Extended ASM with varying population

1 The AS Model

Mathematically, we can formalize the model as follows:
and since $x_l + x_o = 1$:

$$\begin{aligned}\frac{dx_l}{dt} &= s_l x_l^{\alpha}(1 - x_l) - s_o(1 - x_l)^{\alpha} x_l \\ \frac{dx_o}{dt} &= s_o x_o^{\alpha}(1 - x_o) - s_l(1 - x_o)^{\alpha} x_o\end{aligned} \tag{1}$$

Often, the prestige of the languages sums to 1: $s_l + s_o = 1$. Therefore, when the language L has high prestige, the other language O has low prestige and the formulation results in a symmetric form:

$$\frac{dx}{dt} = sx^{\alpha}(1 - x) - (1 - s)(1 - x)^{\alpha} x$$

2 The Bilingual group: double increments

The Bilingual group has two types of increments. The first one derives from (1) and concerns the percentage of monolingual speakers within the Indigenous group that adopt Spanish as a second language. The second increment depends on the probability that out of n new Indigenous, k are already bilingual. The increments of population X_o are described below:

$$X_o(t + dt) = X_o(t) + \Delta X_o\Big|_{ASM} + \Delta X_o\Big|_{growth} \tag{2}$$

The term directly connected to the ASM dynamics is the following:

$$\Delta X_o\Big|_{ASM} = I(t)\frac{dx_o}{dt} = I(t)\left[s_o x_o^a(1 - x_o) - s_l(1 - x_o)^a x_o\right],$$

while the growth term is:

$$\Delta X_o\Big|_{growth} = \frac{d}{dt}(p_o I) = p_o\frac{dI}{dt} + I\frac{dp_o}{dt}$$

The differential equation for the absolute number of bilingual speakers is presented in the following equation:

$$\frac{dX_o}{dt}\Big|_{tot} = I\frac{dx_o}{dt}\Big|_{ASM} + p_o\frac{dI}{dt} + I\frac{dp_o}{dt} \tag{3}$$

Since $x_o = X_o/I$ it follows:

$$\frac{dx_o}{dt} = \frac{d}{dt}\left(\frac{X_o}{I}\right) = \frac{1}{I}\frac{dX_o}{dt} - x_o\frac{1}{I}\frac{dI}{dt}$$

Using (3), we obtain:

$$\left.\frac{dx_o}{dt}\right|_{tot} = \frac{1}{I}\left[I\left.\frac{dx_o}{dt}\right|_{ASM} + p_o\frac{dI}{dt} + I\frac{dp_o}{dt}\right] - \frac{1}{I}x_o\frac{dI}{dt} \tag{4}$$

By the means of (1):

$$\left.\frac{dx_o}{dt}\right|_{tot} = s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o + \frac{1}{I}\left[p_o - x_o + I\frac{dp_o}{dI}\right]\frac{dI}{dt} \tag{5}$$

The probability p_o is described by:

$$p_o = \frac{p_{max}}{1 + Ae^{-\nu m_{si}(t)}} \tag{6}$$

According (6), p depends on I through $m_{si} = \frac{S(t)}{I(t)}$. We use the chain rule to calculate the derivative of p_o with respect to I:

$$\frac{dp_o}{dI} = \frac{dp_o}{dm_{si}}\frac{dm_{si}}{dI}$$

Where

$$\frac{dp_o}{dm_{si}} = p_o\left(1 - \frac{p_o}{p_{max}}\right)\nu, \quad \frac{dm_{si}}{dI} = -\frac{m_{si}(t)}{I}$$

Results in

$$\frac{dp_o}{dm_{si}} = p_o\left(1 - \frac{p_o}{p_{max}}\right)\nu\cdot\left(-\frac{m_{si}(t)}{I}\right) = -p_o\left(1 - \frac{p_o}{p_{max}}\right)\nu\left(\frac{S}{I^2}\right)$$

The calculation proceeds as follows:

$$\begin{aligned}\frac{dx_o}{dt} &= s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o + \frac{1}{I}\left[p_o - x_o + I\frac{dp_o}{dI}\right]\frac{dI}{dt}\\ &= s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o + \left[p_o - x_o - I\left(p_o\left(1-\frac{p_o}{p_{max}}\right)\nu\left(\frac{m_{si}}{I}\right)\right)\right] r_I\left(1-\frac{I}{K_I}\right)\\ &= \underbrace{s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o}_{\text{ASM}} + \underbrace{\left[p_o - x_o - p_o\left(1-\frac{p_o}{p_{max}}\right)\nu m_{si}\right] r_I\left(1-\frac{I}{K_I}\right)}_{\text{natural increment}}\end{aligned}$$

3 Fixed points and Jacobian

The full system used to analyze the fixed points is given by:

$$\begin{cases}(i) & \frac{dx_o}{dt} &= s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o + \left[p_o - x_o - p_o\left(1-\frac{p_o}{p_{max}}\right)\nu m_{si}\right] r_I\left(1-\frac{I}{K_I}\right),\\ (ii) & \frac{dI}{dt} &= r_I I\left(1-\frac{I}{K_I}\right),\\ (iii) & p_o &= \frac{p_{max}}{1+Ae^{-\nu m_{si}}},\\ (iv) & m_{si} &= \frac{S(t)}{I(t)}.\end{cases}$$

In the non-equilibrium case where $\frac{dI}{dt} \neq 0$ (i.e., $I \neq 0$ and $I \neq K_I$), determining the points at which (i) is equal to zero is intricate. To simplify the analysis, we introduce the functions f, and g:

$$f(x_o) = s_o x_o^a (1 - x_o) - s_l (1 - x_o)^a x_o,$$

$$g(x_o, I) = \left[p_o - x_o - p_o \left(1 - \frac{p_o}{p_{max}} \right) \nu m_{si} \right] r_I \left(1 - \frac{I}{K_I} \right),$$

which allows us to express (i) in the compact form

$$f(x_o) = -g(x_o, I).$$

For each value of I within the interval $(0, K_I)$, we fix $I^*(t^*)$ and determine the corresponding value of $S^*(t^*)$. The metric $m_{si}(I^*)$ from (iv) is then utilized to compute the proportion of bilingual speakers, denoted as $p_o(I^*)$, according to (iii). Solving equation (i), where $dx_o/dt = 0$, is equivalent to identifying the family of curves in the (x_o^*, I^*) phase plane that satisfies these conditions:

$$f(x_o^*) = -g(x_o^*, I^*).$$

The interaction between trajectories and the curves of "quasi-equilibrium" points, where $\frac{dx_o}{dt} = 0$, while the group of Indigenous (I) continues to evolve, governs the overall dynamics of the system. The stability of the family of fixed points is determined by the Jacobian of the system.

$$J_{00} = A_{00} + B_{00} + C_{00}$$

$$\begin{cases} A_{00} = (1 - 2x_o) \left[s_o x_o^{a-1} - s_l (1 - x_o)^{a-1} \right] \\ B_{00} = x_o (1 - x_o) \left[(a-1)(s_o x_o^{a-2} + s_l (1 - x_o)^{a-2}) \right] \\ C_{00} = -r_I \left(1 - \frac{I}{K_I} \right) \end{cases} \tag{7}$$

Let

$$g(x_o, I) = h(x_o, I) y(I)$$

Where:

$$h(x_o, I) = \left[p_o - x_o - p_o \left(1 - \frac{p_o}{p_{max}} \right) \nu \frac{S}{I} \right] = \left[p_o \left(1 - \nu \frac{S}{I} \right) + \frac{\nu}{p_{max}} \frac{S}{I} p_o^2 - x_o \right]$$

$$y(I) = r_I \left(1 - \frac{I}{K_I} \right)$$

It follows:

$$\frac{\partial g(x_o, I)}{\partial I} = h(x_o, I) \frac{\partial y(I)}{\partial I} + y(I) \frac{\partial h(x_o, I)}{\partial I}$$

The derivative of $y(I)$ is strightforward:

$$\frac{\partial y(I)}{\partial I} = -\frac{r_I}{K_I}$$

The derivative of $h(x_o, I)$ is complex. The definition of $h(x_o, I)$ is reported below for clarity:

$$h = \left[p_o \left(1 - \nu \frac{S}{I} \right) + \frac{\nu}{p_{max}} \frac{S}{I} p_o^2 - x_o \right]$$

$$\begin{aligned}
\frac{\partial h(x_o,I)}{\partial I} &= \frac{\partial}{\partial I}\left[p_o\left(1-\nu\frac{S}{I}\right)+\frac{\nu}{p_{max}}\frac{S}{I}p_o^2-x_o\right] = \left\{\frac{\partial p_o}{\partial I}\left(1-\nu\frac{S}{I}\right)+\nu p_o\frac{S}{I^2}+\frac{\nu}{p_{max}}\left[\frac{S}{I}\left(2p_o\frac{\partial p_o}{\partial I}\right)-p_o^2\frac{S}{I^2}\right]\right\} \\
&= \left\{\frac{\partial p_o}{\partial I}\left(1-\nu\frac{S}{I}+2p_o\frac{\nu}{p_{max}}\frac{S}{I}\right)+\nu p_o\frac{S}{I^2}-\frac{\nu}{p_{max}}p_o^2\frac{S}{I^2}\right\} \\
&= \left\{\frac{\partial p_o}{\partial I}\left(1-\nu\frac{S}{I}+2p_o\frac{\nu}{p_{max}}\frac{S}{I}\right)+\underbrace{\nu p_o\frac{S}{I^2}\left(1-\frac{p_o}{p_{max}}\right)}_{-\frac{\partial p_o}{\partial I}}\right\} \\
&= \nu\frac{S}{I}\frac{\partial p_o}{\partial I}\left(\frac{2p_o}{p_{max}}-1\right) = \nu\frac{S}{I}\left[\nu\frac{S}{I^2}p_o\left(1-\frac{p_o}{p_{max}}\right)\right]\left(1-\frac{2p_o}{p_{max}}\right) = \nu^2\frac{S^2}{I^3}p_o\left(1-\frac{p_o}{p_{max}}\right)\left(1-2\frac{p_o}{p_{max}}\right)
\end{aligned}$$

$J_{01} = A_{01} + B_{01}$

$$\begin{cases} A_{01} = -\left[p_o\left(1-\nu\frac{S}{I}\right)+\frac{\nu}{p_{max}}\frac{S}{I}p_o^2-x_o\right]\cdot\frac{r_I}{K_I} \\ B_{01} = r_I\left(1-\frac{I}{K_I}\right)\cdot\nu^2\frac{S^2}{I^3}p_o\left(1-\frac{p_o}{p_{max}}\right)\left(1-2\frac{p_o}{p_{max}}\right) \end{cases} \tag{8}$$

$$J_{10} = \frac{\partial(dI/dt)}{\partial x_o} = 0$$

$$J_{11} = \frac{\partial(dI/dt)}{\partial I} = r_I\left(1-2\frac{I}{K_I}\right)$$

This completes the calculation of the Jacobian