Uniform Global Asymptotic Synchronization of Kuramoto Oscillators via Hybrid Coupling

Riccardo Bertollo[a] Elena Panteley[b] Romain Postoyan[c] Luca Zaccarian[a,d]

[a] Dipartimento di Ingegneria Industriale, University of Trento, 38122, Trento, Italy (e-mail: riccardo.bertollo@unitn.it)
[b] CNRS, L2S, Gif-sur-Yvette, France (e-mail: panteley@centralesupelec.fr)
[c] Université de Lorraine, CNRS, CRAN, Nancy, France (e-mail: romain.postoyan@univ-lorraine.fr)
[d] CNRS, LAAS, Université de Toulouse, 31400 Toulouse, France(e-mail: zaccarian@laas.fr, luca.zaccarian@unitn.it)

21st IFAC WC - July 12-17, 2020

Overview of the classical Kuramoto model

Kuramoto (1975) proposed a simple model to describe **systems of oscillators** with "all-to-all" connection layout

$$\dot{\theta}_i = \omega_i + \frac{K}{n} \sum_j \sin(\theta_j - \theta_i), \quad i \in \{1, \dots, n\}$$

Numerous applications in biology [*Michaels et al., Circulation Res. (1987)*], physics and engineering [*Wiesenfeld et al., Phys. Rev. (1993)*], [*Jiang et al., J. Opt. Soc. Am. (1998)*], and more recently in automatic control.

More detailed studies on the model:

- stability properties [*Strogatz, Physica D. (2000)*]
- more complex network structures [*Dörfler and Bullo, SIAM J. Contr. Opt. (2010)*] or coupling gains [*Leonard et al., Proc. Nat. Ac. Sc. (2012)*]

Non-uniform convergence of the classical Kuramoto model

Coupling sine function vanishes at π $\rightarrow$ **multiple equilibria**.

The closer to π the initial phase difference is, the longer it will take for the states to converge to the equilibrium configuration $\rightarrow$ **non-uniform convergence**:

- acceptable to describe certain natural phenomena (*e.g.* synchronizing pendulums)
- less suitable for engineering applications (*e.g.* control of unstable aircrafts)

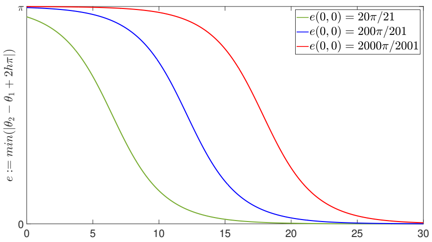


Figure: Evolution of the synchronization error for different initial conditions.

Hybrid approach can avoid multiple equilibria

Multiple synchronization sets unavoidable using a continuous approach.

Hybrid framework → **coupling function domain** can be restricted to a **compact set**.

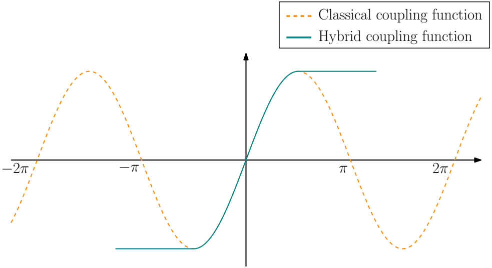


Figure: Graphical representation of a possible hybrid coupling function.

Properties of the hybrid coupling function

In order to obtain a **UGAS (uniformly globally asymptotically stable)** system, the hybrid coupling function needs to satisfy some properties

Property

Coupling function σ is continuous in $\operatorname{dom}\sigma := [-\pi-\delta, \pi+\delta]$ *and satisfies*

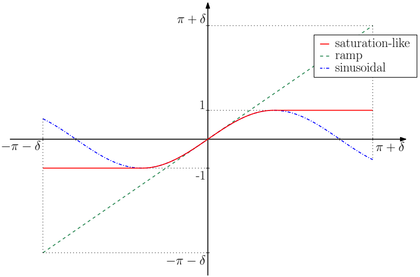
(i) $\sigma(s) = -\sigma(-s)$, *for any* $s \in \operatorname{dom}\sigma$

(ii) $\sigma(s)s > 0$, *for any* $s \in \operatorname{dom}\sigma \setminus \{0\}$

where $\delta \in (0, 2\pi/3]$ is a positive scalar ensuring space regularization, to prevent Zeno solutions.

To ensure that $\operatorname{dom}\sigma$ is compact:

- introduce **jumps in the state variables**
- keep track of their unwinding $\rightarrow$ $\rightarrow$ **additional discrete state k_{12}**



Hybrid model of two coupled oscillators: flow dynamics

We will start from the case of **two coupled oscillators**.

Continuous evolution *(flow)* is a generalization of the original Kuramoto model dynamics

$$\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \begin{bmatrix}\omega + \gamma\sigma(\theta_2 - \theta_1 + 2k_{12}\pi)\\ \omega - \gamma\sigma(\theta_2 - \theta_1 + 2k_{12}\pi)\end{bmatrix}, \qquad x := (\theta_1, \theta_2, k_{12}) \in C.$$

State $k_{12} \in \{-2, -1, 0, 1, 2\}$ and jump rules (to be defined) will ensure that

$$C \subset \{x : |\theta_2 - \theta_1 + 2k_{12}\pi| \leq \pi + \delta\}.$$

To complete the **flow dynamics**, we include the (discrete) state k_{12}

$$\dot{x} = \begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{k}_{12}\end{bmatrix} = f(x) := \begin{bmatrix}\omega + \gamma\sigma(\theta_2 - \theta_1 + 2k_{12}\pi)\\ \omega - \gamma\sigma(\theta_2 - \theta_1 + 2k_{12}\pi)\\ 0\end{bmatrix}, \qquad x \in C.$$

Jump dynamics: keep the state space compact

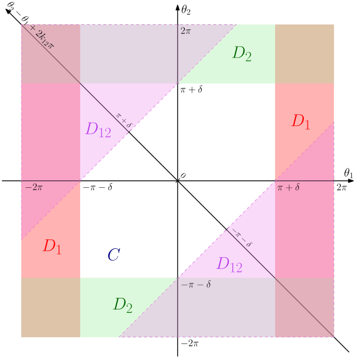
We want to keep $\boldsymbol{\theta_i \in [-\pi - \delta, \pi + \delta]}$, $i \in \{1, 2\}$ → we introduce **jump rules** for the states θ_i

$$\theta_i^+ = g_\theta(\theta_i) := \theta_i - 2\pi\,\mathrm{sgn}(\theta_i), \qquad x \in D_i := \{x : |\theta_i| \in [\pi + \delta, 2\pi]\}.$$

Jumps of k_{12} enforced as well → argument of σ in $f(x)$ remains constant

$$\begin{bmatrix} \theta_1^+ \\ \theta_2^+ \\ k_{12}^+ \end{bmatrix} = g_1(x) := \begin{bmatrix} g_\theta(\theta_1) \\ \theta_2 \\ k_{12} - \mathrm{sgn}(\theta_1) \end{bmatrix}, \quad x \in D_1,$$

$$\begin{bmatrix} \theta_1^+ \\ \theta_2^+ \\ k_{12}^+ \end{bmatrix} = g_2(x) := \begin{bmatrix} \theta_1 \\ g_\theta(\theta_2) \\ k_{12} + \mathrm{sgn}(\theta_2) \end{bmatrix}, \quad x \in D_2.$$



Jump dynamics: ensure that the argument of σ is bounded

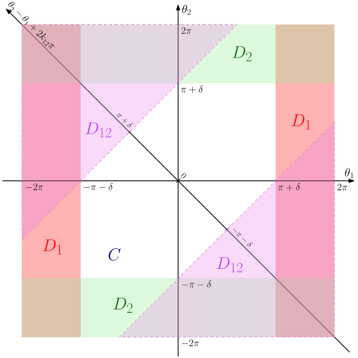
Another jump law only affecting state k_{12}:

- Goal $\rightarrow$ ensure that **argument of $\boldsymbol{\sigma}$ has norm smaller than $\boldsymbol{\pi + \delta}$**
- Jumps triggered in the set

$$D_{12} := \{x : \min_{h \in \mathbb{Z}} |\theta_2 - \theta_1 + 2h\pi| \leq |\theta_2 - \theta_1 + 2k_{12}\pi| - 2\delta\}$$

- Corresponding jump map

$$\begin{bmatrix} \theta_1^+ \\ \theta_2^+ \\ k_{12}^+ \end{bmatrix} \in G_{12}(x) := \begin{bmatrix} \theta_1 \\ \theta_2 \\ \operatorname{argmin}_{h \in \mathbb{Z}} |\theta_2 - \theta_1 + 2h\pi| \end{bmatrix}$$



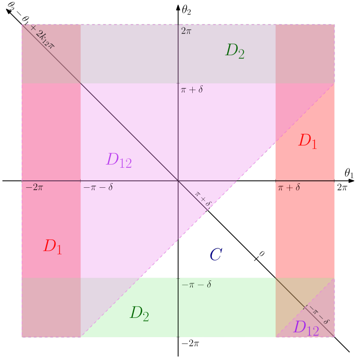
Another jump law only affecting state k_{12}:

- Goal $\rightarrow$ ensure that **argument of $\boldsymbol{\sigma}$ has norm smaller than $\boldsymbol{\pi + \delta}$**
- Jumps triggered in the set

$$D_{12} := \{x : \min_{h\in\mathbb{Z}} |\theta_2 - \theta_1 + 2h\pi| \leq |\theta_2 - \theta_1 + 2k_{12}\pi| - 2\delta\}$$

- Corresponding jump map

$$\begin{bmatrix}\theta_1^+ \\ \theta_2^+ \\ k_{12}^+\end{bmatrix} \in G_{12}(x) := \begin{bmatrix}\theta_1 \\ \theta_2 \\ \underset{h\in\mathbb{Z}}{\operatorname{argmin}} |\theta_2 - \theta_1 + 2h\pi|\end{bmatrix}$$



Overall hybrid model

To **complete the hybrid formulation**, we write the data in the following **general form**

$$\begin{cases} \dot{x} = f(x), & x \in C, \\ x^+ \in G(x), & x \in D. \end{cases}$$

- flow map f already defined
- jump (set-valued) map G defined in terms of its graph

$$\text{gph}\, G := \text{gph}\, g_1 \cup \text{gph}\, g_2 \cup gphG_{12}.$$

- compact state space X divided into flow and jump sets in such a way to prioritize jumps

$$X := [-2\pi, 2\pi]^2 \times \{-2, -1, 0, 1, 2\}$$

$$D := D_1 \cup D_2 \cup D_{12}, \quad C := \overline{X \setminus D}.$$

Structural properties of the hybrid model

Lemma

The proposed hybrid model satisfies the hybrid basic conditions, as defined in [Goebel et al., Princeton University Press (2012)], then it is **well posed**.

Consequence of well posedness:

- sequential compactness of solutions
- intrinsic robustness of asymptotic stability

Lemma

For each initial condition, there exists **at least one complete solution** *to the proposed hybrid system. Moreover, no complete solution* ϕ *exists that never flows (it is discrete) and only jumps according to* G_1 *or* G_2.

Note: second part of this lemma $\equiv$ rationale behind the introduction of δ

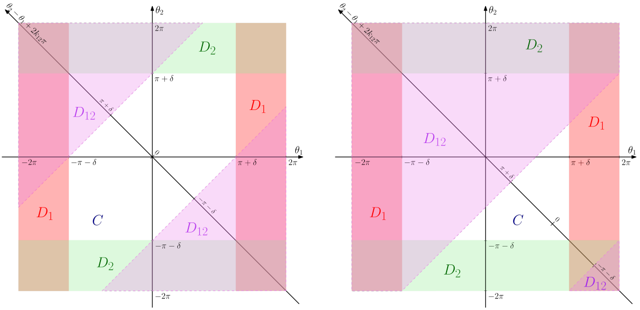
We can prove that $G_{12} \implies$ **argument of σ has norm smaller than $\pi + \delta$**

Lemma

The flow set C satisfies the condition

$$C \subset \{x : |\theta_2 - \theta_1 + 2k_{12}\pi| \leq \pi + \delta\}.$$

Intuitively confirmed by the **graphical representation** of the flow and jump sets



(a) $k_{12} = 0$ (b) $k_{12} = 1$

Figure: Visual example of the flow and jump sets.

Main result: UGAS of the synchronization set

Some solutions are not complete $\rightarrow$ we add prefix "pre" to stability properties (consistent with [*Goebel et al., Princeton University Press (2012)*])

Theorem

The **synchronization set**

$$\mathcal{A} := \{x \in X : \theta_1 = \theta_2 + 2k_{12}\pi,\ k \in \{-1, 0, 1\}\}$$

is **UGpAS** *(uniformly globally pre-asymptotically stable) for the proposed hybrid system.*

UGpAS comprises:

- UGS (uniform global stability)
- **UGpA (uniform global pre-attractivity)** $\rightarrow$ not satisfied by original Kuramoto model

Note: well-posedness (emphasized in one of the previous lemmas) $\implies$ $\mathcal{A}$ is also **robustly** pre-asymptotically stable.

Sketch of the proof

1) Weak Lyapunov function

Consider the **Lyapunov function candidate**

$$V(x) := \int_0^{\mathrm{sat}_{\pi+\delta}(\theta_2-\theta_1+2k_{12}\pi)} \sigma(s)ds.$$

Using this function, we can prove that

$$\begin{aligned} \langle \nabla V(x), f(x) \rangle < 0, &\quad \forall x \in C \setminus \mathcal{A}', \\ V(g) - V(x) \leq 0, &\quad \forall x \in D, g \in G(x), \end{aligned}$$

for a **superset $\mathcal{A}'$ of $\mathcal{A}$**.

Sketch of the proof

2) Invariance principle

No solution ξ_{bad} exists satisfying $V(\xi_{bad}(t,j)) = V(\xi_{bad}(0,0)) \neq 0$, for all $(t,j) \in \operatorname{dom}\xi$

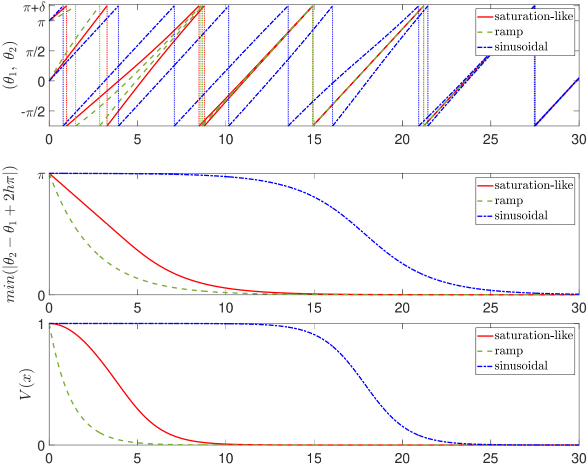
$$\Downarrow$$

set $\mathcal{A}'$ is **UGpAS for the hybrid system**.

3) Reduction theorem

$$\left\{\begin{array}{c} \mathcal{A}' \text{ UGpAS} \\ + \\ \boldsymbol{\mathcal{A}} \textbf{ UGpAS from } \boldsymbol{\mathcal{A}'} \end{array}\right. \implies \boldsymbol{\mathcal{A}} \textbf{ UGpAS}$$

Simulations results: initial phase difference close to π



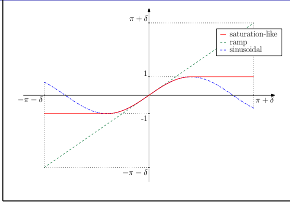


Figure: Simulation results for different coupling functions, with an **initial phase difference close to π**.

Simulations results: small initial phase difference

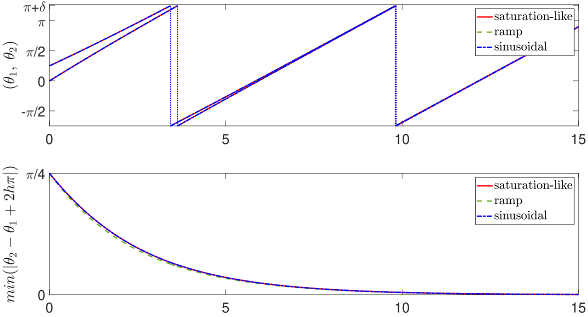


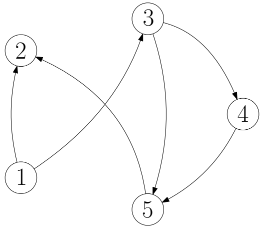
Figure: Simulation results for different coupling functions, with a **small initial phase difference**.

Extension to any connection layout: use of directed graphs

To describe **higher order systems** $\rightarrow$ **undirected graph**

- oscillators $\rightarrow$ nodes
- connections $\rightarrow$ edges

Then we choose an orientation of that graph, to define its **incidence matrix**



$$B = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{bmatrix}$$

Moreover, we can define

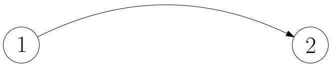
$$\sigma_{ij} := \sigma(\theta_j - \theta_i + 2k_{ij}\pi)$$

and collect these σ_{ij}'s in a vector $\boldsymbol{\sigma}$, ordered like the columns of B.

Incidence matrix used to express flow dynamics compactly

We want to include the two oscillators dynamics as a particular case, where

$$\begin{cases} \dot{\theta}_1 = \omega + \gamma\sigma_{12} \\ \dot{\theta}_2 = \omega - \gamma\sigma_{12} \end{cases}$$



Convention: apply σ_{ij} to $\dot{\theta}_i$ with a "+" sign when the corresponding edge is (i, j)

$$\dot{\theta}_i = f_i(x) := \omega + \gamma \sum_{h \in \mathcal{O}_i} \sigma_{ih} - \gamma \sum_{h \in \mathcal{I}_i} \sigma_{hi}, \qquad \forall i \in \mathcal{V}.$$

Using the incidence matrix $\rightarrow$ **complete flow dynamics** expressed compactly

$$\begin{cases} \dot{\theta} = f_\theta(x) := \omega \mathbf{1}_n - \gamma B \boldsymbol{\sigma} \\ \dot{k} = 0 \end{cases}$$

Jump dynamics: same logic as the two oscillators example

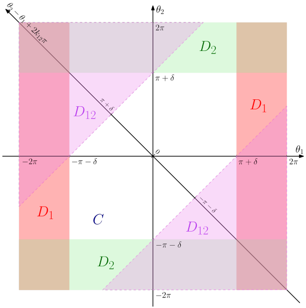
The **jump sets and maps** follow the same idea as the two oscillators case, with a quite convoluted notation

$$x^+ = g_i(x) := \begin{bmatrix} g_{i,\theta}(x) \\ g_{i,k}(x) \end{bmatrix}, \quad x \in D_i$$

$$(g_{i,\theta})_j = \begin{cases} g_\theta(\theta_i), & \text{if } j = i, \\ \theta_j, & \text{otherwise} \end{cases}$$

$$(g_{i,k})_{(u,v)} = \begin{cases} k_{uv} + \text{sgn}(\theta_i), & \text{if } v = i, \\ k_{uv} - \text{sgn}(\theta_i), & \text{if } u = i, \\ k_{uv}, & \text{otherwise} \end{cases}$$

$$D_i := \{x : |\theta_i| \in [\pi + \delta, 2\pi]\}$$



Jump dynamics: same logic as the two oscillators example

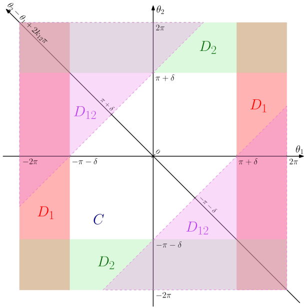
$$x^+ \in G_{ij}(x) := \begin{bmatrix} \theta \\ G_{ij,k}(x) \end{bmatrix}, \quad x \in D_{ij}$$

$$(G_{ij,k})_{(u,v)} = \begin{cases} \operatorname*{argmin}_{h\in\mathbb{Z}} |\theta_j - \theta_i + 2h\pi|, & \text{if } (u,v) = (i,j), \\ k_{uv}, & \text{otherwise} \end{cases}$$

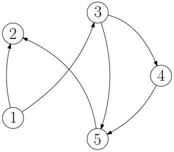
$$D_{ij} = \{x : \min |\theta_j - \theta_i + 2h\pi| \leq |\theta_j - \theta_i + 2k_{ij}\pi| - 2\delta\}$$

$$\text{gph } G := \left(\bigcup_{i\in\mathcal{V}} \text{gph } g_i\right) \cup \left(\bigcup_{(i,j)\in\mathcal{E}} \text{gph } G_{ij}\right)$$

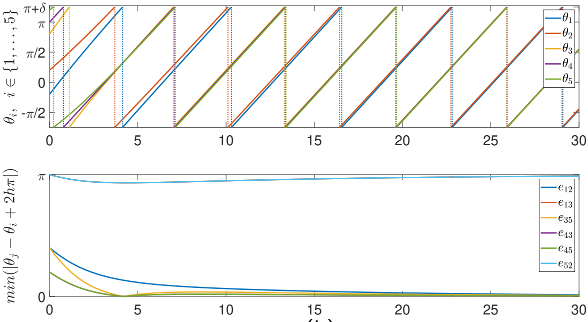
$$D := \left(\bigcup_{i\in\mathcal{V}} D_i\right) \cup \left(\bigcup_{(i,j)\in\mathcal{E}} D_{ij}\right)$$



Simulation results: classical model, connected graph



(a)



(b)

Figure: Simulation results for the graph in (a) (orientation of a **connected graph**) using the **classical Kuramoto model**: evolution of the phases and the phase differences (b).

Simulation results: proposed model, connected graph

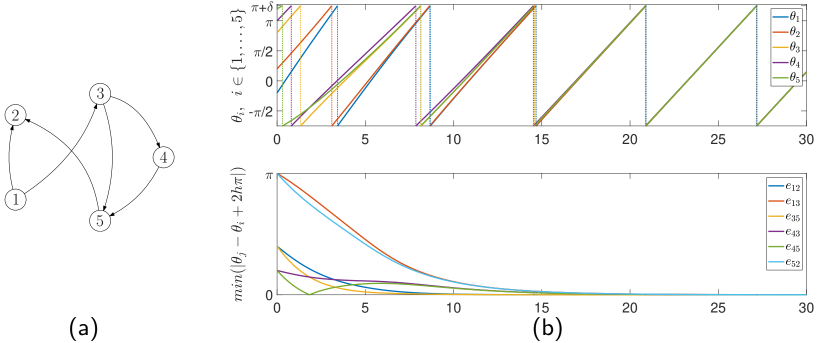


Figure: Simulation results for the graph in (a) (orientation of a **connected graph**) using the proposed generalized **hybrid model**: evolution of the phases and the phase differences (b).

Conclusions

- **Hybrid coupling** has been used to ensure **uniform convergence properties**
- **UGpAS of the synchronization set** has been succesfully proved, in the case of two oscillators, for any coupling function σ satisfying some identified properties
- **Simulations** have confirmed the analytical results
- Both the generality of the proof technique and the simulations results suggest a **possible** succesful **extension** to higher order dynamics
- Other possible future developments:
  - Relaxing requirements for σ
  - Studying the case of different natural frequencies
  - Treating analytically the case of disconnected/directed graphs