Dynamic Models for Bilingualism, Language Adoption, and Language Dynamics

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Overview

Motivation and Background (i)

What is México? A country where...

Colonization, migration, and urban development in México have each played distinct roles in shaping the country's multilingual character and influencing the transmission of languages across generations.

Linguistic diversity in México is under pressure:

Spanish colonization contributed to the Spanish language's predominant status in México. Recent studies [1]–[3] emphasize how factors such as social integration, urbanization, and access to economic and educational opportunities have influenced many Indigenous speakers to adopt Spanish as a second language.

Motivation and Background (ii)

Preservation of language diversity

This shift toward Bilingualism poses considerable challenges for the preservation of Indigenous languages. Understanding the dynamics of language adoption is essential for maintaining linguistic diversity and planning for sustainable cultural integration.

Macro (social) models are required

There is a need for models that combine social dynamics and demographic factors.

Input data

Mexican Censuses

The data is from INEGI[a] (*El Instituto Nacional de Estadística y Geografía*), which conducts periodic censuses to track demographic changes in Mexico. The datasets include information on education levels, ages[b], urbanization, migration, and spoken languages.

[a] `https://www.inegi.org.mx/programas/`
[b] Only individuals older than five years are included.

Data features

Year: from 1895 to 2020; **Population**: Total population of México; **Spanish**: Total number of people from families of Spanish genealogy; **Indigenous** (either **Bilingual** or **Monolingual**).

Research Question(s)

Socio-demographic question

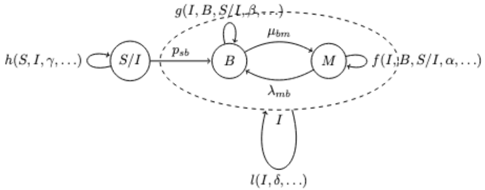
Given the actual nature of México, it is highly likely that an Indigenous individual will learn Spanish for work or educational reasons.
Consequently, social researchers are interested in monitoring the relationship between bilinguals and monolinguals within the Indigenous population to provide valuable support for policy decisions to prevent causes of endangerment to Indigenous languages.

Dynamic systems involved

Language dynamics, competition, and adoption can be approached with birth-death models, prey-predator, oscillating systems,. . .

A complete picture of the system under observation

Internal and external dynamics



- The dashed ellipse contains the two populations B and M, which evolve with f and g. The function l drives the dynamic of I;
- The node S/I is interpreted by an external action that modifies the dynamics of B.

We analyze two types of dynamics: Within the ellipse, we examine the internal changes of individual populations, M and B, which change according to λ_{mb} and μ_{bm}.
The second type captures the action of S/I, p_{sb} on the number of B.

Language Competition Modeling

Abrams-Strogatz Model (ASM)

The Abrams-Strogatz Model [4] was originally used to predict language death from experimental data. The basic scenario accounts for two competitive languages that share the same number of speakers. Macro parameters shape the transition probabilities between these two languages to show whether consensus or coexistence is possible.

Extended ASM

Basic ASM requires a constant population. We extend the ASM to include variations in population and add the probability of belonging to one language versus another.

The Basic ASM Equation

Basic assumptions

ASM considers two competitive languages x_o and x_l summing to $1 : x_o + x_l = 1$

$$\frac{dx_o}{dt} = \underbrace{s_o x_o^a (1 - x_o)}_{\mu} - \underbrace{s_l (1 - x_o)^a x_o}_{\lambda}, \frac{dx_l}{dt} = -\frac{dx_o}{dt}$$

- x_o: proportion of speakers of language o $(1 - x_l$, of language $l)$.
- s_o, s_l: prestige of each language. A macroscopic parameter that measures how much a language is attractive for speakers;
- a: volatility parameter. $a > 1$ *consensus*, where one language dominates (stable), and an unstable state where the two languages *coexist*. Stability changes when $0 < a < 1$.

Basic ASM critical points

$\frac{dx_o}{dt} = 0 \implies -s_l x_l^a x_o + s_o x_o^a x_l = 0 \implies x_l x_o (s_l x_l^{a-1} - s_o x_o^{a-1}) = 0$, with $x_o = 1 - x_l$.

Three critical points: $x_l = 0 (x_o = 1)$, $x_l = 1 (x_o = 0)$, and $\frac{x_l}{1-x_l} = \left(\frac{s_o}{s_l}\right)^{\frac{1}{a-1}} = \beta$.

$x_l = \frac{\beta}{1+\beta}$

Critical points

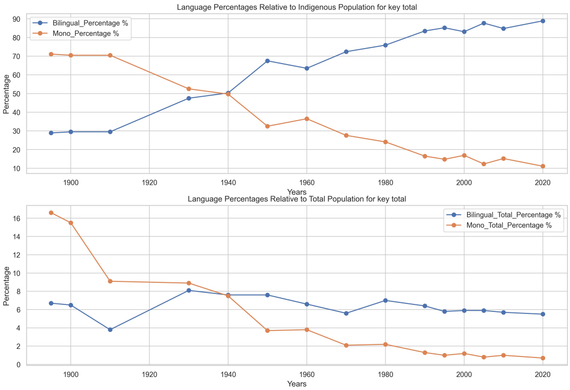
$$\begin{cases} p_0(x_l, x_o) = (0,1), \\ p_1(x_l, x_o) = (1,0), \\ \text{stable if } a > 1 \text{ unstable if } a < 1 \\ p^*(x_l^*, x_o^*) = \left(\frac{\beta}{1+\beta}, \frac{1}{1+\beta}\right) \text{ where } \beta = \left(\frac{s_o}{s_l}\right)^{\frac{1}{a-1}}, \\ \text{ unstable if } a > 1 \text{ stable if } a < 1 \end{cases}$$

Eigenvalues after ε expansion:

$$\begin{cases} p_0(\varepsilon, 1-\varepsilon) \to \lambda_o \approx -s_o + s_l a \varepsilon^{a-1} \\ p_1(1-\varepsilon, \varepsilon) \to \lambda_l \approx -s_l + s_o a \varepsilon^{a-1} \\ p^*(x_l^* + \varepsilon, x_l^* + \varepsilon) \to \\ \lambda_* \approx (a-1) s_o \left(\frac{1}{1+\beta}\right)^{a-1} (1 + \mathcal{O}(\varepsilon)) \end{cases}$$

Trends

Bilingual and monolingual percentages trends calculated on Indigenous (upper plot) and Population (lower plot) categories.

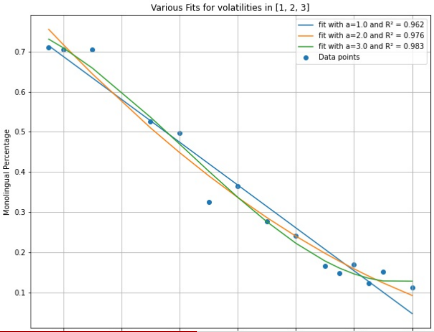


- Monolingual and Bilingual show a typical oscillating path;
- Note that the overall percentage of Indigenous decreases, even if it tends to stabilize during the last censuses.

Volatility estimation

Volatility measures the non-linearity of the process: The value $a = 0.5$ (sublinear interactions) enables minority groups to influence the dynamics. The value $a = 2$ accounts for potential superlinear effects, while $a = 3$ does not capture the data trend

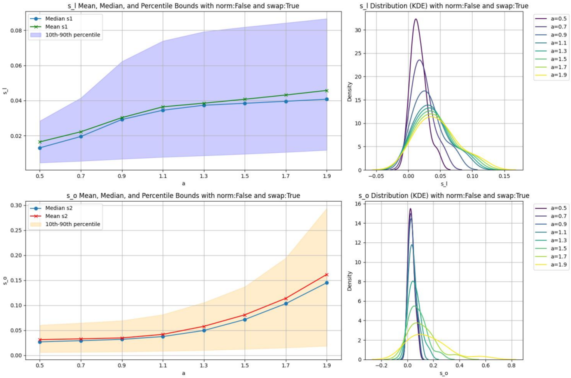
- Bounds for the volatility a: $a \in [0.5, 2]$;
- The picture illustrates the polynomial fit x_l^a for various volatility parameters:
  - Linear and quadratic models best fit the final trends in the data;
  - The lower bound $a = 0.5$, which allows for sublinear interactions, is not shown in the plot.



Prestige estimation

The prestige factors s_l and s_o measure how quickly each population responds to changes in their respective proportions.

- Prestige values s_o and s_l are calculated as: $s_o = \frac{k}{x_l(1-x_l)^a}$ and $s_l = \frac{k}{x_l^a(1-x_l)}$;
- The value of k is related to $\frac{dx_l}{dt}$. Bounded by the value of $k_{max} \approx 1.72 \cdot 10^{-3}$;
- The figures show a clear separation between the prestige values of s_l and s_o;
- Shadowed areas are used to define bounds in optimization algorithms.

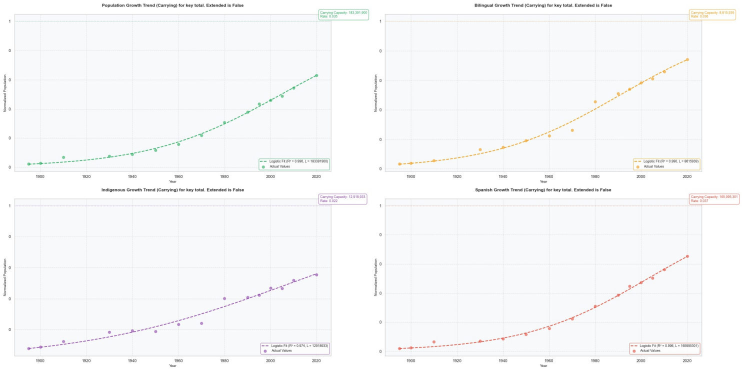


Data-Driven Population Modeling

Logistic pattern. Note the discontinuity around 1970 in Indigenous (and Bilingual)

- Census data cover the period (1895–2020) [1]. The picture shows the normalized curves for the entire population, Spanish, Indigenous, and Bilingual;
- The curves follow a logistic pattern: Logistic growth:

$$X(t) = \frac{K_X}{1 + C_X e^{-r_X t}}$$

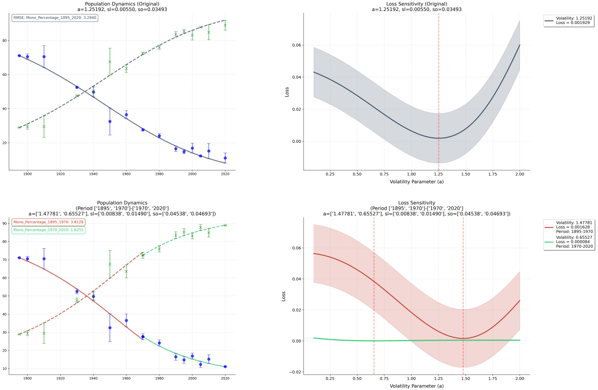


Basic ASM fitting results

Complete $(1895 - 2020)$ vs $1970 - 2020$ process fit. 1970 is the discontinuity point in Logistic curves.

The fitting experiments returned the following results:

- **Period** $1895 - 2020$**:**
  $a = 1.2519 \pm 0.3130$,
  $s_l = 0.0055 \pm 0.0007$,
  $s_o = 0.0349 \pm 0.0087$;
- **Period** $1970 - 2020$**:**
  $a = 0.6553 \pm 0.1638$,
  $s_l = 0.0149 \pm 0.0037$,
  $s_o = 0.0469 \pm 0.0092$.

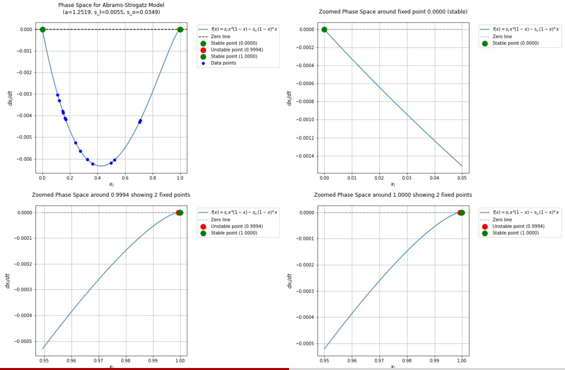


Fixed points and stability (i)

Complete Period: 1895 – 2020

The numerical calculation of the derivatives at the fixed points is in good agreement with theoretical values when $a > 1$: p_0 and p_1 are stable, while p^* is unstable.

$$\frac{dx_l}{dt} = \begin{cases} \left.\dfrac{dx_l}{dt}\right|_{p_0} = \overbrace{-0.0348 \pm 0.0108}^{x_l=0} \\ \left.\dfrac{dx_l}{dt}\right|_{p_1} = \overbrace{-0.0050 \pm 0.0013}^{x_l=1} \\ \left.\dfrac{dx_l}{dt}\right|_{p^*} = \underbrace{0.0014 \pm 0.0004}_{x_l \approx 0.9994} \end{cases}$$

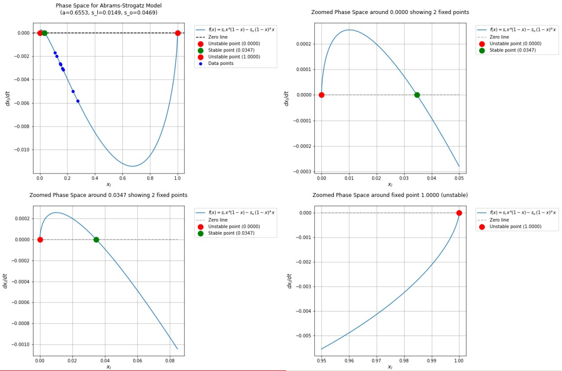


Fixed points and stability (ii)

Period: 1970 – 2020

The numerical calculation of the derivatives at the fixed points agrees with perturbation theory in ε when $a < 1$: p_0 and p_1 become unstable when $\varepsilon \geq 9.2 \cdot 10^{-8}$. p^* is stable.

$$\frac{dx_l}{dt} = \begin{cases} \left.\dfrac{dx_l}{dt}\right|_{p_0} = \overbrace{2.7845 \pm 0.1217}^{x_l=0} \\ \left.\dfrac{dx_l}{dt}\right|_{p_1} = \overbrace{8.8975 \pm 0.1884}^{x_l=1} \\ \left.\dfrac{dx_l}{dt}\right|_{p^*} = \underbrace{-0.0164 \pm 0.0005}_{x_l \approx 0.0347} \end{cases}$$



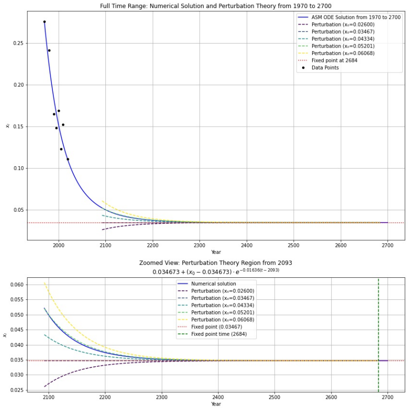
Fixed points and stability (iii)

Perturbation Theory: 1970 – 2700

The value of $x_{th} = 0.0347$ is reached around 2700, while $2x_{th}$ is reached in $t_p = 2093$.
We applied perturbation theory using:

$$x_l(t) = x_{th} + \left(x_o(t_p) - x_{th}\right) e^{-\lambda^*(t - t_p)}$$

with $\lambda^* = -0.0164$.



ASM with a Varying Population I(t)

Model modifications required

The principal modifications are the following:

No more constant population: We relax the constraint $x_o + x_I = 1$ and assume a more general $X_o(t) + X_I(t) = I(t)$;

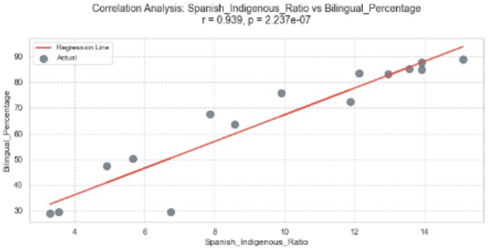
Social lattice: We suppose a double increment for Bilingual speakers: the standard adoption pattern and the fact that social connections may "create" already bilingual offspring with a probability $p = \binom{n}{k} p^k (1-p)^{n-k}$, k bilingual out of n new individuals;

Macro Influence: The metric $m_{si} = \frac{S(t)}{I(t)}$, the ratio between the number of Spanish speakers per Indigenous, measures the Spanish exposure within the Indigenous communities. m_{si} positively correlates with Bilingual births, used to model p.

Macro Influence of the m_{si} Metric

$$msi(t) = \frac{S(t)}{I(t)} = \underbrace{\left[\frac{K_S}{K_I} \cdot \frac{1 + C_I e^{-r_I t}}{1 + C_S e^{-r_S t}}\right]}_{\text{formal definition}}$$

- m_{si} positively correlates with the percentage of Bilingual speakers within the Indigenous communities ($r = 0.94, p << 0.005$):
    - The social connections increase the number of bilingual families;
    - $p = p(m_{si})$;



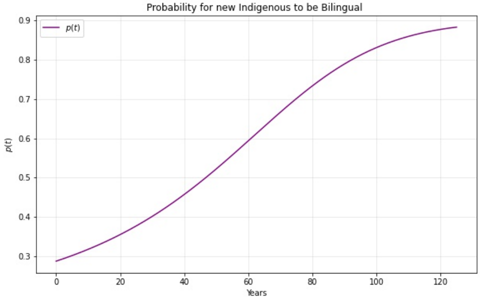
Probability $p = p(m_{si})$

m_{si} positively correlates with the increase of Bilingual speakers: p as an increasing monotonic function of m_{si}. Best fitting p is a scaled logistic curve:

- $p(t) = \frac{p_{max}}{1+Ae^{-\nu m_{si}(t)}}$

  $A \approx 5.47, \nu \approx 0.271, p_{max} \approx 0.97.$

- The exponential component, $e^{-\nu m_{si}(t)}$, stabilizes at K_S/K_I: p approaches its asymptotic value without oscillation or divergence.



Double increment of X_0

$$\Delta X_o\Big|_{\text{Tot.}} = \Delta X_o\Big|_{\text{ASM}} + \Delta X_o\Big|_{\text{Log.}}$$

The Bilingual group x_o has two types of increments:

- The fraction of monolingual speakers that adopt Spanish as a second language:

$$\Delta x_o\Big|_{\text{ASM}} = \Delta(x_o I) = I\Delta x_o\Big|_{\text{ASM}} + x_o \underbrace{\Delta I}_{=0 \text{ in micro-dynamic}} = I\Delta x_o\Big|_{\text{ASM}}$$

- The percentage of Bilingual offspring:

$$\Delta x_o\Big|_{\text{Log.}} = \Delta(pI) = p\Delta I + I\Delta p$$

Complete dynamics for the bilingual fraction x_0

The dynamics consists of two distinct contributes:

$$\frac{dx_o}{dt} = f(x_o) + g(x_o, I) = f(x_o) + h(x_o, I)y(I)$$

Where

$$f(x_o) = \overbrace{s_o x_o^a (1 - x_o) - s_I (1 - x_o)^a x_o}^{\text{ASM}}$$

$$g(x_o, I) = \overbrace{\underbrace{\left[p_o \left(1 - \nu m_{si}\right) + \nu \frac{m_{si}}{p_{max}} p_o^2 - x_o \right]}_{h(x_o, I)} \cdot \underbrace{r_I \left(1 - \frac{I}{K_I}\right)}_{y(I)}}^{\text{Log.}}$$

Complete Dynamic System

Coupled equations for the pair (x_o, I)

$$\begin{cases} (i) & \frac{dx_o}{dt} = f(x_o) + h(x_o, I)y(I) \\ (ii) & \frac{dI}{dt} = r_I I \left(1 - \frac{I}{K_I}\right) \end{cases}$$

- (ii) equals to 0 implies $I = 0$ or $I = K_I$. $I = 0$ means no population at all, while setting $I = K_I$ in (i) reduces (i) to the basic ASM model:

$$\frac{dx_o}{dt} = f(x_o) = s_0 x_o^a (1 - x_o) - s_I x_o (1 - x_o)^a = 0$$

- Interesting to study (i)=0 when $\frac{dI}{dt} \neq 0$.

Quasi-equilibrium System

Quasi-equilibrium pair (x_o, I)

For each value of I within the interval $(0, K_I)$:

- (i) Fix $I^*(t^*)$: (ii) Determine the corresponding value of $S^*(t^*)$;
- (iii) Calculate the metric $m_{si}(I^*)$; (iv) Calculate $p_o(I^*)$

Solving $dx_o/dt = 0$, is equivalent to identifying the family of curves in the (x_o^*, I^*) phase plane that satisfy these conditions:

$$f(x_o^*) = -g(x_o^*, I^*) = -h(x_o^*, I^*)y(I^*).$$

The interaction between trajectories and these curves of "quasi-equilibrium" points, where $dx_o/dt = 0$, while $dI/dt \neq 0$, controls the overall system dynamics.

Fixed points and stability (iv)

Modified stability

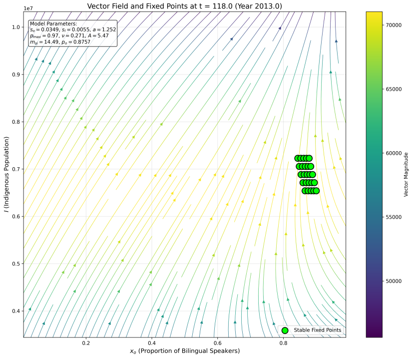
The stability of the system is verified using the following algorithm:

- (i) Pick a t and calculate I;
- (ii) Create a set of I^*, s.t. $I^* \in [-n\Delta I, +n\Delta I]$, $n = \pm 2, \pm 1, 0$ and ΔI is a small fraction of I;
- (iii) For each I^* calculate S^*, m_{si}^*, and p^*;
  - (iii-a) Find x_o^* s.t. $f(x_o^*) + h(x_o^*, I^*)y(I^*) = 0$;
  - (iii-b) Evaluate the eigenvalues of the Jacobian at the point (x_o^*, I^*)

Fixed points and stability (v)

Complete census period: 1895 – 2020

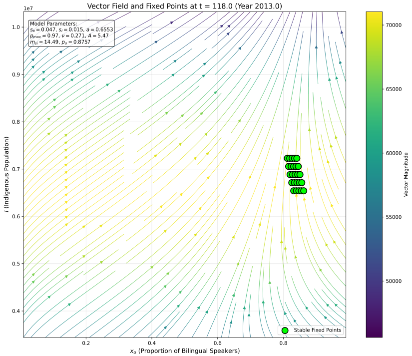
Volatility $a \sim 1.26$ implies a consensus status with $x_o \approx 1$
The complete system with varying I start stabilizing around $t = 2013$, with the average value for $x_0 \approx 0.8720$.
The system's eigenvalues are -0.0014 and -0.0383. The eigenvalue with the largest magnitude, 0.0383 models ASM (it was -0.0348) while the eigenvalue -0.0014 is associated with the process of natural increment.



Fixed points and stability (vi)

Census period: 1970 – 2020

Volatility $a \sim 0.66$ implies a coexistence status between x_o and x_I. The complete system with varying I start stabilizing around $t = 2013$, with the average value for $x_0 \approx 0.8505$.
The system's eigenvalues are -0.0673 and -0.0031. The eigenvalue with the value, 0.0031 models ASM (it was -0.0015) while the eigenvalue -0.0673 is associated with the process of natural increment.



Conclusions and Outlook

Empirical Observations

- The Extended model predicts earlier stabilization than the original ASM;
- Transition in x_o stability begins in 2013 in both scenarios;
- The extended model combines internal social dynamics with natural increment;
- It captures bilingualism more realistically;

Future works

- Explore regional heterogeneity and Indigenous peculiarities (spatial diffusion?)
- Introduce x_l (monolingual) logistic growth.

`https://github.com/riccardodg/language_dynamics`

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From Birth-Death Microscopy to ASM Macro-Dynamics

Birth-Death Process

$$dP(n,t) = \lambda_{n-1}P(n-1,t) + \mu_{n+1}P(n+1,t) - (\lambda_n + \mu_n)P(n,t) \quad P(n) = P(0)\Pi_{k=1}^{n}\frac{\mu_k}{\lambda_{k-1}}$$

Microscopic transition rates (finite N)

$$\lambda_n = s_X(N-n)\left(\frac{n}{N}\right)^{\alpha}, \qquad \mu_n = s_Y n\left(\frac{N-n}{N}\right)^{\alpha}$$

Mean-field limit ($N \to \infty$)

Let $x = \frac{n}{N}$: $\frac{dx}{dt} = s_X(1-x)x^{\alpha} - s_Y x(1-x)^{\alpha}$

Albedo and Random Telegraph

Albedo Model

$$\frac{dP_h}{dt} = \omega_{hc} P_h + \omega_{ch} P_c, \quad \frac{dP_c}{dt} = -\frac{dP_h}{dt}$$

where the ωs oscillate.

Random Telegraph

$$\frac{dP_+}{dt} = -\lambda P_+ + \mu P, \quad \frac{dP_-}{dt} = \lambda P_+ - \mu P_-$$

$$P_+ = \frac{\mu}{\lambda + \mu}, \quad P_- = \frac{\lambda}{\lambda + \mu}$$

$$\langle X(t) \rangle = P_+ - P_-, \quad \frac{d}{dt}\langle X(t) \rangle = -(\lambda + \mu)\langle X(t) \rangle + (\mu - \lambda)$$

ASM in Complex Systems

Connections to other used models

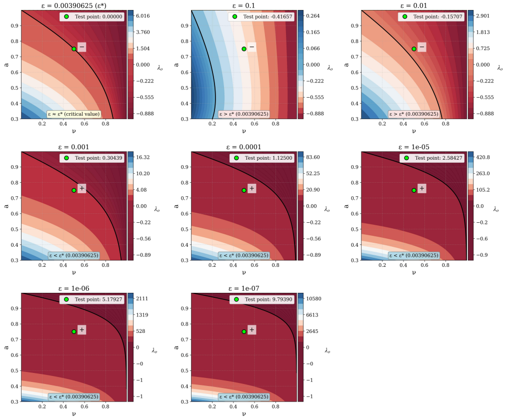
ASM is similar to the mean field for the Voter Model, [5], which is obtained when $s_o = s_l$ and $a = 1$. $dx_i/dt = 0$, reflecting the fact that the total number of o and l is conserved.
If $a = 1$, but $s_o \neq s_l$, we recover the Lotka-Volterra, [6], [7] competition or replicator formulation:

$$\frac{dx_l}{dt} = rx\left(1 - x\right), r = s_l - s_o$$

ASM is also used in different topologies (namely, lattices)[8], accounting for multiple languages, [9], and studying how geographic distribution and mobility affect the dynamics of linguistic competition [10].

Epsilon divergence when $a < 1$

Each subplot corresponds to a specific ε value The Figure is generated from the parameters $a = 0.75, s_o = 0.75$, and $s_l = 0.25$; ε_o^* results ≈ 0.004 and $\lambda_0 >$ when $\varepsilon < \varepsilon^*$



Derivation of double increment (i)

The second increment depends on the probability that out of n new Indigenous, k are already bilingual. The increments of population X_o are described below:

$$X_o(t+dt) = X_o(t) + \Delta X_o\Big|_{ASM} + \Delta X_o\Big|_{growth} \tag{1}$$

The term directly connected to the ASM dynamics is the following:

$$\Delta X_o\Big|_{ASM} = \frac{d}{dt}\left(Ix_o)\right) = I(t)\frac{dx_o}{dt} = I(t)\left[s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o\right] + x_o\frac{dI}{dt}$$

In the internal dynamics $dI/dt = 0$ while the growth term is:

$$\Delta X_o\Big|_{growth} = \frac{d}{dt}\left(p_o I\right) = p_o\frac{dI}{dt} + I\frac{dp_o}{dt}$$

The differential equation for the absolute number of bilingual speakers is presented in the following equation:

$$\frac{dX_o}{dt}\Big|_{tot} = I\frac{dx_o}{dt}\Big|_{ASM} + p_o\frac{dI}{dt} + I\frac{dp_o}{dt} \tag{2}$$

Derivation of double increment (ii)

Since $x_o = X_o/I$ it follows:

$$\frac{dx_o}{dt} = \frac{d}{dt}\left(\frac{X_o}{I}\right) = \frac{1}{I}\frac{dX_o}{dt} - x_o\frac{1}{I}\frac{dI}{dt}$$

Using (2), we obtain:

$$\left.\frac{dx_o}{dt}\right|_{tot} = \frac{1}{I}\left[I\left.\frac{dx_o}{dt}\right|_{ASM} + p_o\frac{dI}{dt} + I\frac{dp_o}{dt}\right] - \frac{1}{I}x_o\frac{dI}{dt} \tag{3}$$

By ASM equation:

$$\left.\frac{dx_o}{dt}\right|_{tot} = s_o x_o^a(1-x_o) - s_l(1-x_o)^a x_o + \frac{1}{I}\left[p_o - x_o + I\frac{dp_o}{dI}\right]\frac{dI}{dt} \tag{4}$$

Derivation of double increment (iii)

The probability p_o is described by:

$$p_o = \frac{p_{max}}{1 + Ae^{-\nu m_{si}(t)}} \tag{5}$$

Since $p = p(m_{si}$, we use the chain rule to calculate the derivative of p_o with respect to I:

$$\frac{dp_o}{dI} = \frac{dp_o}{dm_{si}} \frac{dm_{si}}{dI}$$

Where

$$\frac{dp_o}{dm_{si}} = p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu, \quad \frac{dm_{si}}{dI} = -\frac{m_{si}(t)}{I}, \quad \frac{dp_o}{dI} = p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu \cdot \left(-\frac{m_{si}(t)}{I}\right) = -p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu \left(\frac{S}{I^2}\right)$$

The calculation proceeds as follows:

$$\frac{dx_o}{dt} = \underbrace{s_o x_o^a (1 - x_o) - s_I (1 - x_o)^a x_o}_{\text{ASM}} + \underbrace{\left[p_o - x_o - p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu m_{si}\right] r_I \left(1 - \frac{I}{K_I}\right)}_{\text{natural increment}}$$

Jacobian of the extended system (i)

The full system used to analyze the fixed points is given by:

$$\begin{cases} (i) & \frac{dx_o}{dt} = s_o x_o^a (1 - x_o) - s_l (1 - x_o)^a x_o + \left[p_o - x_o - p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu m_{si}\right] r_I \left(1 - \frac{I}{K_I}\right), \\ (ii) & \frac{dI}{dt} = r_I I \left(1 - \frac{I}{K_I}\right), \\ (iii) & p_o = \frac{p_{max}}{1 + Ae^{-\nu m_{si}}}, \\ (iv) & m_{si} = \frac{S(t)}{I(t)}. \end{cases}$$

To simplify the analysis, we introduce the functions f, and g:

$$f(x_o) = s_o x_o^a (1 - x_o) - s_l (1 - x_o)^a x_o, \quad g(x_o, I) = \left[p_o - x_o - p_o \left(1 - \frac{p_o}{p_{max}}\right) \nu m_{si}\right] r_I \left(1 - \frac{I}{K_I}\right),$$

which allows us to express (i) in the compact form

$$f(x_o) = -g(x_o, I).$$

Jacobian of the extended system (ii): J_{00}

$$J_{00} = A_{00} + B_{00} + C_{00} \quad \text{where:} \begin{cases} A_{00} = (1-2x_o)\left[s_o x_o^{a-1} - s_I(1-x_o)^{a-1}\right] \\ B_{00} = x_o\,(1-x_o)\left[(a-1)(s_o x_o^{a-2} + s_I(1-x_o)^{a-2})\right] \\ C_{00} = -r_I\left(1-\frac{I}{K_I}\right) \end{cases}$$

Let

$$g(x_o, I) = h(x_o, I)y(I)$$

Where:

$$h(x_o, I) = \left[p_o - x_o - p_o\left(1 - \frac{p_o}{p_{max}}\right)\nu\frac{S}{I}\right] = \left[p_o\left(1 - \nu\frac{S}{I}\right) + \frac{\nu}{p_{max}}\frac{S}{I}p_o^2 - x_o\right], \quad y(I) = r_I\left(1 - \frac{I}{K_I}\right)$$

It follows:

$$\frac{\partial g(x_o, I)}{\partial I} = h(x_o, I)\frac{\partial y(I)}{\partial I} + y(I)\frac{\partial h(x_o, I)}{\partial I}, \quad \frac{\partial y(I)}{\partial I} = -\frac{r_I}{K_I}$$

Jacobian of the extended system (iii): J_{01}, J_{10}, and J_{11}

The definition of $h(x_o, I)$ is reported below for clarity:

$$h = \left[p_o\left(1 - \nu\frac{S}{I}\right) + \frac{\nu}{p_{max}}\frac{S}{I}p_o^2 - x_o\right]$$

$$\frac{\partial h(x_o, I)}{\partial I} = \nu\frac{S}{I}\left[\nu\frac{S}{I^2}p_o\left(1 - \frac{p_o}{p_{max}}\right)\right]\left(1 - \frac{2p_o}{p_{max}}\right) = \nu^2\frac{S^2}{I^3}p_o\left(1 - \frac{p_o}{p_{max}}\right)\left(1 - 2\frac{p_o}{p_{max}}\right)$$

$J_{01} = A_{01} + B_{01}$

$$\begin{cases} A_{01} = -\left[p_o\left(1 - \nu\frac{S}{I}\right) + \frac{\nu}{p_{max}}\frac{S}{I}p_o^2 - x_o\right]\cdot\frac{r_I}{K_I} \\ B_{01} = r_I\left(1 - \frac{I}{K_I}\right)\cdot\nu^2\frac{S^2}{I^3}p_o\left(1 - \frac{p_o}{p_{max}}\right)\left(1 - 2\frac{p_o}{p_{max}}\right) \end{cases} \tag{6}$$

$$J_{10} = \frac{\partial(dI/dt)}{\partial x_o} = 0$$

$$J_{11} = \frac{\partial(dI/dt)}{\partial I} = r_I\left(1 - 2\frac{I}{K_I}\right)$$

Instability of “coexistence” in 2010