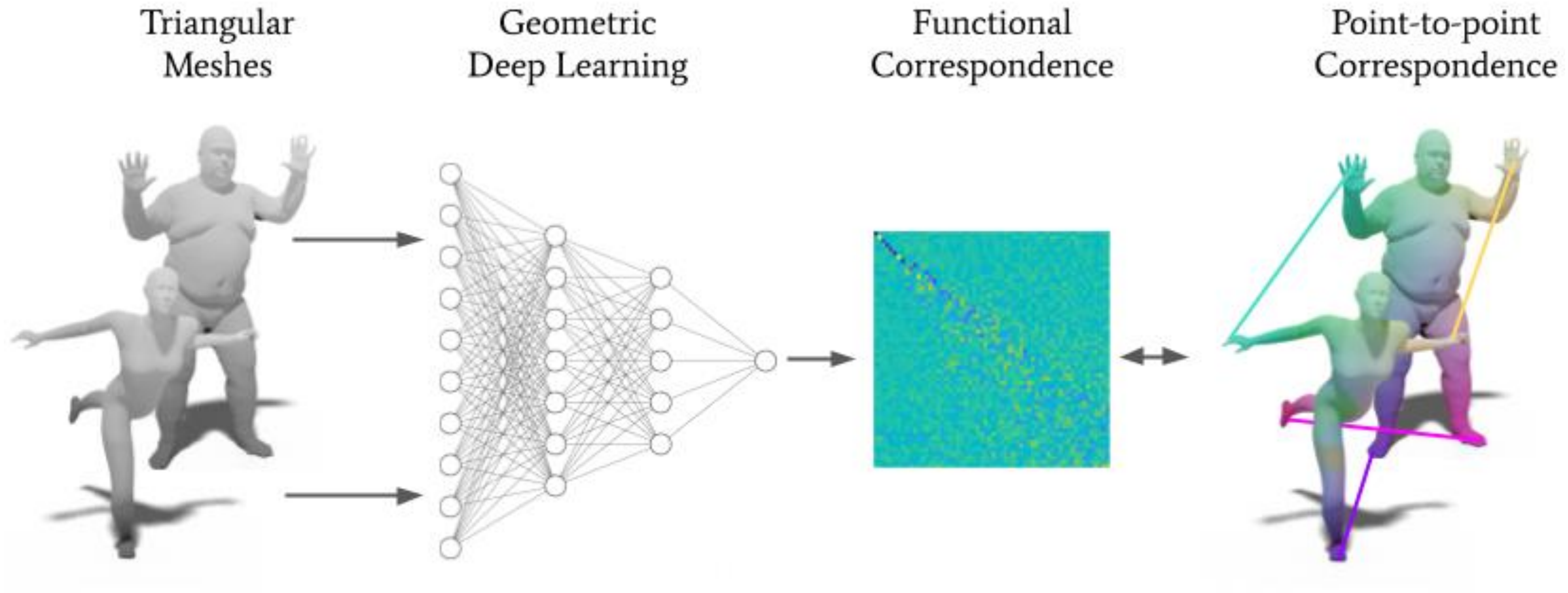
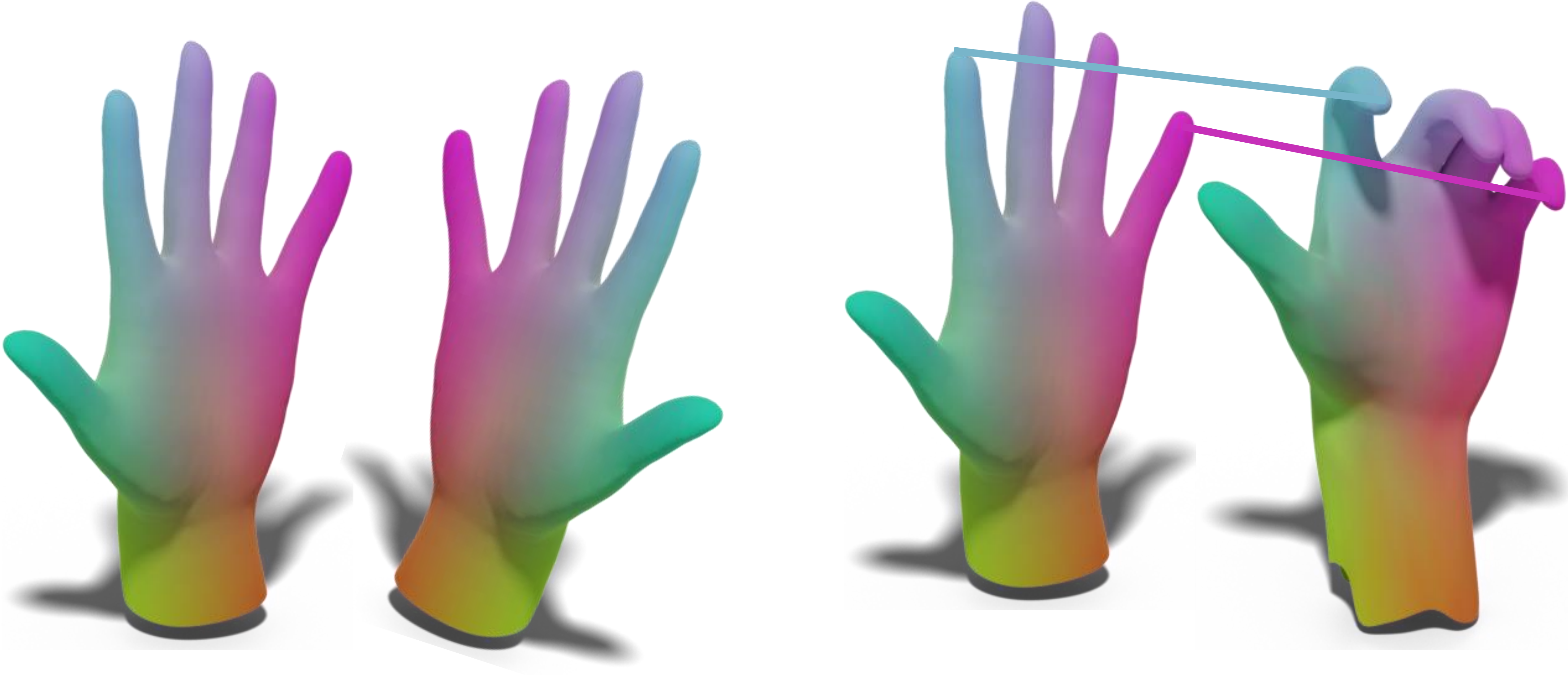
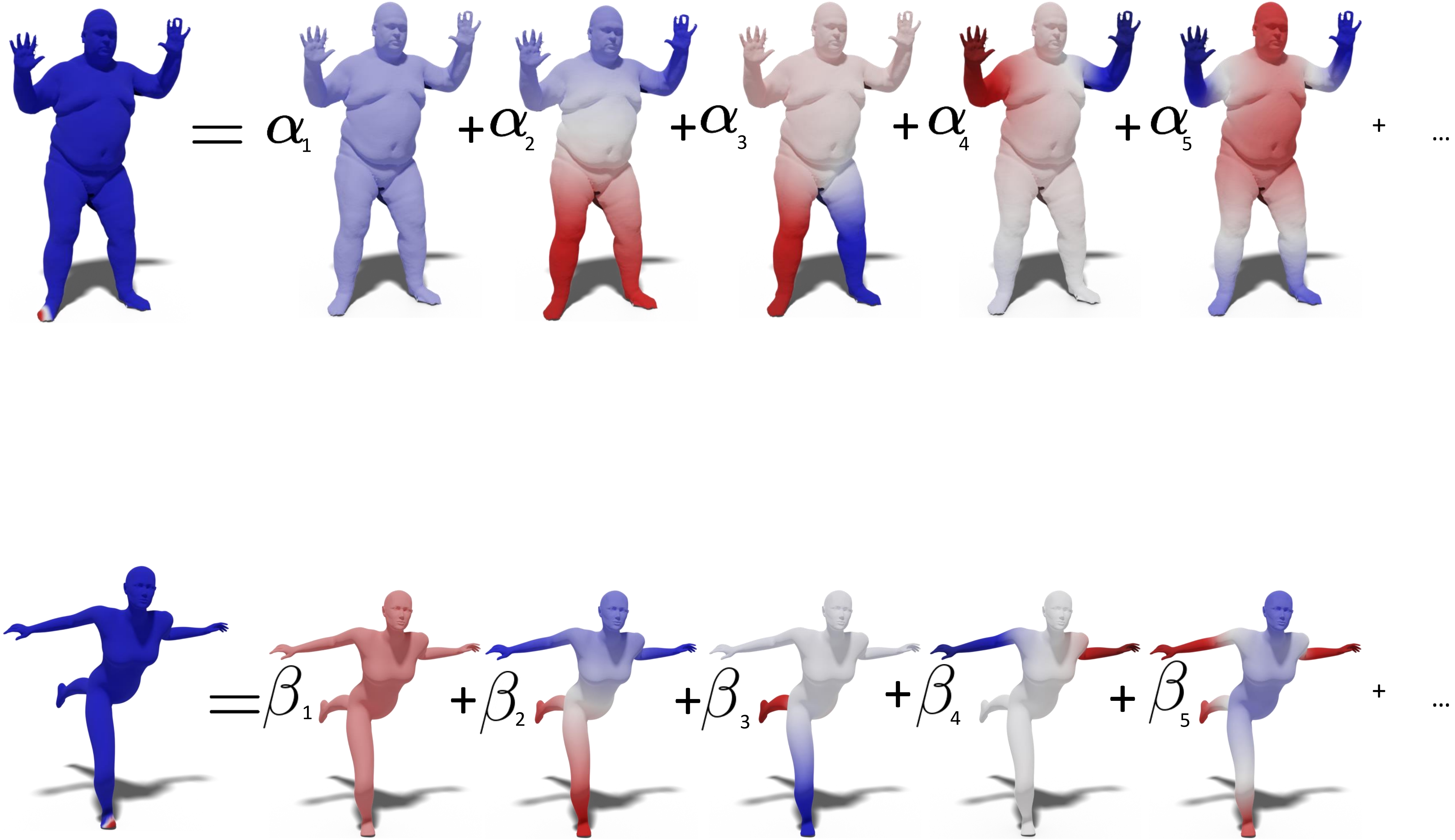
Functional Correspondence from Discrete Geometry to Learning

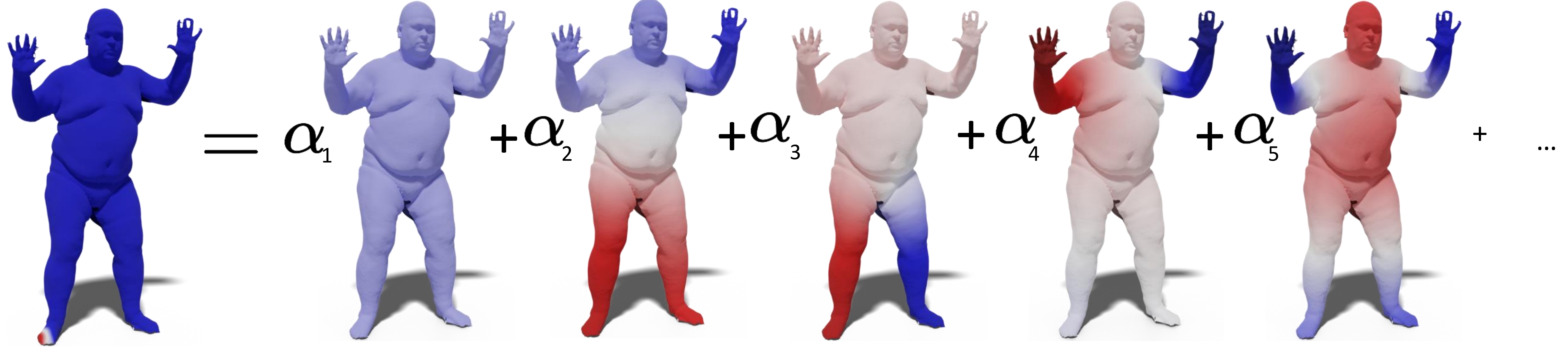
Riccardo Marin, Emanuele Rodolà



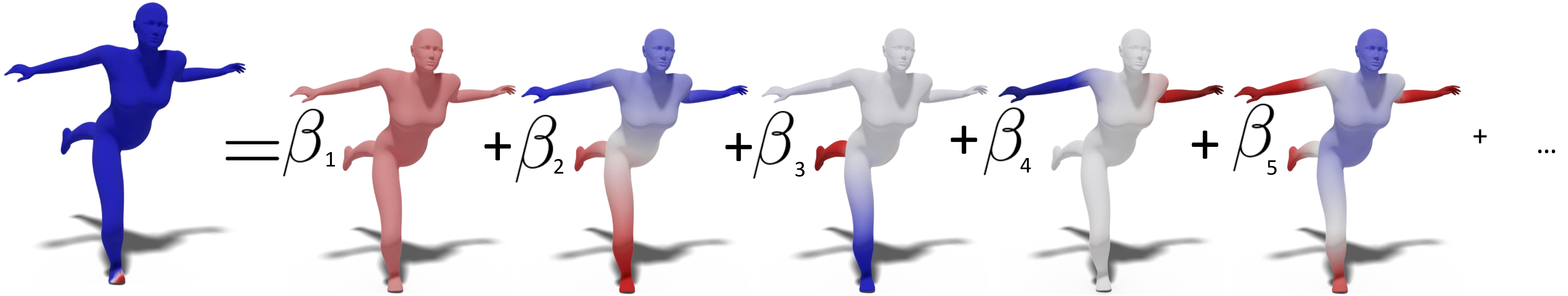
Genoa, 1st July 2023, Symposium on Geometry Processing

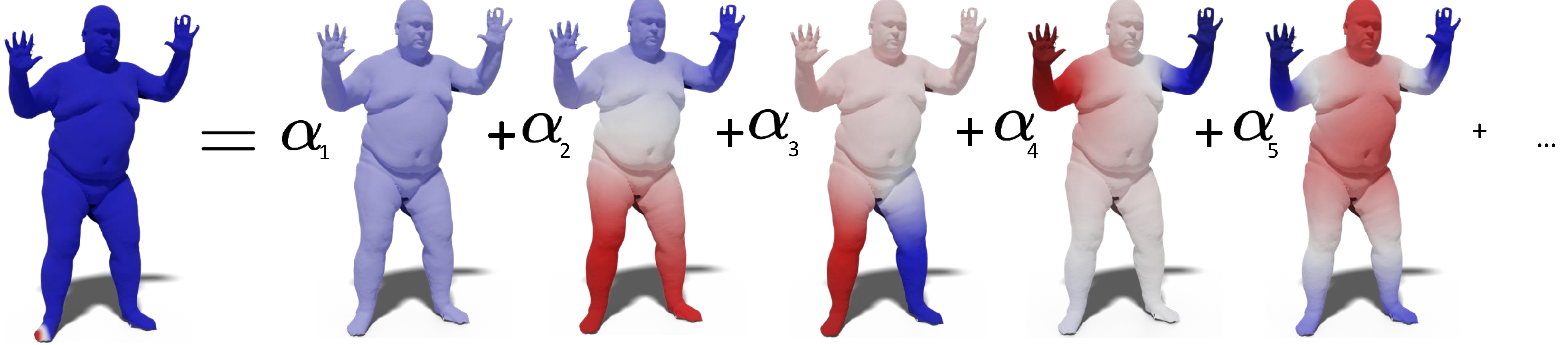



$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots$
$= \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \ldots$


$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots$

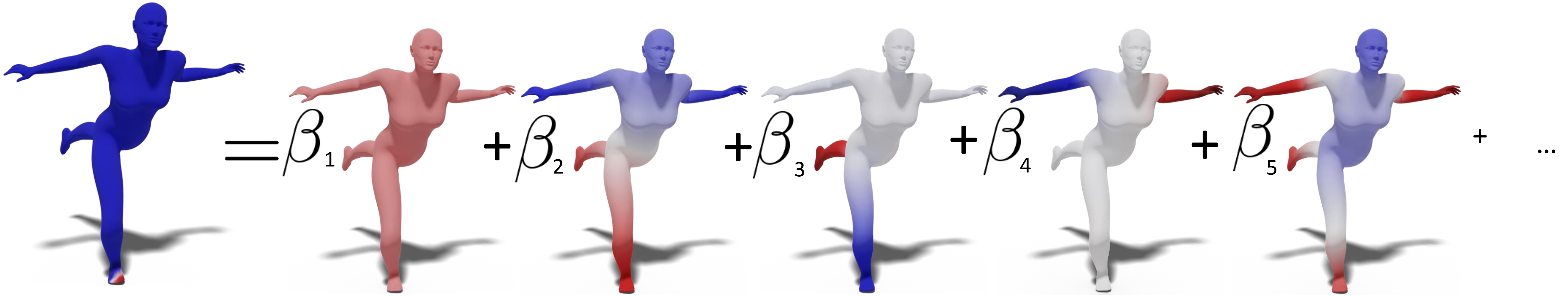
$$B = CA$$

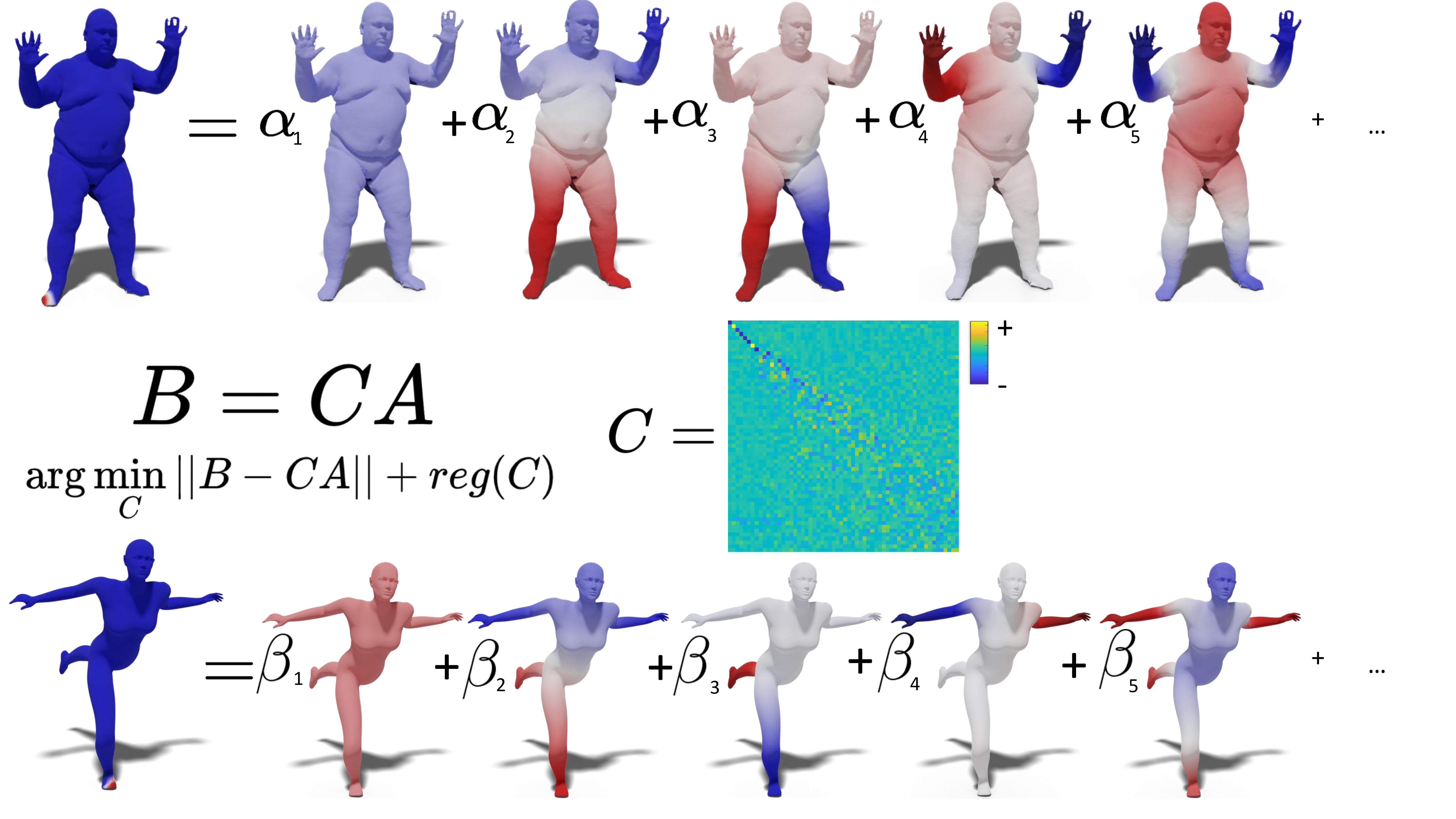

$= \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \ldots$



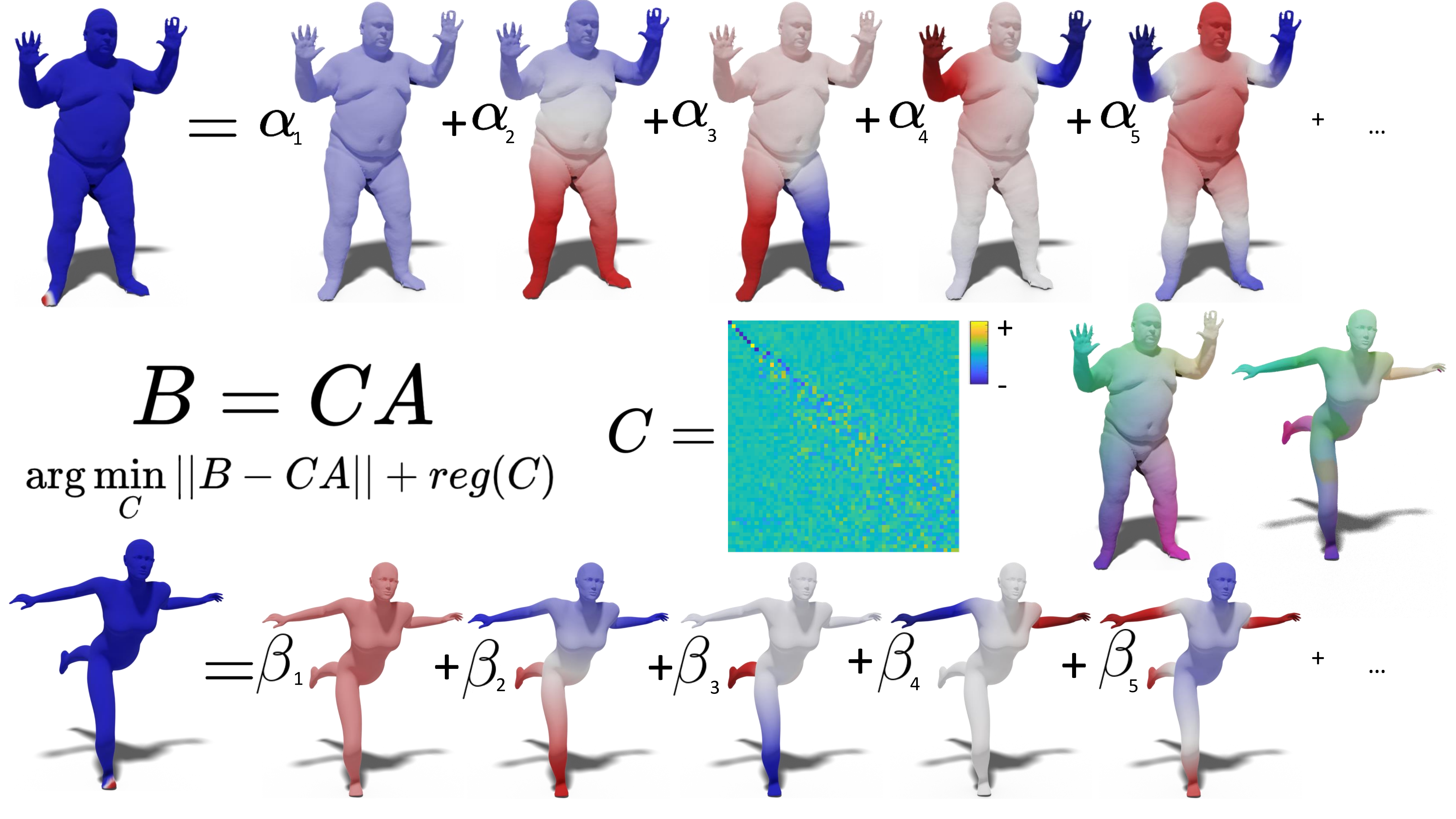
$$B = CA$$

$$\arg\min_{C} ||B - CA|| + reg(C)$$





$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \dots$
$B = CA$
$\arg\min_{C} ||B - CA|| + reg(C)$
$C =$
+
-
$= \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \dots$


$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \ldots$
$B = CA$
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+
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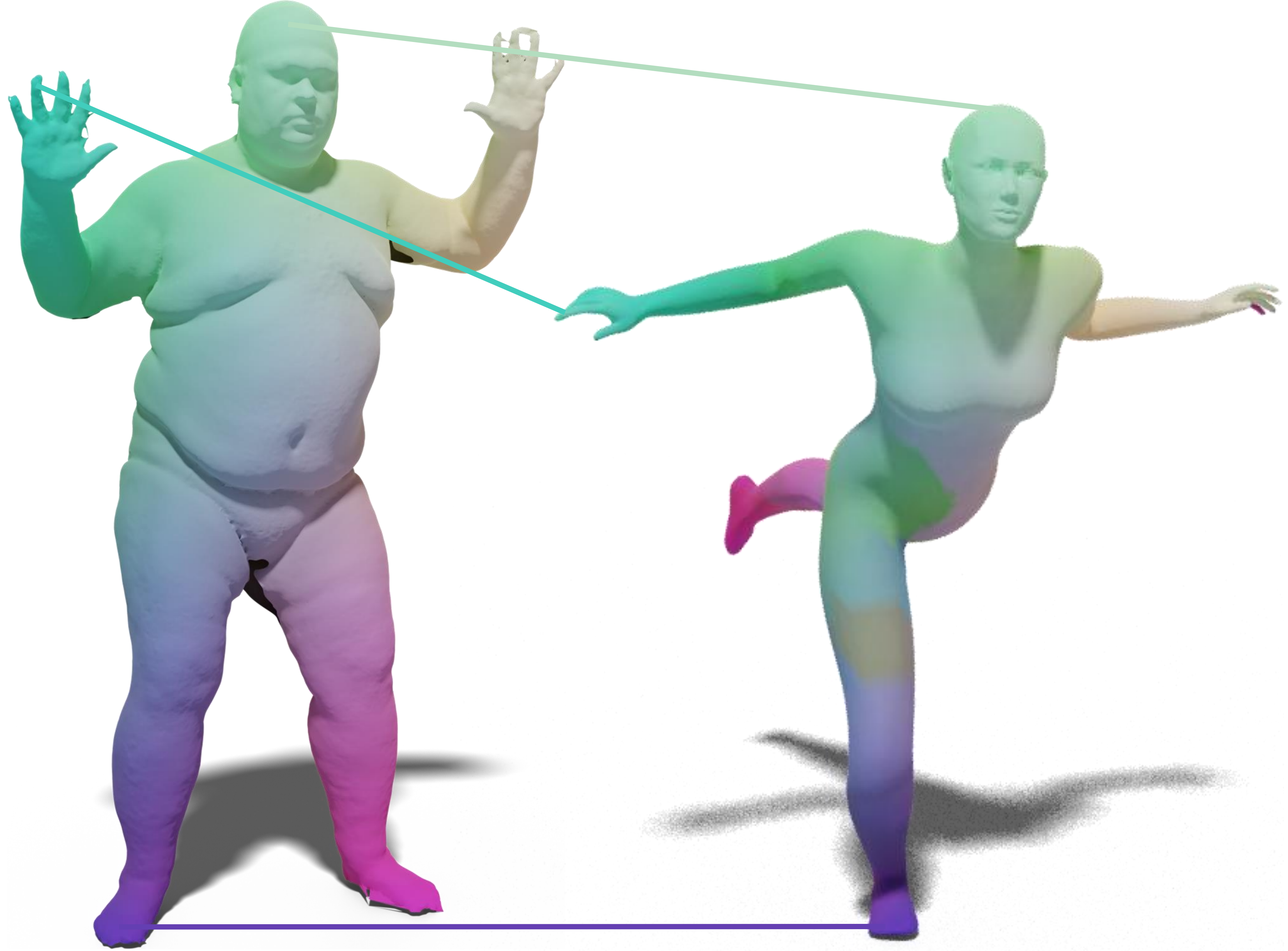
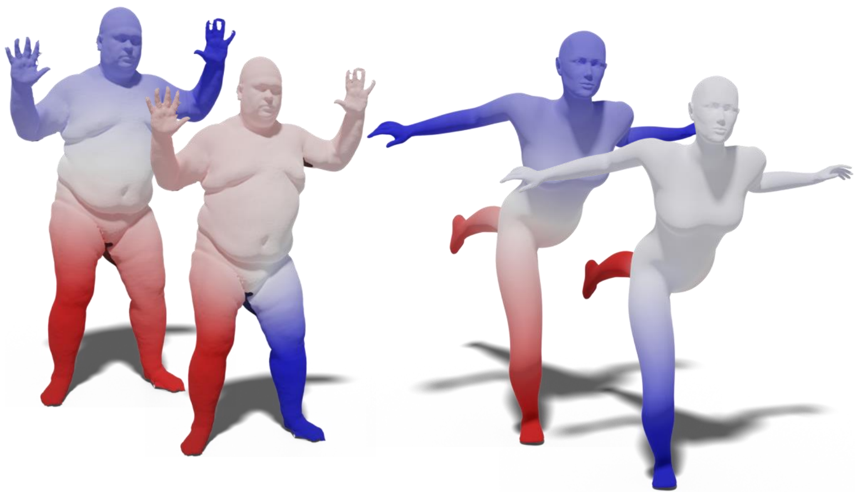
$= \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \ldots$

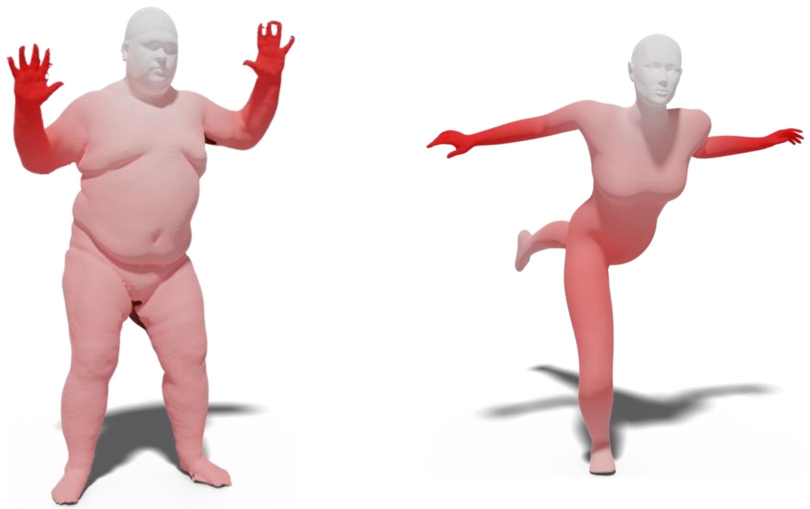


Key Ingredients

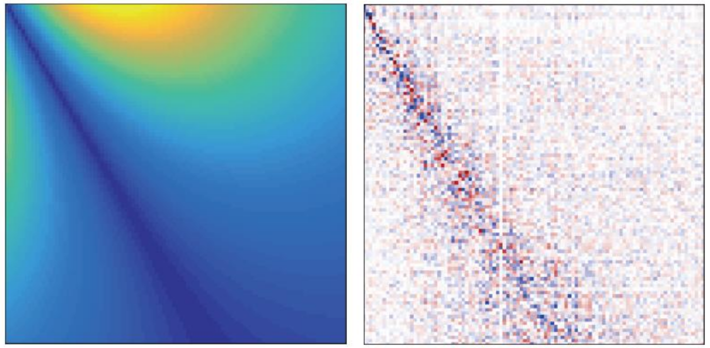
Choice of the basis



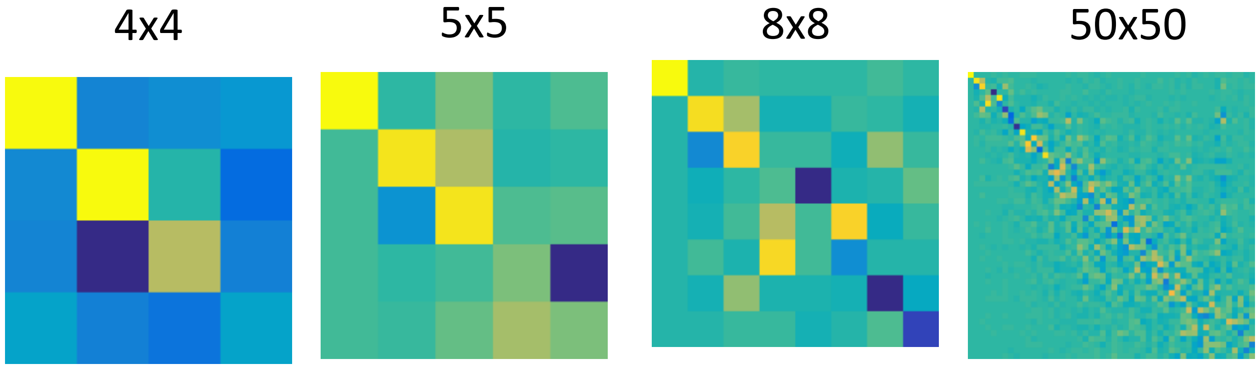
Informative Descriptors



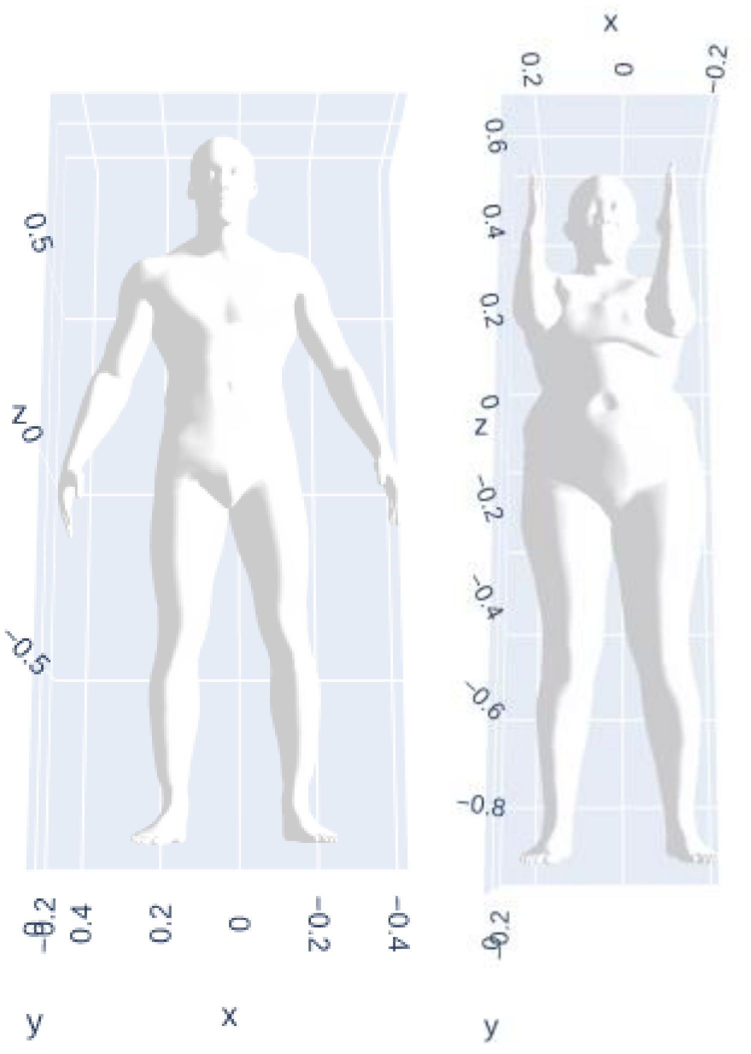
Regularizations



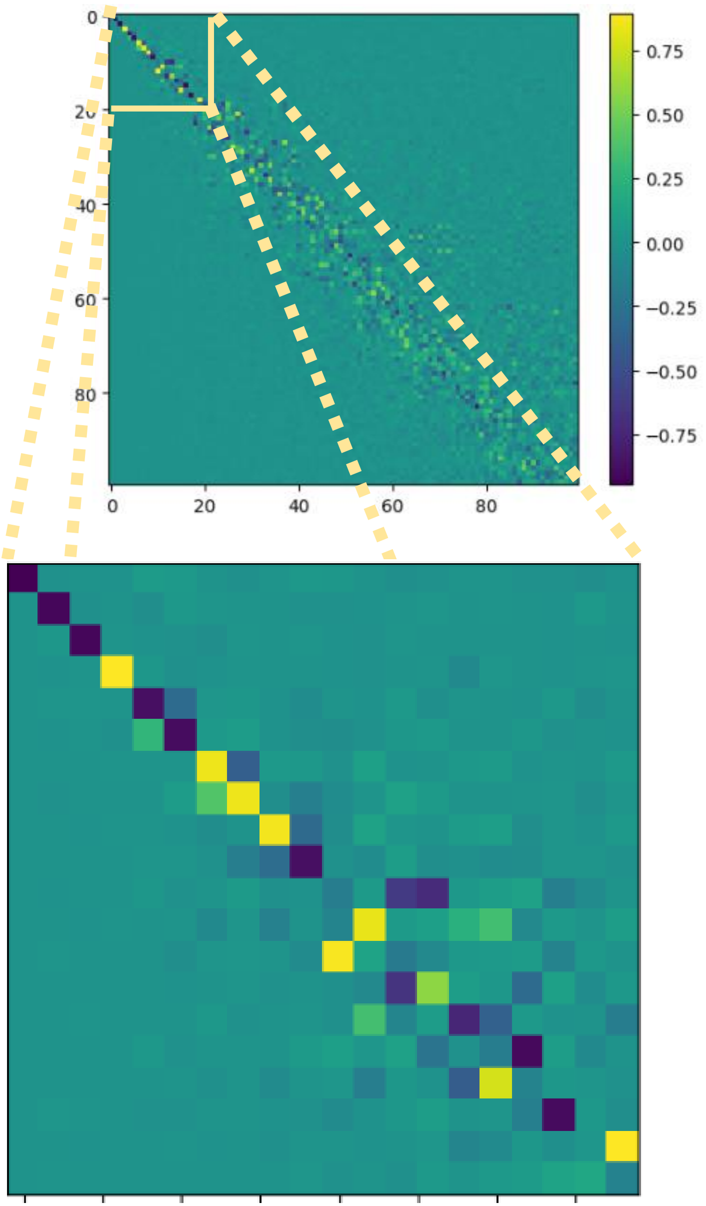
Refinements



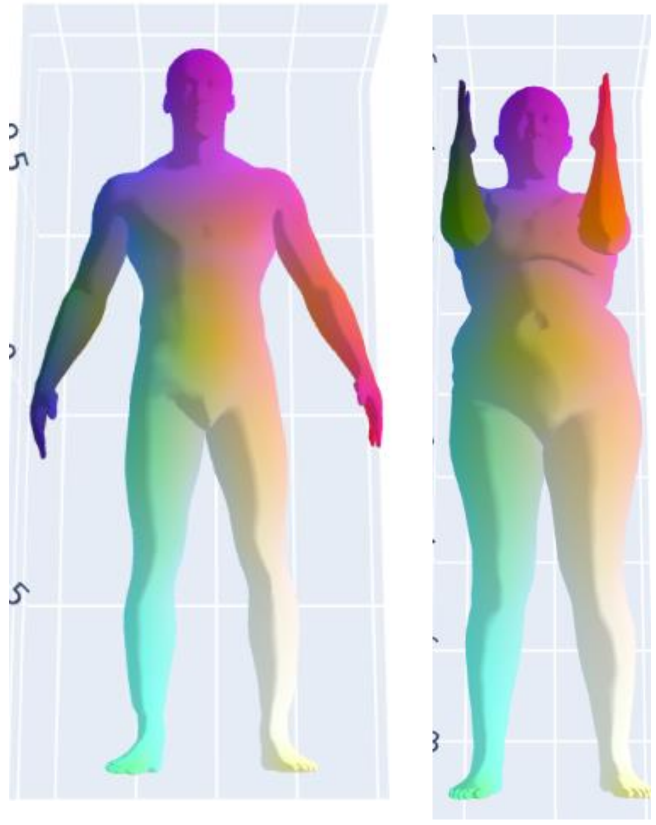
Input



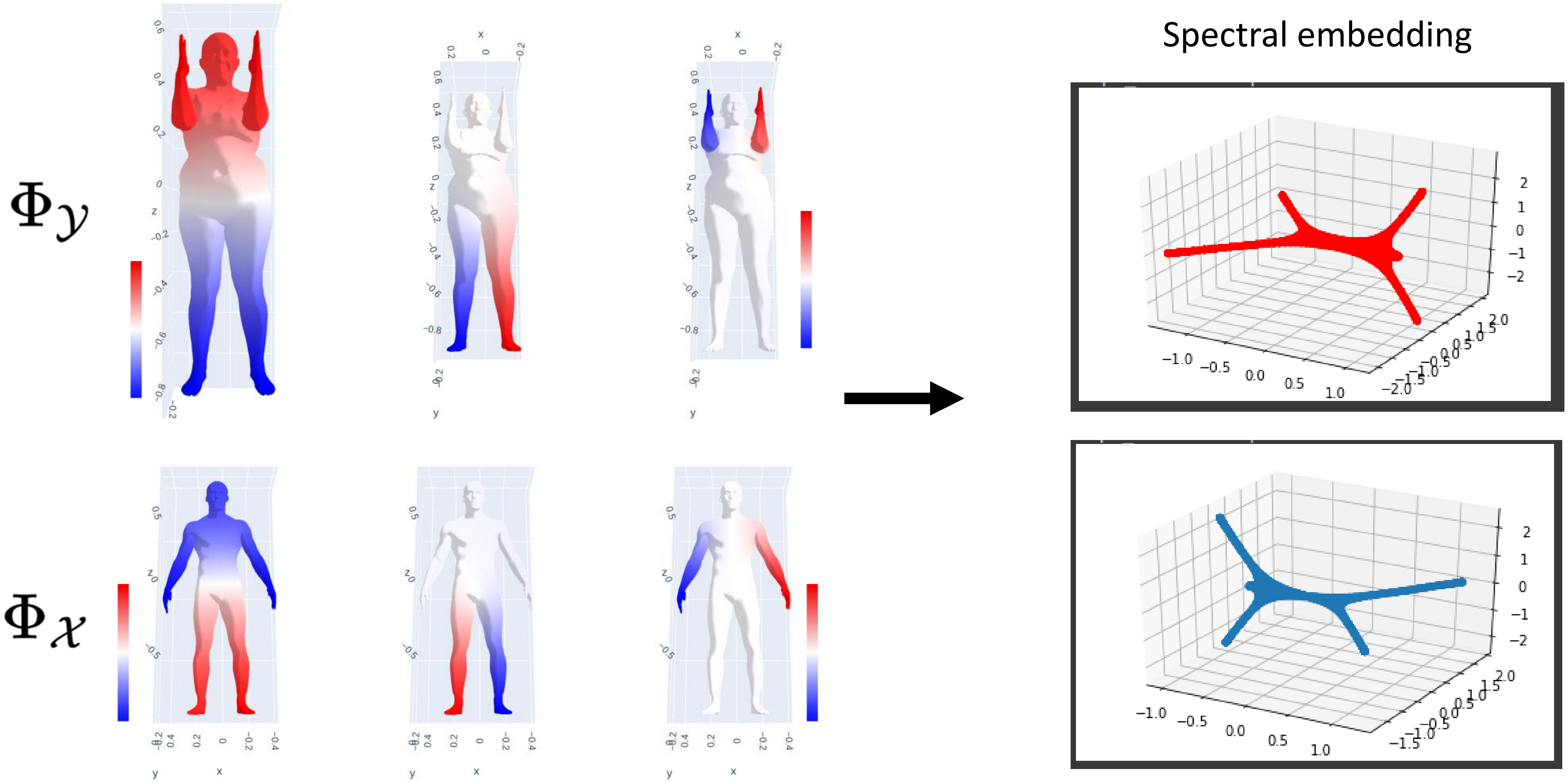
Functional Map



P2P Correspondence

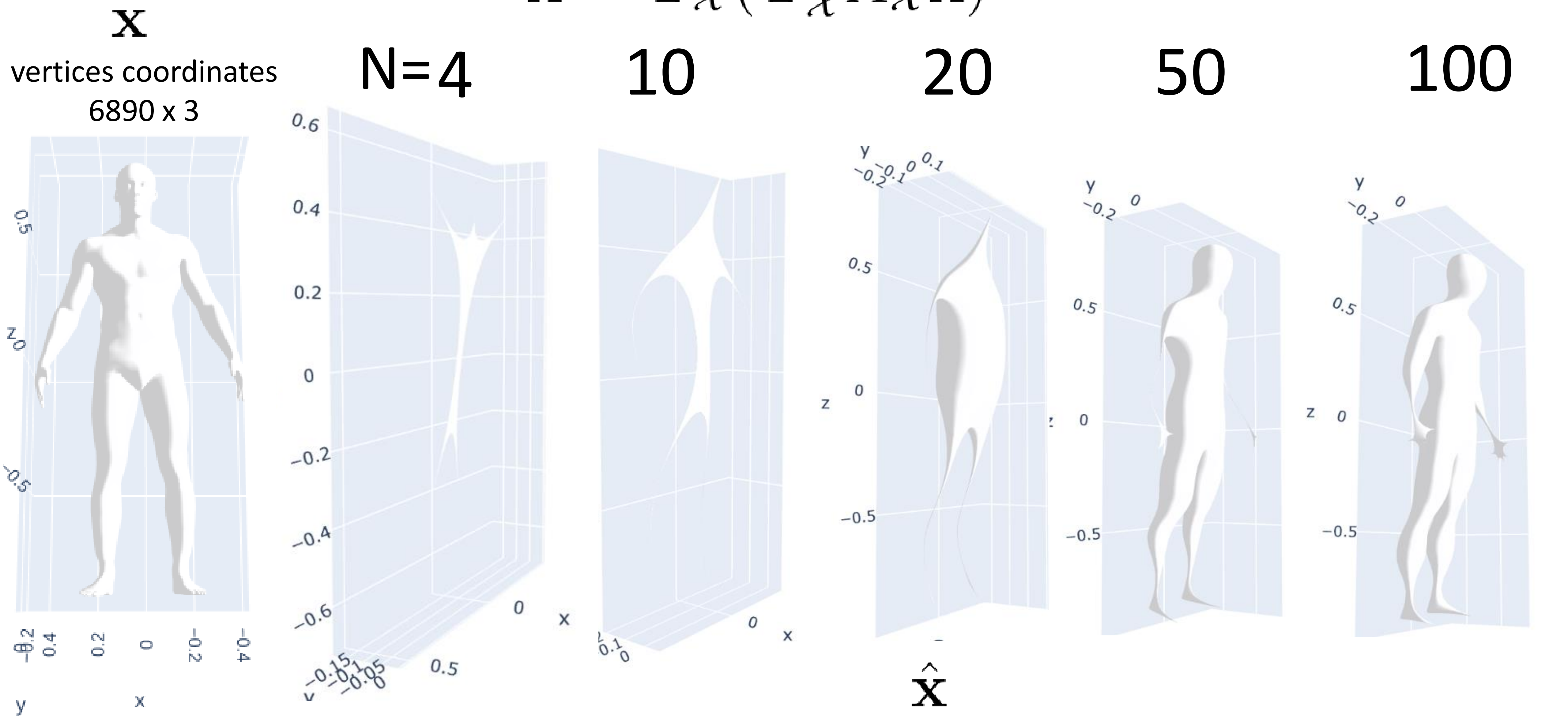


LBO Eigenfunctions

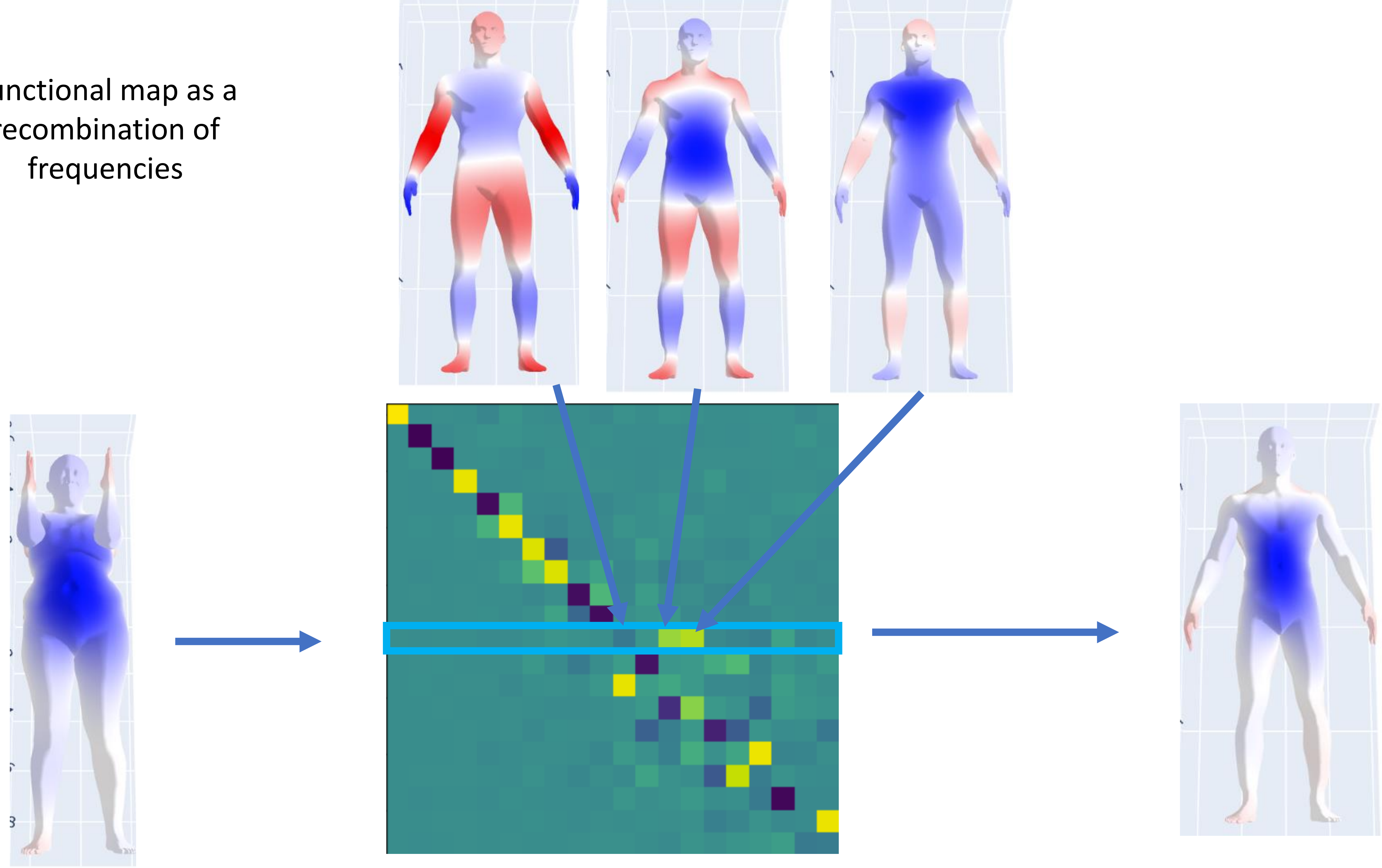


Coordinates representation in the the first n LBO basis vectors

$$\hat{\mathbf{x}} = \mathbf{\Phi}_{\mathcal{X}}(\mathbf{\Phi}_{\mathcal{X}}^{T} A_{\mathcal{X}} \mathbf{x})$$

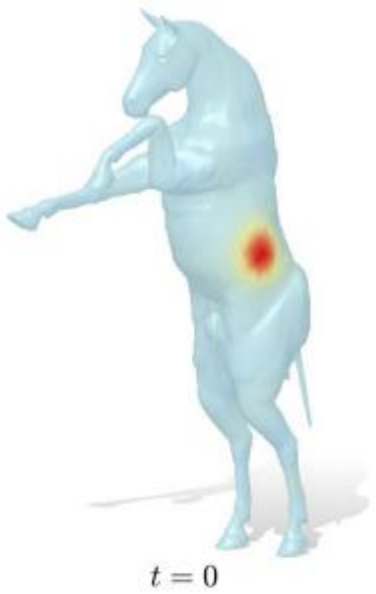


Functional map as a recombination of frequencies

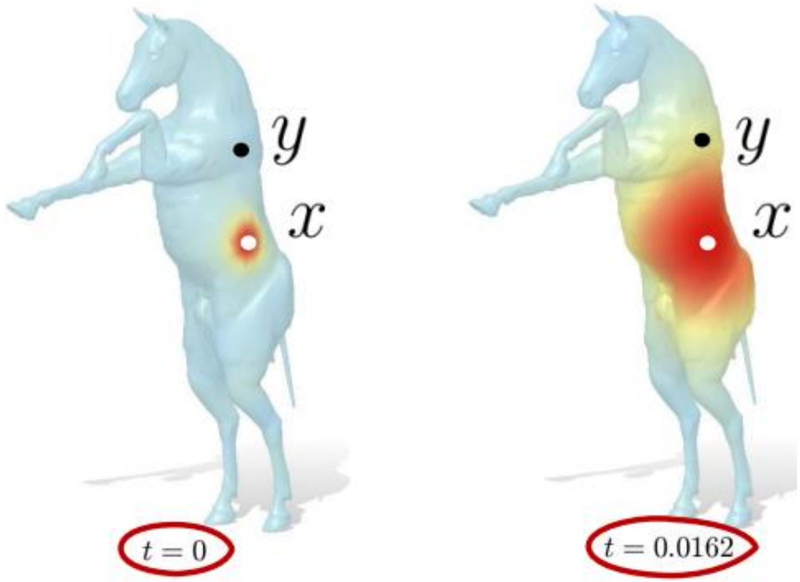


Informative Descriptors

Heat Diffusion
from a single point



Heat kernel
The heat transfer
between two points at
a given time t



$$k_t(x, y) = \sum_{l=0}^{\infty} e^{-t\lambda_l} \phi_l(x) \phi_l(y)$$

HKS(x)
How much heat
remains in the point x
at time t

$$k_t(x, x) = \sum_{l=0}^{\infty} e^{-t\lambda_l} \phi_l(x) \phi_l(x)$$

$$\boldsymbol{HKS}(x) = [k_{t_1}(x, x), k_{t_2}(x, x), \ldots, k_{t_Q}(x, x)] \quad t_1 < t_2 < \ldots t_Q \in \mathbb{R}$$

Sun, Ovsjanikov, Guibas, A Concise and Provably Informative Multi-Scale Signature Based on Heat Diffusion, SGP 2009

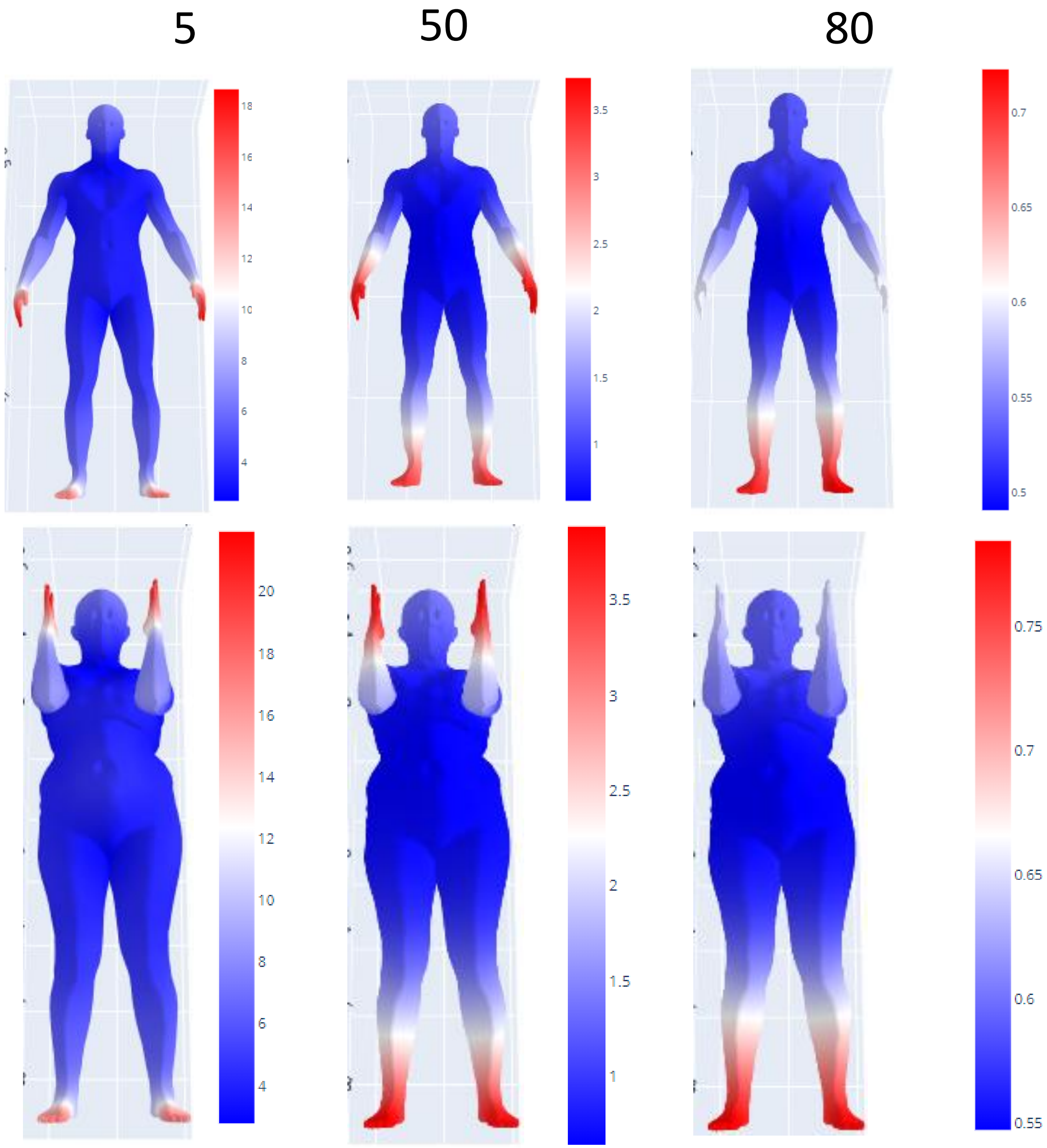
HKS

Pro

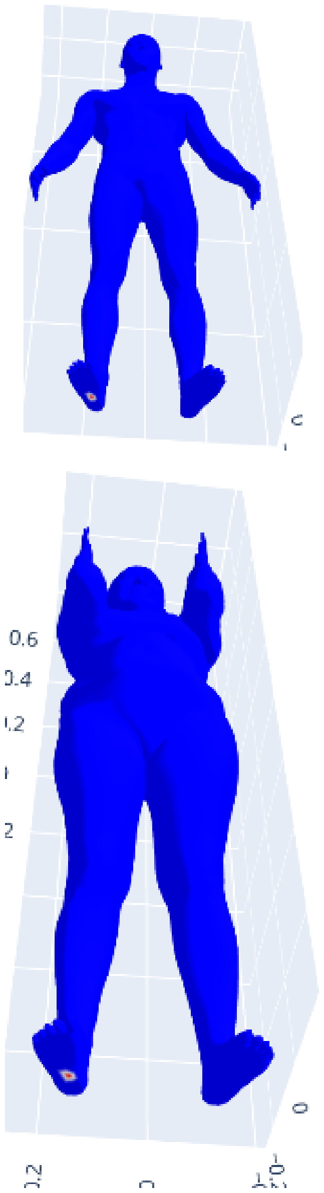
Descriptors are intrinsic and consistent

Con

They do not distinguish symmetries



Landmarks

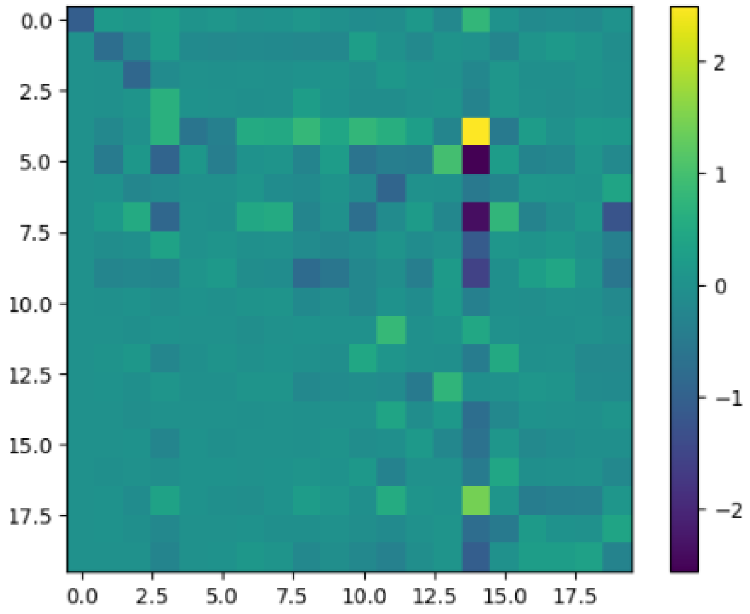


Optimization

$$\arg\min_{C} ||B - CA||$$

Descriptors

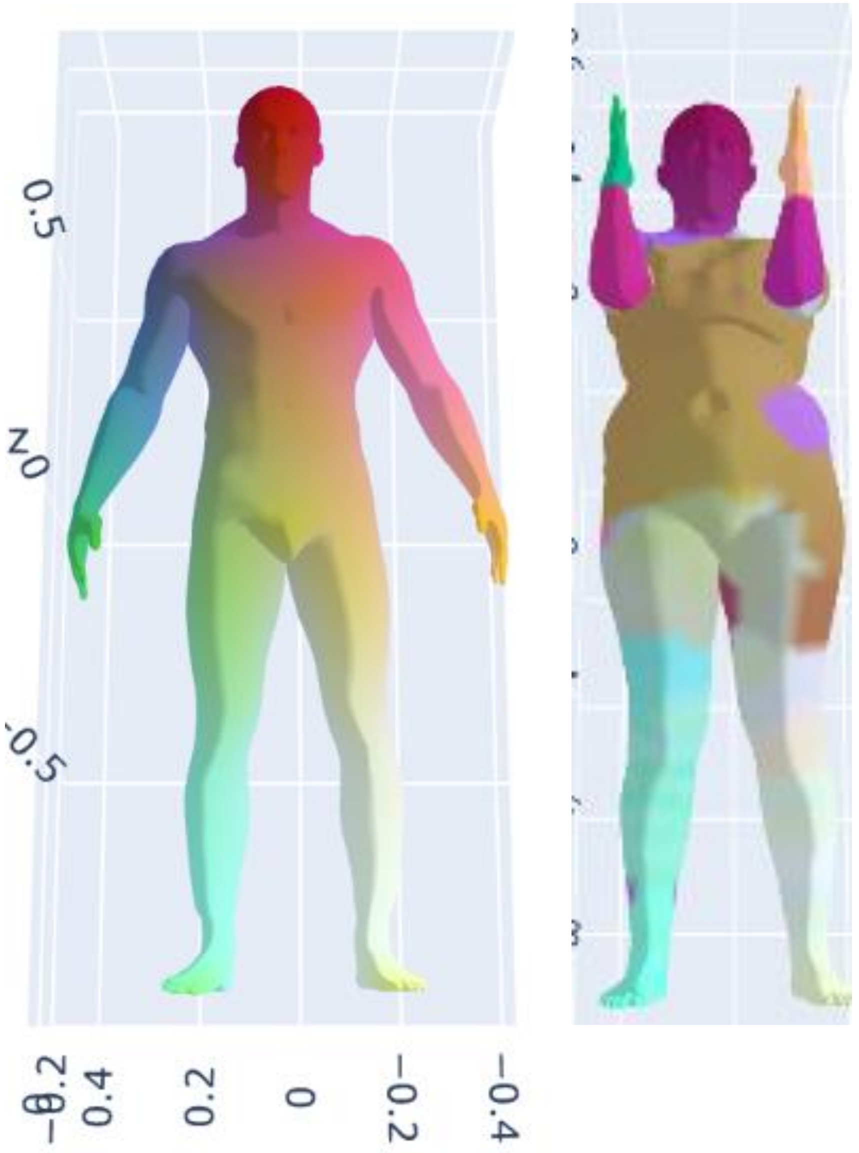
6 random landmarks +
100 HKS descriptors

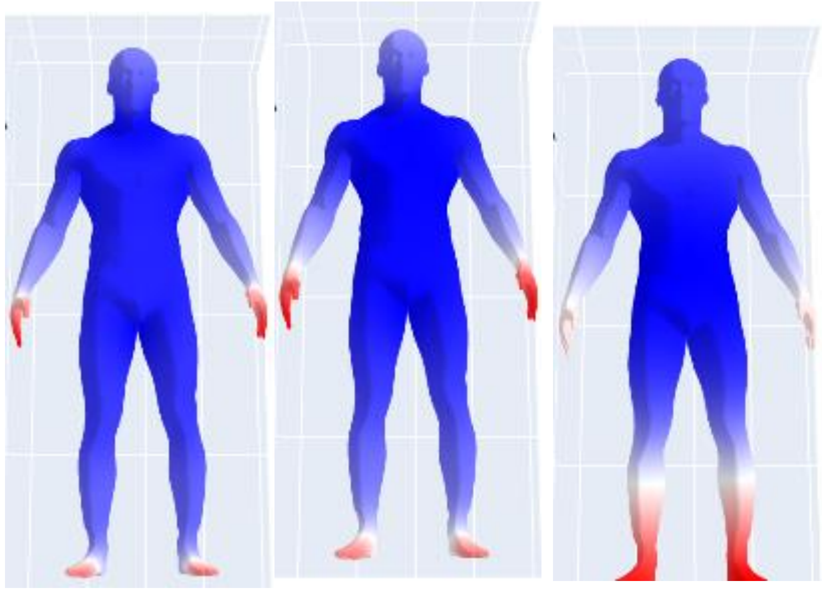


20 x 20
Functional maps

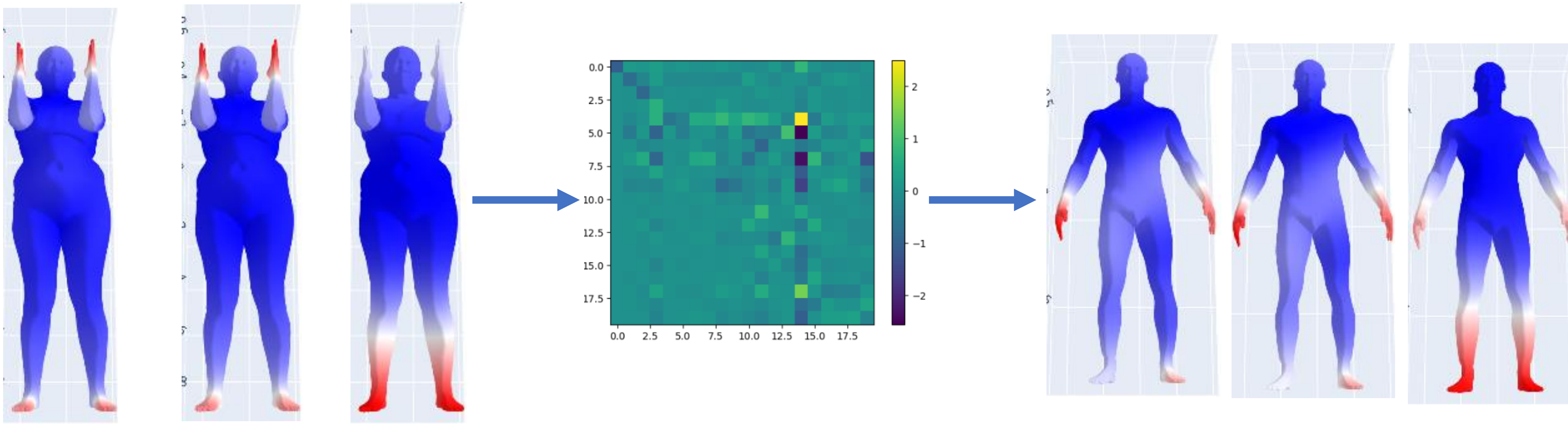
Transferring Delta
Functions

Induced
Correspondence





Descriptors Transfer



Regularization

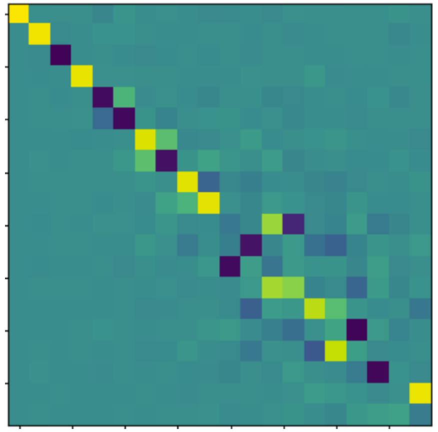
Commutativity with
the (eigenvalues of the) Laplacian

$$||\Delta_N C - C\Delta_M|| = 0$$

<u>Lemma</u>:

The mapping is isometric if and only if the functional map matrix commutes with the Laplacian

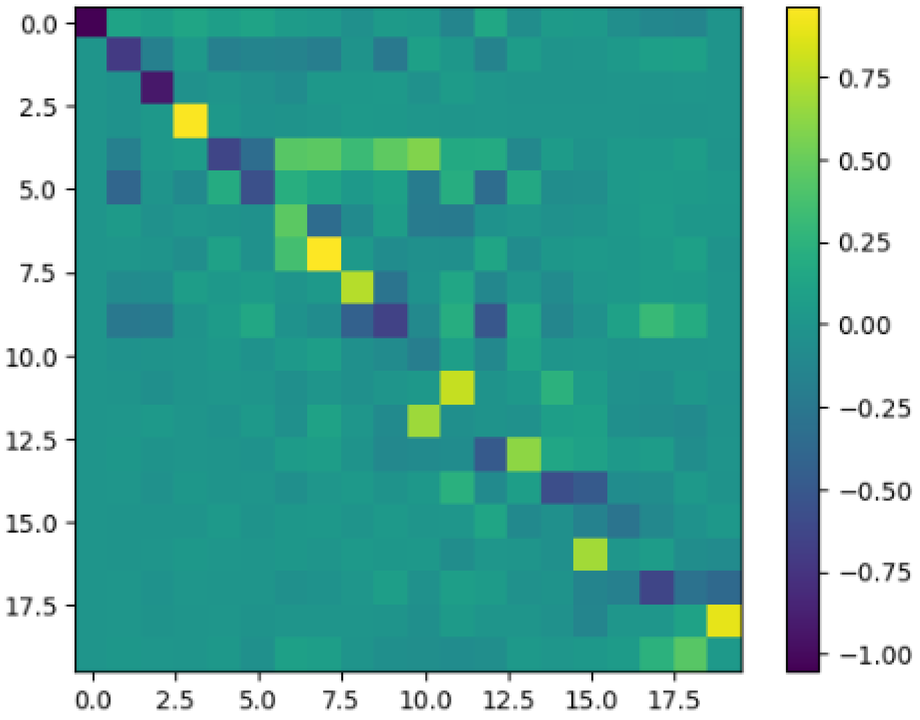
If the two shapes are near isometric -> the C should be diagonal



Optimization

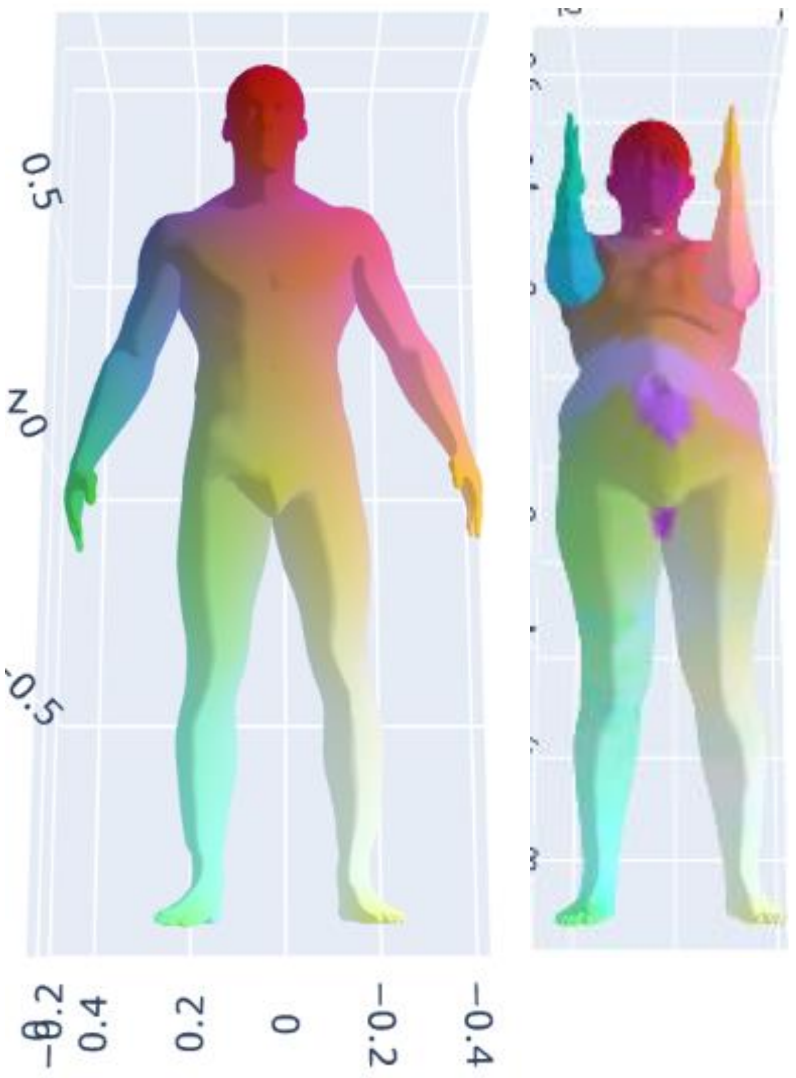
$$\arg\min_{C} ||B - CA|| + reg(C)$$

6 landmarks + 100 HKS descriptors
+ Commutativity with Laplacian

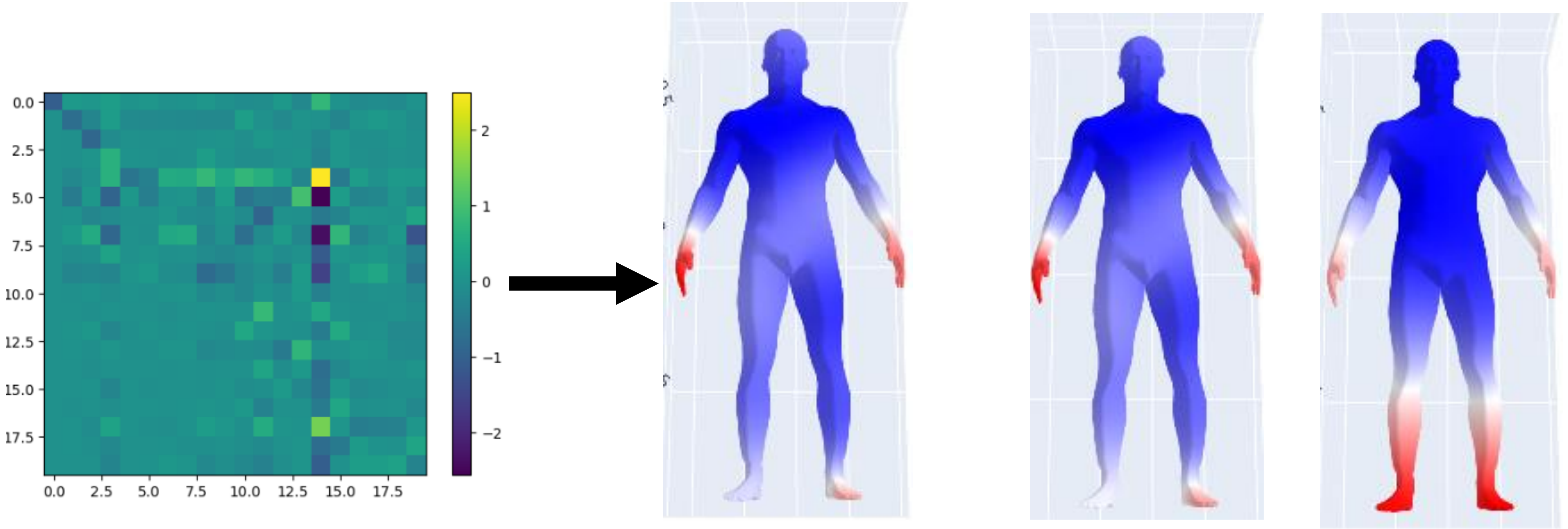


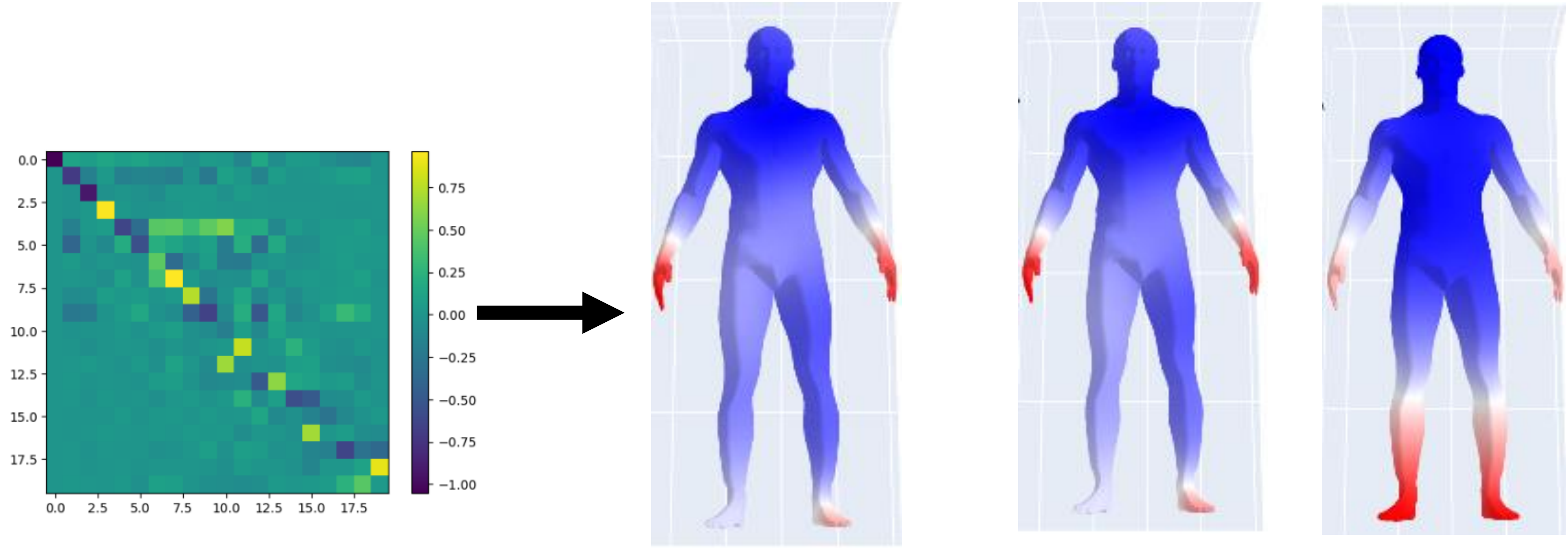
20 x 20
Functional maps

Transferring Delta
Functions

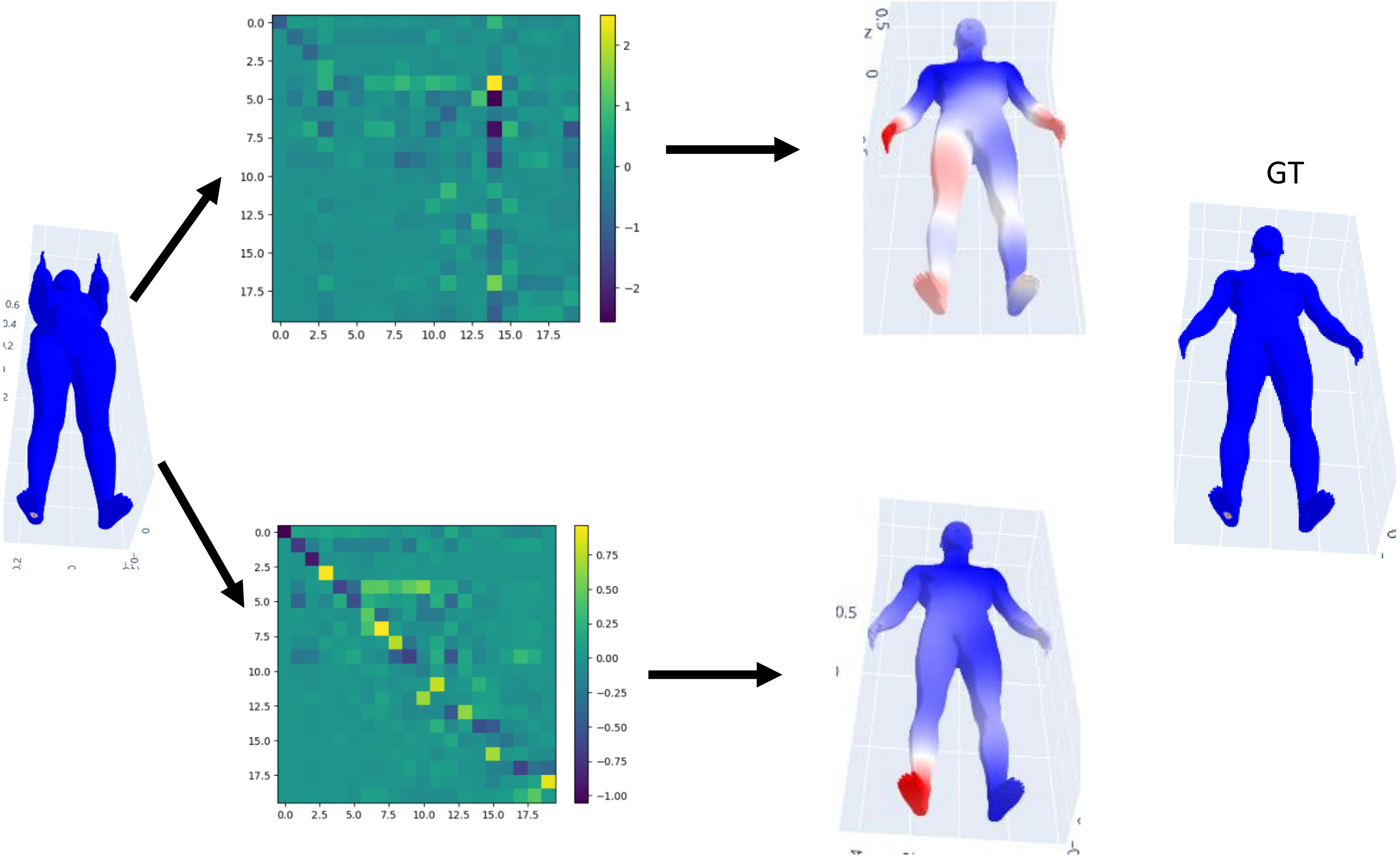


Descriptors transfer
is almost the same





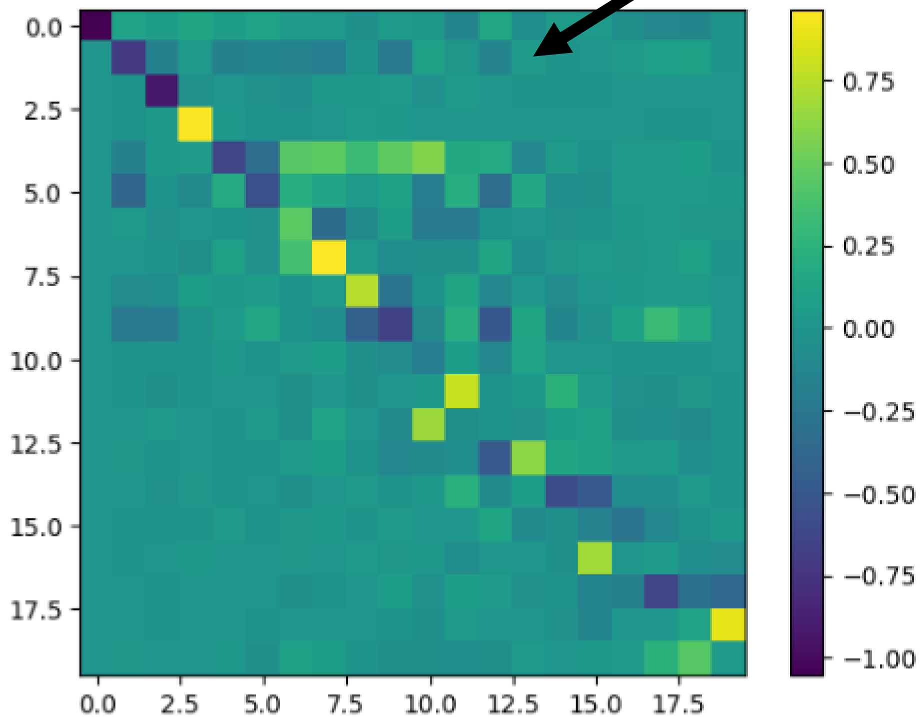
Landmark transfer is much improved



Optimization

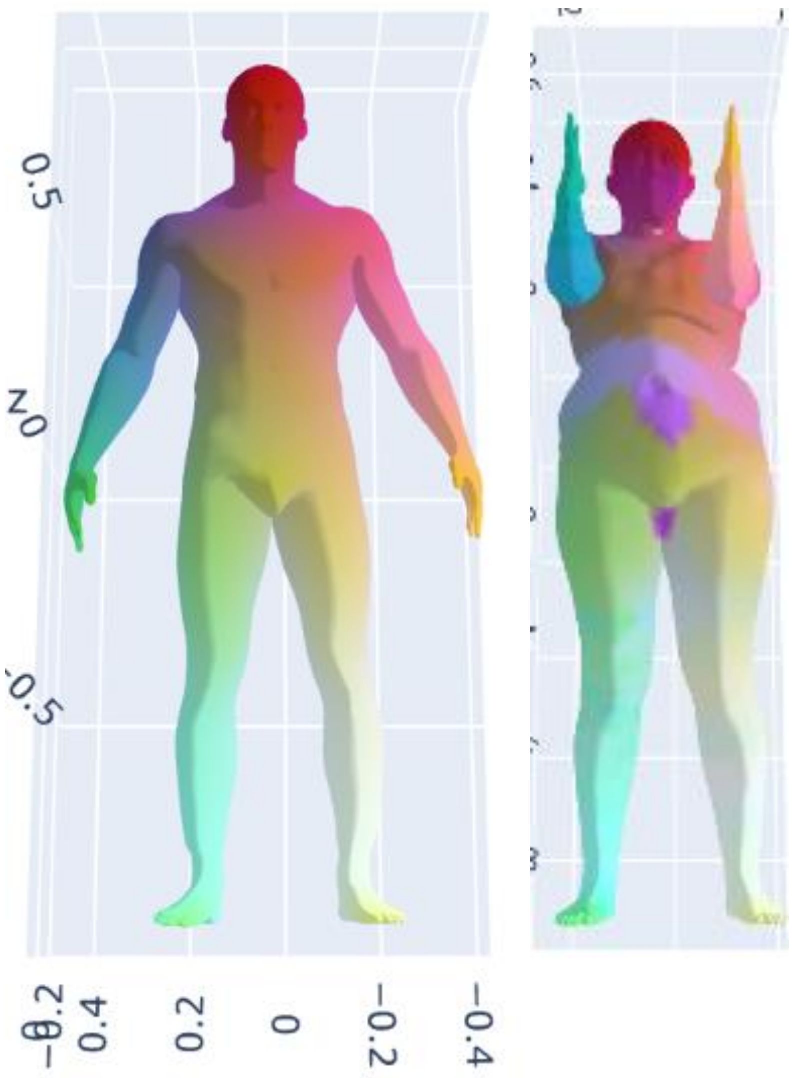
$$\arg\min_{C} ||B - CA|| + reg(C)$$

6 landmarks + 100 HKS descriptors
+ Commutativity with Laplacian



20 x 20
Functional maps

Transferring Delta
Functions



Refinement

<u>Lemma</u>

The mapping is locally volume preserving, if and only if the functional map is orthonormal

$$\|\mathbf{C}^T\mathbf{C}\| = \mathbf{I}$$

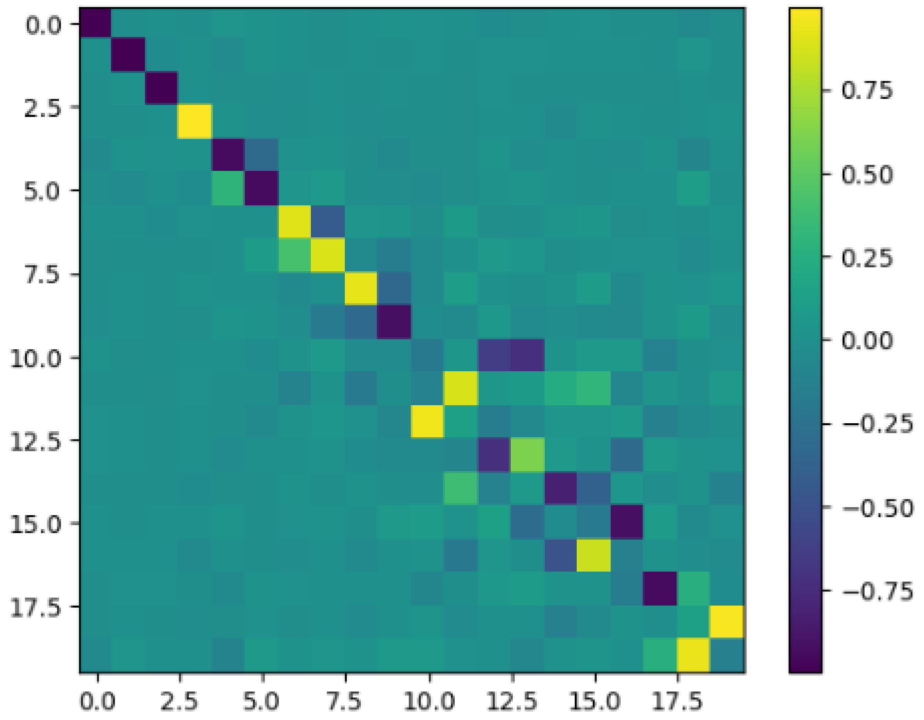
<u>Idea</u>

Finding the best orthonormal matrix that aligns the two spectral embeddings

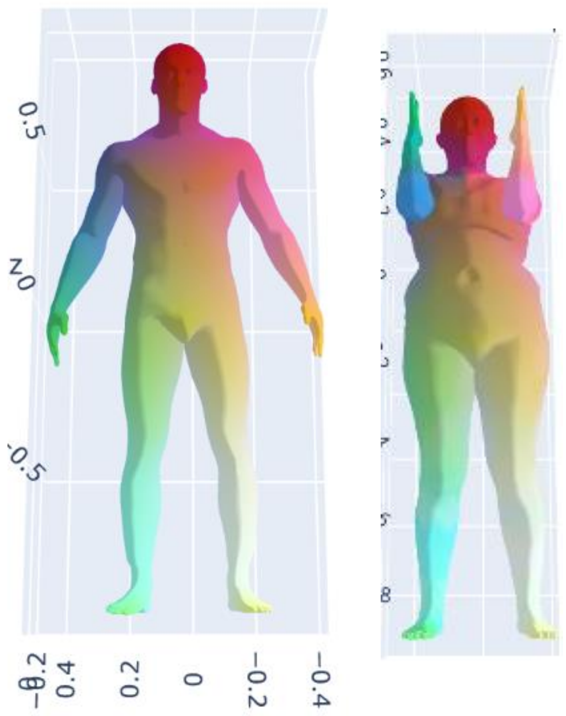
Iterative **Closest** Point

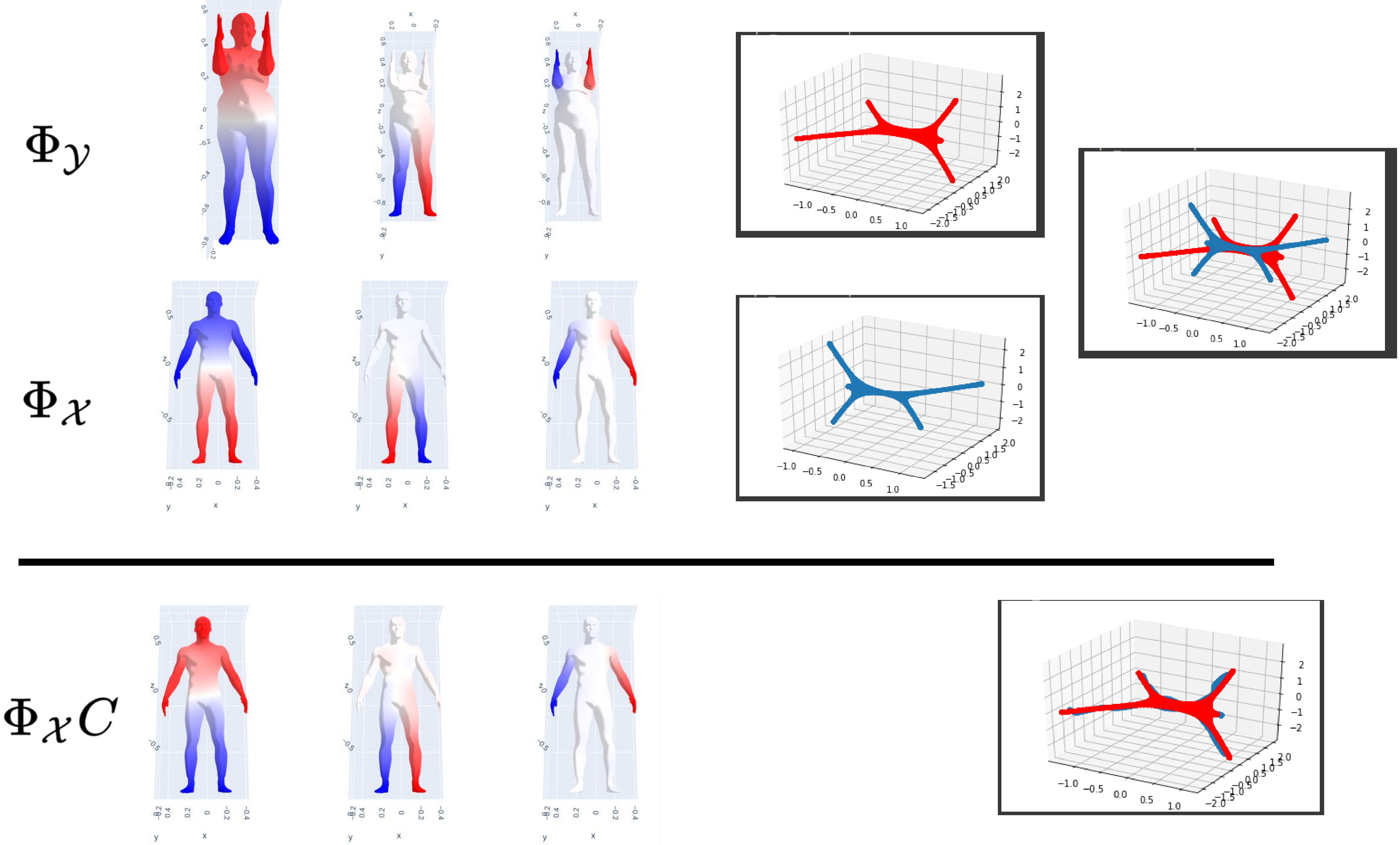
(ICP)

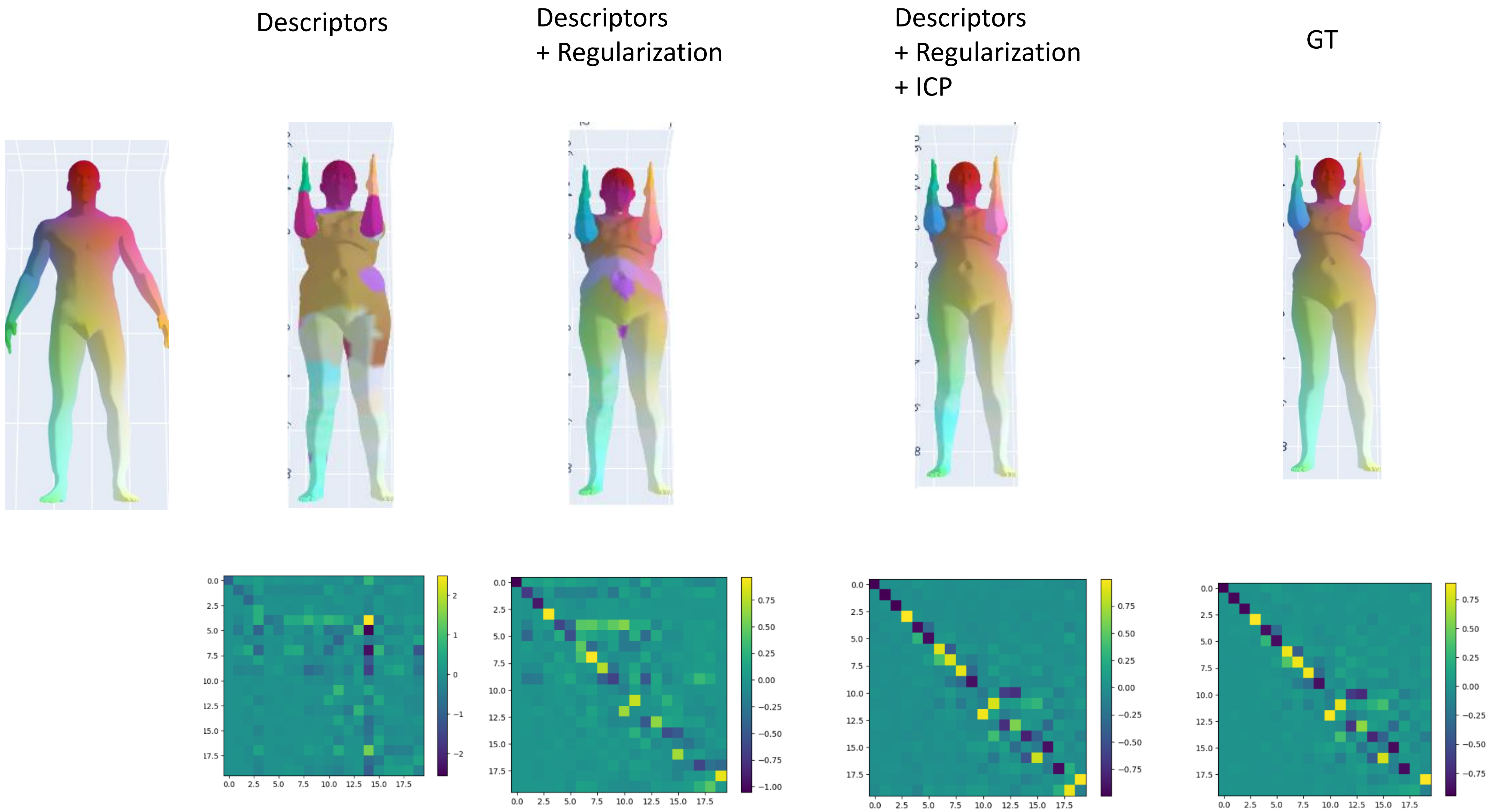
6 landmarks + 100 HKS descriptors
+ Commutativity with Laplacian
+ ICP



Transferring Delta Functions

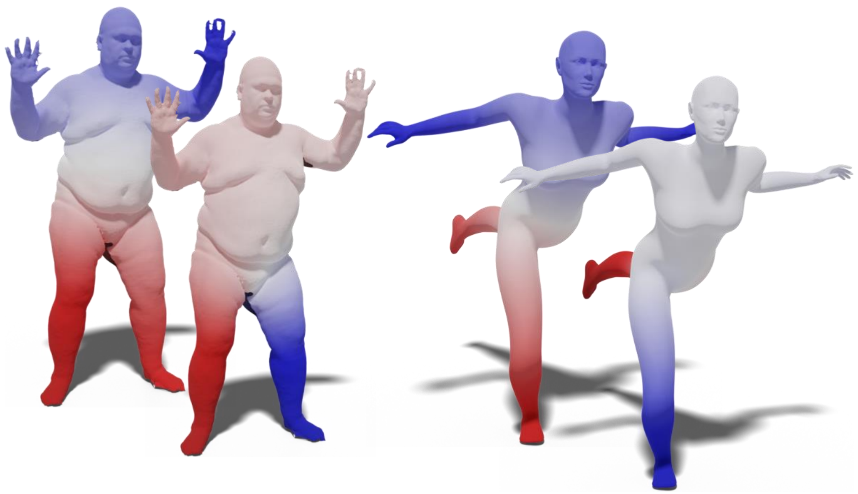



$\Phi_{\mathcal{Y}}$
$\Phi_{\mathcal{X}}$
$\Phi_{\mathcal{X}}C$

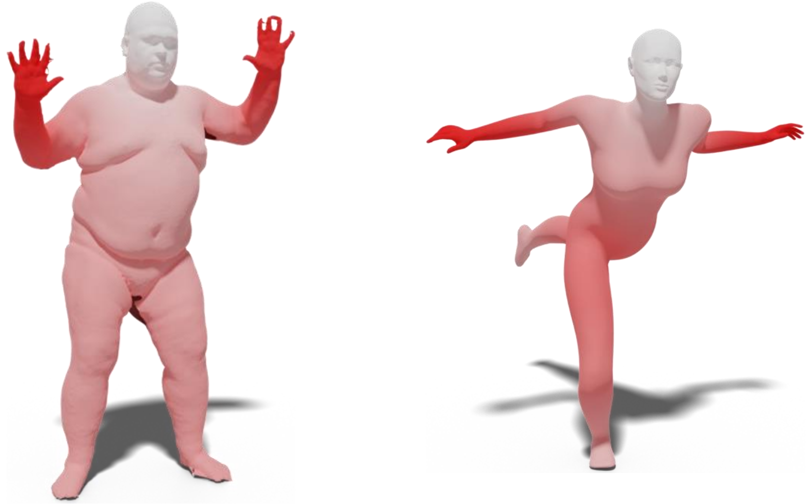

Descriptors
Descriptors
+ Regularization
Descriptors
+ Regularization
+ ICP
GT

Key Ingredients

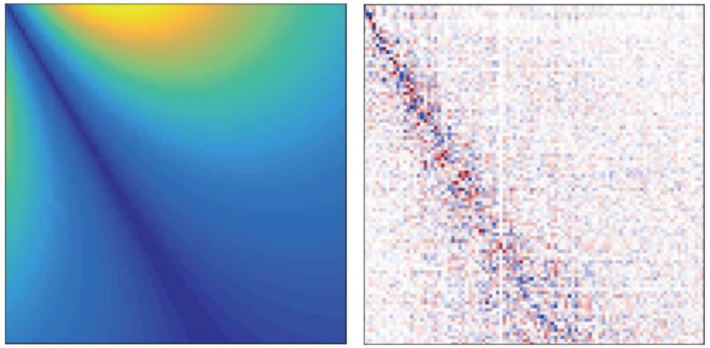
Choice of the basis



Informative Descriptors

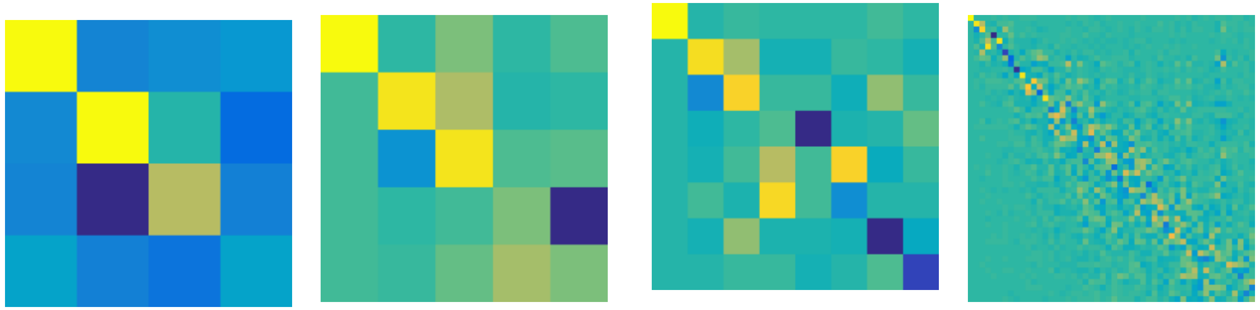


Regularizations

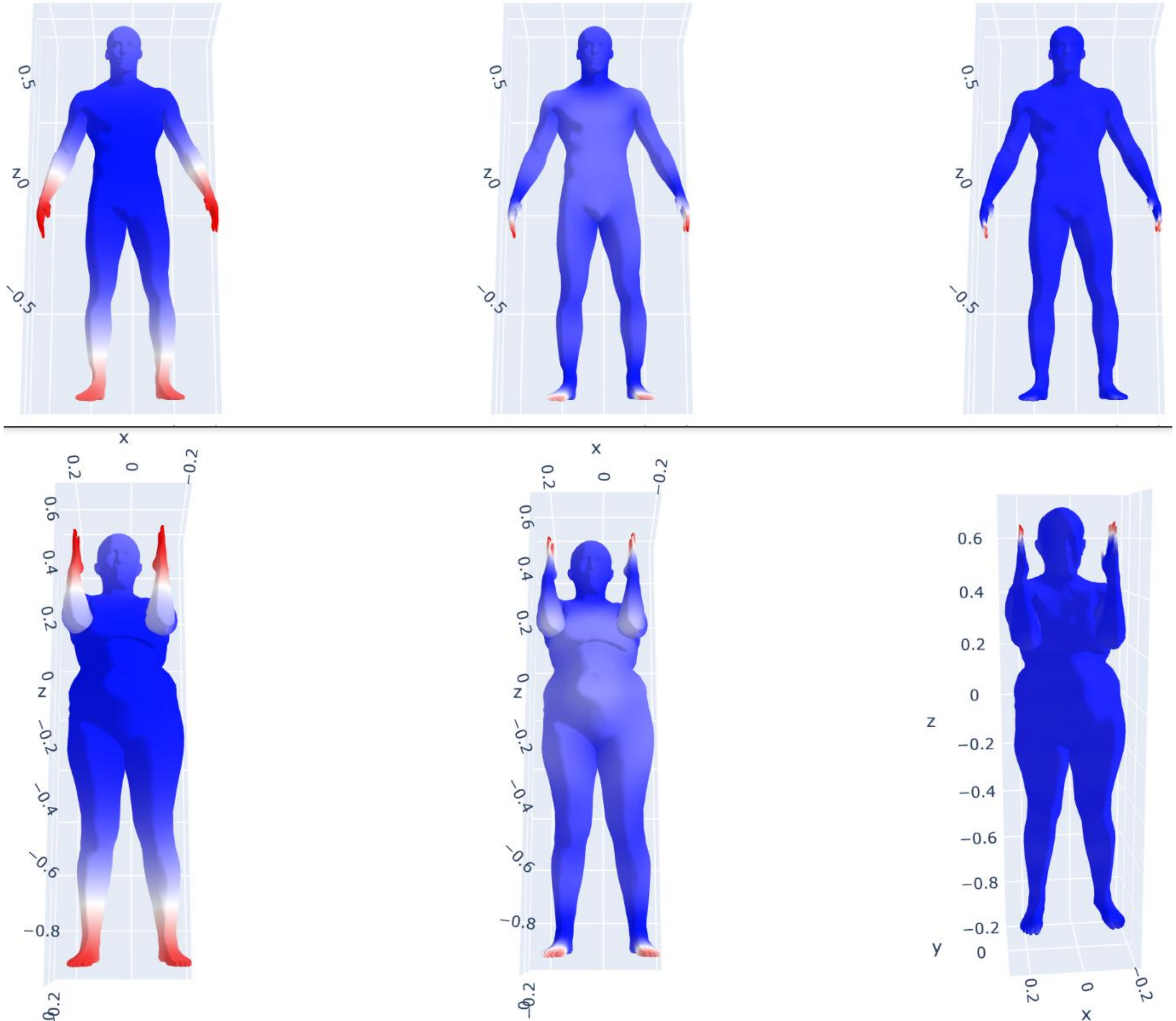


Refinements

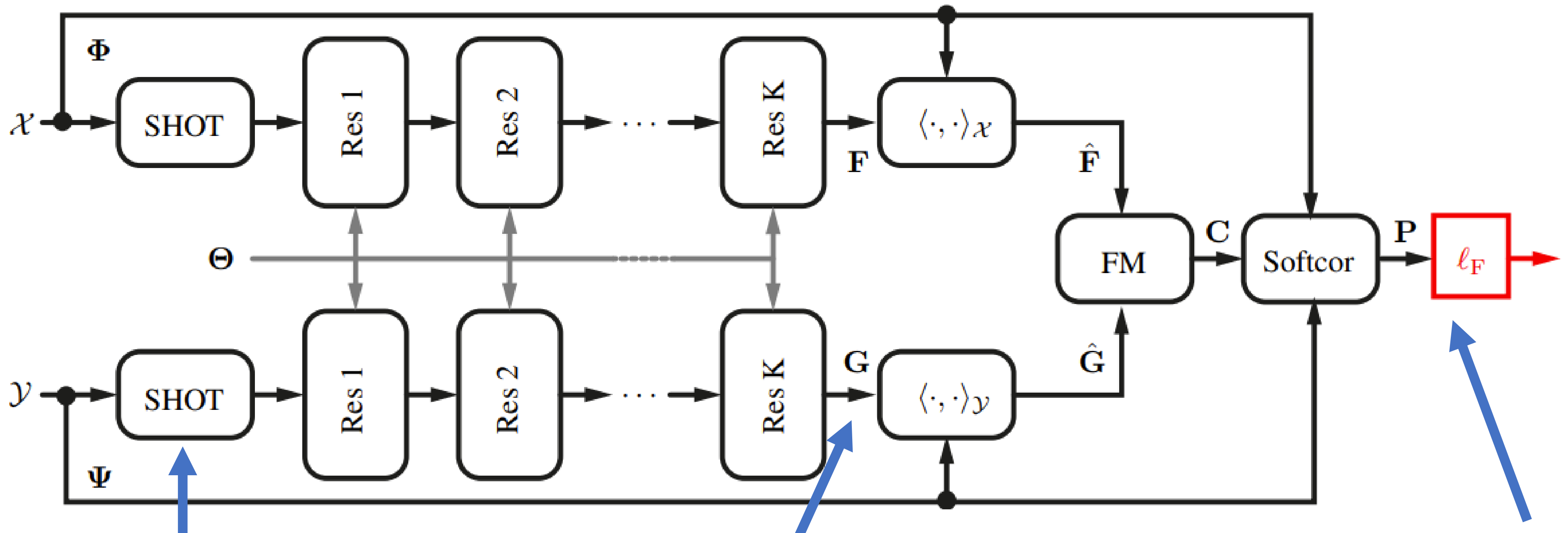
4x4 5x5 8x8 50x50



Informative Descriptors



Siamese Residual Network



Input:
Descriptors

Output:
Refined descriptors

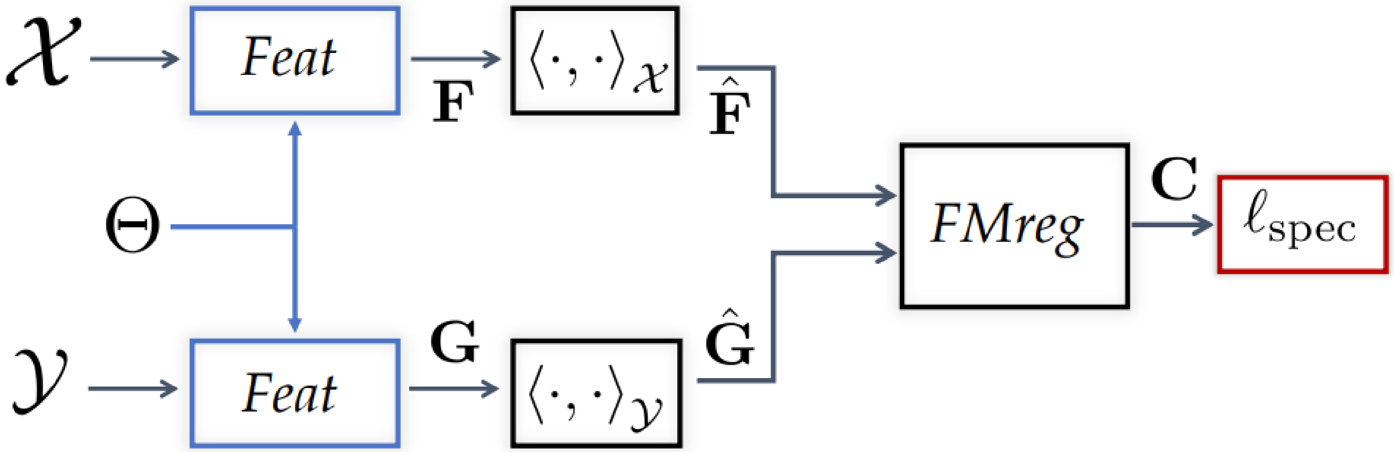
Loss at training time:
A soft correspodence matrix,
Multiplied by the geodesic distance

$$\ell_F = \|\mathbf{P} \circ \mathbf{D}_{\mathcal{Y}}\|_F$$

$$\mathbf{P} = |\mathbf{\Psi} \mathbf{C} \mathbf{\Phi}^\top \mathbf{A}|^{\wedge}$$

Litany, Remez, Rodolà, Bronstein, Bronstein; Deep Functional Maps: Structured Prediction for Dense Shape Correspondence; ICCV17

Point cloud feature extractor



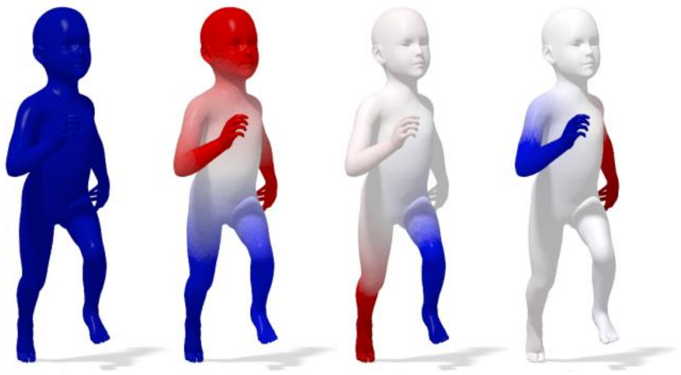
$$l_{\text{spec}} = \left\|\mathbf{C} - \mathbf{C}^{gt}\right\|^2$$

Loss directly on the functional map C

Donati, Sharma, Ovsjanikov; Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence; CVPR2020

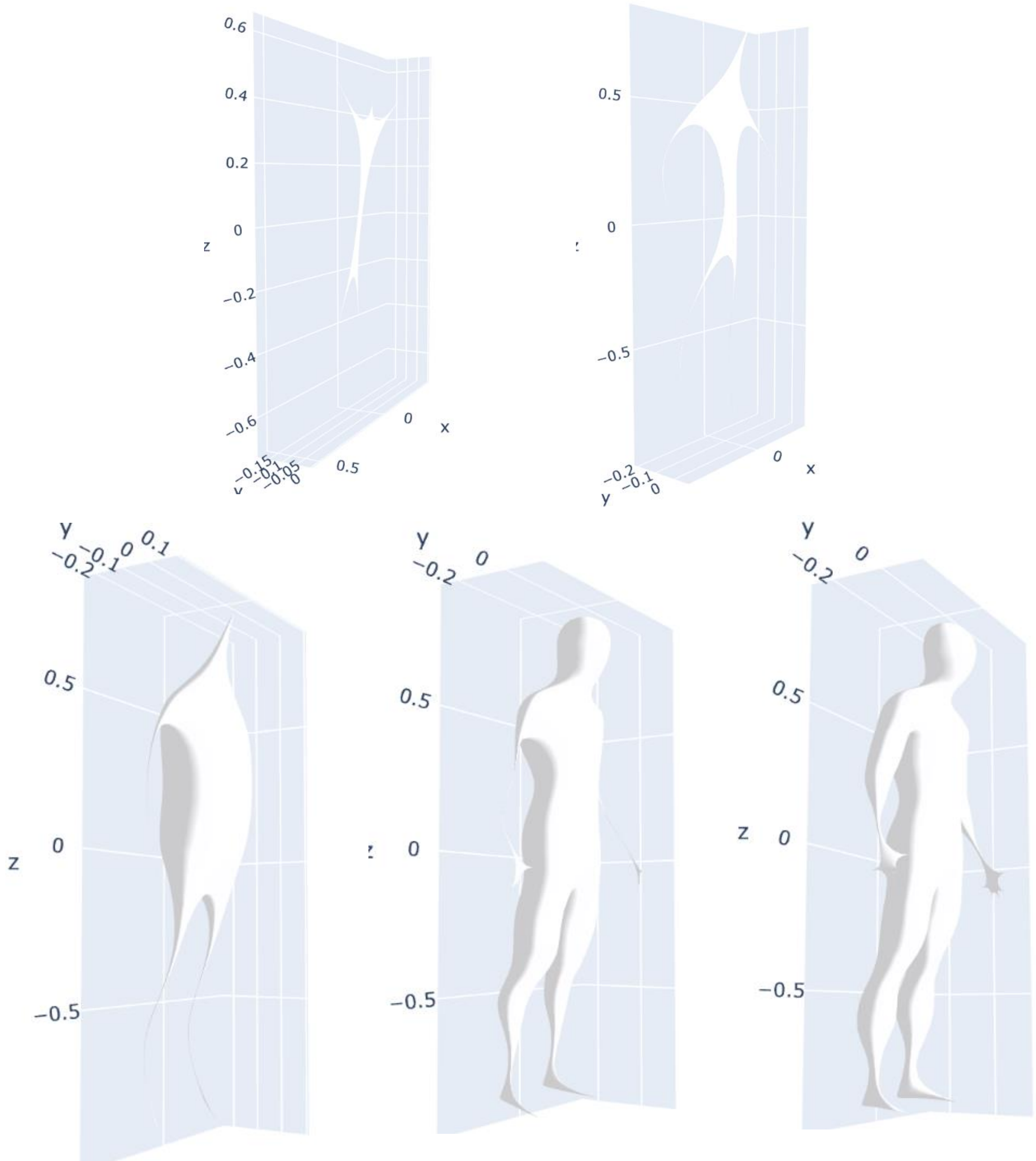
LBO Problems

Global



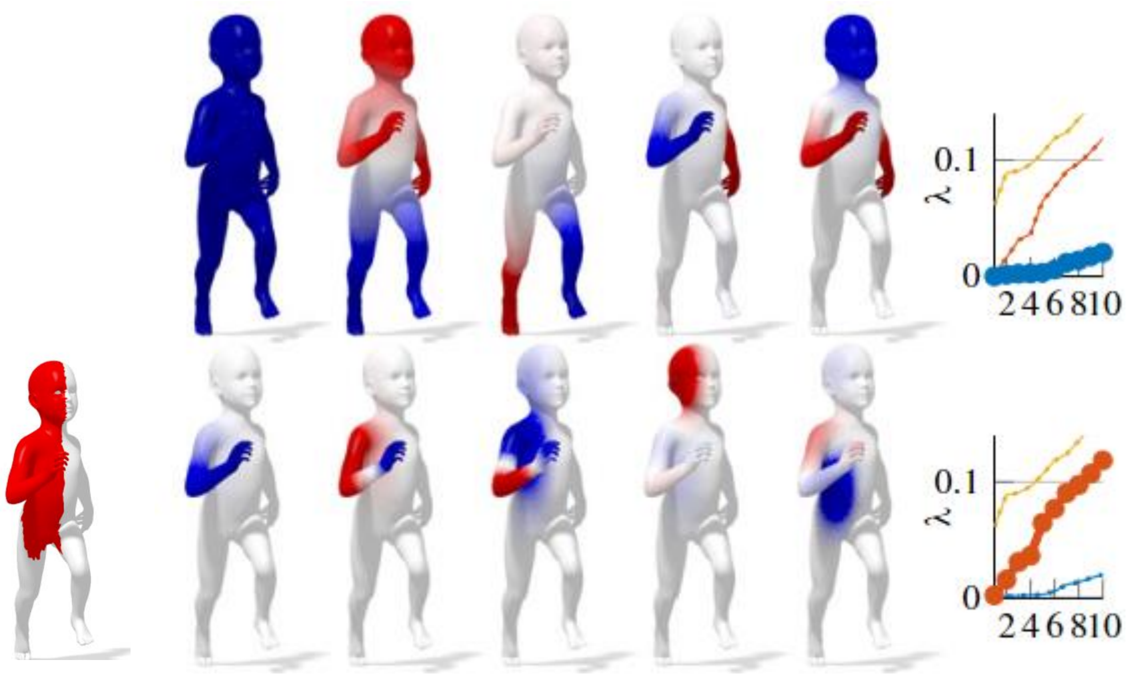
Local region

Low-Pass



Locality – Localized Manifold Harmonics

LBO Global



LBO Local

LMH

Local + Orthogonal

$$\mathcal{E}(\psi_j) = \mathcal{E}_S(\psi_j) + \mu_R \mathcal{E}_R(\psi_j) + \mu_\perp \mathcal{E}_\perp(\psi_j),$$

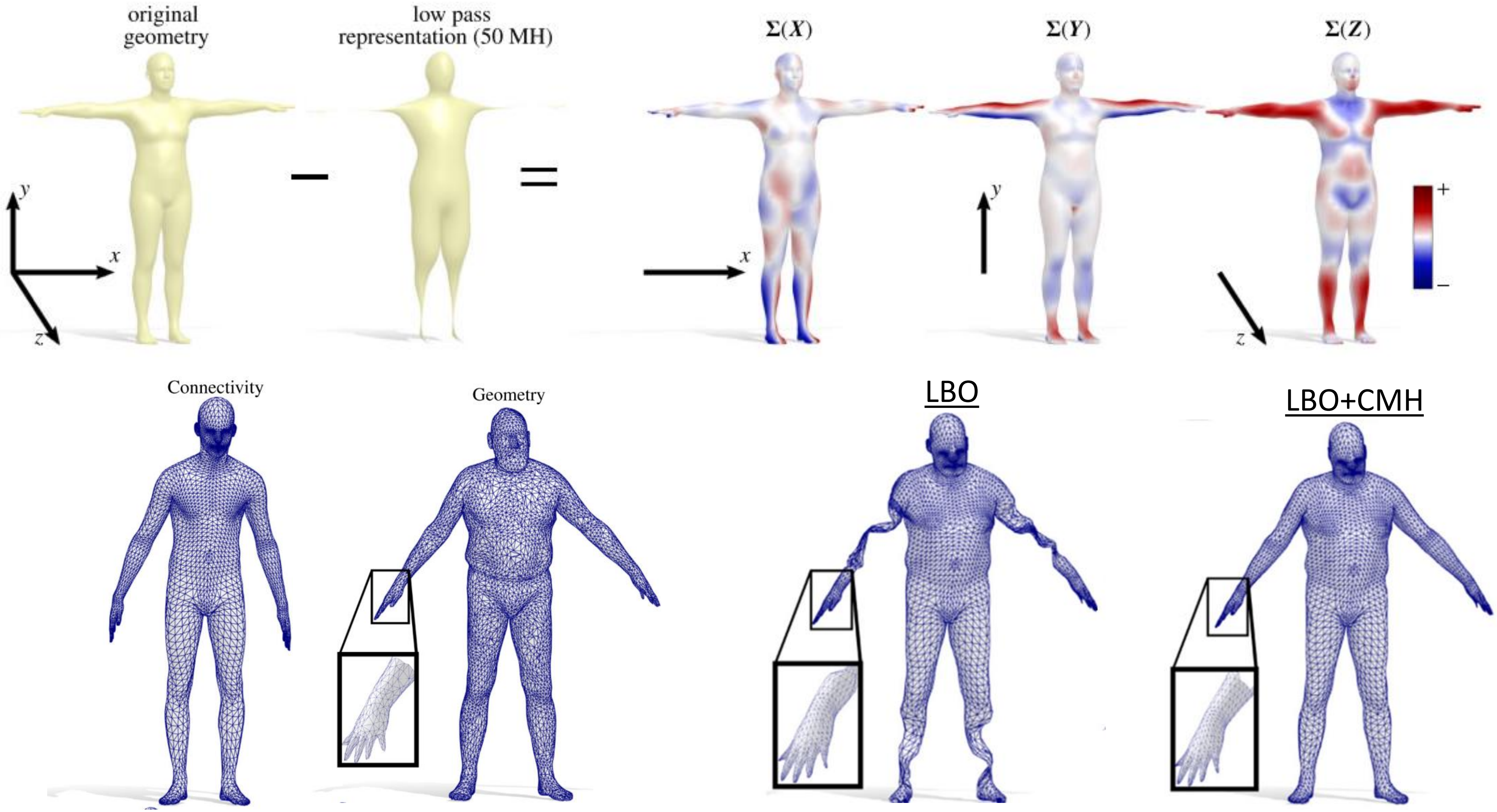
Smooth

Local

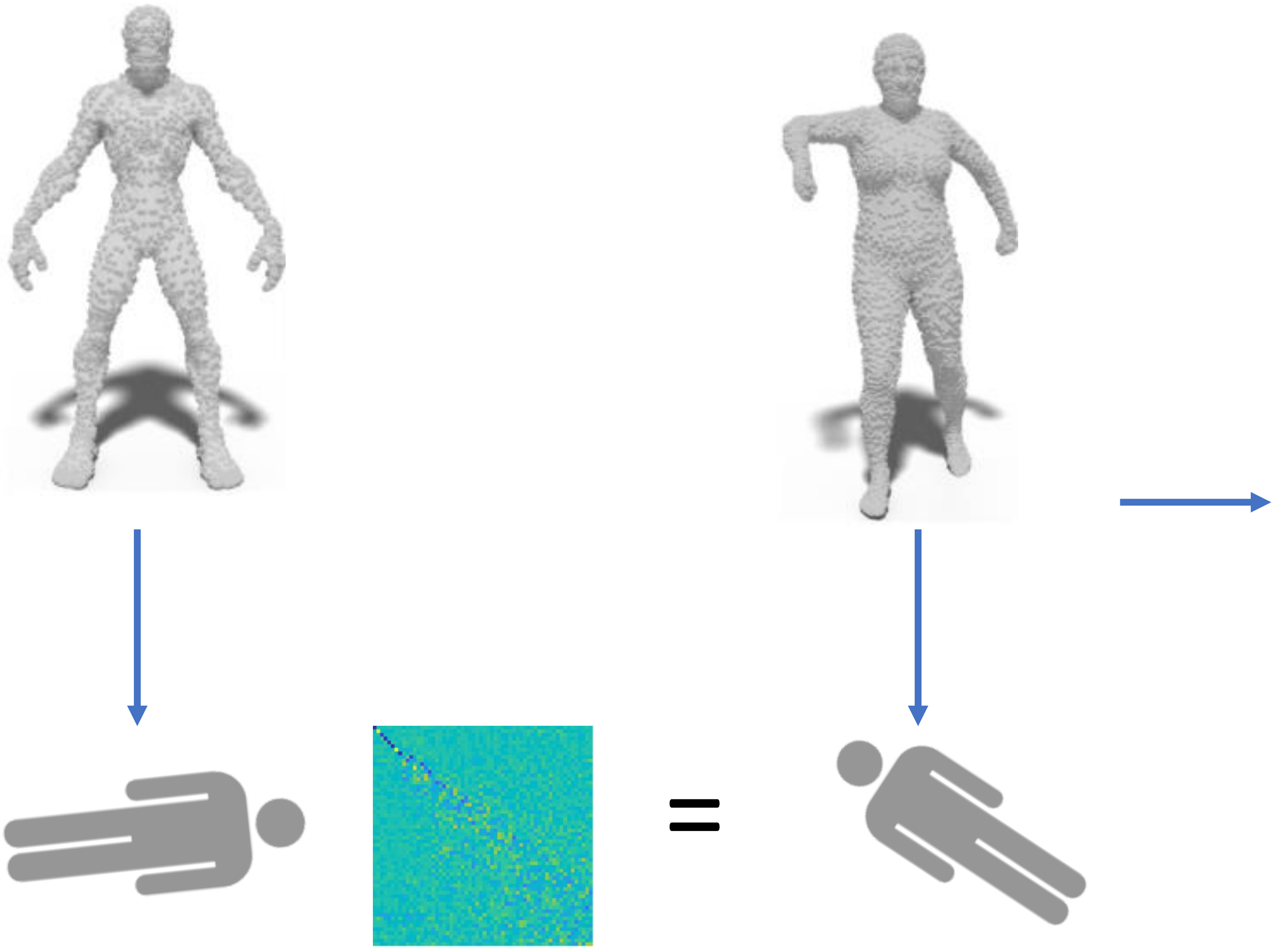
Orthogonal

Melzi, Rodolà, Castellani, Bronstein; Localized Manifold Harmonics for Spectral Shape Analysis, 2017 CGF

Low-pass – Coordinates Manifold Harmonics

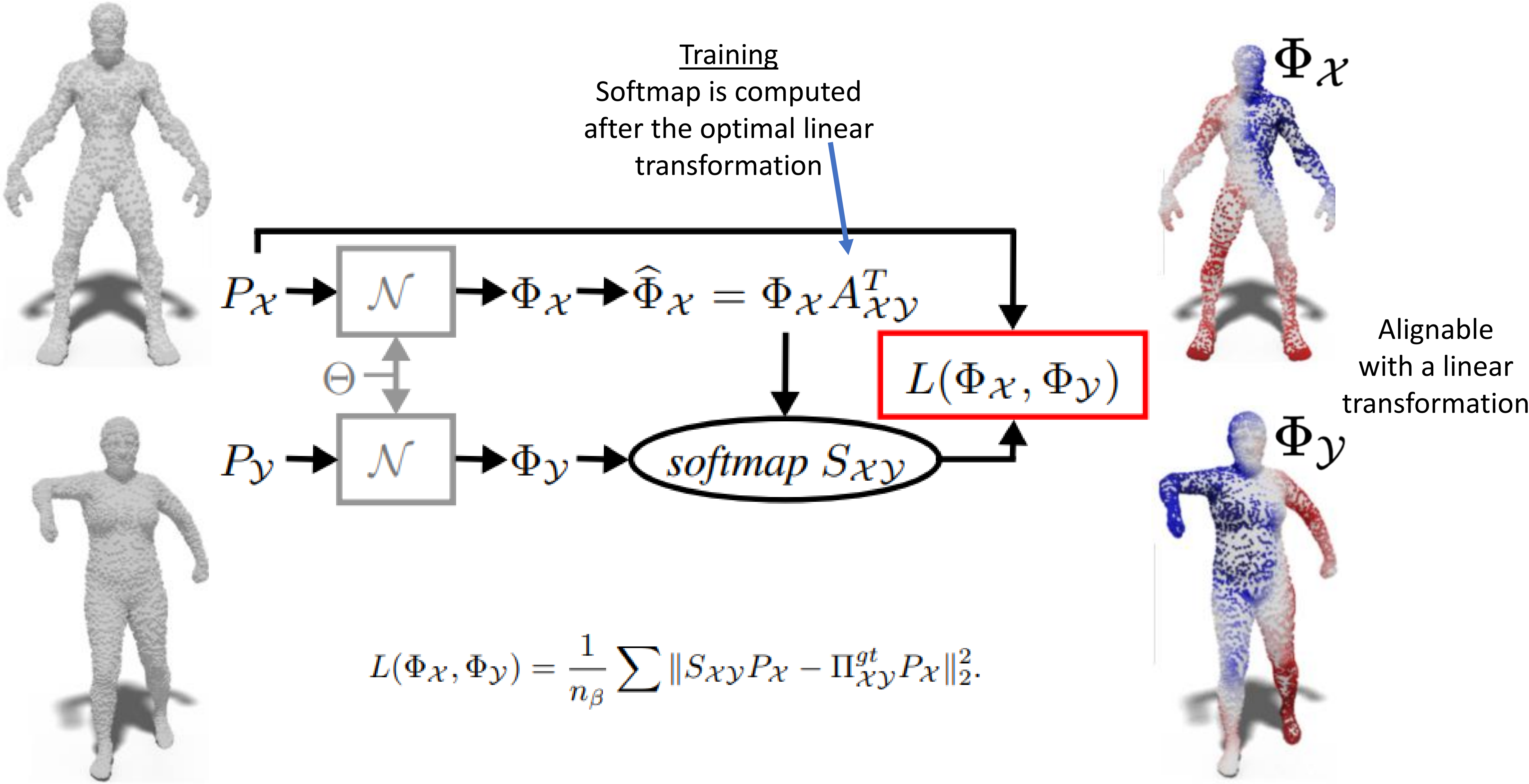


Melzi, Marin, Musoni, Bardon, Tarini, Castellani; Intrinsic/extrinsic embedding for functional remeshing of 3D shapes; C&G 2020



We look for a convenient embedding, s.t. matching can be solved by a linear transformation

Basis Learning – Linearly-Invariant Embedding

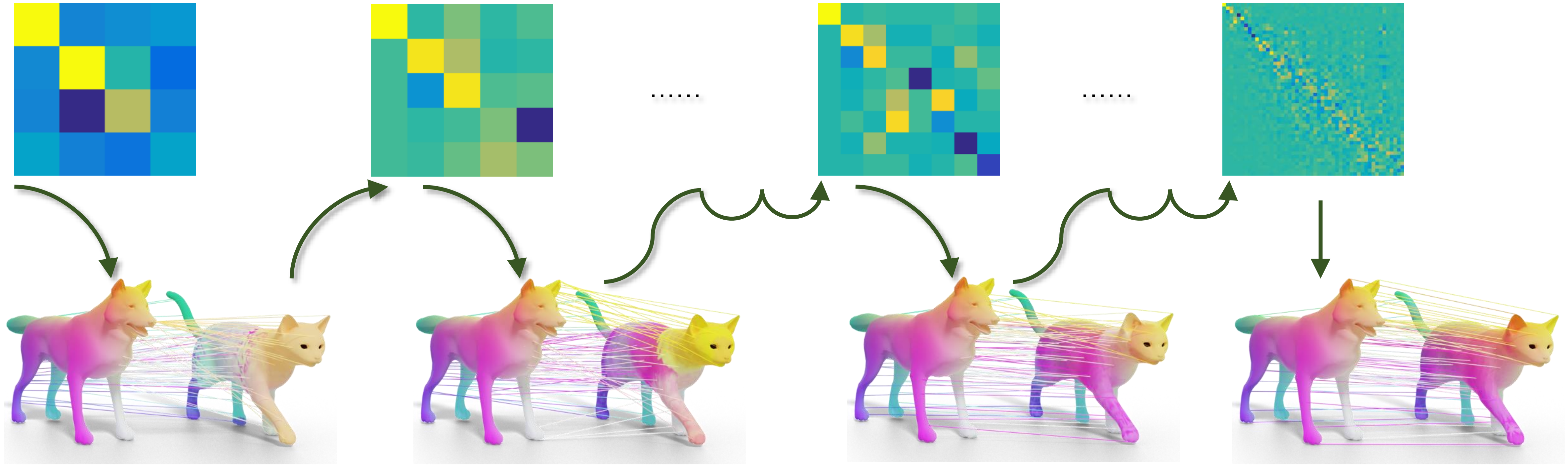


$$L(\Phi_\mathcal{X}, \Phi_\mathcal{Y}) = \frac{1}{n_\beta} \sum \| S_{\mathcal{XY}} P_\mathcal{X} - \Pi^{gt}_{\mathcal{XY}} P_\mathcal{X} \|_2^2.$$

Marin, Rakotosaona, Melzi, Ovsjanikov; Correspondence Learning via Linearly-invariant embedding; NeurIPS 2020

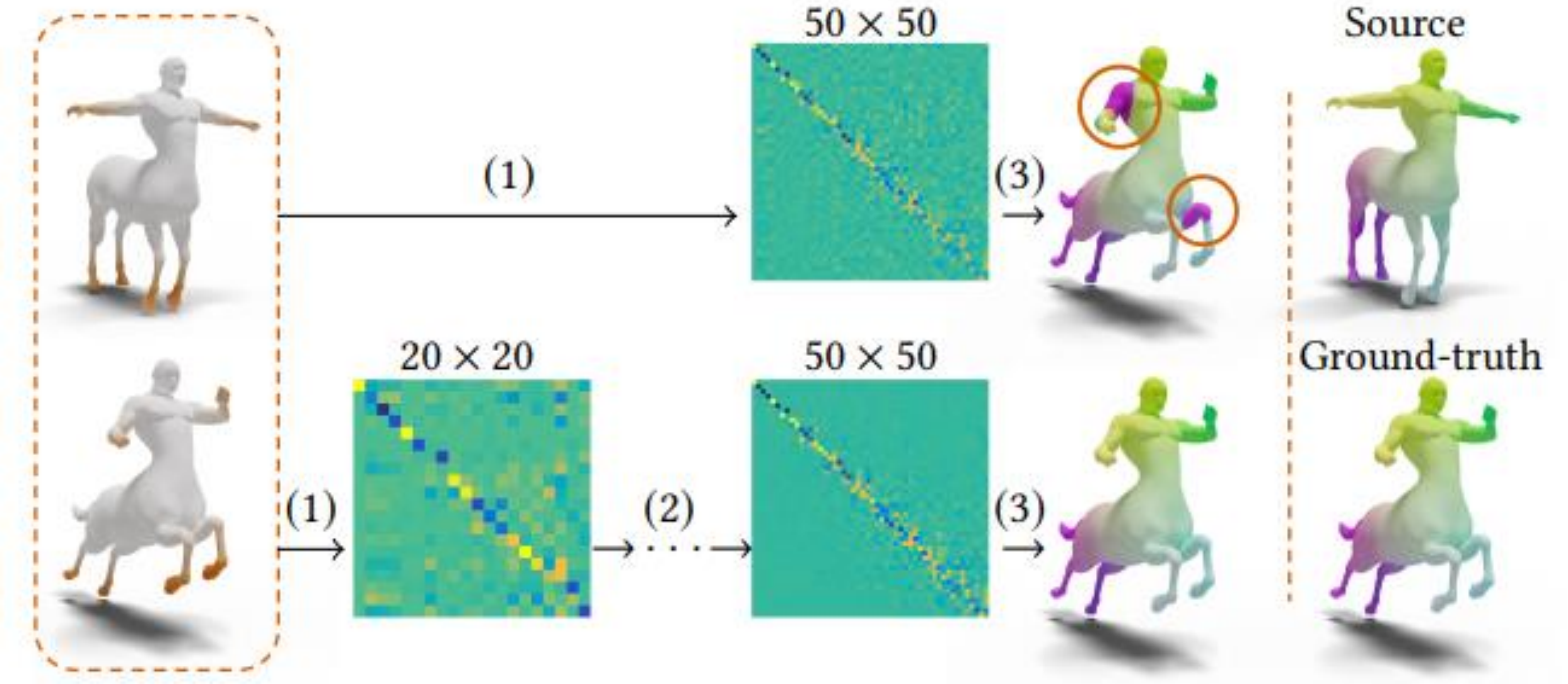
Refinement - ZoomOut

(1) Given $k = k_0$ and an initial C_0 of size $k_0 \times k_0$.
(2) Compute $\arg\min_{\Pi} \|\Pi\Phi^k_{\mathcal{N}} C_k^T - \Phi^k_{\mathcal{M}}\|_F^2$.
(3) Set $k = k + 1$ and compute $C_k = (\Phi^k_{\mathcal{M}})^+ \Pi \Phi^k_{\mathcal{N}}$.
(4) Repeat the previous two steps until $k = k_{\max}$.



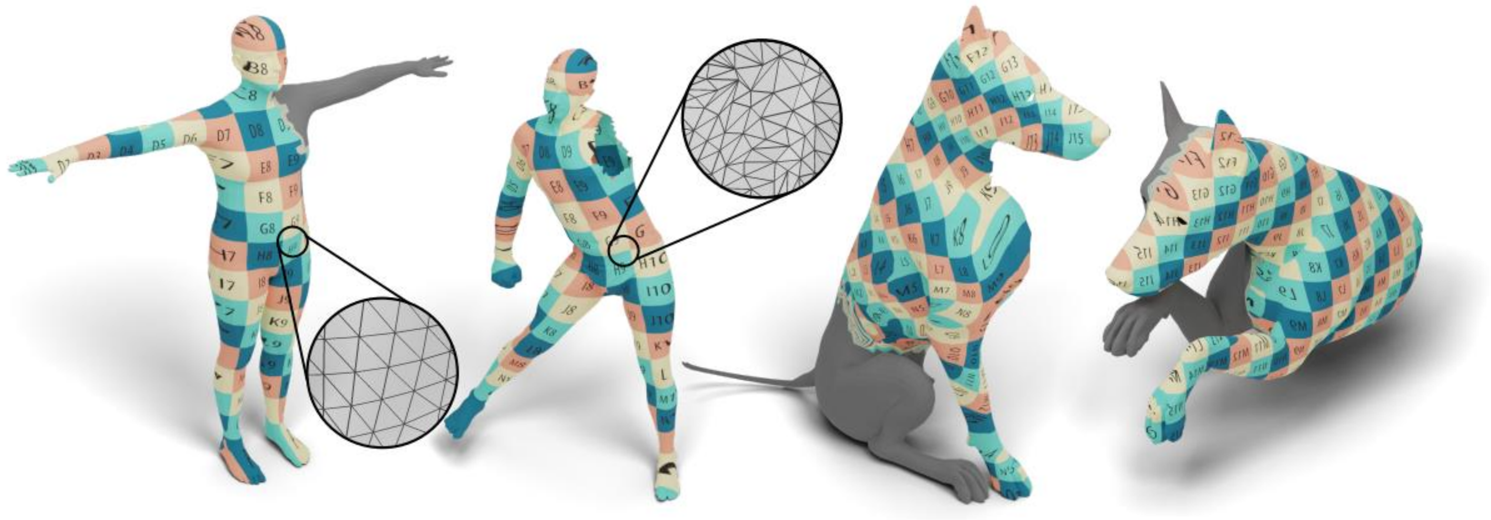
Melzi, Ren, Rodolà, Sharma, Wonka, Ovsjanikov; ZoomOut: Spectral Upsampling for Efficient Shape Correspondence; TOG 2019

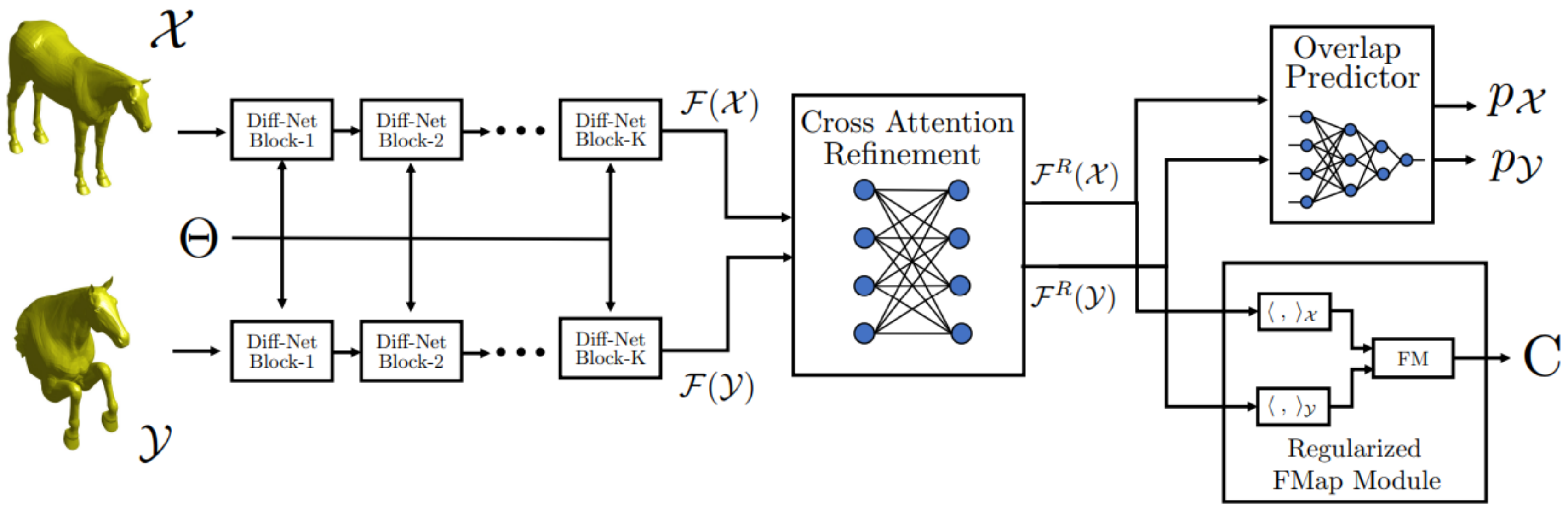
Refinement - ZoomOut



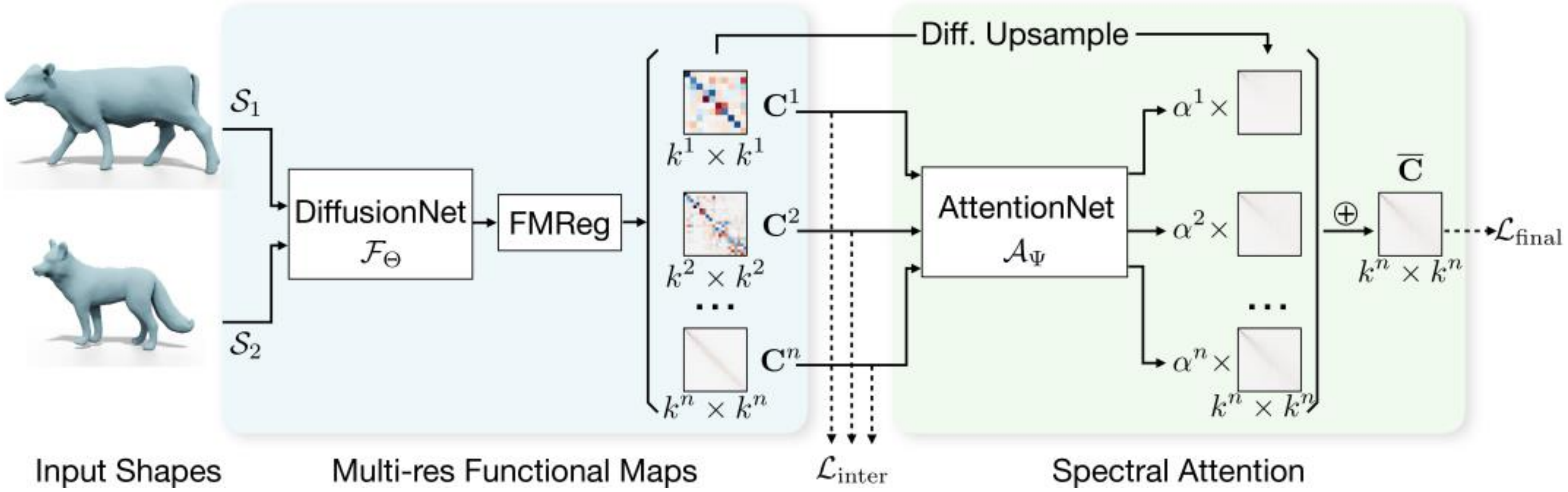
Melzi, Ren, Rodolà, Sharma, Wonka, Ovsjanikov; ZoomOut: Spectral Upsampling for Efficient Shape Correspondence; TOG 2019

DPFM: Deep Partial Functional Maps





Attaiki, Pai, Ovsjanikov; DPFM: Deep Partial Functional Maps; 3DV 2021



$$\overline{\mathbf{C}} = \sum_{i=1}^{n} \alpha^i \widehat{\mathbf{C}}^i.$$

$$L(\mathbf{C}) = \|\mathbf{C} - \mathbf{C}^{\text{gt}}\|^2$$

$$\mathcal{L}_{\text{inter}} = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{k^n}{k^i}\right)^2 L(\mathbf{C}^i), \qquad \mathcal{L}_{\text{final}} = L(\overline{\mathbf{C}})$$

Lei Lei, Donati, Ovsjanikov; Learning Multi-resolution Functional Maps with Spectral Attention for Robust Shape Matching, NeurIPS 2022

Where to start?

Python library for Functional Maps

RobinMagnet / pyFM

https://github.com/RobinMagnet/pyFM

Collection of slides, demos, examples, and pointers to other repos

riccardomarin / Tutorial_Spectral_Geometry

https://github.com/riccardomarin/Tutorial_Spectral_Geometry

https://github.com/riccardomarin/SpectralShapeAnalysis

https://github.com/melzismn/fmap

https://github.com/AriannaRampini/InverseSpectralGeometry_3DVTutorial

https://github.com/lcosmo/isospectralization

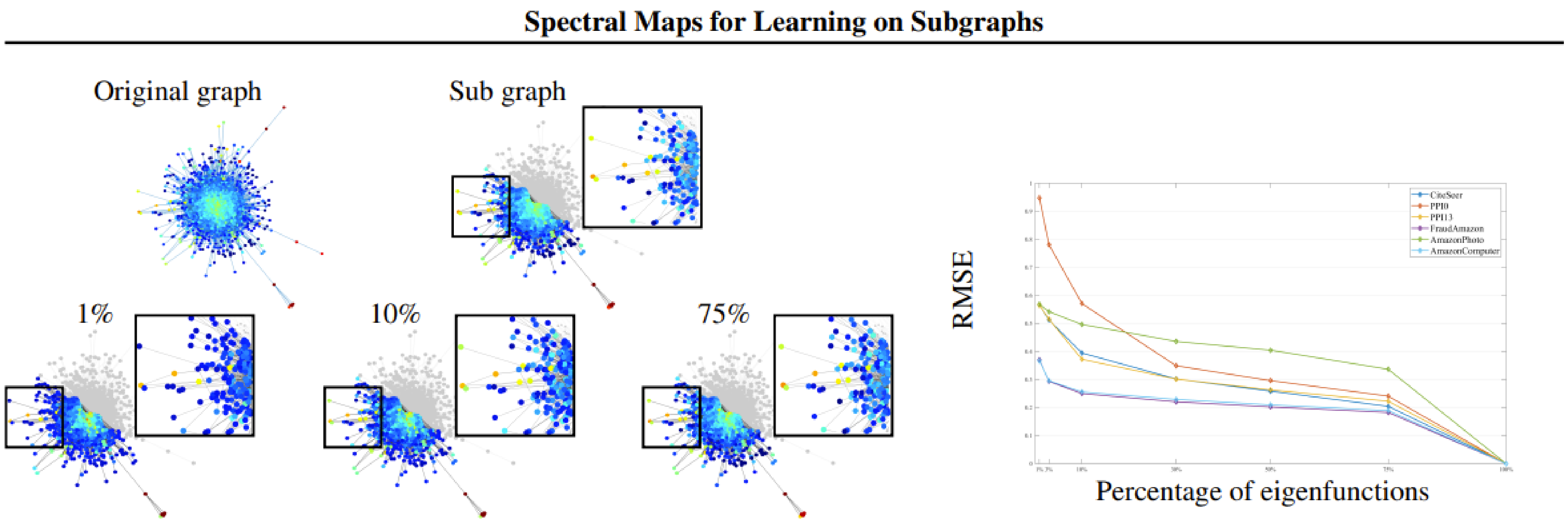
Other Courses

SGP Summer School Presentations

2017 5 Shape Correspondence and Functional Maps

2020 3 The functional representation of 3D shapes and High-Frequencies

Conclusions



Beyond Geometry Processing and matching

Pegoraro, Marin, Rampini, Melzi, Cosmo, Rodolà; Spectral Maps for Learning on Subgraphs, ArXiv 2023

Thanks!
Questions?

Thanks to Simone Melzi for HKS images, and to Marco Pegoraro for HKS code.