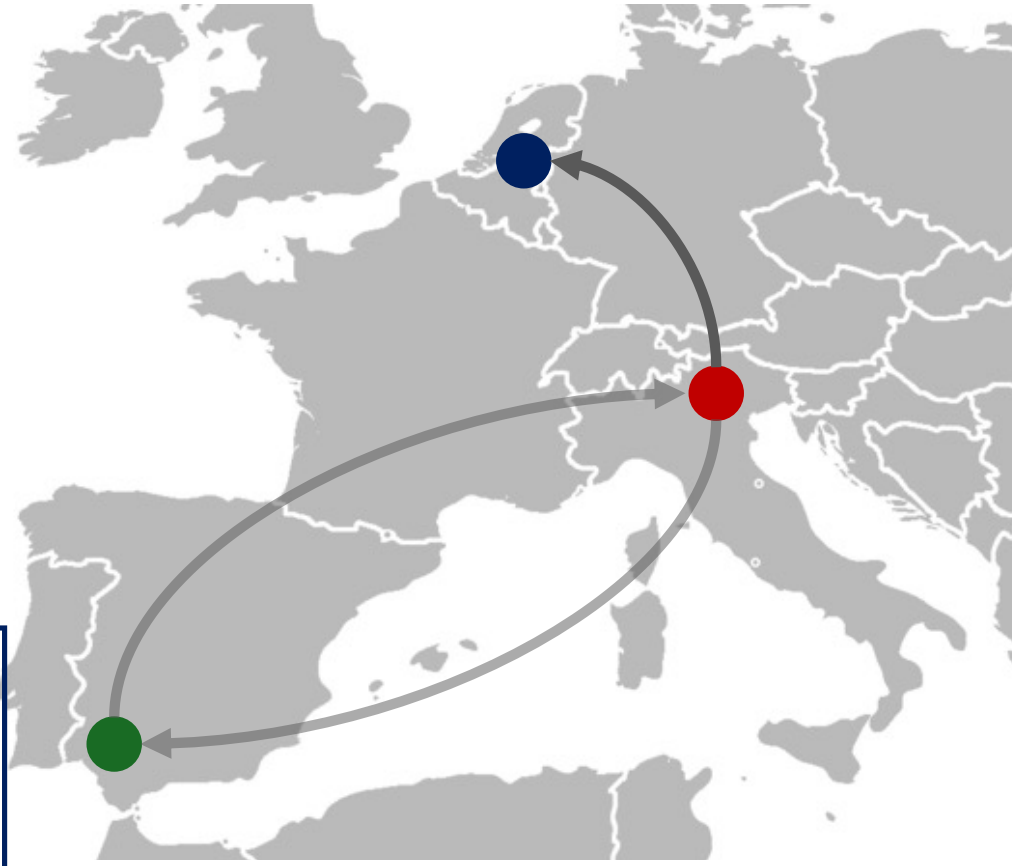

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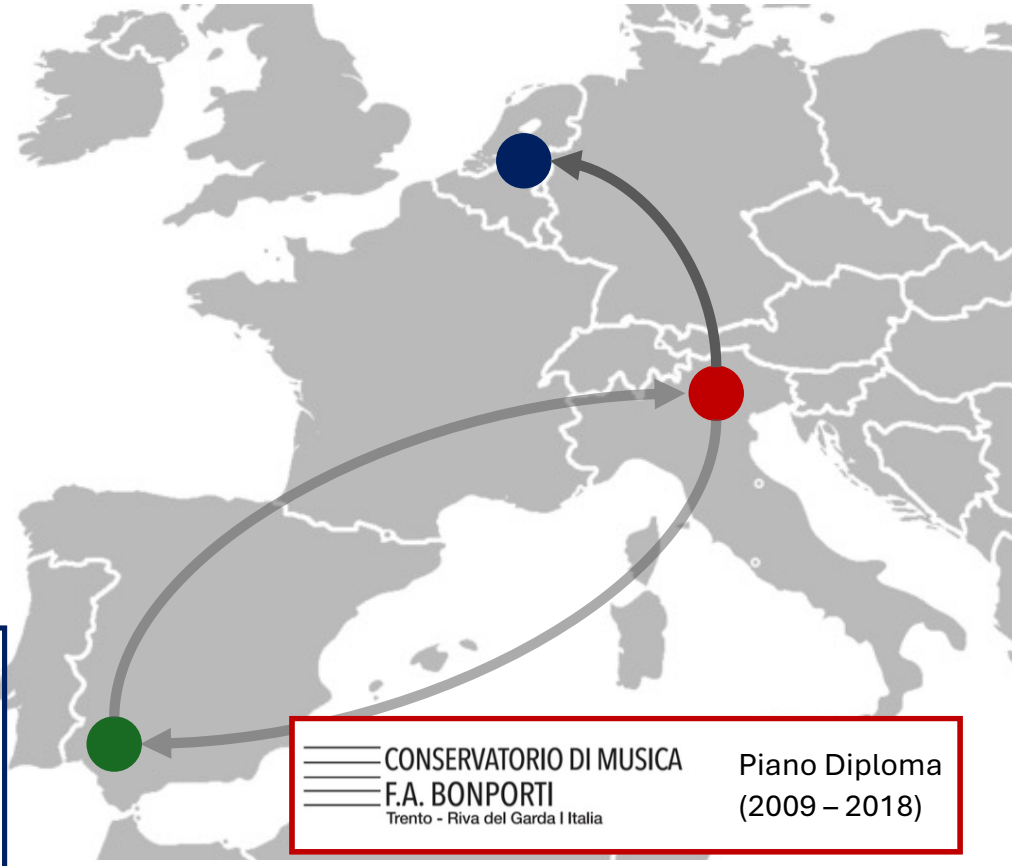
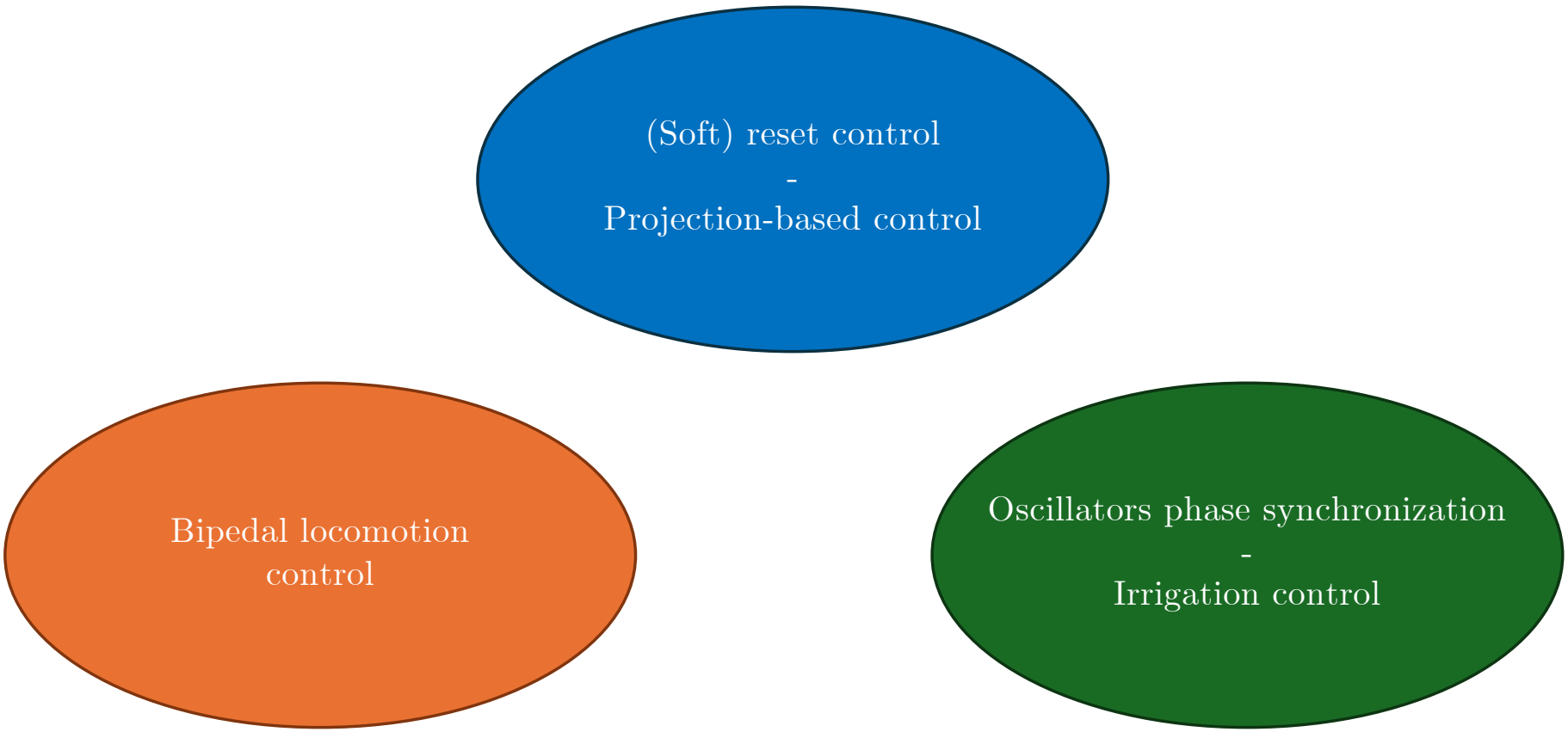


CONSERVATORIO DI MUSICA
F.A. BONPORTI
Trento - Riva del Garda | Italia

Piano Diploma
(2009 – 2018)

My research so far



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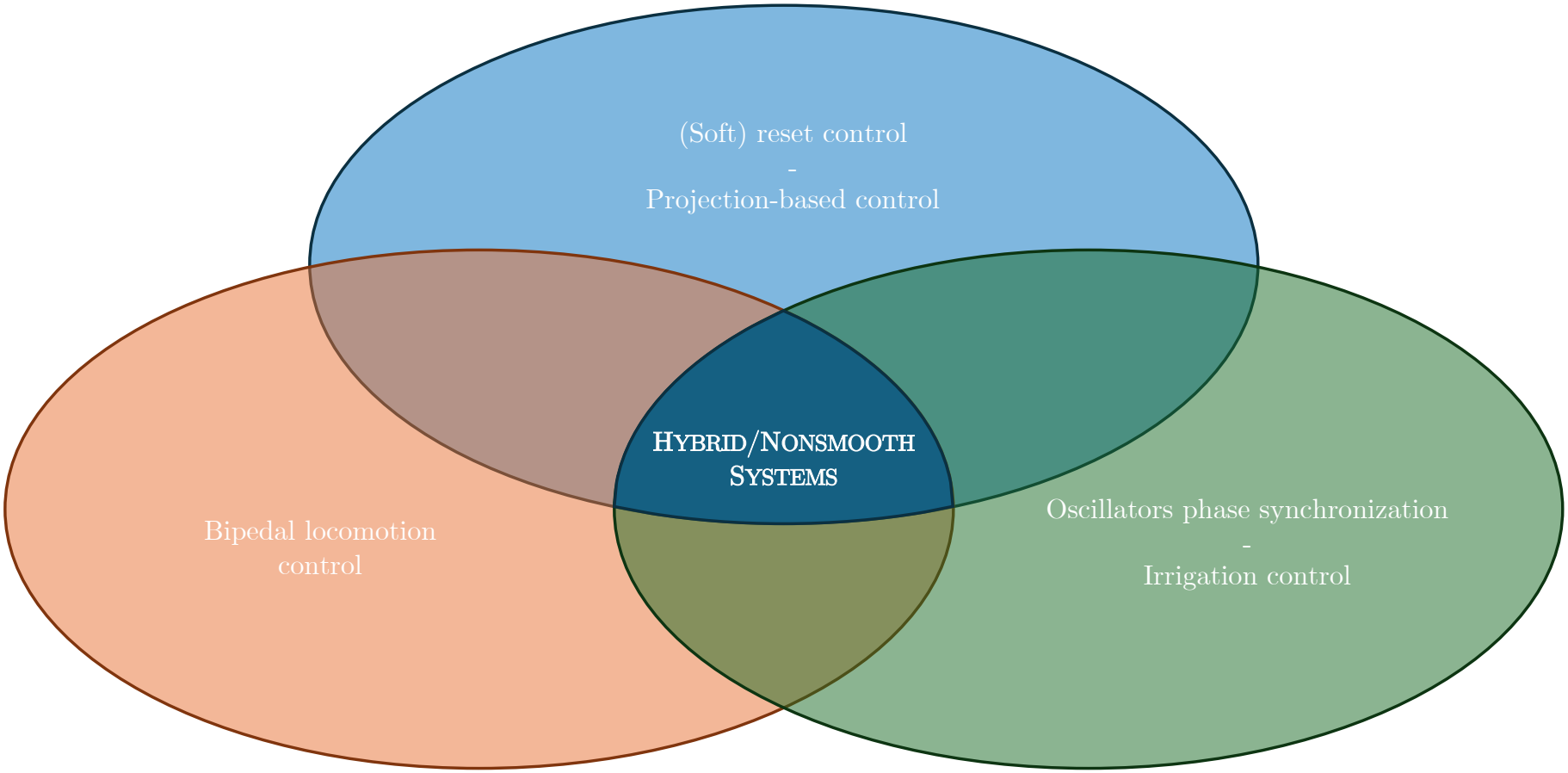
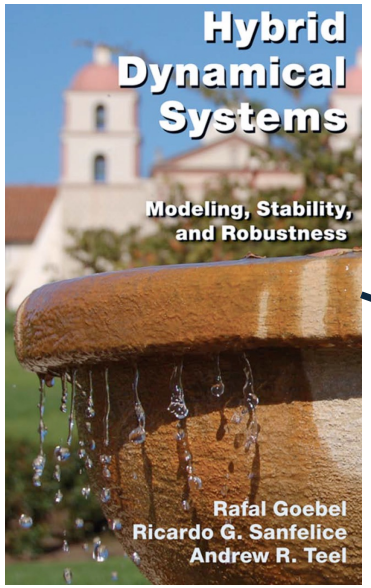
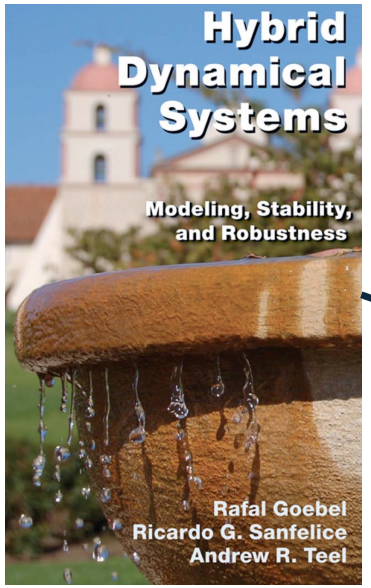


Hybrid dynamical systems



$$\begin{cases} \dot{x} \in F(x,u), & (x,u) \in C \\ x^+ \in G(x,u), & (x,u) \in D \end{cases}$$

Hybrid dynamical systems



Cyber-physical → Sampled-data
Event-triggered

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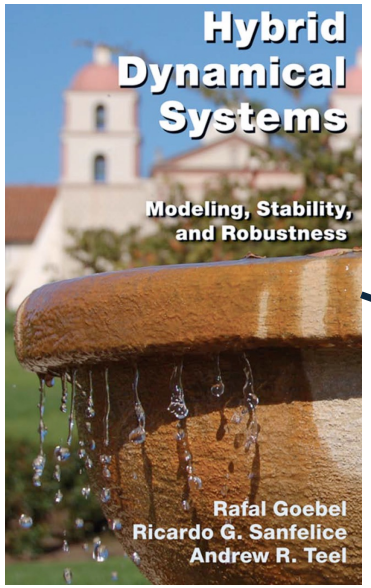

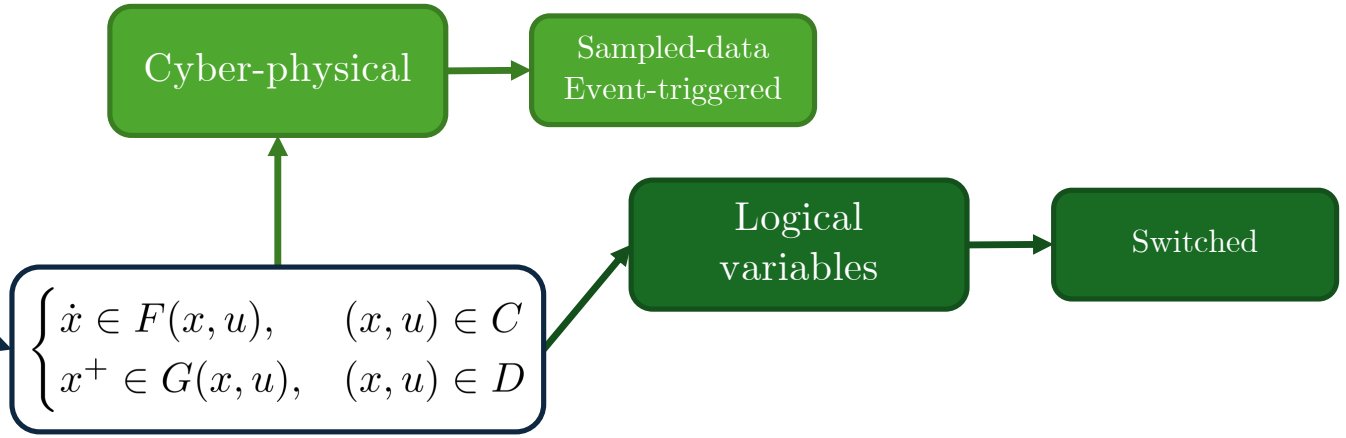

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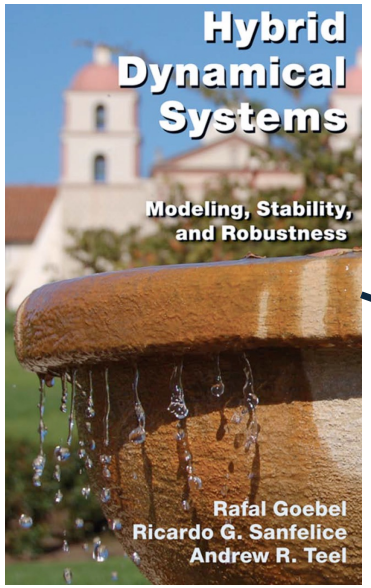

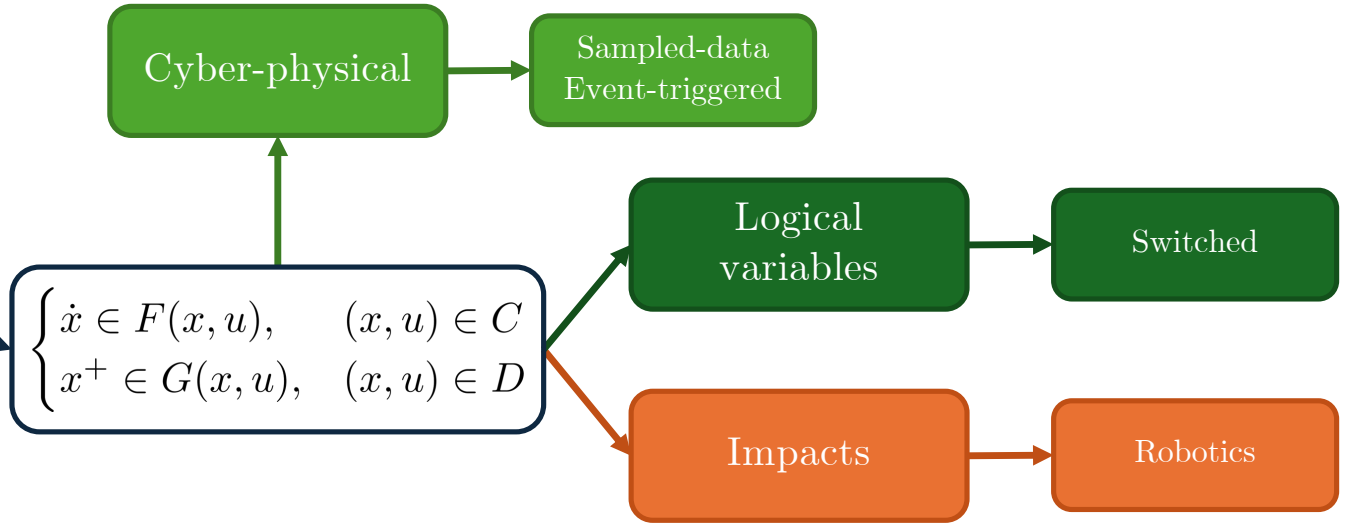

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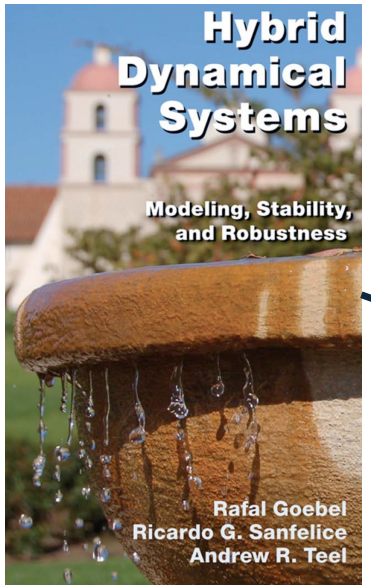

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- Cyber-physical
  - Sampled-data
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- Logical variables
  - Switched
- Impacts
  - Robotics
- Hybrid controller
  - Resets
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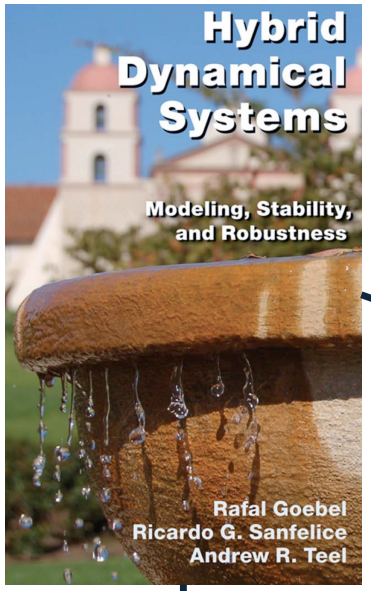
Hybrid dynamical systems



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- Cyber-physical → Sampled-data, Event-triggered
- Logical variables → Switched
- Impacts → Robotics
- Hybrid controller → Resets, Projections
- Lyapunov techniques

First Order Reset Elements

Inspired by the Clegg integrator [1958]

$$\begin{cases} \dot{x}_c = e, & \text{“normally”} \\ x_c^+ = 0, & \text{“at the right moment”} \end{cases}$$

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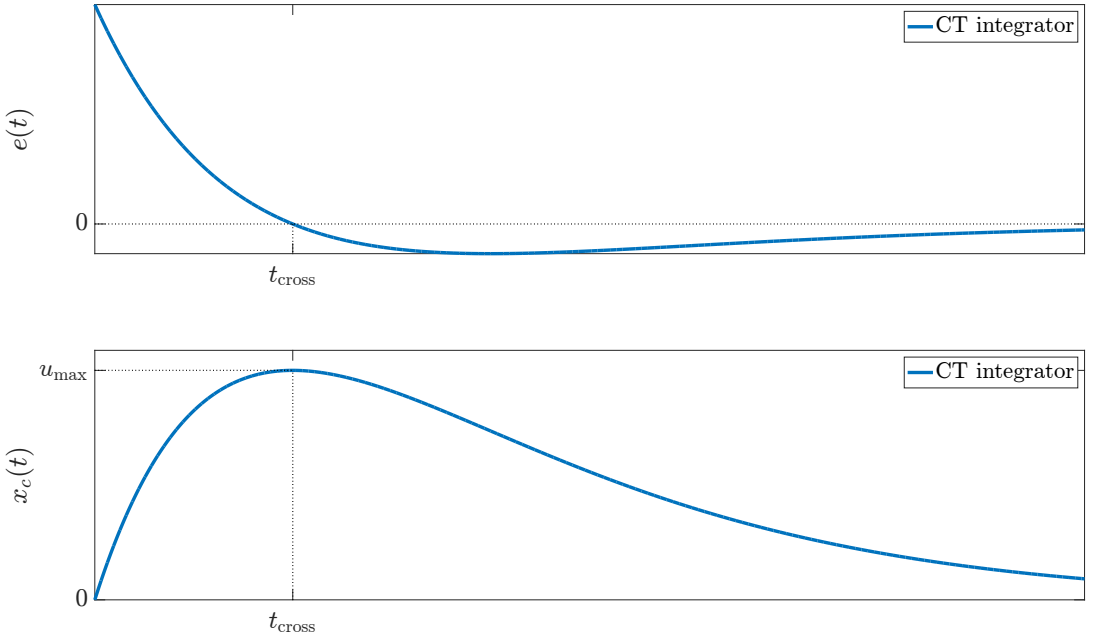

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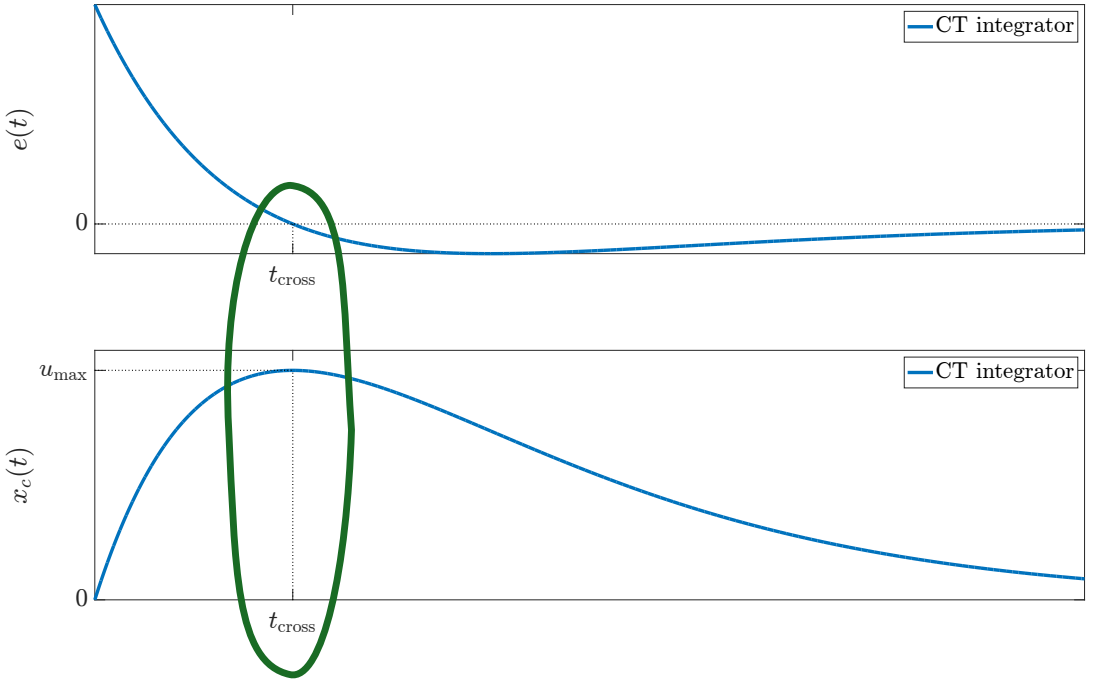

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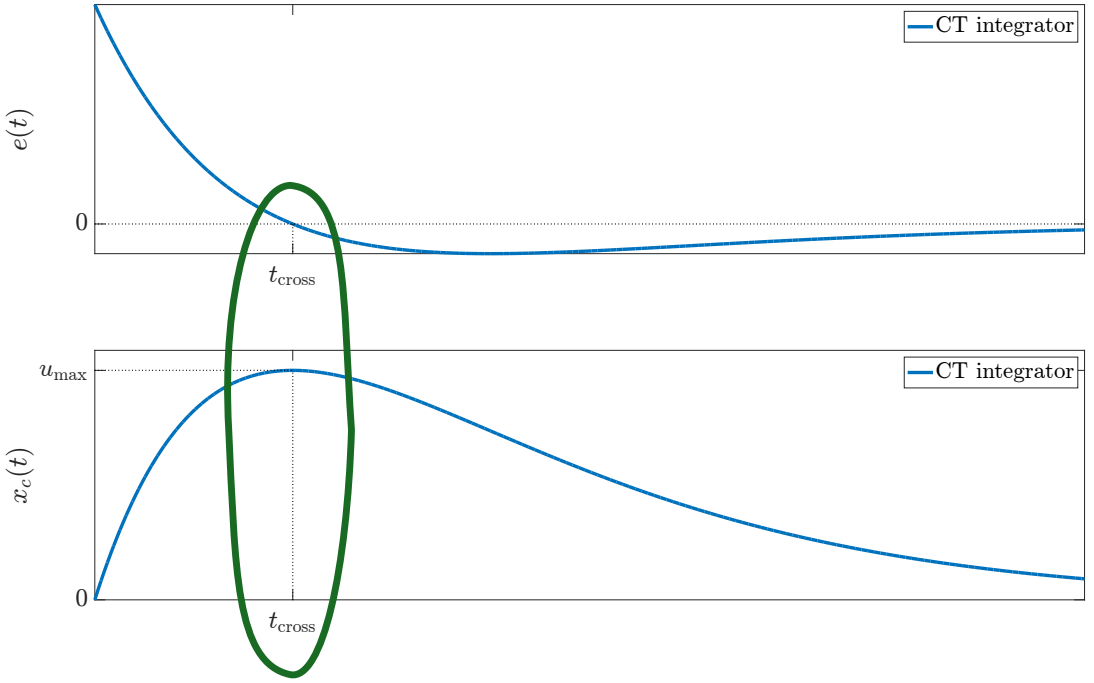

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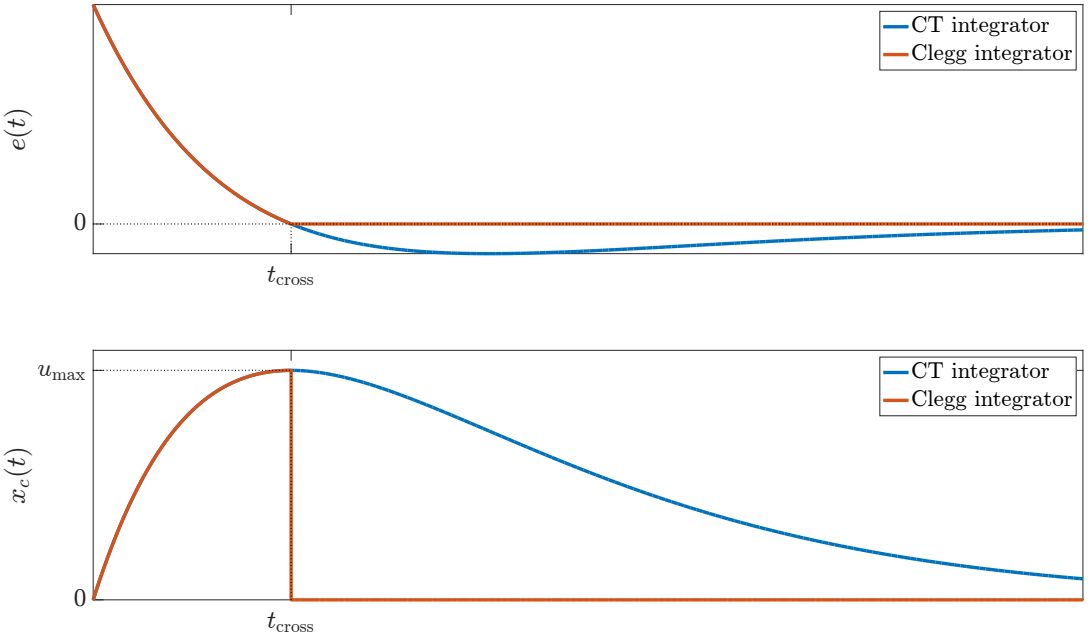

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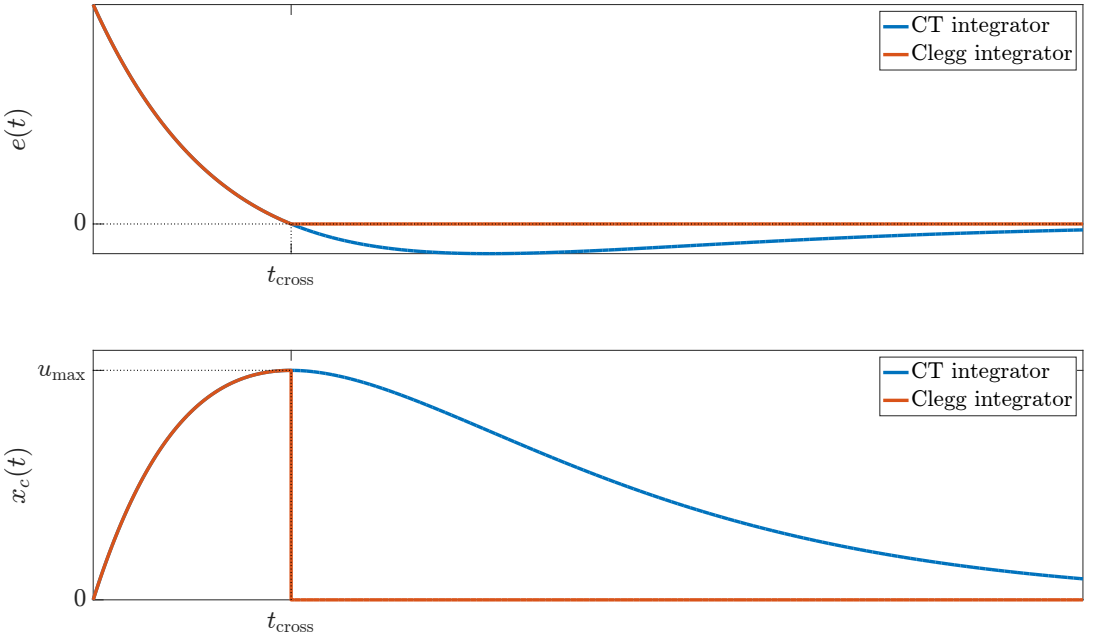

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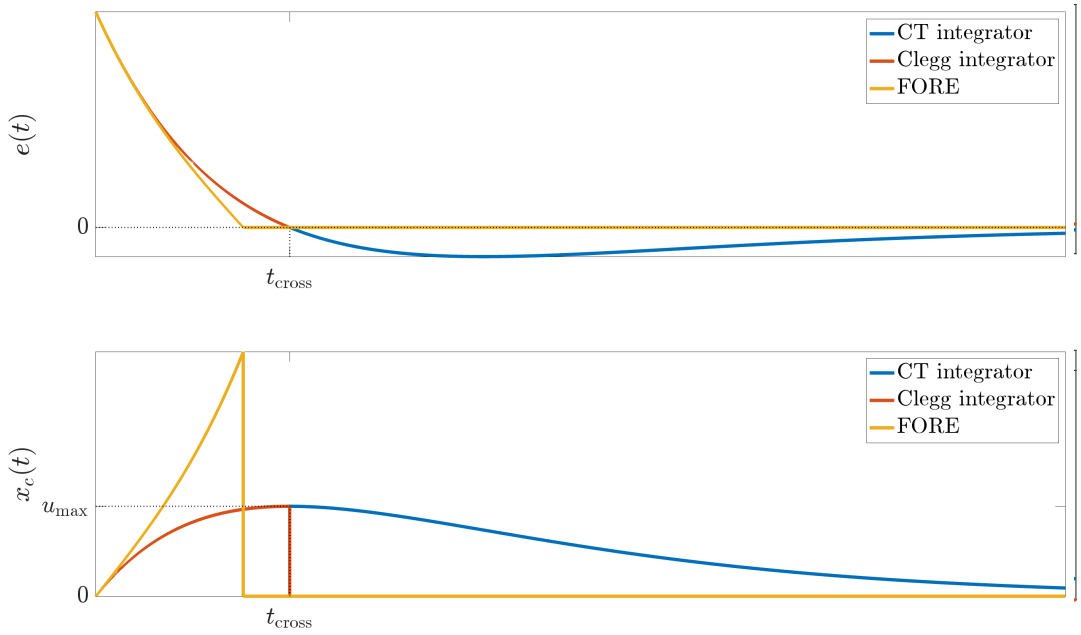

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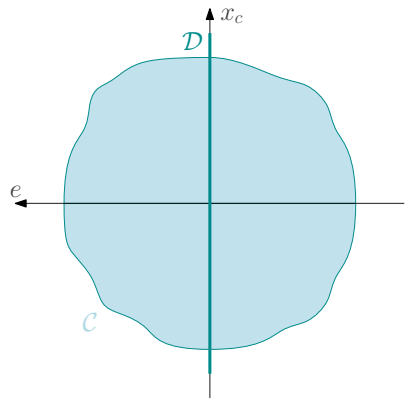

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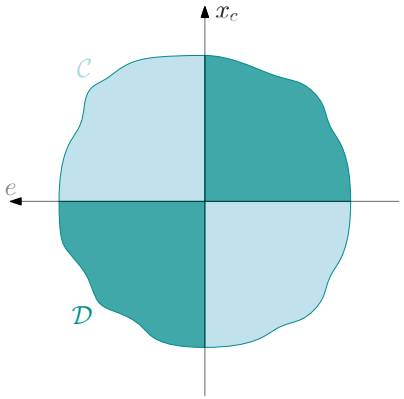

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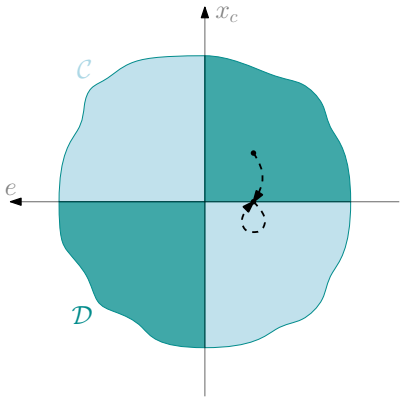

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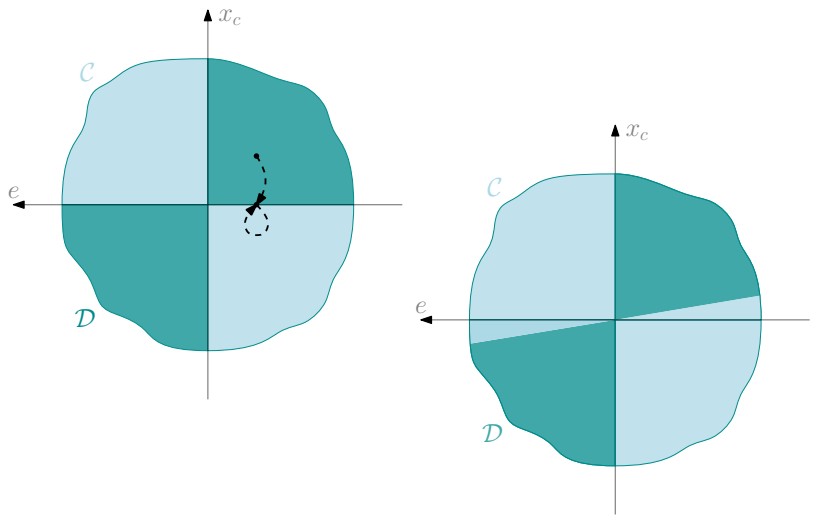

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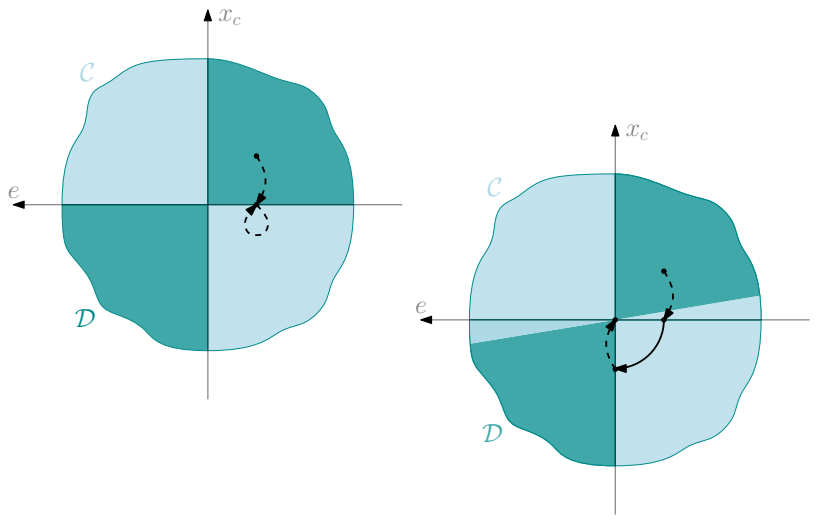

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Soft-reset control [1]

Start from generic reset-based closed loop...

$$\begin{cases} \dot{x} = Ax, & x^\top M x \leq 0 \\ x^+ = Rx, & x^\top M x \geq 0 \end{cases}$$

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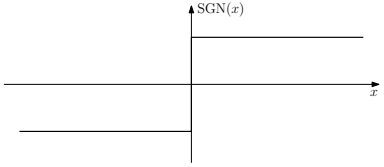
Krasovskii regularization:

$$\mathrm{SGN}(x) := \begin{cases} \mathrm{sgn}(x), & \text{if } x \neq 0 \\ [-1, 1], & \text{if } x = 0 \end{cases}$$



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$$x^\top Mx \leq 0 \rightarrow \mathrm{SGN}(x^\top Mx) + 1 = 0$$

$$F(x) = Ax$$

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$$\begin{cases} \dot{x} = Ax, & x^\top Mx \leq 0 \\ x^+ = Rx, & x^\top Mx \geq 0 \end{cases}$$

...and define a differential inclusion

$$\dot{x} \in F(x) := Ax + \gamma(\mathrm{SGN}(x^\top Mx) + 1)(Rx - x)$$

$$x^\top Mx \leq 0 \rightarrow \mathrm{SGN}(x^\top Mx) + 1 = 0$$
$$F(x) = Ax$$

$$x^\top Mx \geq 0 \rightarrow \mathrm{SGN}(x^\top Mx) + 1 = 2$$
$$\gamma \gg 1 \rightarrow \dot{x} = -\gamma(x - Rx)$$

[1] A.R. Teel. "Continuous-time implementation of reset control systems". *Trends in Nonlinear and Adaptive Control (Z.P. Jiang, C. Prieur, and A. Astolfi, editors)*. 2021.

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Regulates how closely the CT solutions resemble the ones of the original HS!

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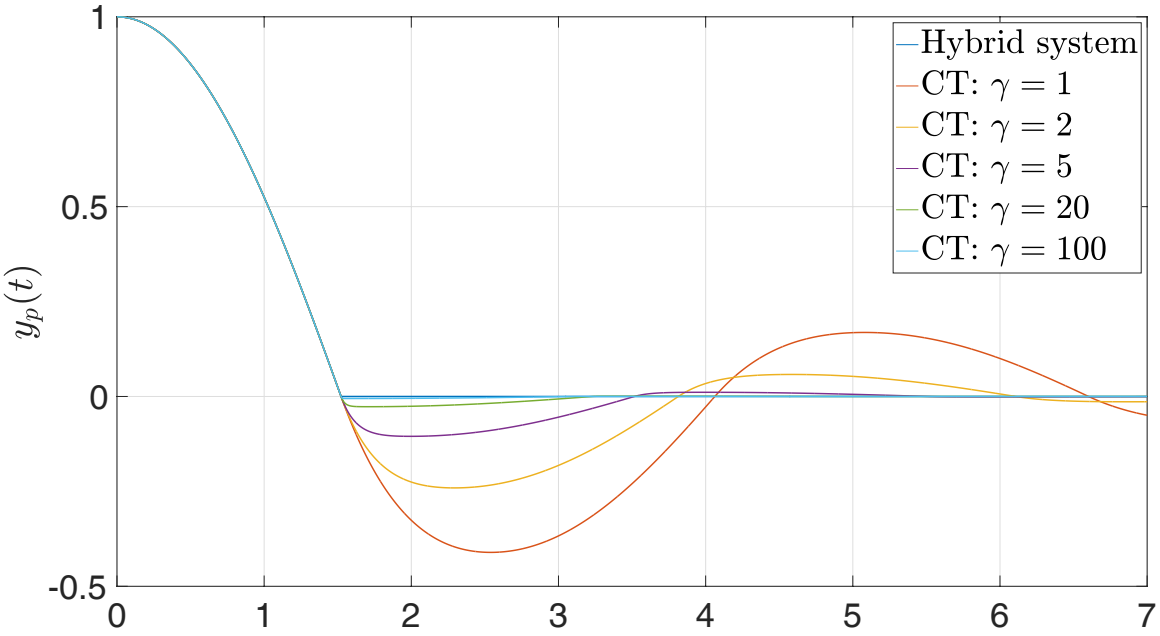

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Stability with soft-reset control

$$\dot{x} \in F(x) := Ax + \gamma(\mathrm{SGN}(x^\top Mx) + 1)(Rx - x)$$

Theorem

If $\exists \varepsilon > 0$ and a Lyapunov function V such that

1. V is **Lipschitz continuous**, homogeneous of degree 2, positive definite, strongly convex;
2. defined $C_\varepsilon := \{x \in \mathbb{R}^b : x^\top Mx \leq \varepsilon x^\top x\}$, we have

$$\begin{aligned} &\langle \nabla V, Ax \rangle \leq 0, && \textbf{for almost all } x \in C_\varepsilon, \\ &V(Rx) - V(x) \leq 0, && \text{for all } x \in D; \end{aligned}$$

3. $x \in D$ implies $Rx \in C$;
4. no continuous solution of the hybrid system keeps V constant and nonzero;

then the origin is GES for $\dot{x} \in F(x)$, for $\gamma > 0$ large enough.

Why do we need nonsmoothness?

FORE + integrator closed loop:

$$\dot{x} = \begin{bmatrix} \dot{x}_c \\ \dot{e} \end{bmatrix} = \begin{bmatrix} 0.1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_c \\ e \end{bmatrix} = Ax$$

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Quadratic Lyapunov function:

$$V = x^\top P x = x^\top \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} x$$

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Flow condition:

$$\dot{V}(x) := \langle \nabla V, Ax \rangle = 2x^\top APx = 2x^\top \begin{bmatrix} 0.1p_{11} - p_{12} & p_{11} \\ 0.1p_{12} - p_{22} & p_{12} \end{bmatrix} x$$

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$$\dot{V}\left([0,1]^\top\right) \leq 0 \quad \dot{V}\left([1,0]^\top\right) \leq 0$$

$$\downarrow \qquad \downarrow$$

$$p_{12} \leq 0 \qquad p_{12} - 0.1p_{11} \geq 0$$

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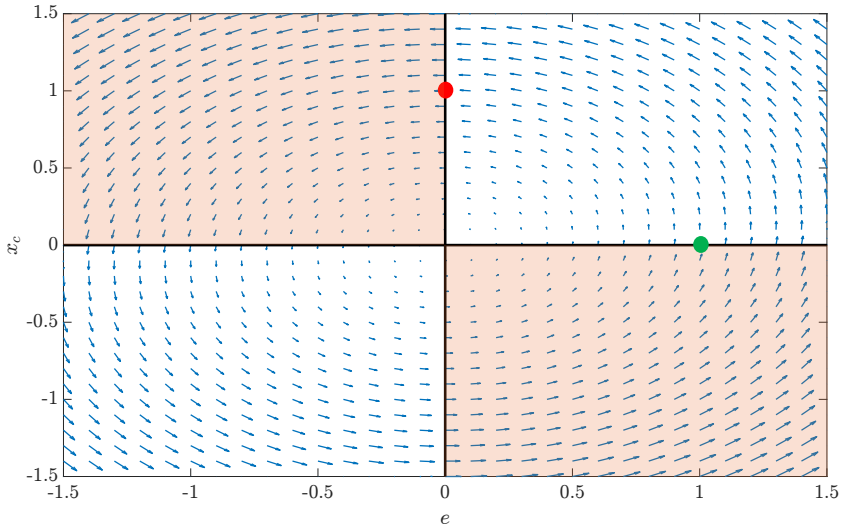

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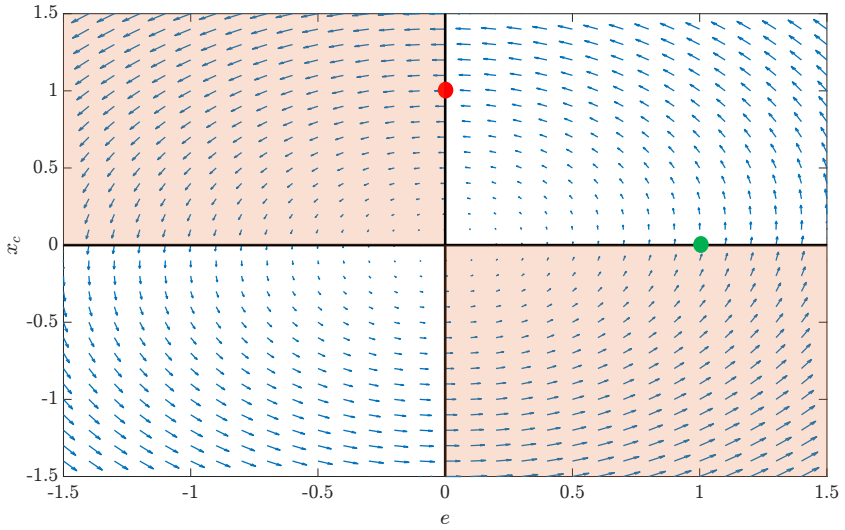
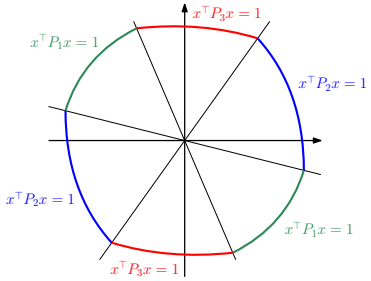
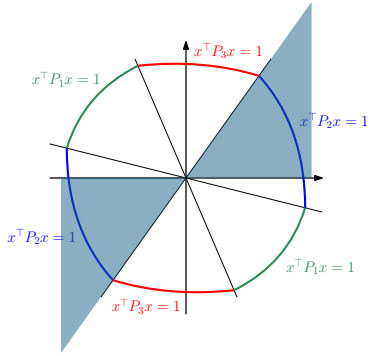


Piecewise-quadratic (max-of-quadratics) functions!



Numerical sufficient conditions

$$V(x) = \max_{i \in \{1,\dots,q\}} x^\top P_i x, \quad P_i \succ 0$$



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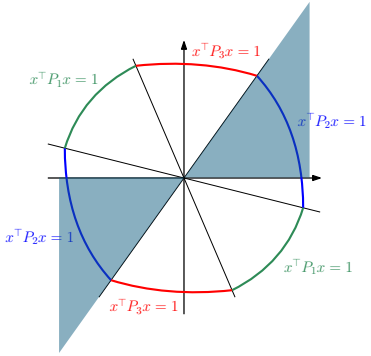

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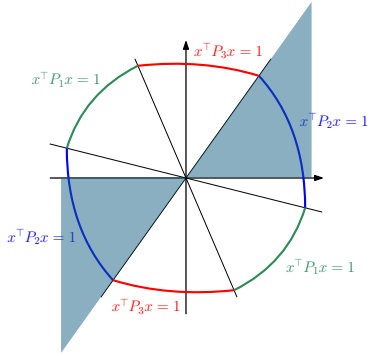
$A^\top P_i + P_i A \preceq 0, \quad \forall i \in \{1, \dots, q\}$?

Still conservative!



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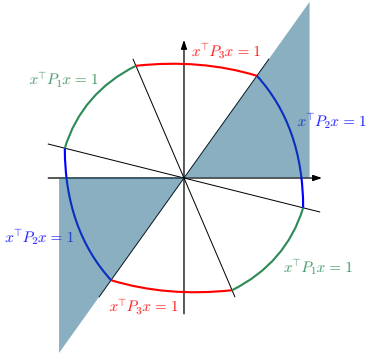

We need to check

$$\begin{cases} x \in C_\varepsilon \\ x^\top (P_i - P_j) x \geq 0, \ \forall j \end{cases} \implies \langle P_i x, Ax \rangle \leq 0$$

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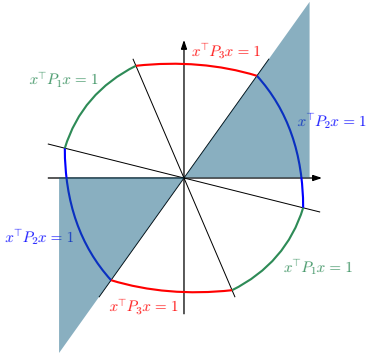

S-procedure

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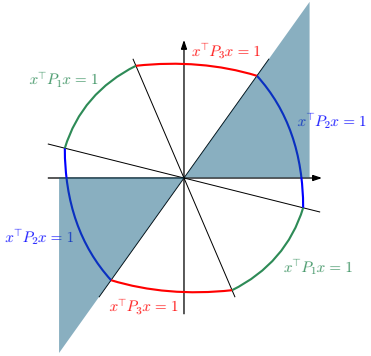

S-procedure

Applied once: necessary and sufficient

$$x^\top Q x \geq 0 \implies x^\top S x \geq 0$$

iff

$$\exists \mu > 0 : S - \mu Q \succeq 0$$

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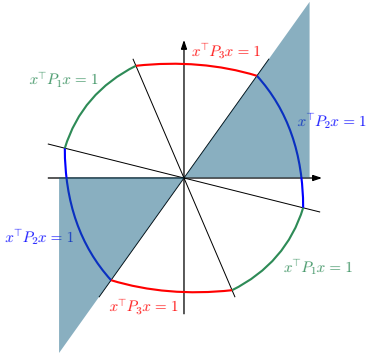

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Applied multiple times: only sufficient

$$x^\top Q_1 x \geq 0, \dots, x^\top Q_q x \geq 0 \implies x^\top S x \geq 0$$

if

$$\exists \mu_1, \dots, \mu_q > 0 : S - \sum_{i=1}^{q} \mu_i Q_i \succeq 0$$

Numerical sufficient conditions

$$V(x) = \max_{i \in \{1,\dots,q\}} x^\top P_i x, \quad P_i \succ 0$$

How do we check the Lyapunov conditions numerically?

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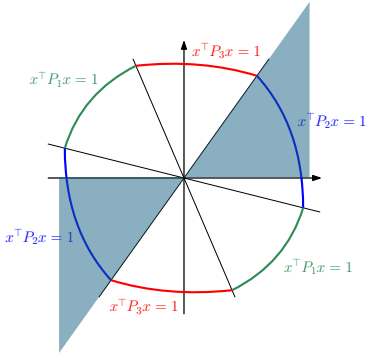

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$$\begin{cases} x \in C_\varepsilon \\ x^\top (P_i - P_j) x \geq 0, \ \forall j \end{cases} \implies \langle P_i x, Ax \rangle \leq 0$$

Many numerical tests, none feasible

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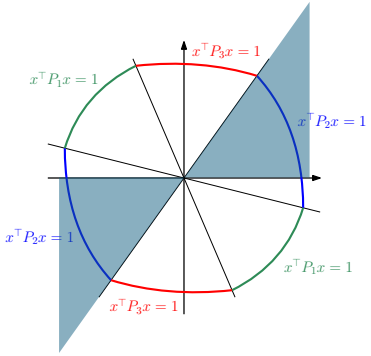

Sum Of Squares (SOS) programming

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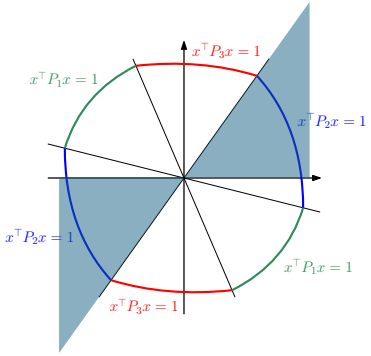

Sum Of Squares (SOS) programming

Same concept, but with $\mu(x)$

$$x^\top Q x \geq 0 \implies x^\top S x \geq 0$$

iff

$$\exists \mu(x) \in SOS : x^\top S x - \mu(x) x^\top Q x \in SOS$$

Numerical sufficient conditions

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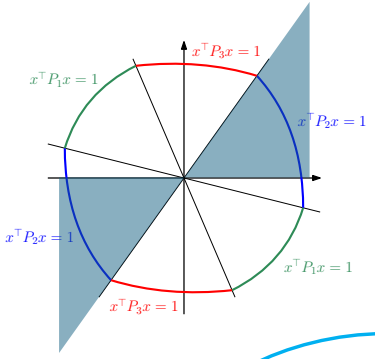

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$$-x^\top (A^\top P_i + P_i A) x - 2\alpha x^\top P_i x - r_{i,F}(x)(1 - x^\top x) - \sum_{i=1}^{q} \mu_{ij}(x) x^\top (P_i - P_j) x - \mu_{i0}(x) x^\top (M - \varepsilon I) x \in SOS$$

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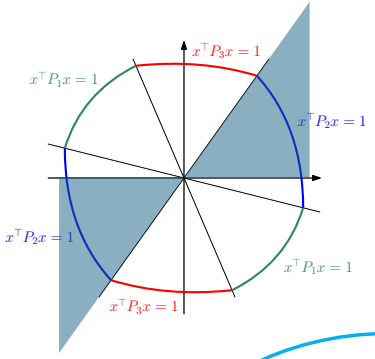

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$\dot{V}(x) \leq -2\alpha V(x)$

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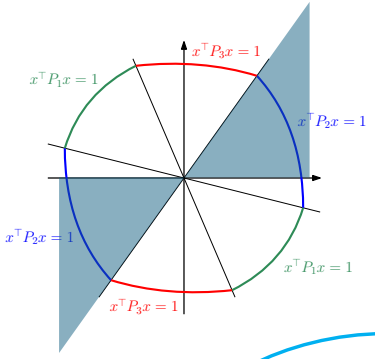

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$\dot{V}(x) \leq -2\alpha V(x)$ $V(x) = x^\top P_i x$

$$-x^\top (A^\top P_i + P_i A) x - 2\alpha x^\top P_i x - r_{i,F}(x)(1 - x^\top x) - \sum_{i=1}^{q} \mu_{ij}(x) x^\top (P_i - P_j) x - \mu_{i0}(x) x^\top (M - \varepsilon I) x \in SOS$$

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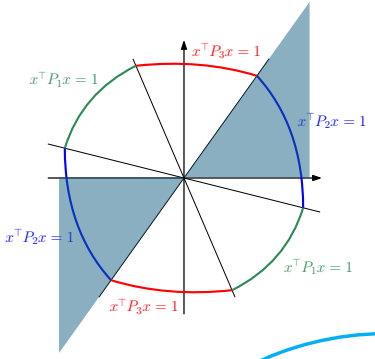

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$\dot{V}(x) \leq -2\alpha V(x)$ $V(x) = x^\top P_i x$ $x \in C_\varepsilon$

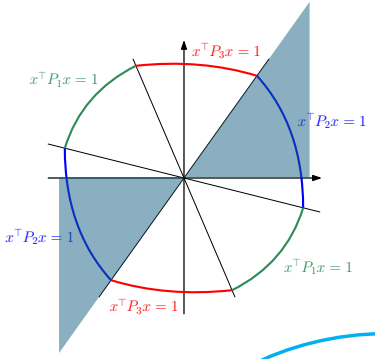
$$-x^\top (A^\top P_i + P_i A) x - 2\alpha x^\top P_i x - r_{i,F}(x)(1 - x^\top x) - \sum_{i=1}^{q} \mu_{ij}(x) x^\top (P_i - P_j) x - \mu_{i0}(x) x^\top (M - \varepsilon I) x \in SOS$$

Numerical sufficient conditions

$$V(x) = \max_{i \in \{1,\dots,q\}} x^\top P_i x, \quad P_i \succ 0$$

How do we check the Lyapunov conditions numerically?

$A^\top P_i + P_i A \preceq 0, \quad \forall i \in \{1,\dots,q\}?$
Still conservative!



Sum Of Squares (SOS) programming

Same concept, but with $\mu(x)$

$$x^\top Q x \geq 0 \implies x^\top S x \geq 0$$

iff

$$\exists \mu(x) \in SOS : x^\top S x - \mu(x) x^\top Q x \in SOS$$

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Check only on the unit sphere

$$-x^\top (A^\top P_i + P_i A) x - 2\alpha x^\top P_i x - r_{i,F}(x)(1 - x^\top x) - \sum_{i=1}^{q} \mu_{ij}(x) x^\top (P_i - P_j) x - \mu_{i0}(x) x^\top (M - \varepsilon I) x \in SOS$$

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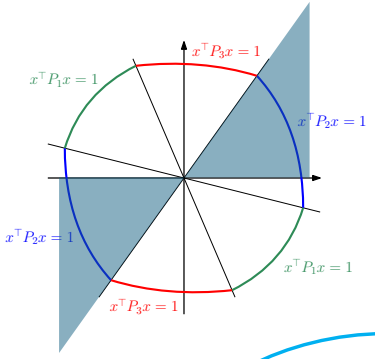

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We cannot optimize over P_i because this term would be bilinear!

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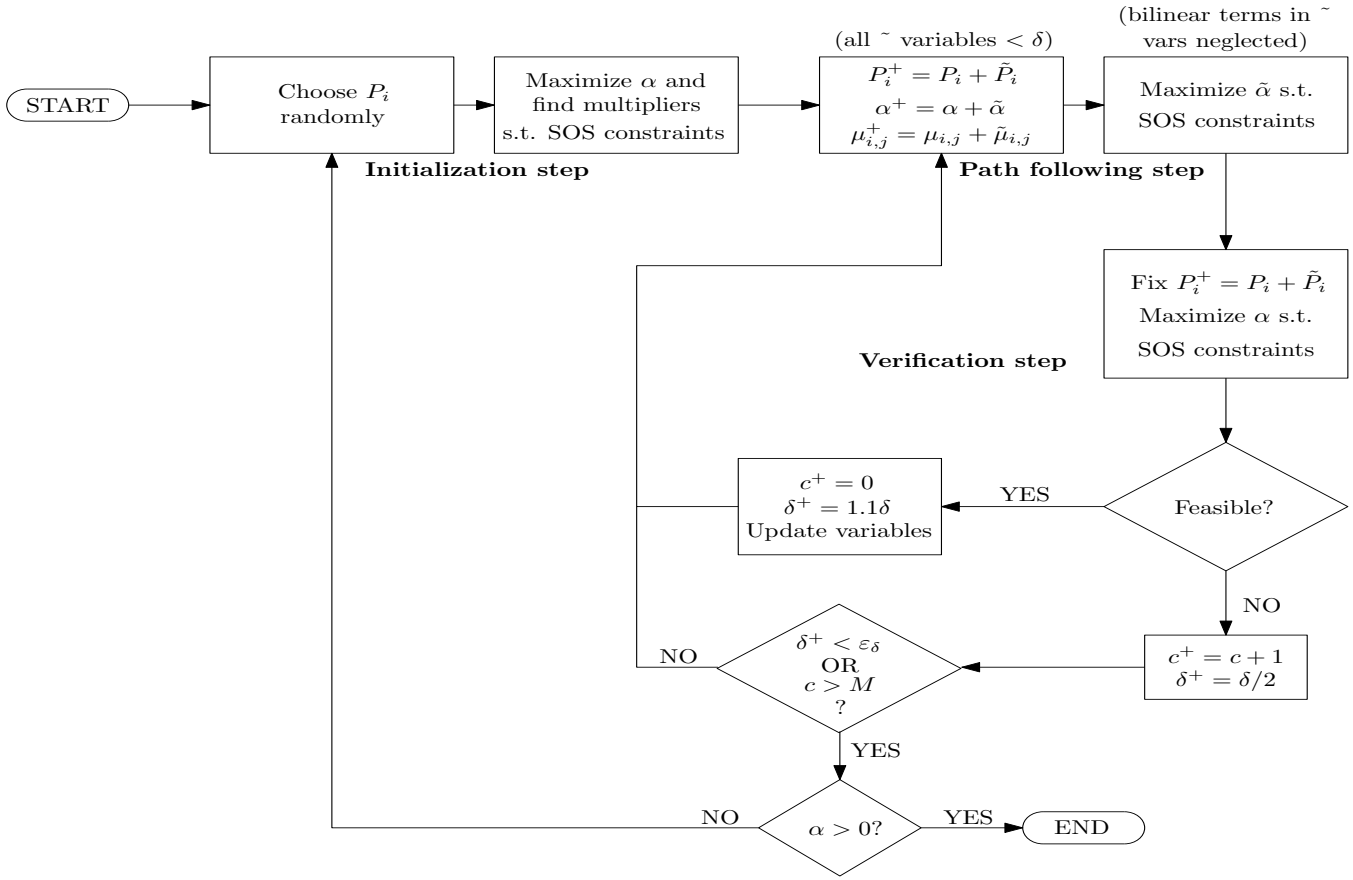
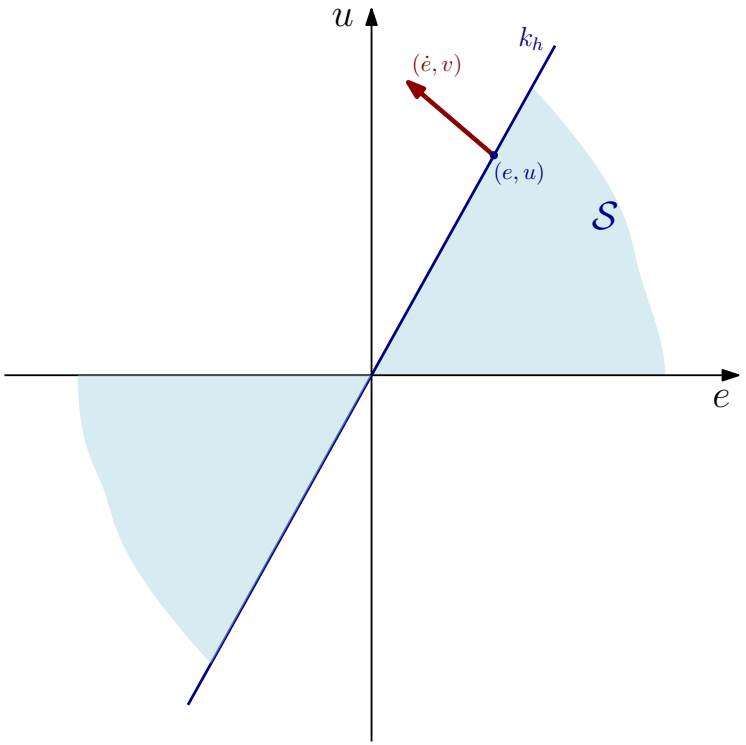
Recursive algorithm for synthesis



[1] A. Hassibi, J. How, S. Boyd, "A path-following method for solving BMI problems in control". *Proceedings of the American Control Conference.* 1999.

To (soft-)reset or to project?

Reset control works well because it keeps $ex_c \geq 0$



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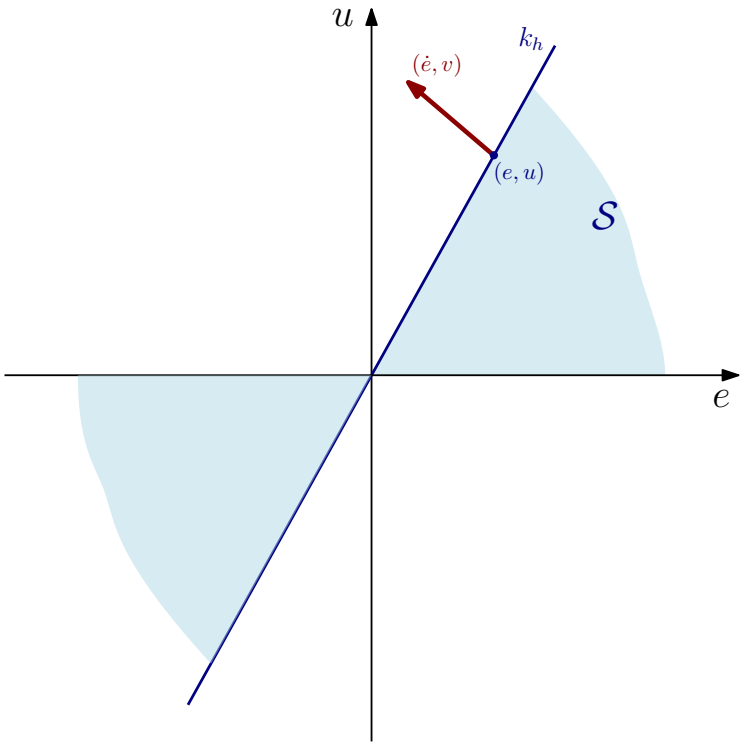

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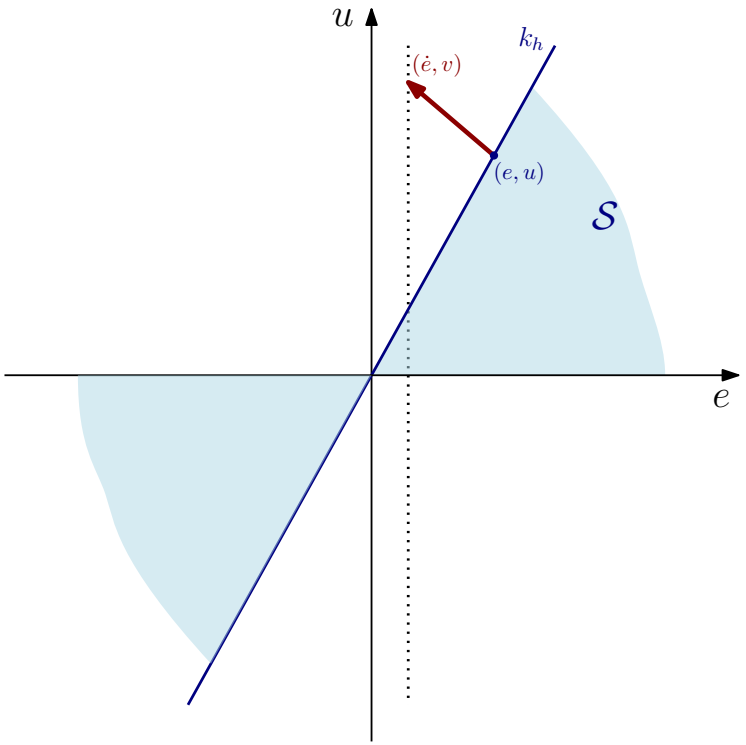

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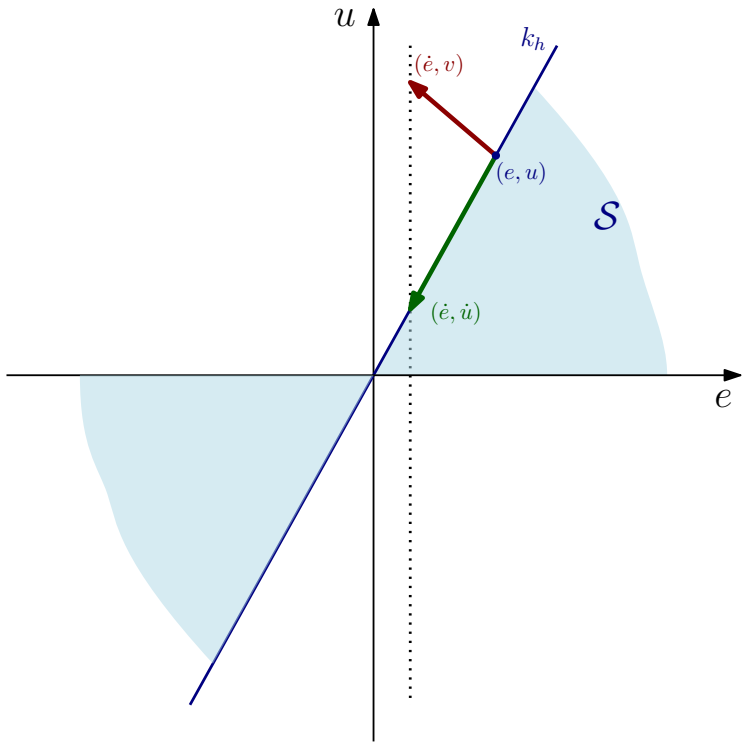

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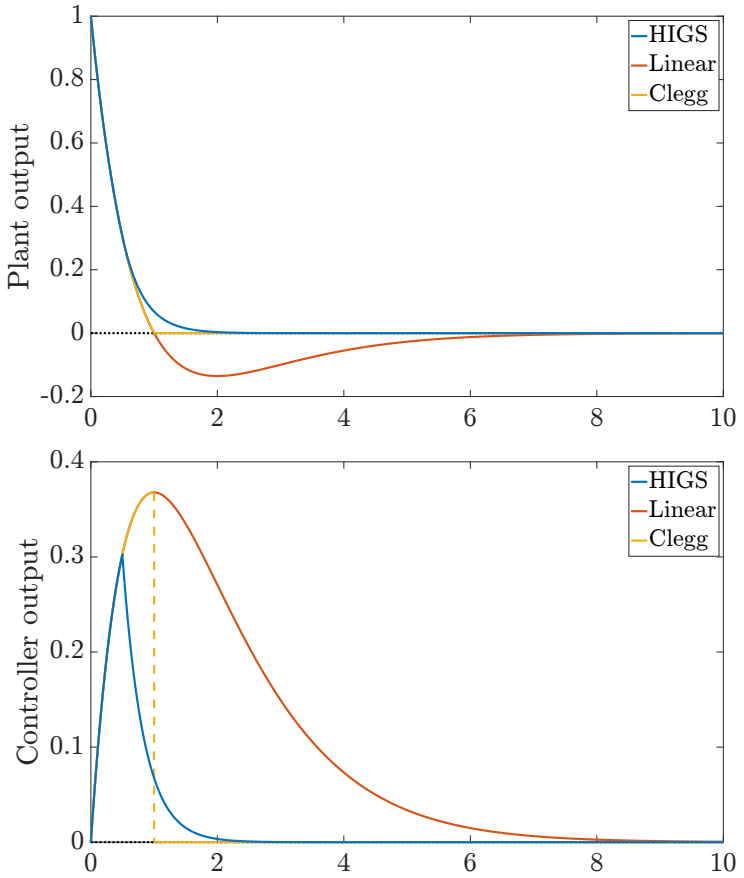

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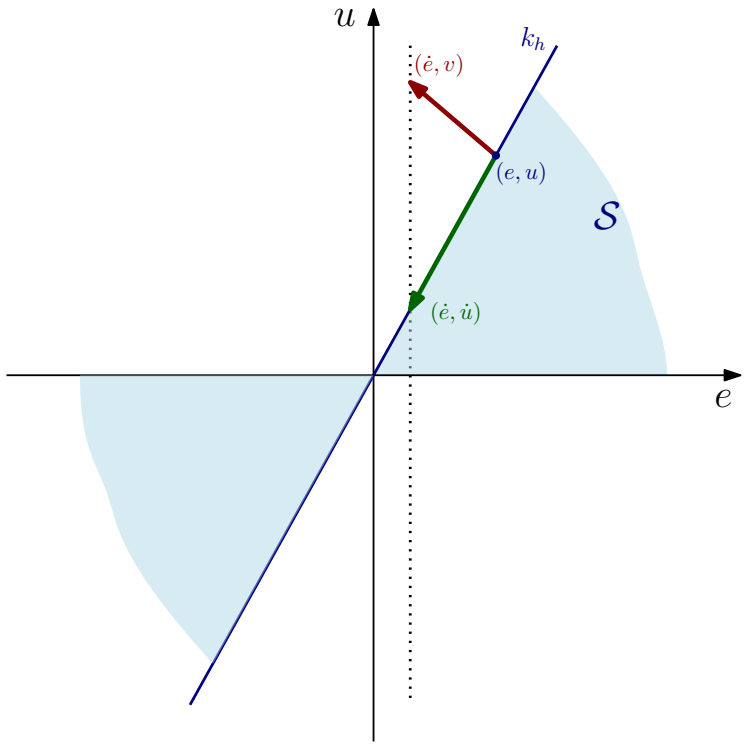
Can be any "nominal" dynamics

$$\dot{u} = \begin{cases} \boldsymbol{v}, & (\boldsymbol{v}, \dot{e}) \in T_{\mathcal{S}}(e, u) \\ k_h \dot{e}, & (\boldsymbol{v}, \dot{e}) \notin T_{\mathcal{S}}(e, u) \end{cases}$$



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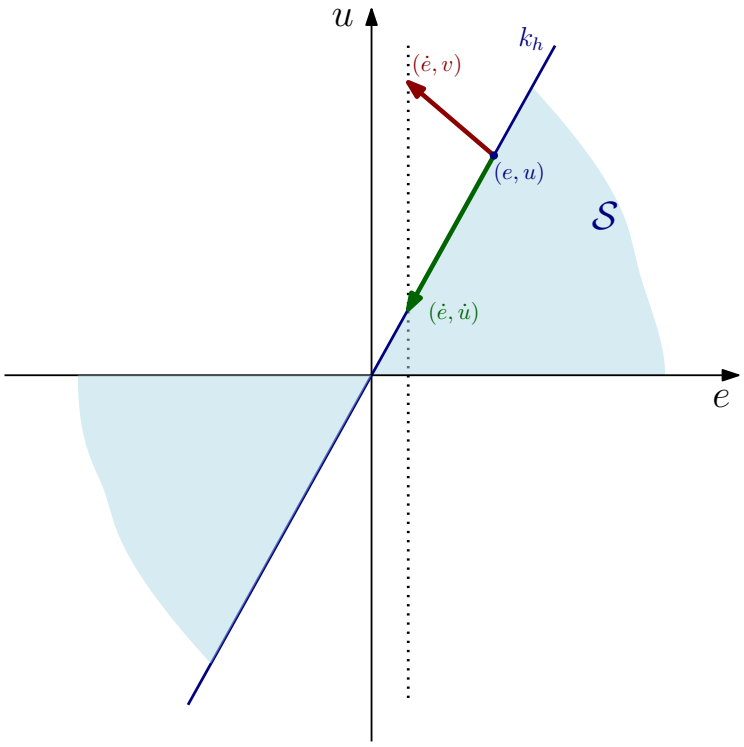
First-Order Projection Elements (FOPE)

$$\dot{u} = \begin{cases} au + be, & (au + be, \dot{e}) \in T_{\mathcal{S}}(e,u) \\ k_h \dot{e}, & (au + be, \dot{e}) \notin T_{\mathcal{S}}(e,u) \end{cases}$$



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Analysis is not trivial

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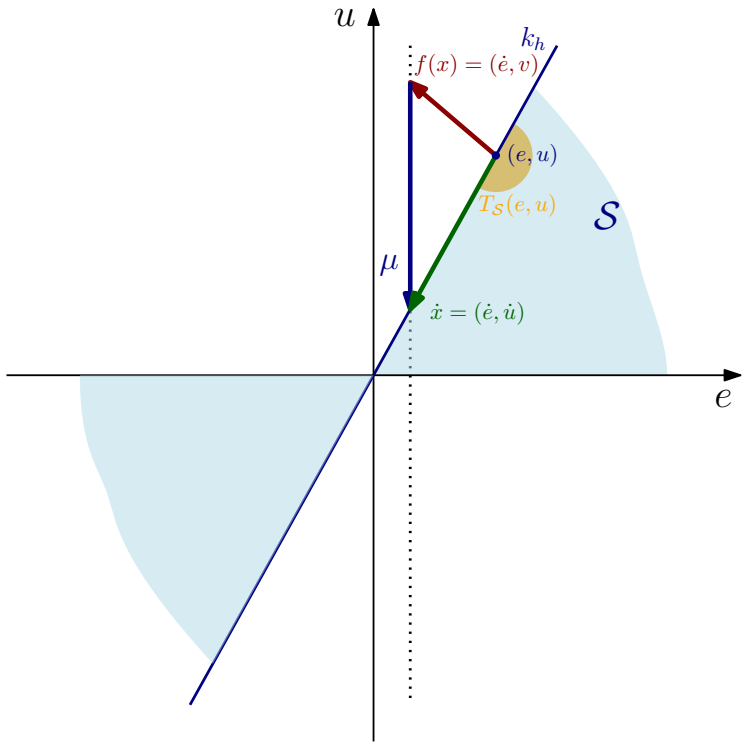

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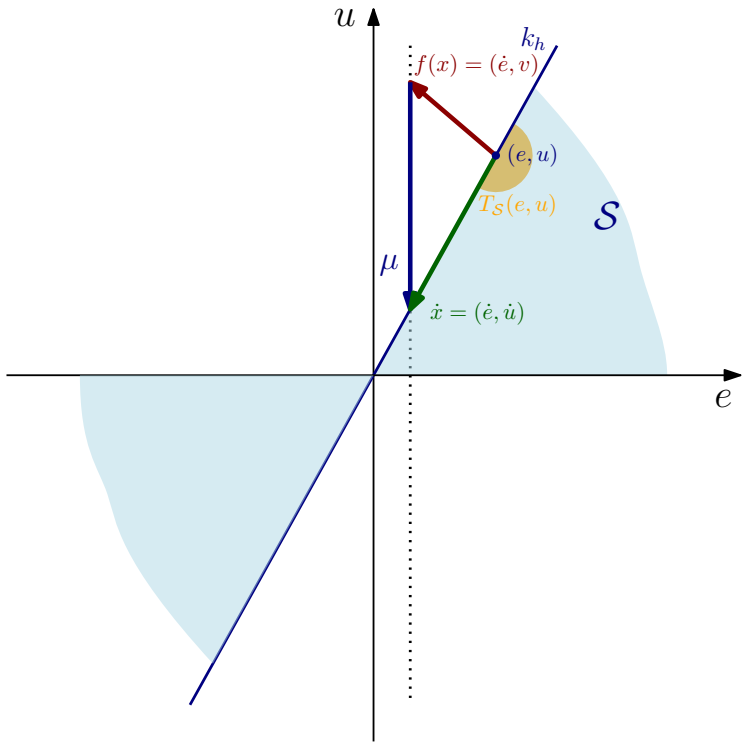

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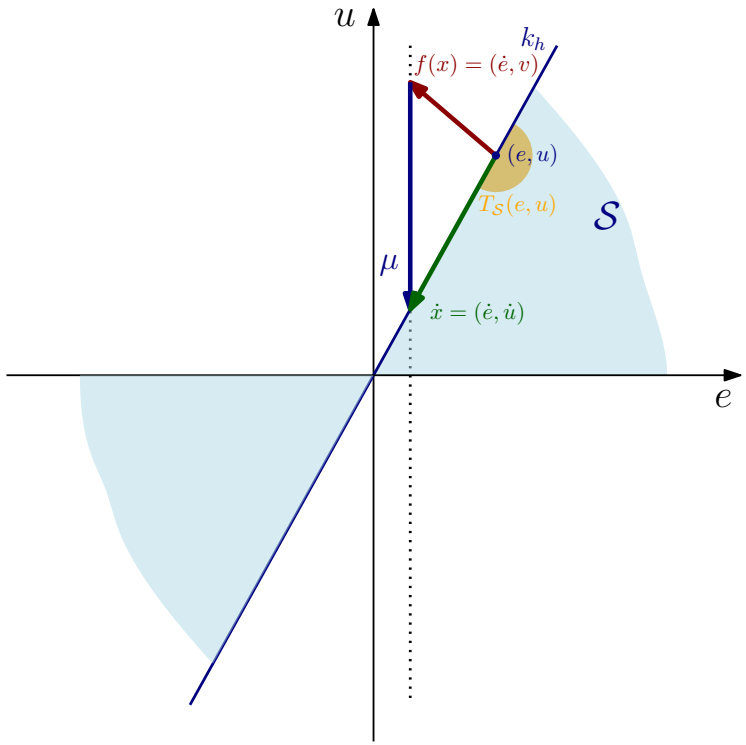
Extended projected dynamical systems (ePDS) [1]

$$\dot{x} = \Pi_{\mathcal{S},\mathcal{E}} f(x) := f(x) + \arg\min \|\mu\|^2$$

$$\text{s.t.} \begin{cases} f(x) + \mu \in T_{\mathcal{S}}(x) \\ \boldsymbol{\mu \in \mathcal{E}} \end{cases}$$



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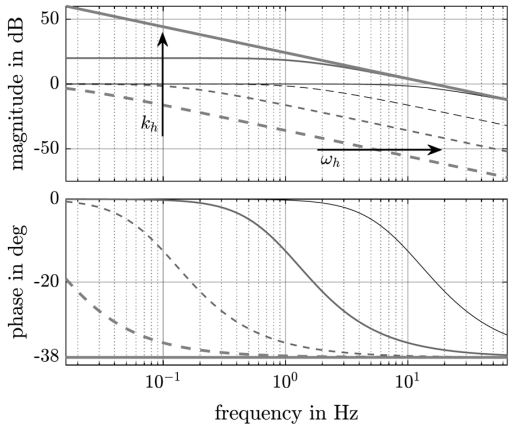

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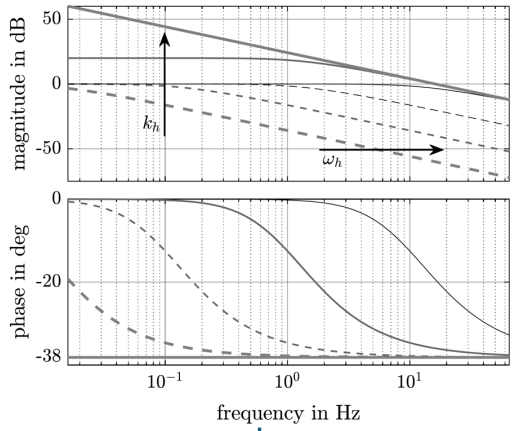
HIGS **improves performance** w.r.t. linear integrators [2]
(phase lag reduced to 38.15°)



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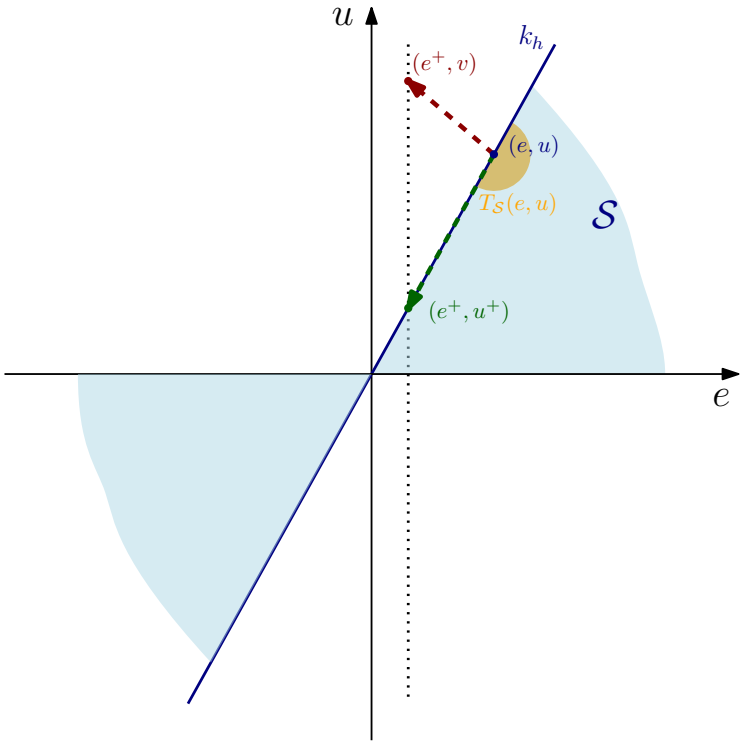

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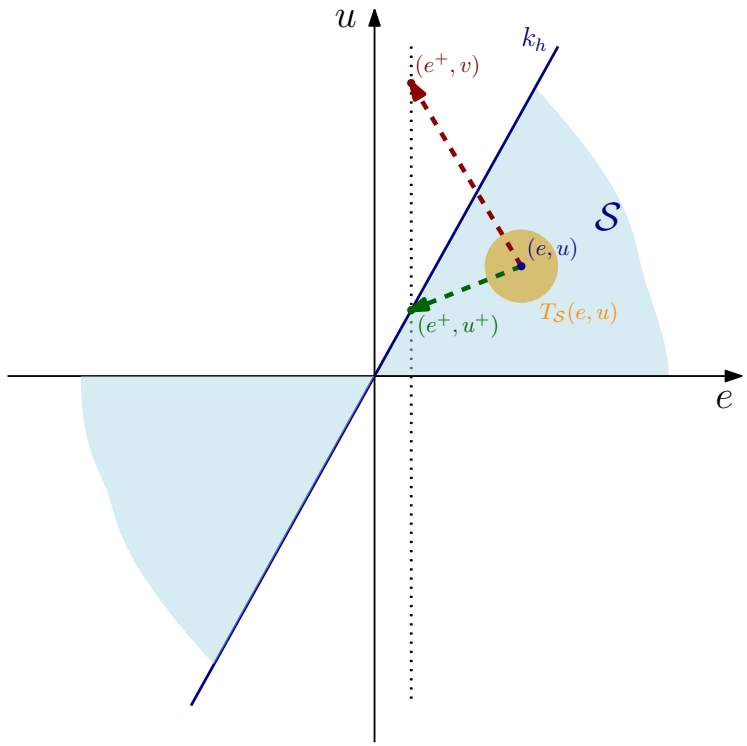

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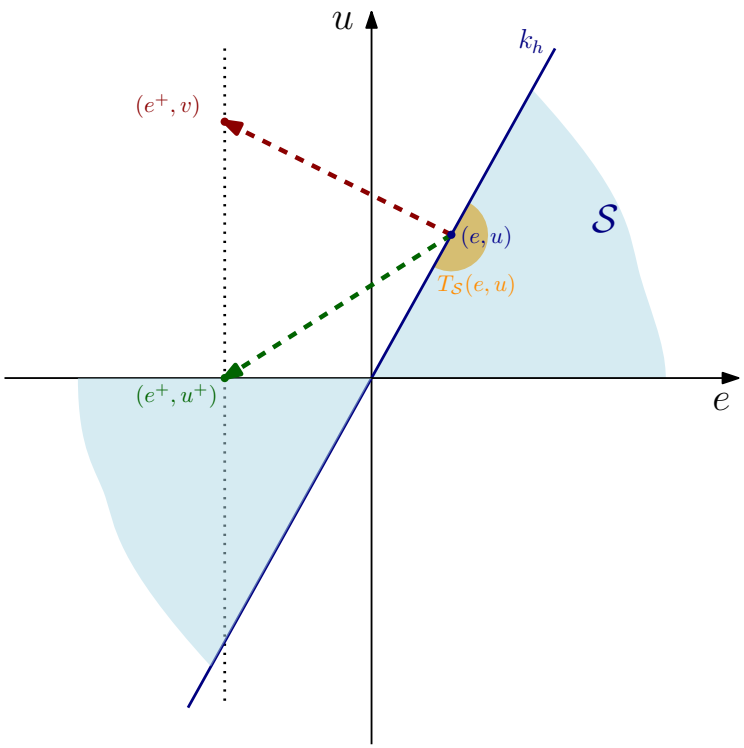

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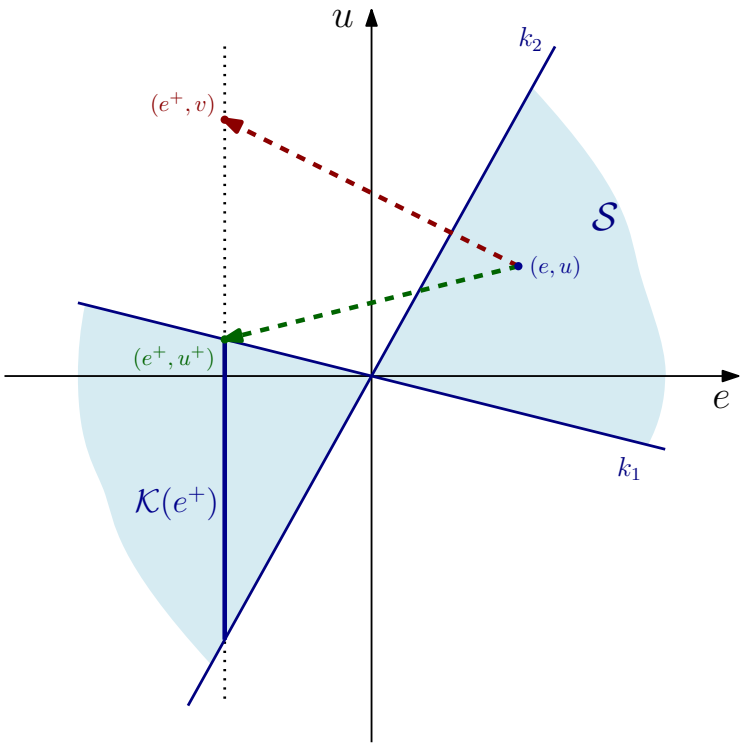

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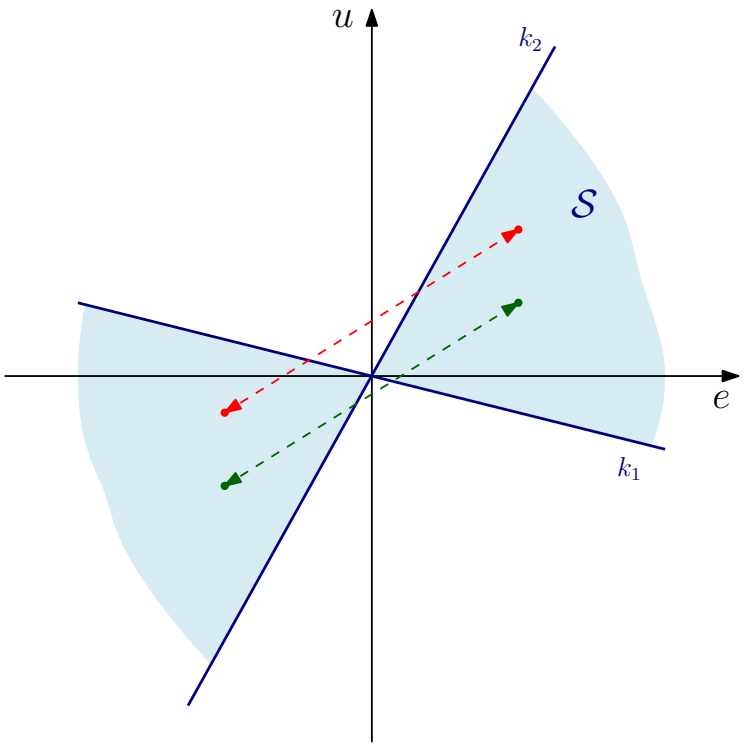

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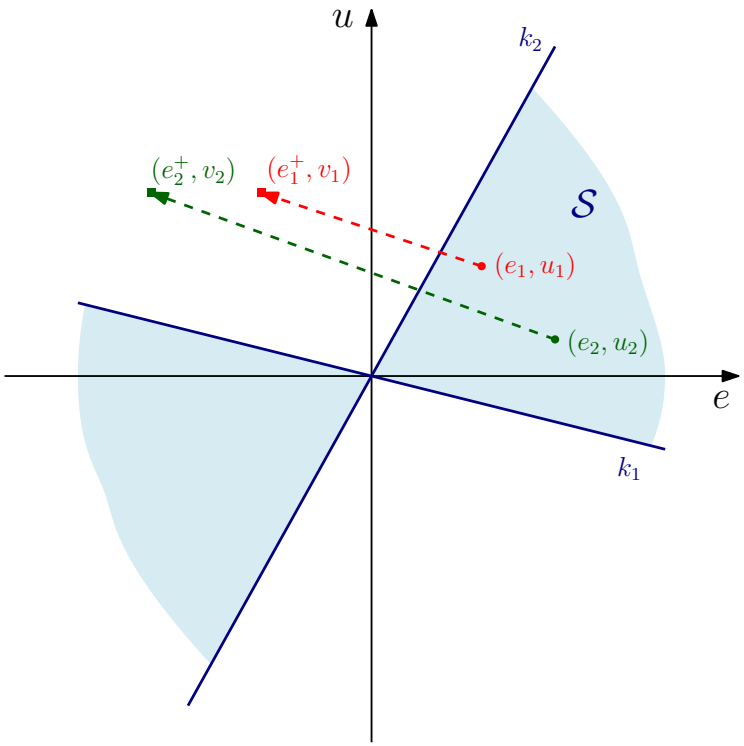

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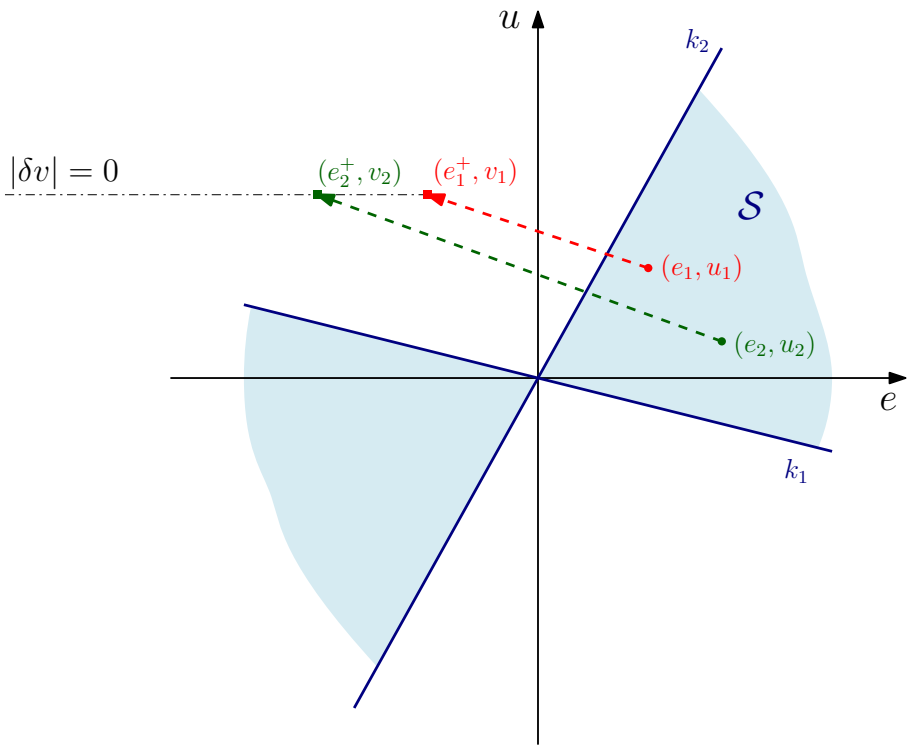

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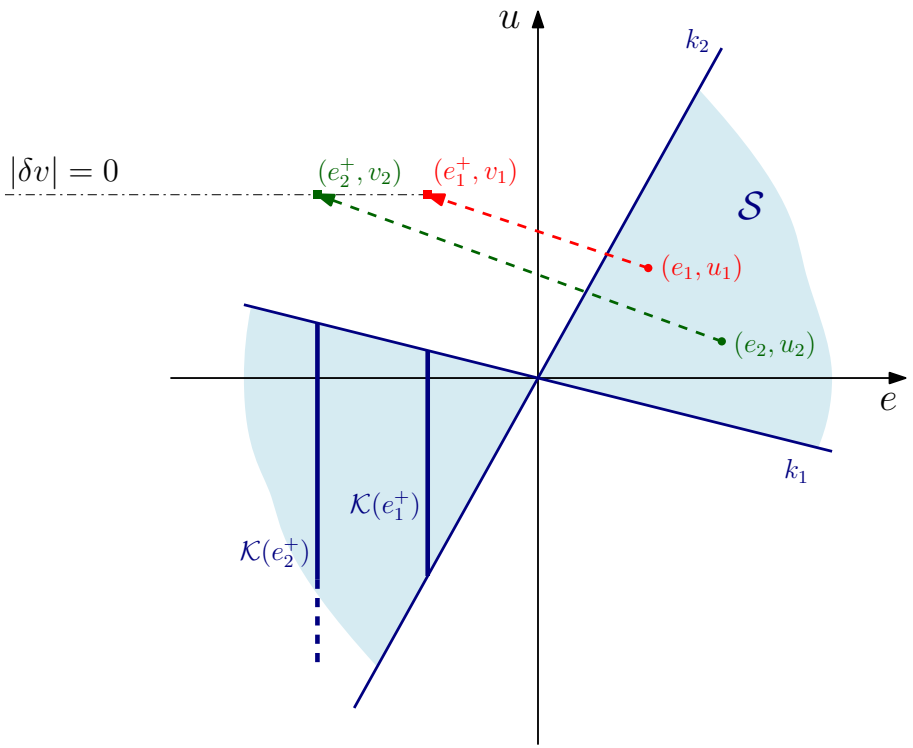

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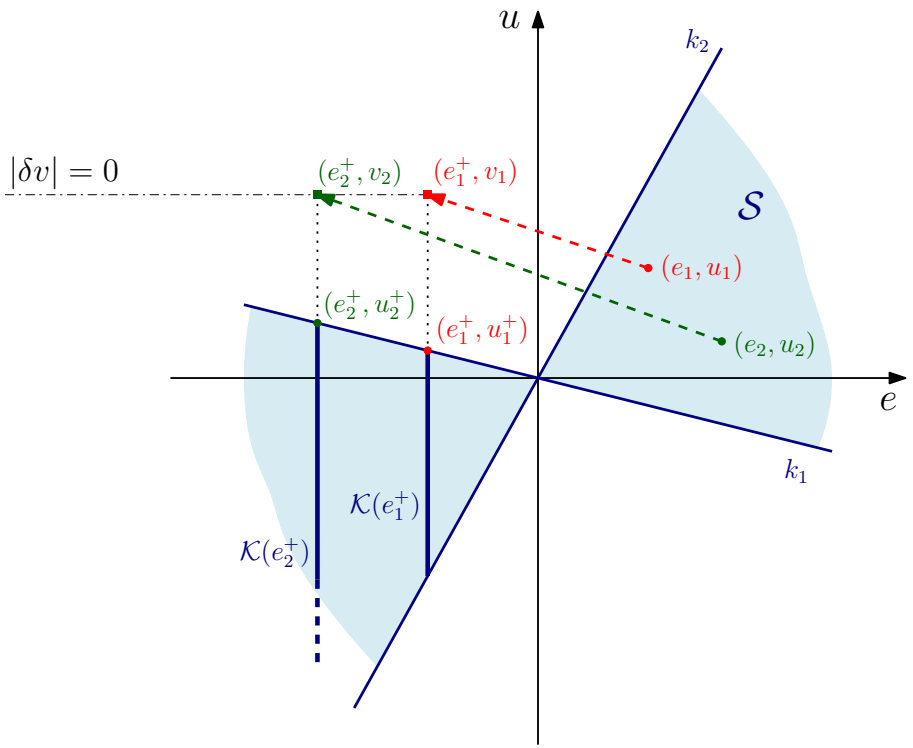

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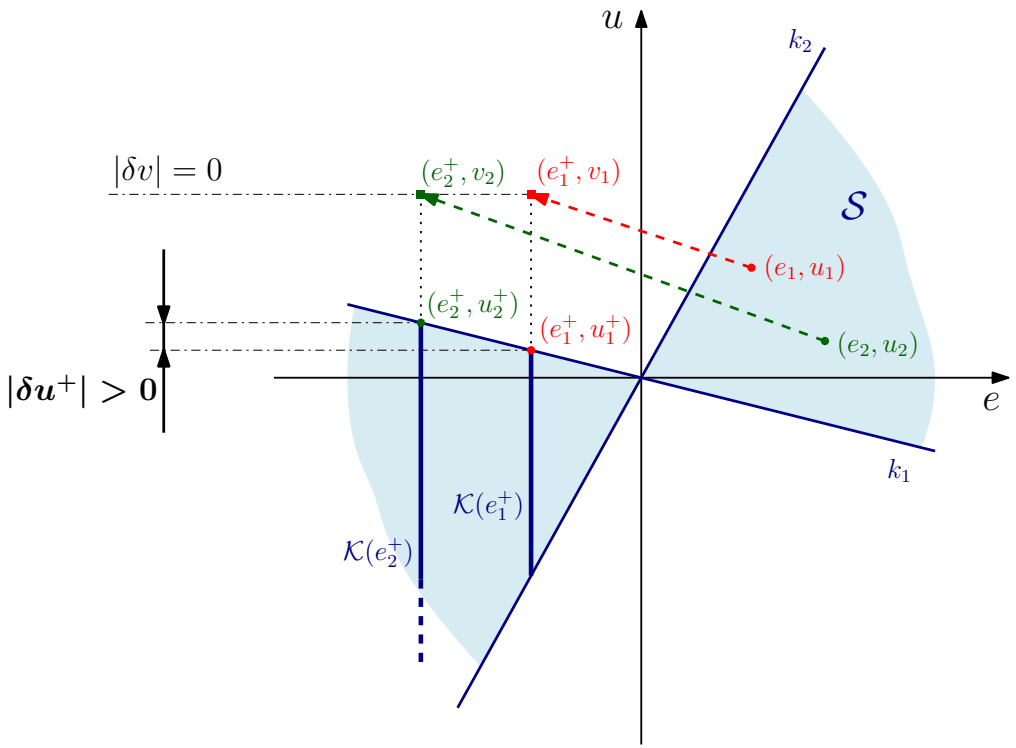

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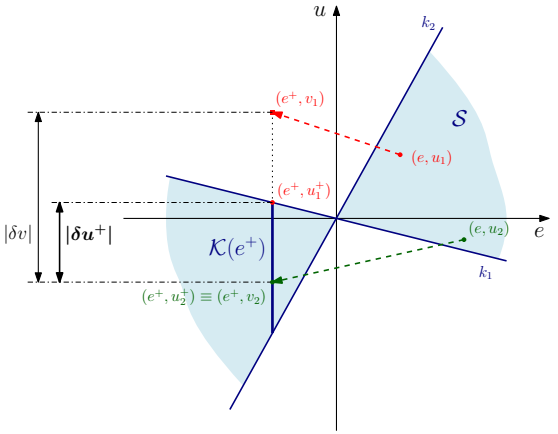

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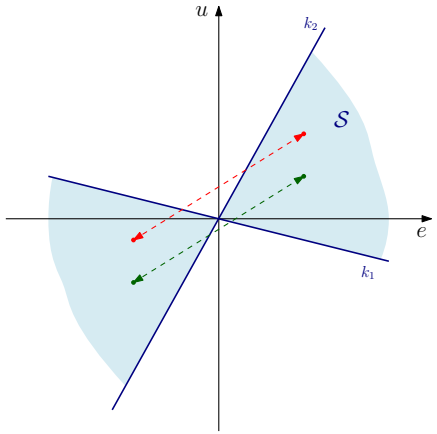

Some results in discrete time

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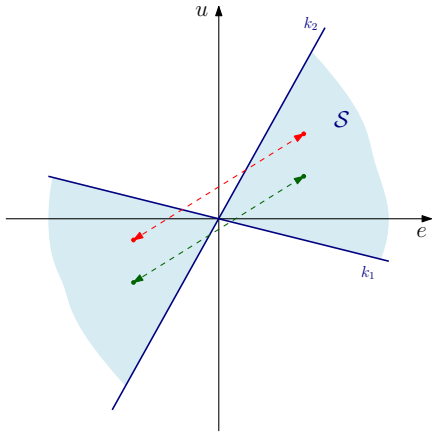

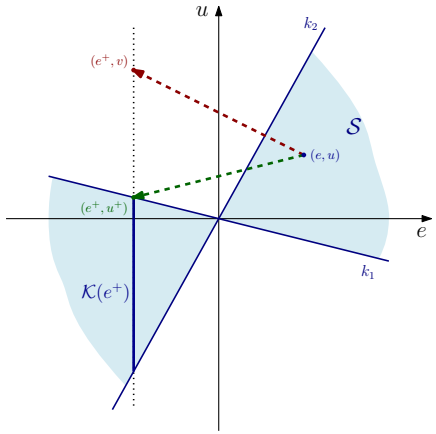

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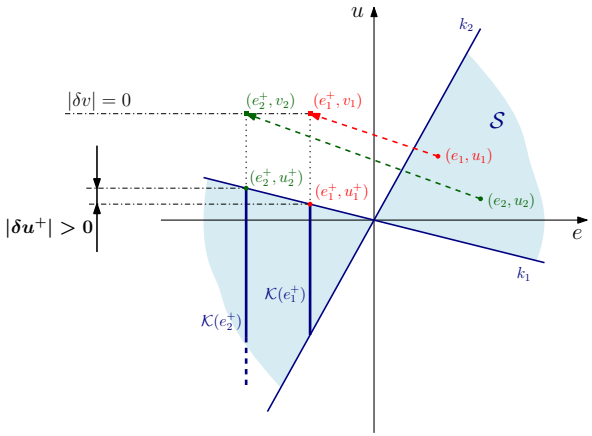

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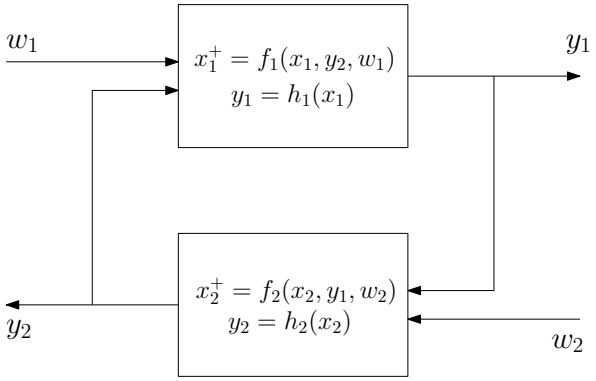

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More general analysis: ePDS, and connection to regular PDS

Extended projected dynamical systems (ePDS) [1]

$$\dot{x} = \Pi_{\mathcal{S},\mathcal{E}} f(x) := f(x) + \arg\min \|\mu\|^2$$
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  - [1] proves completeness of solutions for continuous norms, but we proved the result in this particular discontinuous case → possible extension of the result

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Application to feedback optimization with state constraints

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- Modelling of partially elastic impacts as a “hybrid singular perturbation” of inelastic impacts, for control purposes?

Other (older) projects

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Presented next:

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- Stabilization of bipedal gait using hybrid dynamics and reference spreading

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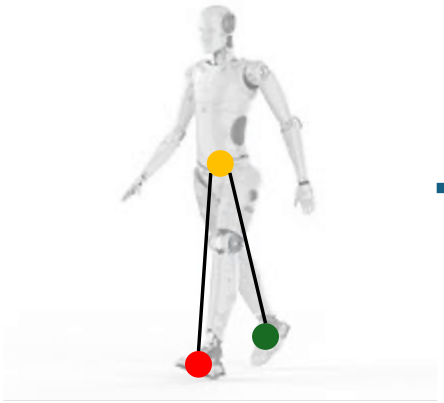
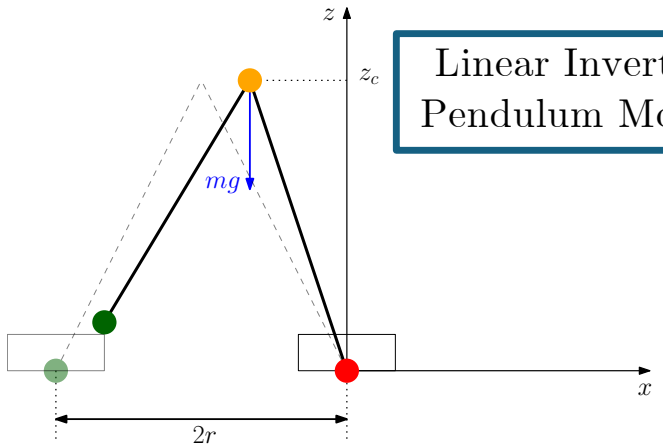
- Distributed observers with different timescales (sampled-data systems with hybrid systems formalism)

Bipedal gait stabilization using reference spreading





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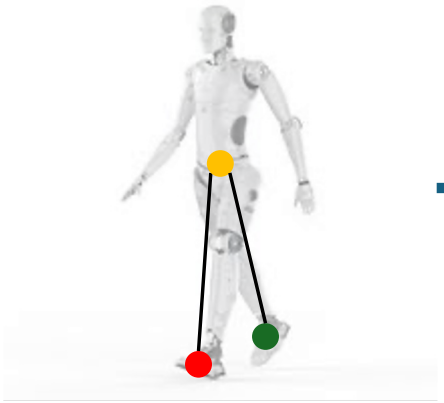
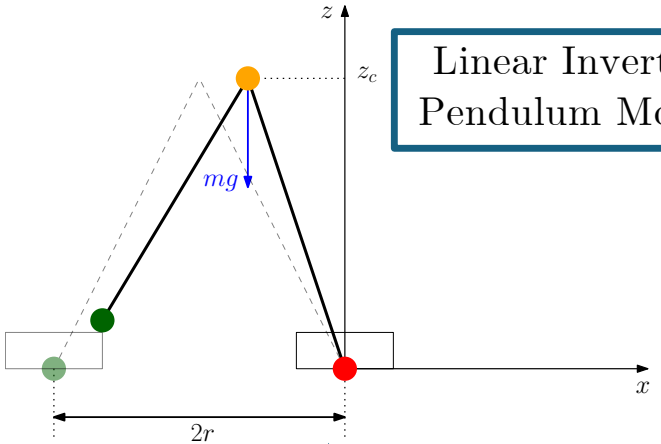
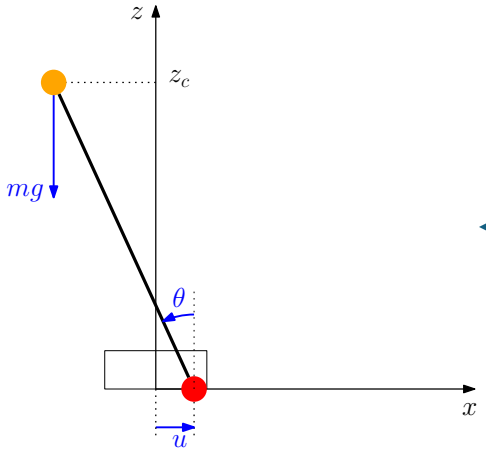


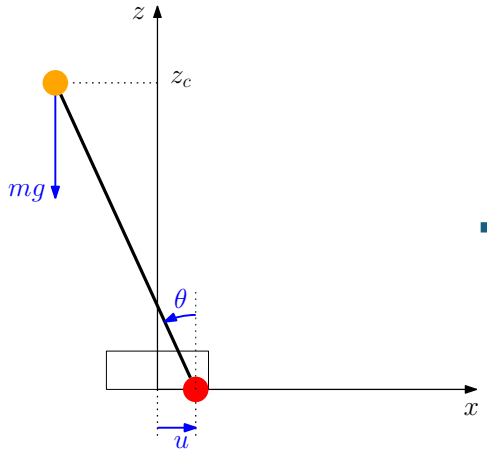


Linear Inverted Pendulum Model



Control action: moving the pivot

Bipedal gait stabilization using reference spreading



Equations of motion:

$$\ddot{\theta} = -\frac{g}{z_c}\sin\theta = -\omega^2 \sin\theta$$

Small angles linearization:

$$\ddot{x}_p = -\omega^2(x_p - u)$$

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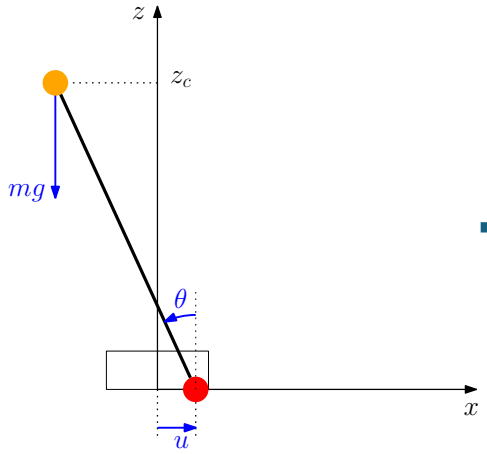

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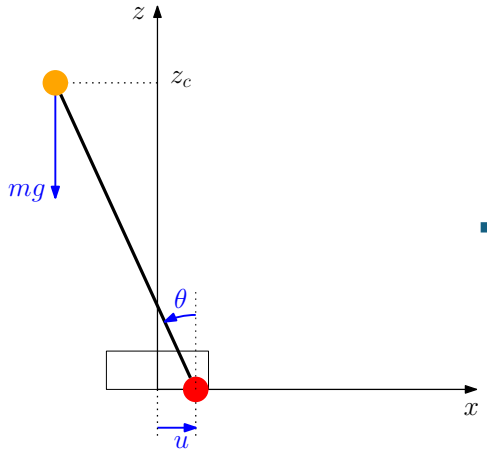

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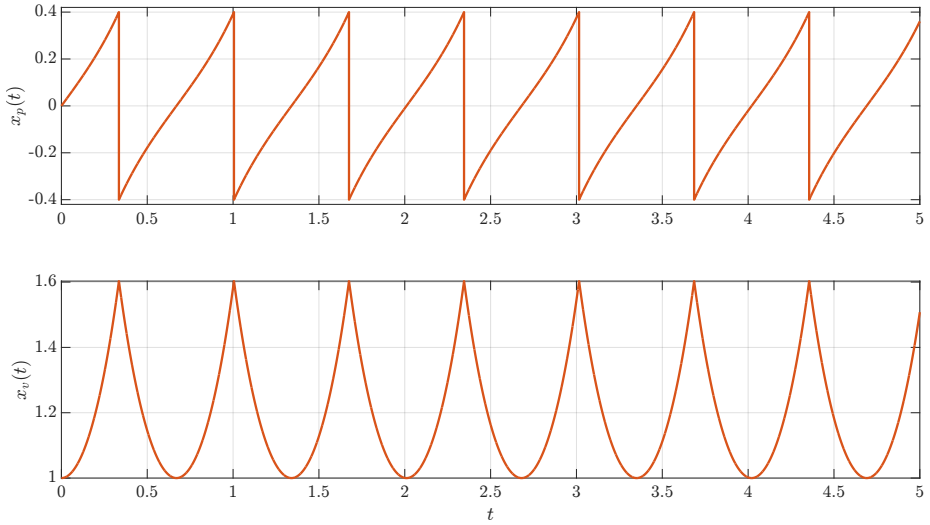
Hybrid dynamical system!

$$x = \begin{bmatrix} x_p \\ x_v \end{bmatrix} \quad \rightarrow \quad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -\omega^2 \end{bmatrix} u, & x_p \in [-r, r] \\ x^+ = x - \begin{bmatrix} 2r \\ 0 \end{bmatrix}, & x_p = r \end{cases}$$

Bipedal gait stabilization using reference spreading

The hybrid LIPM has a periodic trajectory (no inputs)

Goal: stabilize the gait on the periodic reference



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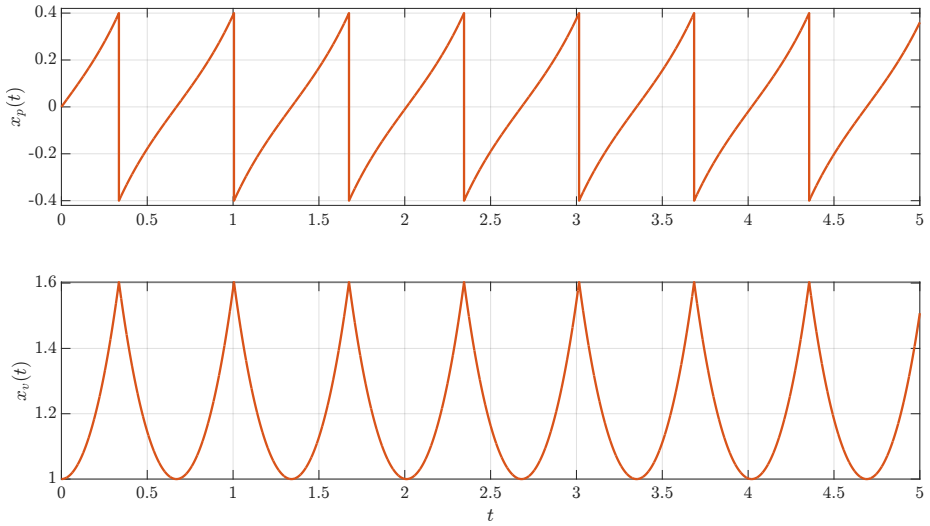
Issues when robot and reference are **standing on different feet**



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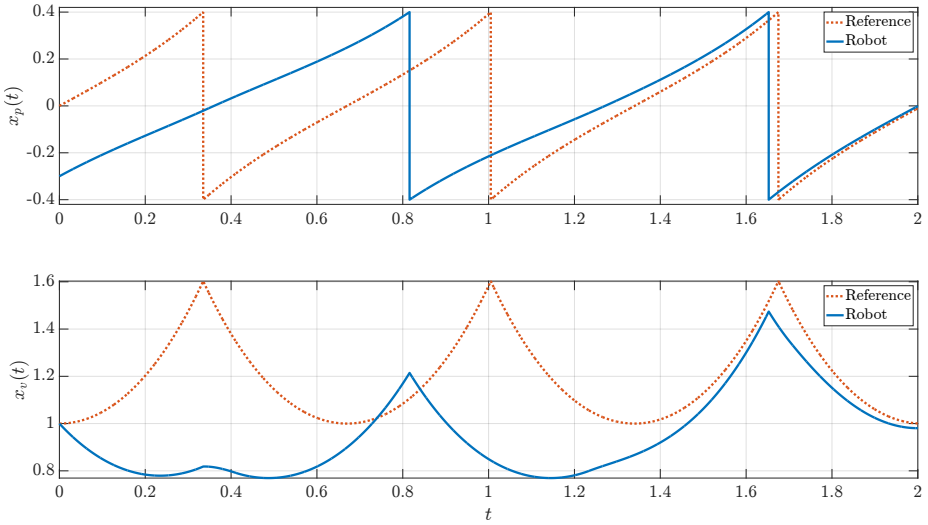

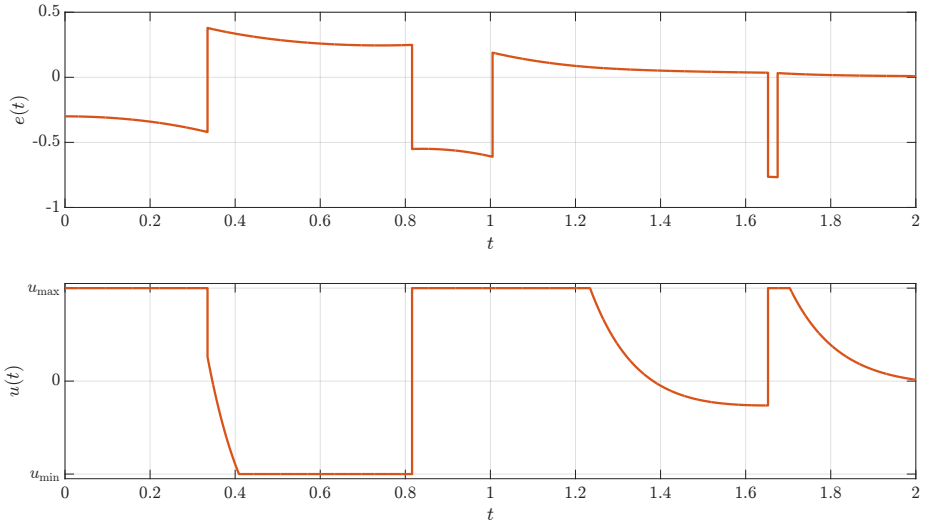

Bipedal gait stabilization using reference spreading

New idea:

- Introduce a timer state to parametrize the reference
  - Reset the timer when the robot changes foot

$$\begin{cases} \dot{\tau} = 1, & x_p \in [-r, r] \\ \tau^+ = \tau - T, & x_p = r \end{cases}$$

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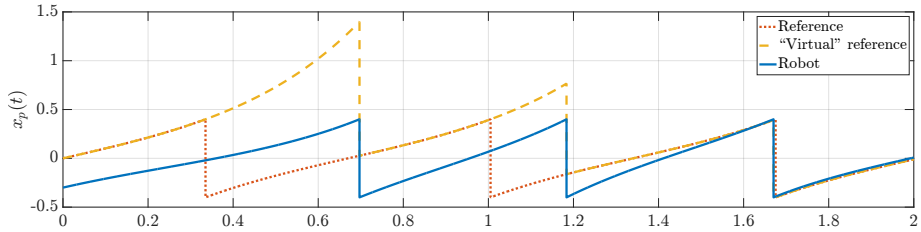

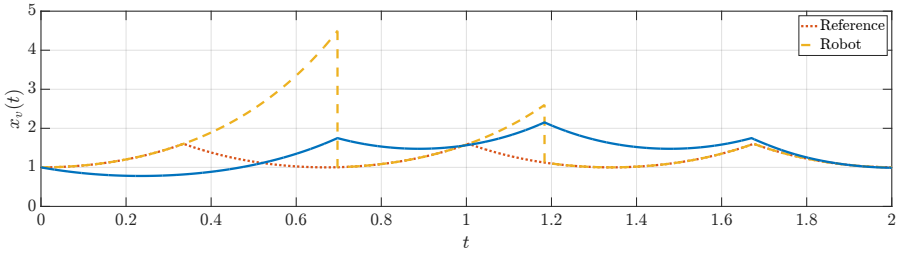

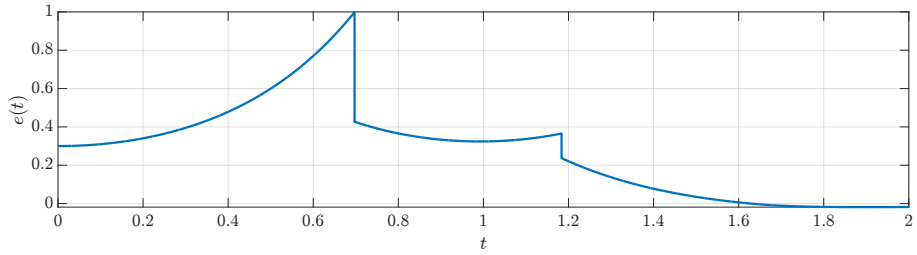

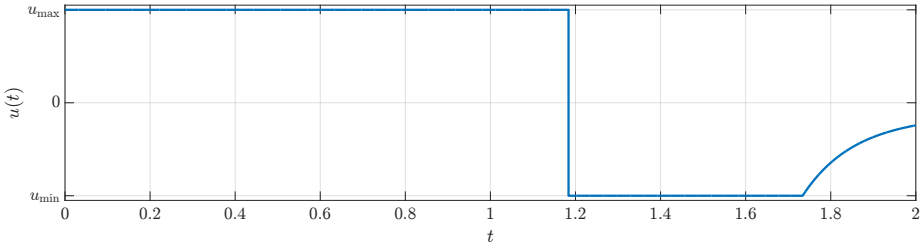

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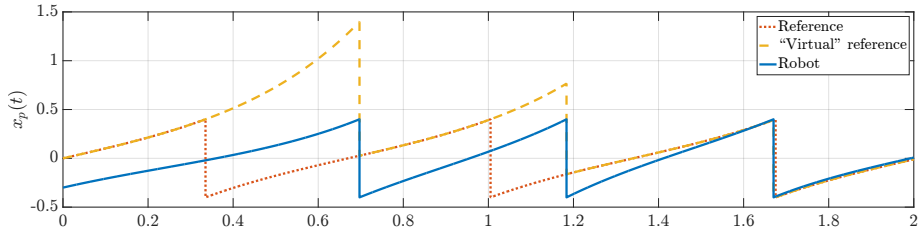

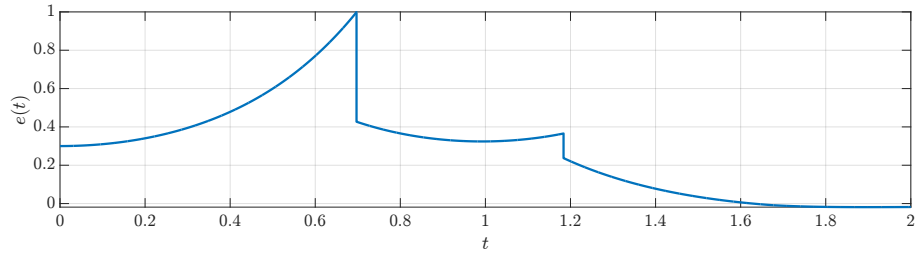

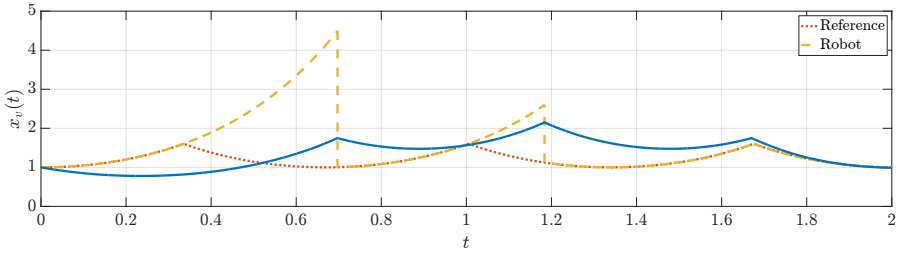

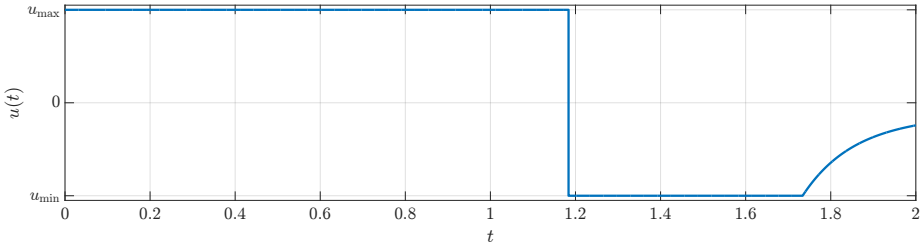

With such a "regular" error trajectory, the control synthesis problem becomes easier!
We defined **LMIs** to synthesize a stabilizing saturated full-state feedback, and we identified **necessary and sufficient conditions for feasibility**