Bayesian model of a *gravitational microlensing* event

Nicole Grillo, Riccardo Loddo, Margherita Pini

1 Introduction

A gravitational lens is a substantial amount of matter between an observer and an object which bends its light. Microlensing effects create distorted images of the source and change in its magnification but the reduced angular scales do not allow to see them clearly. These effects can be identified when the light of a background object changes in time, due to the change in the relative position of the source and the lens. The purpose of this experience is to establish a Bayesian model of a *gravitational microlensing* event. The data will be analyzed through Bayesian inference and with the Python programming language.

2 Model

2.1 Physical model

The observed brightness L_{obs} is related to the brightness of the lens L_{lens} and that of the source L_{source} through the equation $L_{\text{obs}} = L_{lens} + A \cdot L_{source}$, where A is the *amplification factor*:

$$A(u) = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}. \tag{1}$$

A is a function of u, which is defined as the distance r between the lens and the source divided by the *Einstein radius* of the problem r_{E}, that is the radius of the *Einstein ring* in the event of perfect alignment; u is in turn parameterized over time t:

$$u(t) = \sqrt{u_{\text{min}}^2 + \left(\frac{t - t_{\text{min}}}{t_{\text{E}}}\right)^2} \tag{2}$$

where u_{min}, the minimum value of $u(t)$, determines the alignment condition, t_{min} is the time in which the passage from u_{min} occurs and t_{E} is the *Einstein time* (the time it takes the physical system to cross r_{E}). Finally, introducing the *blending factor* f_{bl}, which shows the proportion of the source light on the total one:

$$f_{\text{bl}} = \frac{L_{source}}{L_{source} + L_{lens}} \tag{3}$$

and exploiting the relation between magnitude m and luminosity L, the following logarithmic equation is obtained:

$$m_{\text{obs}} = m_{\text{tot}} - 2.5 \cdot \log_{10}(1 - f_{\text{bl}} + A \cdot f_{\text{bl}}), \tag{4}$$

where m_{obs} is the magnitude related to L_{obs} while m_{tot} is the magnitude related to $(L_{lens} + L_{source})$ (without any amplification). To summarize, the observed magnitude m_{obs} depends on five parameters: m_{tot}, f_{bl}, t_{min}, u_{min}, t_{E}.

2.2 Statistical model

The analysis is performed using Bayesian inference. Let $\{D_n\} = \{m_n;\ t_n, m_{\text{err},n}\}$ be the data vector, which consists of the magnitude data m_n measured at time t_n with error $m_{\text{err},n}$. Let $\mathbf{p}$ be the collection of the five parameters on which the model depends: $\mathbf{p} = (m_{\text{tot}}, f_{\text{bl}}, t_{\text{min}}, u_{\text{min}}, t_{\text{E}})$. The *posterior distribution* for $\mathbf{p}$ has been obtained from the data using Bayes' Theorem:

$$P(\mathbf{p}|D_n) = \frac{P(D_n|\mathbf{p}) \cdot P(\mathbf{p})}{\int P(D_n|\mathbf{p'}) \cdot P(\mathbf{p'}) d\mathbf{p'}}. \tag{5}$$

Assuming that each magnitude data m_n, measured at time t_n, is Gaussianly distributed around the $m_{model,n}$ value measured through Eq. 4 (which in turn refers to the Eq. 1, Eq. 2 and Eq. 3) with standard deviation equal to its uncertainty $m_{\text{err},n}$, and assuming that the measurements are independent from each other, the *likelihood function* takes the form:

$$P(D_n|\mathbf{p}) = \prod_i \frac{1}{\sqrt{2\pi m_{err,i}^2}} \cdot \exp\left[-\frac{1}{2}\left(\frac{m_i - m_{model,i}}{m_{err,i}}\right)^2\right]. \tag{6}$$

The *prior function*, assuming that each parameter p_i (i=1,..,5) is uniformly distributed within an interval $\Delta p_i = (p_{i,\text{min}}, p_{i,max})$, becomes:

$$P(\mathbf{p}) = \begin{cases} \frac{1}{(\Delta m_{\text{tot}} \cdot \Delta f_{\text{bl}} \cdot \Delta t_{\text{min}} \cdot \Delta u_{\text{min}} \cdot \Delta t_{\text{E}})} & \text{if } \mathbf{p} \in (\Delta m_{\text{tot}} \times \Delta f_{\text{bl}} \times \Delta t_{\text{min}} \times \Delta u_{\text{min}} \times \Delta t_{\text{E}}) \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In section 3.2 the intervals Δp_i will be illustrated, explaining how they have been identified. Finally, the denominator (*evidence*) in Eq. 5 provides the normalization for the *posterior distribution*.

3 Data

3.1 Data source

The analysis is conducted starting from the data available on the project OGLE Early Warning System (EWS), designed for detection of microlensing events in progress[1]. The specific data set chosen for the simulation is the one present at the link `http://ogle.astrouw.edu.pl/ogle4/ews/2014/blg-0021.html`.

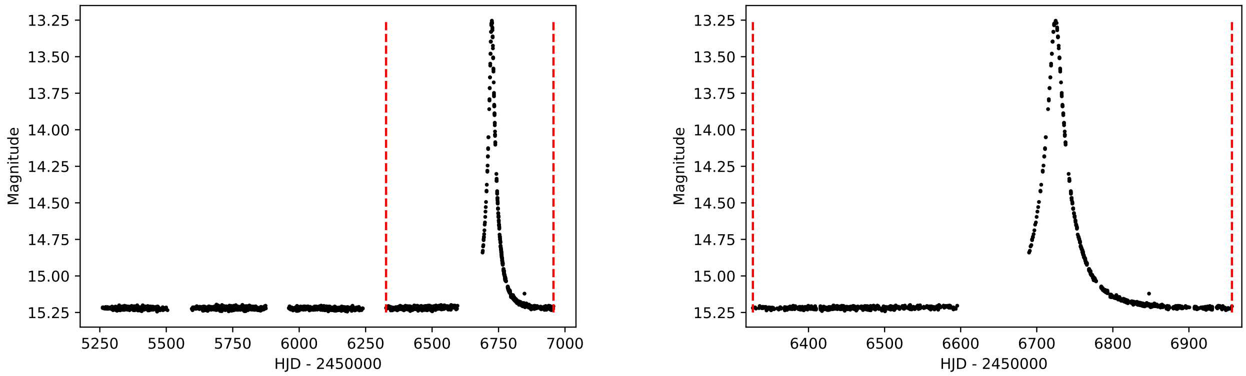

***Figure** 1: Data scatter plot. On the left, the complete dataset, on the right the complete restricted dataset near the peak. The dashed lines define the analyzed time interval. For convenience it is decided to report times translated by a factor of 2450000, so that $\tilde{t} = t - 2450000$ HJD.*

3.2 Data analysis

The analysis is developed on the first three columns of the data set: the Heliocentric Julian Date, the magnitude and the magnitude error. The analyzed data refer to the time interval (2456326.87826, 2456956.52001) HJD, from the beginning of the observation period that includes the peak in magnitude to the end of the data set (Fig. 1), since the previous data are not relevant to the study of the phenomenon.
The *prior* intervals in Eq. 7 are chosen by making considerations on the values in Fig.1: the magnitude extremes are chosen in order to include the height of the peak, $f_{\rm bl}$ is by definition a function of domine (0, 1], $t_{\rm min}$ extremes are reasonably selected from the interval where the peak occurs while $t_{\rm E}$ is taken as half the temporal width of the magnitude peak. Finally, $u_{\rm min}$ extremes are selected knowing that $\frac{r}{r_E} < 1$ in order to have an observable magnification peak. In Table 1 all the extremes are listed.

	$m_{\rm tot}$	$f_{\rm bl}$	$t_{\rm min}$ [HJD]	$u_{\rm min}$	$t_{\rm E}$ [days]
Min	13.00	0.00	2456700	0.00	30.00
Max	15.50	1.00	2456800	1.00	70.00

***Table** 1: Prior intervals.*

The analysis is then carried out using the *emcee*[1] library, which is based on Monte Carlo algorithm, therefore it provides unnormalized distributions and it does not require knowing the value of the *evidence*, which has the role of a rescaling factor.

[1]An MIT licensed pure-Python implementation of Goodman & Weare's Affine Invariant Markov chain Monte Carlo (MCMC) Ensemble sampler.

4 Results

The *unnormalized posterior distributions* for each of the five parameters and their mutual correlations are shown in Fig. 2. The final results are plotted using the `corner`[2] module of *emcee*.

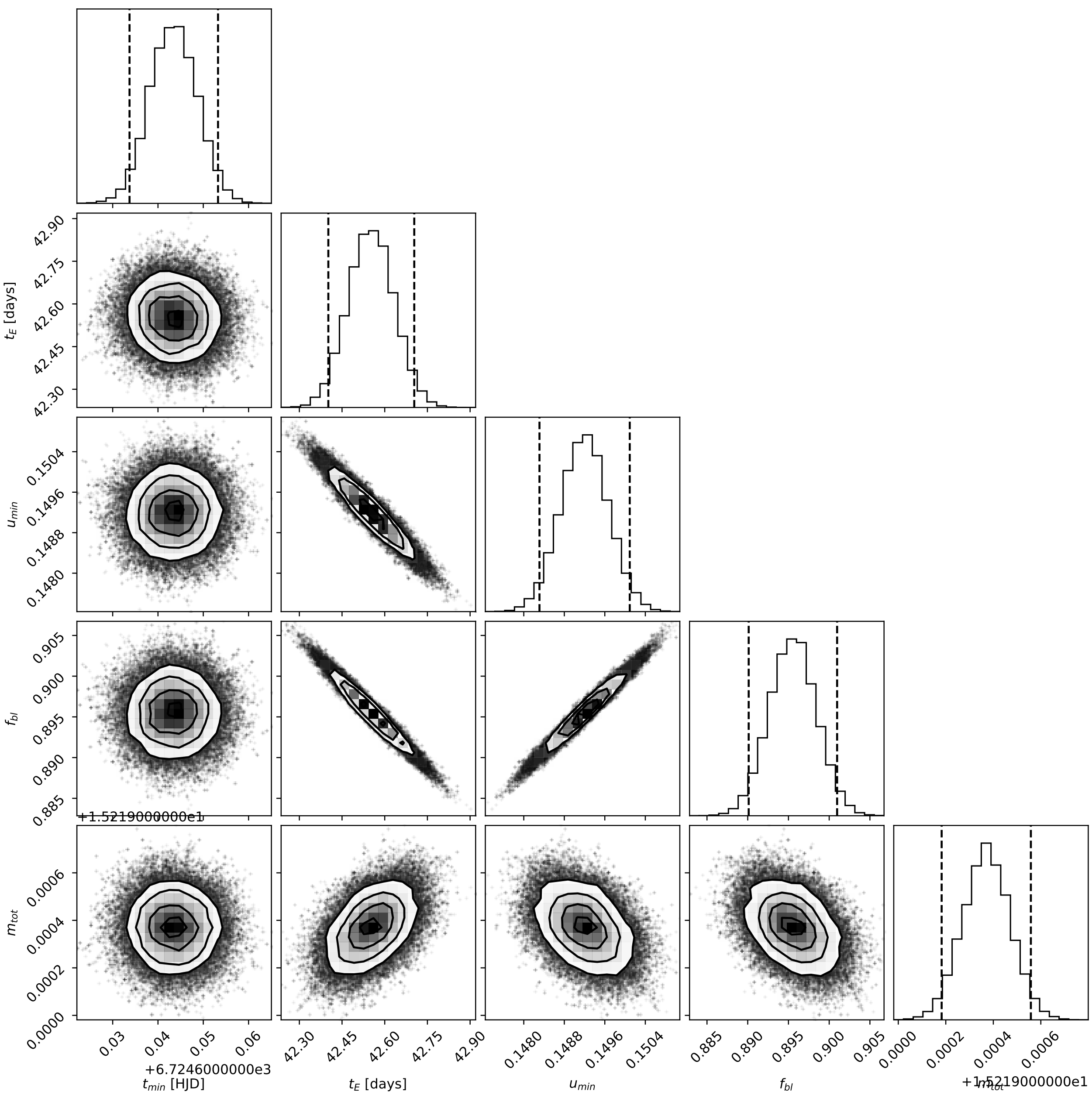

***Figure** 2: Corner plot showing the emcee sampling of the unnormalized posterior distribution for the Bayesian model parameters $m_{\rm tot}$, $f_{\rm bl}$, $t_{\rm min}$, $u_{\rm min}$, $t_{\rm E}$. The dashed lines indicate the 3rd and 97th percentile of the marginalized distribution of each parameter (shown at the top of each column). The graphs below show the correlation between the different parameters. For convenience it is decided to report t_{min} translated by a factor of 2450000, so that $\tilde{t_{min}} = t_{min} - 2450000$ HJD.*

[2]Copyright 2013-2021 Dan Foreman-Mackey.

In analogy with the Gaussian distributions, on each histogram in Fig. 2 the confidence interval is identified as the one from the 3rd to the 97th percentile, where the probability to find the parameter is 94%.

Parameter	m_{tot}	f_{bl}	t_{min} [HJD]	u_{min}	t_{E} [days]
Lower limit	15.2192	0.8902	2456724.634	0.1483	42.399
Upper limit	15.2196	0.9010	2456724.653	0.1501	42.704

***Table** 2: Resulting confidence intervals for the parameters.*

5 Conclusions

The distributions found for the five parameters are bell-shaped. The dimensional charts in Fig. 2 show a graphic representation of the correlation between the different parameters. Particularly, t_{min} tends to be independent from each of the parameters. The pairs (t_{E}, u_{min}) and (t_{E},f_{bl}) are in strong inverse correlation, while (u_{min}, f_{bl}) are in positive correlation. Finally, m_{tot} is slightly positively correlated to t_{E} and inversely correlated to u_{min} and f_{bl}.

References

[1] Udalski et al. 2015, Acta Astron., 65, 1, `http://ogle.astrouw.edu.pl`.