

# Type theory in Lean - 4

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- Formation rule: there is a well formed type  $\mathbb{N}$ .

$$(\mathbb{N} : \text{Type})$$

Its terms are called *natural numbers*.

# Constructors

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$$(\text{succ } n : \mathbb{N})$$

In particular, we have a function

$$\text{succ} : \mathbb{N} \rightarrow \mathbb{N}$$

The fact that  $\text{succ}$  takes a natural number and gives another natural number is what makes  $\mathbb{N}$  an *inductive* type.

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Let  $u$  be a universe and let  $(M : \mathbb{N} \rightarrow \text{Sort } u)$  be a function.  $M$  will give the codomain of the dependent function we want to define, in Lean it is usually called the *motive*.

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We want to define a term

$$\left( f : \prod_{(n:\mathbb{N})} M\ n \right)$$



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Note that there is no need to tell to `rec` what  $M$  is, Lean will guess it from the type of  $s$  (we say that  $M$  is an *implicit variable*).

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In particular, if  $(n : \mathbb{N})$ , then

$$(\text{rec } z\ s\ n : M\ n)$$

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$$\prod_{(M:\mathbb{N} \rightarrow \text{Sort } u)} M\ 0 \rightarrow \left( \prod_{(n:\mathbb{N})} M\ n \rightarrow M\ (\text{succ } n) \right) \rightarrow \prod_{(n:\mathbb{N})} M\ n$$

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To be precise, this is the type of `rec.{ $u$ }`, the eliminator for the universe  $u$ . Since universes are not terms, we cannot take a further product over universes, and there is no universe big enough to contain all the `Sort  $u$` 's, so this is unavoidable.



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Let  $M$ ,  $z$  and  $s$  be as above. We have

$$\text{rec } z \ s \ 0 \equiv z$$

and, if  $(n : \mathbb{N})$ ,

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we need to fix a term  $(z : A)$  and a (non-dependent) function  $(s : \mathbb{N} \rightarrow A \rightarrow A)$ . We get

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such that

$$\text{rec } z \ s \ 0 \equiv z \text{ and } \text{rec } z \ s \ (\text{succ } n) \equiv s \ n \ (\text{rec } z \ s \ n)$$

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## Slogan

*Using `rec`, one can define functions*

$$(f : \mathbb{N} \rightarrow A)$$

*by recursion in the usual way.*

Let's go back to the dependent version of the eliminator, but in the special case where the motive  $(M : \mathbb{N} \rightarrow \text{Prop})$  takes values in  $\text{Sort } 0 = \text{Prop}$ .

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Let's construct such a  $p$ .

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We have that  $(M\ 0 : \text{Prop})$ , so  $z$  is now a proof that  $M\ 0$  holds.  
On the other hand, we also have

$$\left( \prod_{(n:\mathbb{N})} M\ n \rightarrow M\ (\text{succ } n) : \text{Prop} \right)$$

So  $s$  corresponds to a proof of the proposition

$$\forall (n : \mathbb{N}), M\ n \rightarrow M\ (\text{succ } n)$$

that is,  $M\ n$  implies  $M\ (\text{succ } n)$  for all  $(n : \mathbb{N})$ .

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In practice, to prove that  $M\ n$  holds for all  $(n : \mathbb{N})$ , we need to prove that  $M\ 0$  holds and that  $M\ n$  implies  $M\ (\text{succ } n)$  for all  $(n : \mathbb{N})$ .

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### Slogan

*Using `rec`, one can prove propositions on  $\mathbb{N}$  by induction in the usual way.*

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Lean allows a much more convenient notation, called *pattern matching*, where to specify a function  $f$  with domain  $\mathbb{N}$  (in particular to prove a theorem about natural numbers) one has to:

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We will see the precise syntax in the examples.

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so the function ( $s : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}$ ) is given by

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First of all we need the image of 0, that is  $\text{add } 0 : \mathbb{N} \rightarrow \mathbb{N}$ . This will of course be the identity function, so

$$\text{add } 0 \ n \equiv n$$

for all  $(n : \mathbb{N})$ .

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One can of course use `rec` directly, but using pattern matching will be much simpler, since one has not to write explicitly the function

$$(s : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N}))$$

that says how to specify  $\text{add} (\text{succ } n)$  given  $\text{add } n$ .



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We will explain in the following lectures why definitional equality implies equality, a notion that at the moment we have not defined.

Using lambda abstraction, we have terms

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for all  $(a\ b : \mathbb{N})$  are true, but not definitionally. One can prove such results seeing them as dependent functions and using the eliminator explicitly, but Lean has a much nicer syntax, using the `induction` tactic. Under the hood, one has to use the eliminator.

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This is proved using that two definitionally equal terms are equal.

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where  $(a : \mathbb{N})$ . In other words we need to prove that  $a + 0 = a$  implies that  $\text{succ } a + 0 = \text{succ } a$  as expected. Since we need to construct a function, we can use lambda abstraction again. In Lean this is easily done using the `intro` and `rw` tactics, but we will see that  $a + 0 = a$  is an inductive proposition, so to construct such a function one can use the constructor for `=`.

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- We define  $\text{pred } (\text{succ } n)$  to be  $n$ .

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In particular one can prove that succ is injective.

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So we suppose that  $0 = 1$  and we need to prove False.

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In particular

$$f\ 0 \equiv A \text{ and } f\ 1 \equiv B$$

hold definitionally.

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It remains to find two terms that we are able to prove they are different. We now show that  $\text{True} \neq \text{False}$ .

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### Remark

*We didn't reason by contradiction.*

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We have  $\text{False} = f \ 0 = f \ (\text{succ } n) = \text{True}$ , so we are done as before.