Solution Exercise 5.4 (by Kristoffer H. Hellton?)

Natural cubic splines have the additional constraints of being linear beyond the boundary knots. Start from the truncated power series:

$$f(X) = \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3$$

For the left boundary knot

$$f(X) = \sum_{j=0}^{3} \beta_j X^j, \quad X < \xi_1$$

and we need the constraints $\beta_2 = 0$ and $\beta_3 = 0$ for the function to be linear.

For the right boundary knot

$$\begin{aligned} f(X) &= \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)^3, \quad \xi_K \leq X \\ &= \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k X^3 - \sum_{k=1}^{K} \theta_k \xi_k 3X^2 + \sum_{k=1}^{K} \theta_k \xi_k^2 3X - \sum_{k=1}^{K} \theta_k \xi_k^3 \end{aligned}$$

and we need the constraints $\theta_k = 0$ and $\sum_{k=1}^{K} \xi_k \theta_k = 0$ for the function to be linear.

The truncated power series representation

$$f(X) = \sum_{j=0}^{3} \beta_j X^j + \sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3$$

with the constraints on the coefficients

$$\beta_2 = 0, \quad \beta_3 = 0, \quad \sum_{k=1}^{K} \theta_k = 0, \quad \sum_{k=1}^{K} \xi_k \theta_k = 0$$

Taking into account first the β restrictions, we can construct a new basis with the first two basis functions as

$$f(X) = \beta_0 \cdot \underbrace{1}_{N_1(x)} + \beta_1 \underbrace{X}_{N_2(x)} + 0 \cdot X^2 + 0 \cdot X^3 + \cdots$$

For the θ constraints, we utilize that

$$\sum_{k=1}^{K-2} \theta_k = -\theta_{K-1} - \theta_K, \qquad \sum_{k=1}^{K-2} \xi_k \theta_k = -\xi_{K-1} \theta_{K-1} - \xi_K \theta_K$$

Take out the last two terms of the trucated basis functions:

$$\sum_{k=1}^{K} \theta_k (X - \xi_k)_+^3 = \sum_{k=1}^{K-2} \theta_k (X - \xi_k)_+^3 + \theta_{K-1} (X - \xi_{K-1})_+^3 + \theta_K (X - \xi_K)_+^3,$$

and use the θ constraints to show that the two last terms can be rewritten as sums over the $N-2$ first terms.

Start with the second last term and "multiply by 1" and "add 0"

$$\begin{aligned}
\theta_{K-1}(X-\xi_{K-1})^3_+ &= \frac{(X-\xi_K)^3_+}{(\xi_K-\xi_{K-1})}\left(\theta_{K-1}\xi_K-\theta_{K-1}\xi_{K-1}\right)\\
&= \frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\left(\theta_{K-1}\xi_K-\theta_{K-1}\xi_{K-1}+\underbrace{\theta_K\xi_K-\theta_K\xi_K}_{0}\right)\\
&= \frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\left(\xi_K(\theta_{K-1}+\theta_K)-\xi_{K-1}\theta_{K-1}-\xi_K\theta_K\right)\\
&= \frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\left(-\xi_K\sum_{k=1}^{K-2}\theta_k+\sum_{k=1}^{K-2}\theta_k\xi_k\right)\\
&= -\frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\\
&= -\sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}
\end{aligned}$$

Then, take the last term, and do the same

$$\begin{aligned}
\theta_K(X-\xi_K)^3_+ &= \frac{(X-\xi_K)^3_+}{(\xi_K-\xi_{K-1})}\left(\theta_K\xi_K-\theta_K\xi_{K-1}+\underbrace{\theta_{K-1}\xi_{K-1}-\theta_{K-1}\xi_{K-1}}_{0}\right)\\
&= \frac{(X-\xi_K)^3_+}{(\xi_K-\xi_{K-1})}\left(-\xi_{K-1}(\theta_{K-1}+\theta_K)+\xi_{K-1}\theta_{K-1}+\xi_K\theta_K\right)\\
&= \frac{(X-\xi_K)^3_+}{(\xi_K-\xi_{K-1})}\left(\xi_{K-1}\sum_{k=1}^{K-2}\theta_k-\sum_{k=1}^{K-2}\theta_k\xi_k\right)\\
&= (X-\xi_K)^3_+\sum_{k=1}^{K-2}\theta_k\frac{\xi_K-\xi_k}{(\xi_K-\xi_{K-1})}\\
&= (X-\xi_K)^3_+\sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\frac{\xi_{K-1}-\xi_k+\xi_K-\xi_K}{(\xi_K-\xi_{K-1})(\xi_K-\xi_k)}\\
&= (X-\xi_K)^3_+\sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{1}{\xi_K-\xi_{K-1}}-\frac{1}{\xi_K-\xi_k}\right)
\end{aligned}$$

Then, we combine the two expressions

$$\begin{aligned}
\sum_{k=1}^{K}\theta_k(X-\xi_k)^3_+ &= \sum_{k=1}^{K-2}\theta_k(X-\xi_k)^3_+ + \theta_{K-1}(X-\xi_{K-1})^3_+ + \theta_K(X-\xi_K)^3_+,\\
&= \sum_{k=1}^{K-2}\theta_k(X-\xi_k)^3_+ - \sum_{k=1}^{K-2}\theta_k\left(\xi_K-\xi_k\right)\frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\\
&+ (X-\xi_K)^3_+\sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{1}{\xi_K-\xi_{K-1}}-\frac{1}{\xi_K-\xi_k}\right)\\
&= \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\frac{(X-\xi_k)^3_+}{\xi_K-\xi_k} - \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\frac{(X-\xi_{K-1})^3_+}{(\xi_K-\xi_{K-1})}\\
&+ \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{(X-\xi_K)^3_+}{\xi_K-\xi_{K-1}}-\frac{(X-\xi_K)^3_+}{\xi_K-\xi_k}\right)\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
&= \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{(X-\xi_k)^3_+}{\xi_K-\xi_k}-\frac{(X-\xi_{K-1})^3_+}{\xi_K-\xi_{K-1}}+\frac{(X-\xi_K)^3_+}{\xi_K-\xi_{K-1}}-\frac{(X-\xi_K)^3_+}{\xi_K-\xi_k}\right)\\
&= \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(\frac{(X-\xi_k)^3_+-(X-\xi_K)^3_+}{\xi_K-\xi_k}-\left(\frac{(X-\xi_{K-1})^3_+-(X-\xi_K)^3_+}{\xi_K-\xi_{K-1}}\right)\right)
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{K}\theta_k(X-\xi_k)^3_+ = \sum_{k=1}^{K-2}\theta_k(\xi_K-\xi_k)\left(d_k(X)-d_{K-1}(X)\right),$$

where

$$d_k(X)=\frac{(X-\xi_k)^3_+-(X-\xi_K)^3_+}{\xi_K-\xi_k}$$