Solution Exercise 5.4 (by Kristoffer H. Hellton?)

Natural cubic splines have the additional constraints of being linear beyond the boundary knots. Start from the truncated power series:

$$f(X) = \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3$$

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For the left boundary knot

$$f(X) = \sum_{j=0}^{3} \beta_j X^j, \quad X < \xi_1$$

and we need the constraints $\beta_2 = 0$ and $\beta_3 = 0$ for the function to be linear.

For the right boundary knot

$$
\begin{aligned}
f(X) &= \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)^3, \quad \xi_K \leq X \\
&= \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k X^3 - \sum_{k=1}^{K} \theta_k \xi_k 3X^2 + \sum_{k=1}^{K} \theta_k \xi_k^2 3X - \sum_{k=1}^{K} \theta_k \xi_k^3
\end{aligned}
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f(X) &= \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)^3, \quad \xi_K \leq X \\
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and we need the constraints $\theta_k = 0$ and $\sum_{k=1}^{K} \xi_k \theta_k = 0$ for the function to be linear.

The truncated power series representation

$$f(X) = \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3$$

with the constraints on the coefficients

$$\beta_2 = 0, \quad \beta_3 = 0, \quad \sum_{k=1}^{K} \theta_k = 0, \quad \sum_{k=1}^{K} \xi_k \theta_k = 0$$

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Taking into account first the β restrictions, we can construct a new basis with the first two basis functions as

$$f(X) = \beta_0 \cdot \underbrace{1}_{N_1(x)} + \beta_1 \underbrace{X}_{N_2(x)} + 0 \cdot X^2 + 0 \cdot X^3 + \cdots$$

For the θ constraints, we utilize that

$$\sum_{k=1}^{K-2} \theta_k = -\theta_{K-1} - \theta_K, \qquad \sum_{k=1}^{K-2} \xi_k \theta_k = -\xi_{K-1}\theta_{K-1} - \xi_K \theta_K$$

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$$\sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3 = \sum_{k=1}^{K-2} \theta_k (X - \xi_k)_+^3 + \theta_{K-1}(X - \xi_{K-1})_+^3 + \theta_K (X - \xi_K)_+^3,$$

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and use the θ constraints to show that the two last terms can be rewritten as sums over the $N-2$ first terms.

Start with the second last term and "multiply by 1" and "add 0"

$$\theta_{K-1}(X - \xi_{K-1})^3_+ =$$

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\theta_{K-1}(X-\xi_{K-1})_+^3 &= \frac{(X-\xi_K)_+^3}{(\xi_K-\xi_{K-1})}\left(\theta_{K-1}\xi_K-\theta_{K-1}\xi_{K-1}\right) \\
&= \frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}\left(\theta_{K-1}\xi_K-\theta_{K-1}\xi_{K-1}+\underbrace{\theta_K\xi_K-\theta_K\xi_K}_{0}\right)
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&= \frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}\left(\xi_K(\theta_{K-1}+\theta_K)-\xi_{K-1}\theta_{K-1}-\xi_K\theta_K\right)
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\theta_{K-1}(X-\xi_{K-1})_+^3 &= \frac{(X-\xi_K)_+^3}{(\xi_K-\xi_{K-1})}\left(\theta_{K-1}\xi_K-\theta_{K-1}\xi_{K-1}\right)\\
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&= \frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}\left(-\xi_K\sum_{k=1}^{K-2}\theta_k+\sum_{k=1}^{K-2}\theta_k\xi_k\right)
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&= -\frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}\sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\\
&= -\sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}
\end{aligned}$$

Then, take the last term, and do the same

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$$\theta_K(X-\xi_K)^3_+ = \frac{(X-\xi_K)^3_+}{(\xi_K-\xi_{K-1})}\left(\theta_K\xi_K - \theta_K\xi_{K-1} + \underbrace{\theta_{K-1}\xi_{K-1} - \theta_{K-1}\xi_{K-1}}_{0}\right)$$

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&= (X-\xi_K)_+^3 \sum_{k=1}^{K-2}\theta_k \frac{\xi_K-\xi_k}{(\xi_K-\xi_{K-1})}
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&= (X-\xi_K)_+^3\sum_{k=1}^{K-2}\theta_k\frac{\xi_K-\xi_k}{(\xi_K-\xi_{K-1})}\\
&= (X-\xi_K)_+^3\sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\frac{\xi_{K-1}-\xi_k+\xi_K-\xi_K}{(\xi_K-\xi_{K-1})(\xi_K-\xi_k)}
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&= (X-\xi_K)_+^3 \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{1}{\xi_K-\xi_{K-1}} - \frac{1}{\xi_K-\xi_k}\right)
\end{aligned}$$

Then, we combine the two expressions

$$\sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3 = \sum_{k=1}^{K-2} \theta_k (X - \xi_k)_+^3 + \theta_{K-1}(X - \xi_{K-1})_+^3 + \theta_K (X - \xi_K)_+^3,$$

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&= \sum_{k=1}^{K-2} \theta_k (X - \xi_k)_+^3 - \sum_{k=1}^{K-2} \theta_k \left(\xi_K - \xi_k\right) \frac{(X - \xi_{K-1})_+^3}{(\xi_K - \xi_{K-1})}
\end{aligned}$$

$$\vdots$$

Then, we combine the two expressions

$$\begin{aligned}\sum_{k=1}^{K}\theta_k(X-\xi_k)_+^3 &= \sum_{k=1}^{K-2}\theta_k(X-\xi_k)_+^3+\theta_{K-1}(X-\xi_{K-1})_+^3+\theta_K(X-\xi_K)_+^3,\\ &= \sum_{k=1}^{K-2}\theta_k(X-\xi_k)_+^3-\sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\frac{(X-\xi_{K-1})_+^3}{(\xi_K-\xi_{K-1})}\\ &+(X-\xi_K)_+^3\sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{1}{\xi_K-\xi_{K-1}}-\frac{1}{\xi_K-\xi_k}\right)\end{aligned}$$

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Then, we combine the two expressions

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\begin{aligned}
\sum_{k=1}^{K} \theta_k (X-\xi_k)_+^3 &= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 + \theta_{K-1}(X-\xi_{K-1})_+^3 + \theta_K (X-\xi_K)_+^3, \\
&= \sum_{k=1}^{K-2} \theta_k (X-\xi_k)_+^3 - \sum_{k=1}^{K-2} \theta_k \left(\xi_K - \xi_k\right) \frac{(X-\xi_{K-1})_+^3}{(\xi_K - \xi_{K-1})} \\
&+ (X-\xi_K)_+^3 \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k) \left(\frac{1}{\xi_K - \xi_{K-1}} - \frac{1}{\xi_K - \xi_k} \right) \\
&= \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k) \frac{(X-\xi_k)_+^3}{\xi_K - \xi_k} - \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k) \frac{(X-\xi_{K-1})_+^3}{(\xi_K - \xi_{K-1})} \\
&+ \sum_{k=1}^{K-2} \theta_k (\xi_K - \xi_k) \left(\frac{(X-\xi_K)_+^3}{\xi_K - \xi_{K-1}} - \frac{(X-\xi_K)_+^3}{\xi_K - \xi_k} \right) \\
&\vdots
\end{aligned}
$$

$$\vdots$$

$$= \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(\frac{(X-\xi_k)_+^3}{\xi_K - \xi_k} - \frac{(X-\xi_{K-1})_+^3}{\xi_K - \xi_{K-1}} + \frac{(X-\xi_K)_+^3}{\xi_K - \xi_{K-1}} - \frac{(X-\xi_K)_+^3}{\xi_K - \xi_k} \right)$$

$$\vdots$$

$$= \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(\frac{(X-\xi_k)_+^3}{\xi_K-\xi_k} - \frac{(X-\xi_{K-1})_+^3}{\xi_K-\xi_{K-1}} + \frac{(X-\xi_K)_+^3}{\xi_K-\xi_{K-1}} - \frac{(X-\xi_K)_+^3}{\xi_K-\xi_k} \right)$$

$$= \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(\frac{(X-\xi_k)_+^3 - (X-\xi_K)_+^3}{\xi_K-\xi_k} - \left(\frac{(X-\xi_{K-1})_+^3 - (X-\xi_K)_+^3}{\xi_K-\xi_{K-1}} \right) \right)$$

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$$= \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(\frac{(X - \xi_k)_+^3}{\xi_K - \xi_k} - \frac{(X - \xi_{K-1})_+^3}{\xi_K - \xi_{K-1}} + \frac{(X - \xi_K)_+^3}{\xi_K - \xi_{K-1}} - \frac{(X - \xi_K)_+^3}{\xi_K - \xi_k} \right)$$

$$= \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(\frac{(X - \xi_k)_+^3 - (X - \xi_K)_+^3}{\xi_K - \xi_k} - \left(\frac{(X - \xi_{K-1})_+^3 - (X - \xi_K)_+^3}{\xi_K - \xi_{K-1}} \right) \right)$$

Therefore,

$$\sum_{k=1}^{K} \theta_k(X - \xi_k)_+^3 = \sum_{k=1}^{K-2} \theta_k(\xi_K - \xi_k) \left(d_k(X) - d_{K-1}(X) \right),$$

where

$$d_k(X) = \frac{(X - \xi_k)_+^3 - (X - \xi_K)_+^3}{\xi_K - \xi_k}$$