

# Spectral Shape Analysis for 3D matching

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**SAPIENZA**  
UNIVERSITÀ DI ROMA

**ADVANCED TECHNIQUES  
FOR FUNCTIONAL MAP**

# Functional map and the size of the basis

Slide credit M. Ovsjanikov



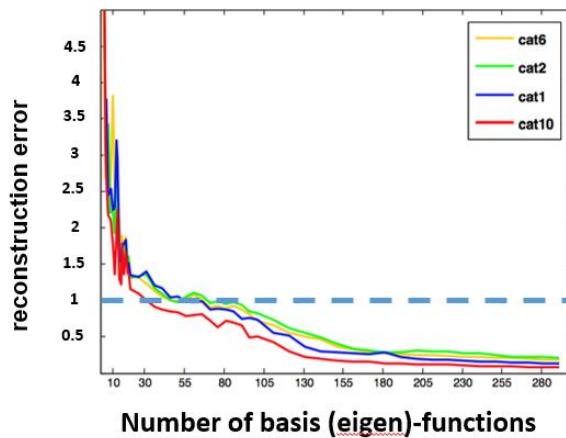
source

Cat10

Cat1

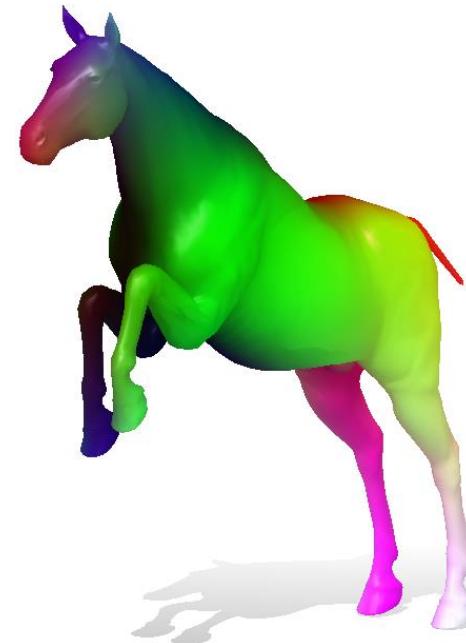
Cat2

Cat6



# Fmaps and the low-pass filter

$60 \times 60$

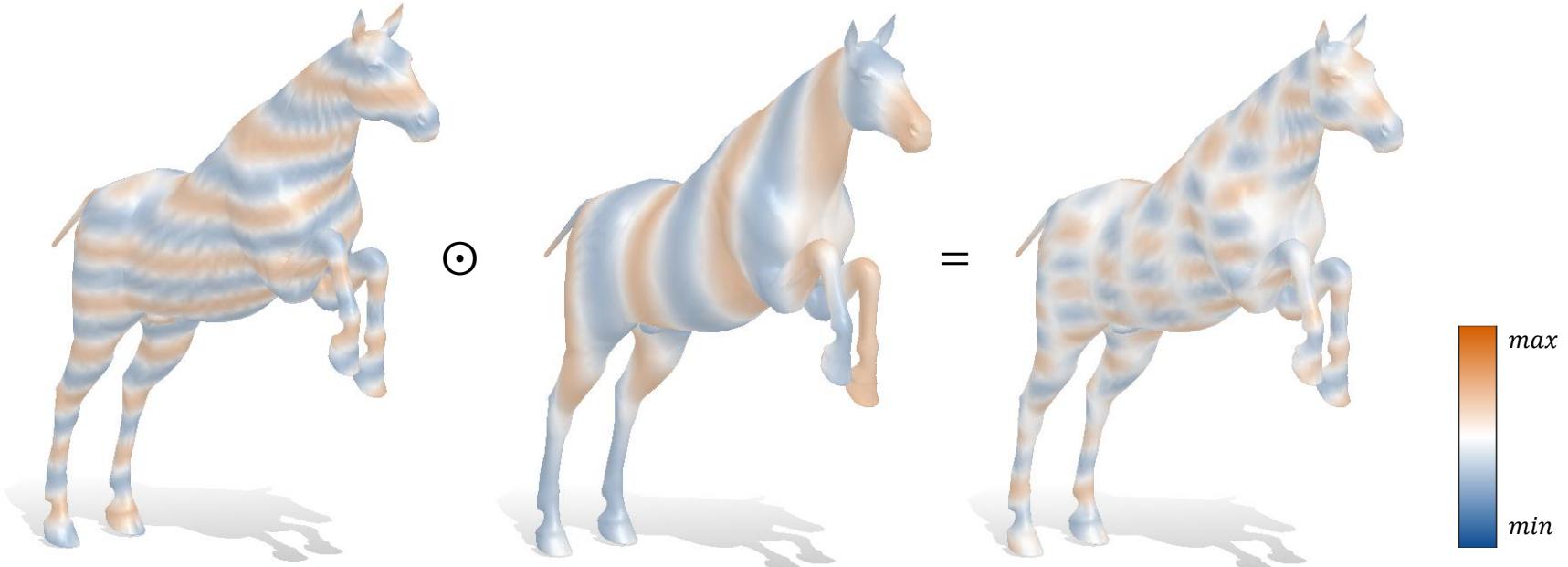


# The trade-off

		+	-
Low-pass	Easy to optimize (fewer probe functions needed)	Low-pass representation (poor functional transfer)	
High-frequencies	Better representation of details (good functional Transfer)	Hard to optimize (more probe functions needed)	

# **Proposed solutions**

# Exploiting the Algebra of functions



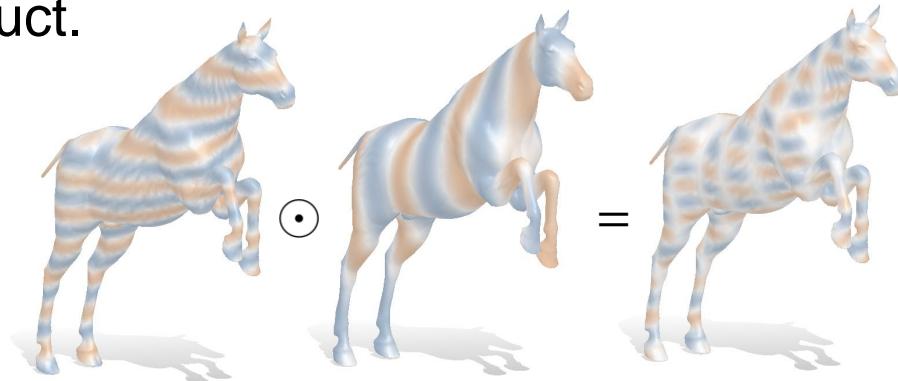
# Pointwise Products

Theorem: A FMap that comes from a point-to-point map  
should preserves the pointwise products of  
functions.

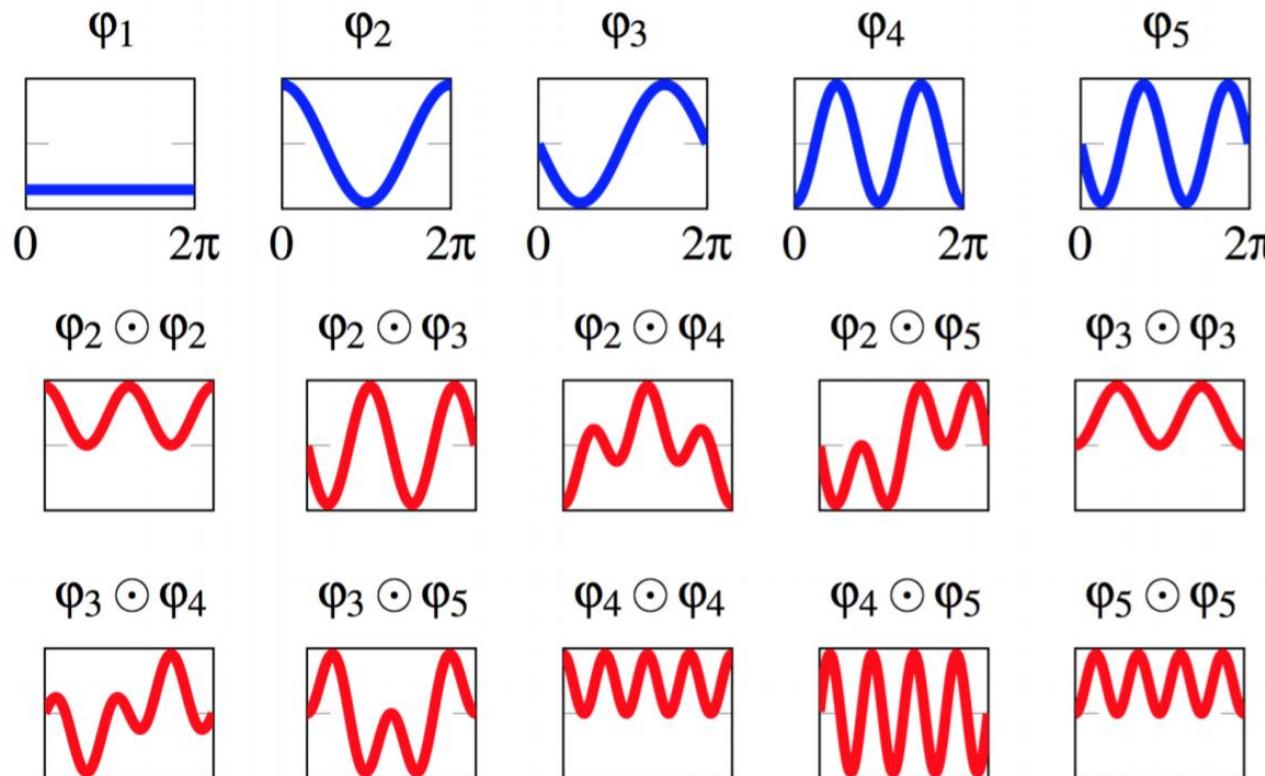
$$T_F(f_1 \odot f_2) = T_F(f_1) \odot T_F(f_2)$$

where  $\odot$  is the pointwise product.

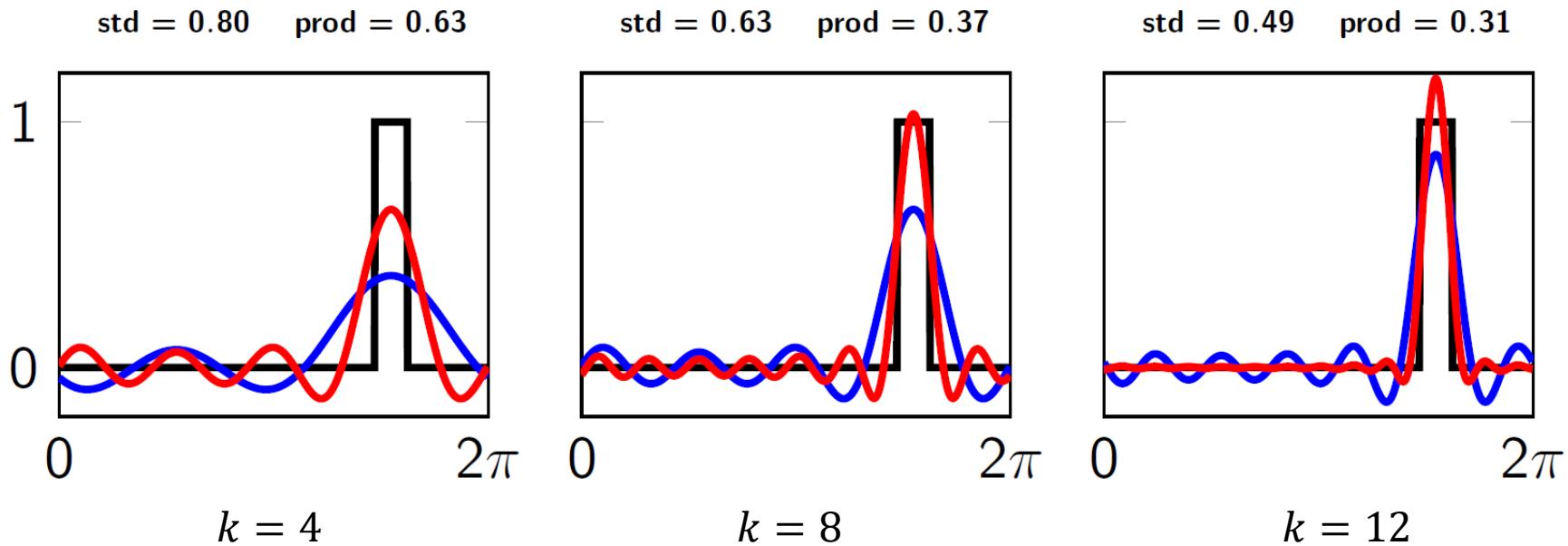
This Theorem tell us that  
these products are strictly  
related to the functional maps!



# Pointwise Products in 1D



# Pointwise Products and high frequencies



# FMaps for pointwise products

$$\tilde{\mathbf{C}}(\mathbf{C}) = \begin{bmatrix} \mathbf{C} & \mathcal{R}(\mathbf{C}) \\ 0 & \mathbf{C}(2:k, 2:k) \otimes \mathbf{C}(2:k, 2:k) \end{bmatrix}$$

**NO FURTHER OPTIMIZATION  
REQUIRED**

where  $\otimes$  is the Kronecker product and

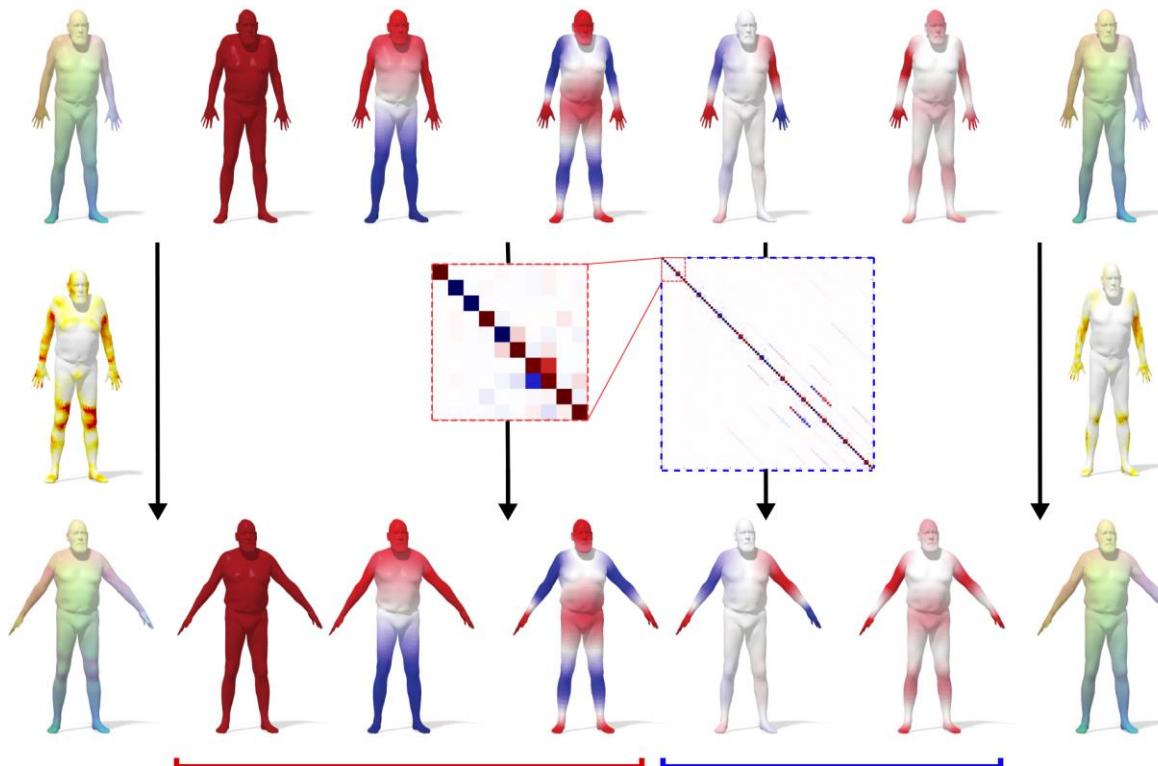
$$\mathcal{R}(\mathbf{C}) = \left[ \phi_1 \mathbf{C}(1,2:k) \otimes \mathbf{C}(1:k, 2:k) + \phi_1 [\mathbf{C}(2:k, 2:k)]^T \otimes \mathbf{C}(1,2:k) \right]$$

$$\phi_1 = \phi_1(1,1)$$

$\mathbf{C}(1,2:k)$  is the row 1 of  $\mathbf{C}$  columns from 2 to  $k$

# A visualization

Fourier basis on  $\mathcal{M}$  products on  $\mathcal{M}$



Improved functional  
mappings via product  
preservation, Nogneng,  
Melzi, Rodolà, Castellani, M.  
Bronstein, Ovsjanikov, CGF  
2018

Fourier basis on  $\mathcal{N}$  products on  $\mathcal{N}$

# Pointwise Products

Given  $\mathcal{M}$  and  $\mathcal{N}$  and their truncated Fourier bases:

$$\Phi_{\mathcal{M}} = \{\phi_1, \dots, \phi_k\} \text{ and } \Psi_{\mathcal{N}} = \{\psi_1, \dots, \psi_k\}$$

We consider the following dictionaries on  $\mathcal{M}$  and  $\mathcal{N}$ :

$$\tilde{\Phi} = \{\phi_1, \phi_2, \dots, \phi_k, \phi_2 \odot \phi_2, \phi_2 \odot \phi_3, \dots, \phi_{k-1} \odot \phi_k, \phi_k \odot \phi_k\}$$

$$\tilde{\Psi} = \{\psi_1, \psi_2, \dots, \psi_k, \psi_2 \odot \psi_2, \psi_2 \odot \psi_3, \dots, \psi_{k-1} \odot \psi_k, \psi_k \odot \psi_k\}$$

The functions in the dictionaries  $\tilde{\Phi}$  and  $\tilde{\Psi}$  are not linear independent.

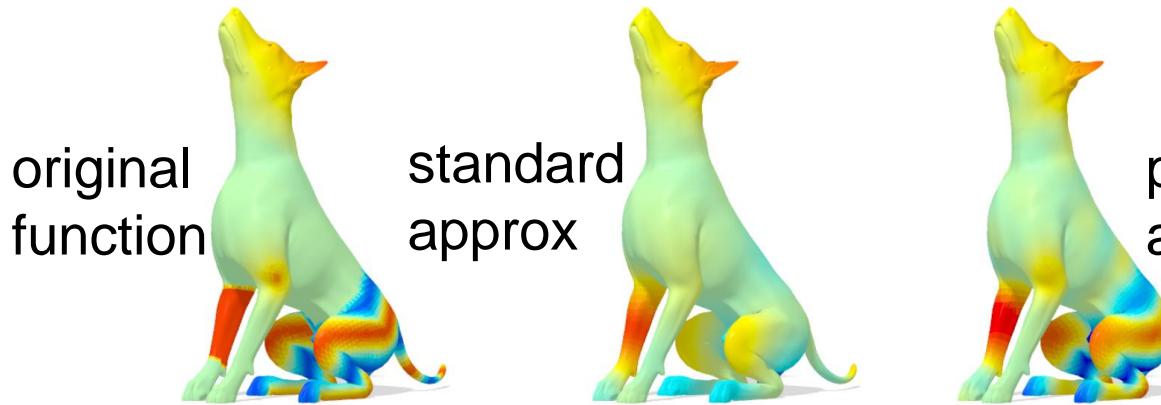
# The proposed solutions

Given a function  $f \in F(\mathcal{M})$  we would represent it as:

$$f \approx \tilde{\Phi}\alpha = \sum_{i=1}^k \alpha_i \phi_i + \sum_{i,j=2}^k a_{i,j} \phi_i \odot \phi_j$$

- **Approach A:**  $\alpha = \underset{\alpha}{\operatorname{argmin}} \|\tilde{\Phi}\alpha - f\|_2 + \varepsilon \|\alpha\|_p$  for  $p = 1$  or  $p = 2$
- **Approach B:**  $\alpha = V\Sigma^\dagger U^\top \sqrt{A_{\mathcal{M}}} f$  where  $U\Sigma V^\top = \sqrt{A_{\mathcal{M}}} \tilde{\Phi}$

# Qualitative results



Improved functional mappings via product preservation, Nogneng, Melzi, Rodolà, Castellani, M. Bronstein, Ovsjanikov, CGF 2018

# Comparison adding frequencies

original function



standard transfer



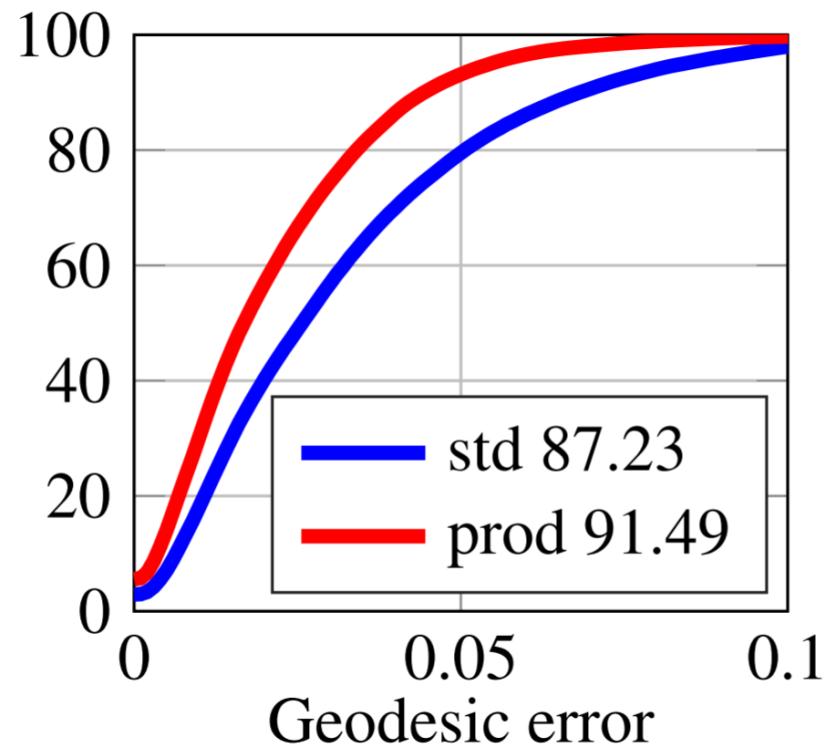
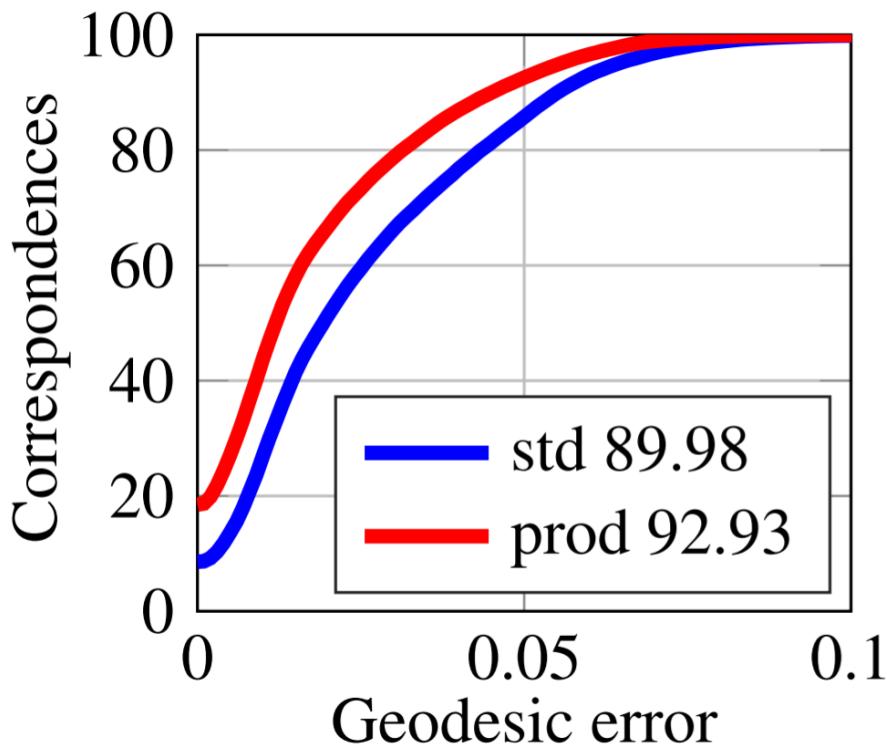
product transfer



$k = C2/65$

Improved functional mappings via product preservation, Nogneng, Melzi, Rodolà, Castellani, M. Bronstein, Ovsjanikov, CGF 2018

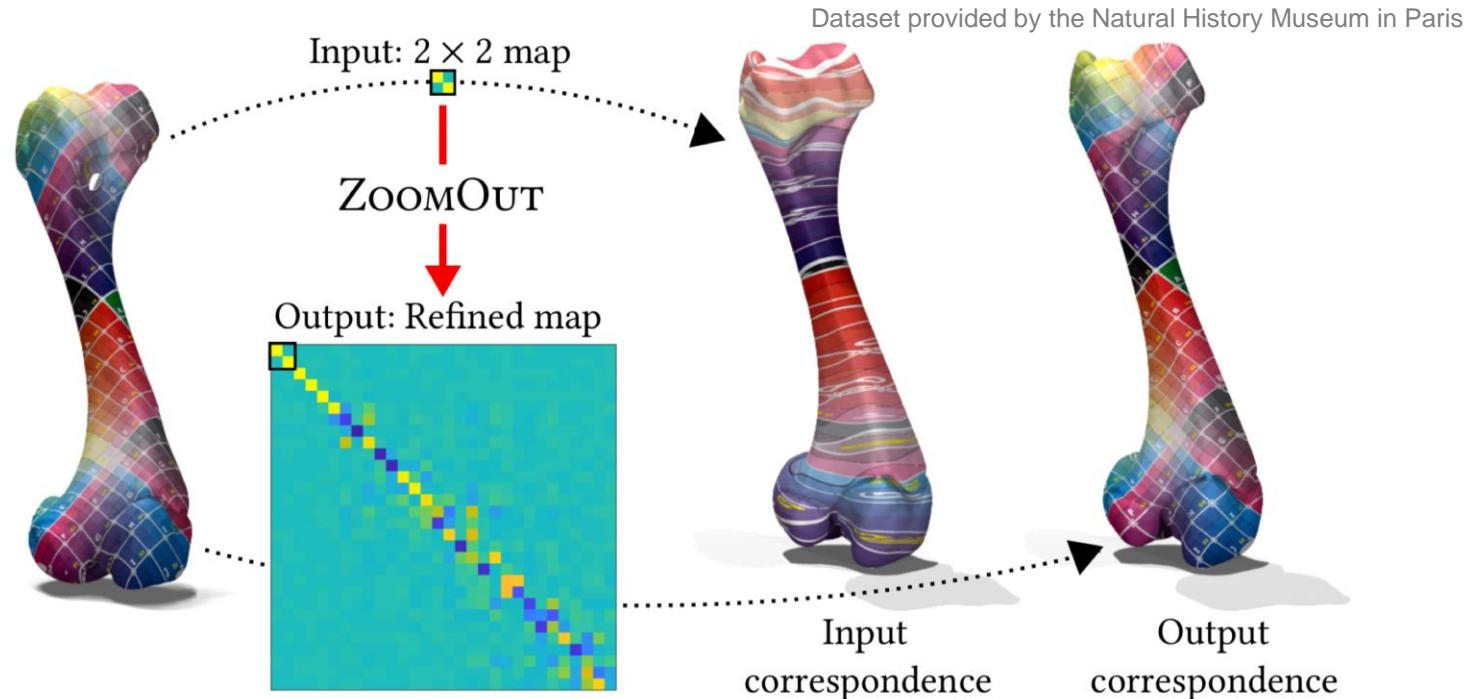
# Quantitative results



Improved functional mappings via product preservation, Nogneng, Melzi, Rodolà, Castellani, M. Bronstein, Ovsjanikov, CGF 2018

# Exploiting the connection with point-to-point-map

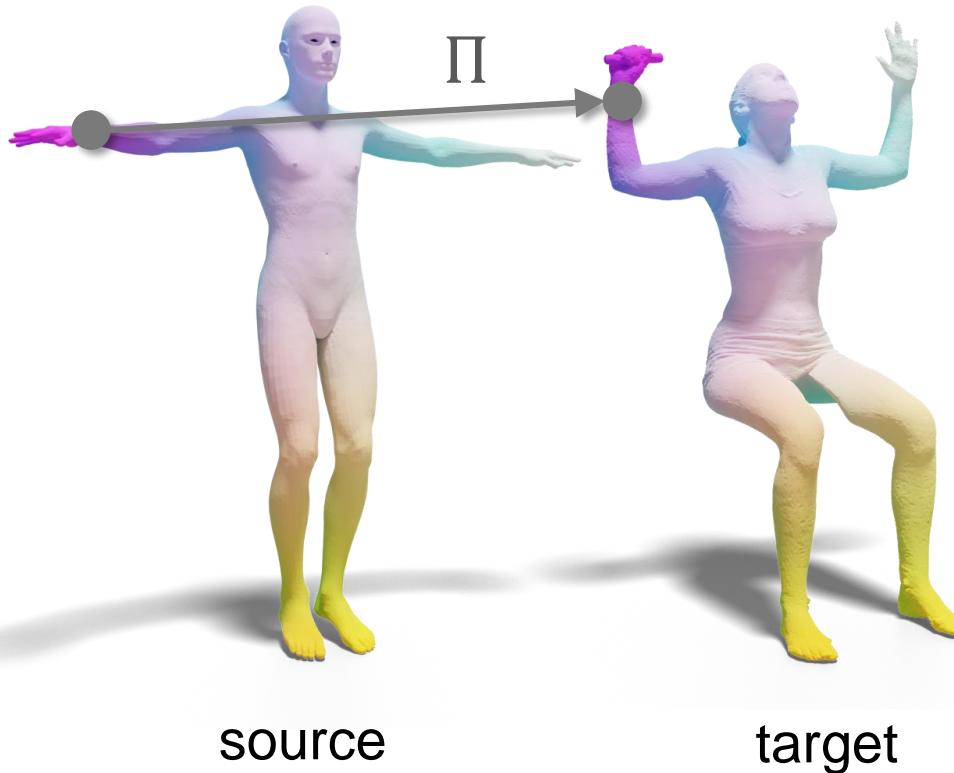
Slide credit M. Ovsjanikov



ZoomOut: Spectral Upsampling for Efficient Shape Correspondence, Melzi, Ren, Sharma, Rodolà, Wonka, Ovsjanikov, SIGGRAPH Asia 2019

# Pointwise correspondence $\Pi$

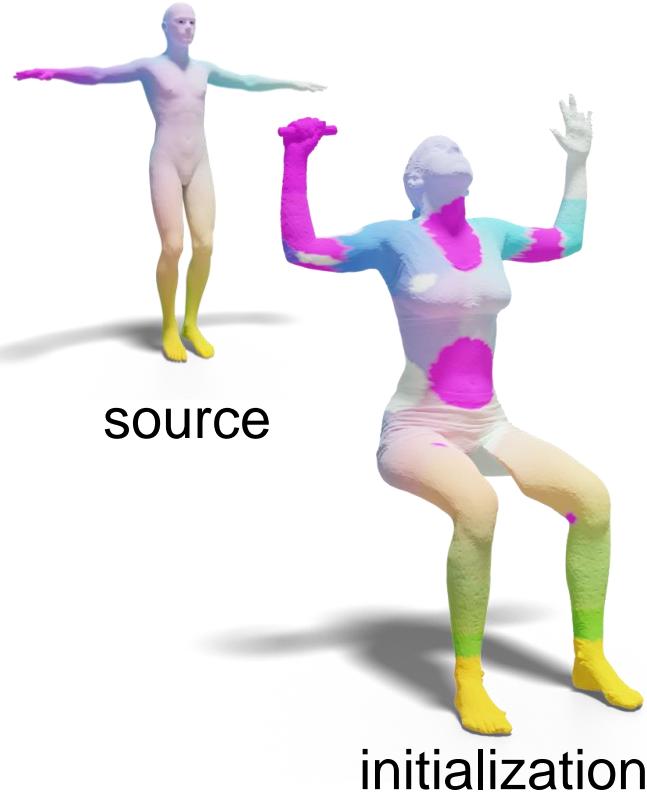
Slide credit J. Ren



The idea is to refine  
 $\Pi$  to obtain a more  
accurate map

# Map refinement

Slide credit J. Ren



## ICP-based

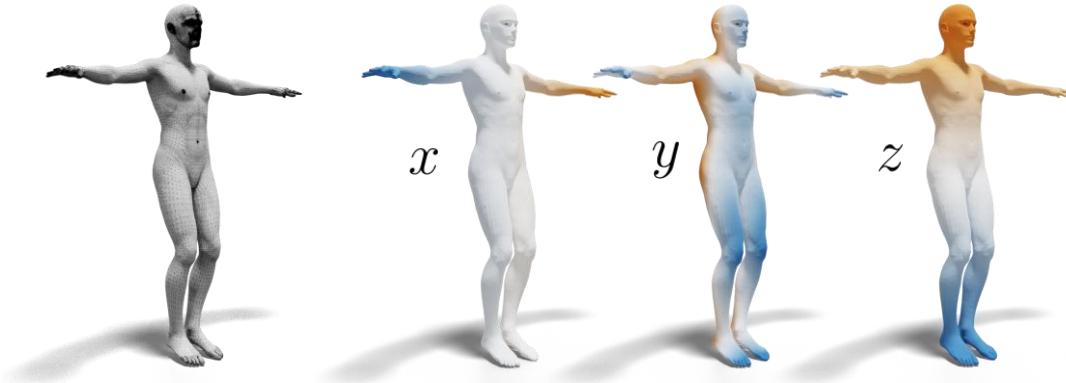
- Spatial (Besl, McKay 1992)
  - Spectral (Ovsjanikov et al. 2012)
  - Spatial & Spectral (Ren et al. 2018)
- ⋮

## Deblurring and Denoising

(Ezuz and Ben-Chen 2017)

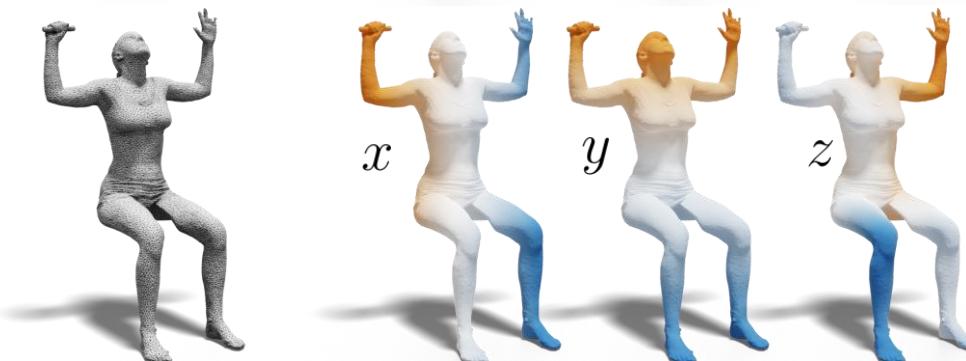
# ICP in the spatial domain

Slide credit J. Ren



$$\min_{\substack{R,t,\Pi \\ R^T R = I}} \sum_{x_i \in S_1} \|Rx_i + t - \Pi(x_i)\|_F^2$$

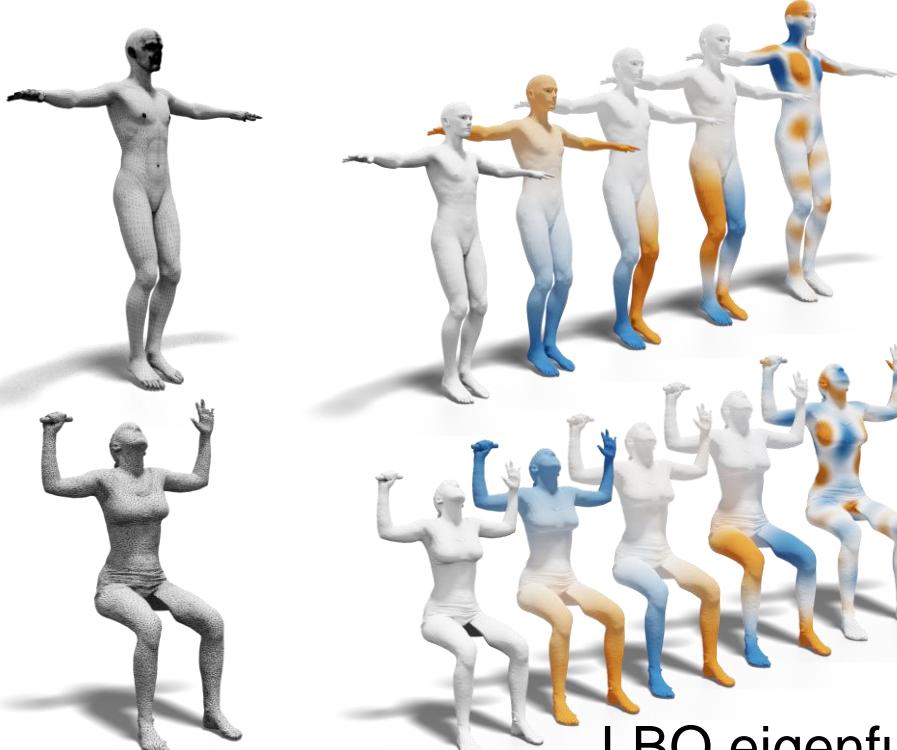
Optimize for a 3D rotation to align the vertex coordinates



A method for registration of 3-D shapes, Besl, McKay, IEEE PAMI 1992

# ICP in the spectral domain

Slide credit J. Ren

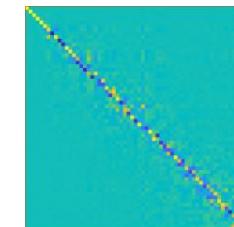


LBO eigenfunctions

$$\min_{C, \Pi} \|\Phi_1 C - \Phi_2(\Pi, :) \|_F^2$$
$$C^T C = I$$

Optimize for  $C$  as a rotation  
in the spectral domain.

Iteratively solve for  $C$  and  $\Pi$   
( $C$  with fixed size =  $k$ )



# ICP and precise maps

ICP:  $\min_{C,\Pi} \| \Phi_1 C - \Phi_2(\Pi, :) \|_F^2$

$$C^T C = I$$

$$\min_{C,\Pi} \| \Phi_1^\dagger \Phi_1 C - \Phi_1^\dagger \Phi_2(\Pi, :) \|_F^2$$

$$\min_{C,\Pi} \| C - \Phi_1^\dagger \Phi_2(\Pi, :) \|_F^2$$

$$\min_{C,P} \| C - \Phi_1^\dagger P_{12} \Phi_2 \|_F^2$$

$$\underset{P_{12}}{\text{minimize}} \quad R(P_{12}) + \| C_{12} - \Phi_1^\dagger P_{12} \Phi_2 \|_F^2$$

$$\text{subject to} \quad P_{12} \in \mathcal{P}_{12}$$

The functional maps must correspond to vertex-to-point maps

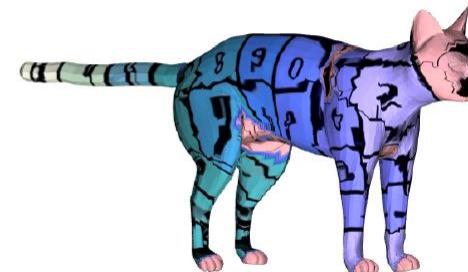
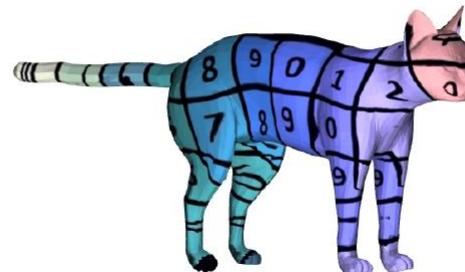
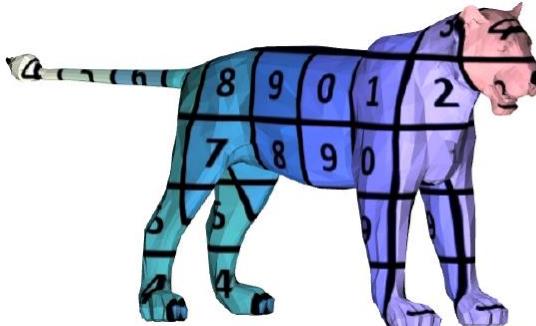
# Functional maps deblurring

The functional maps must correspond to vertex-to-point maps

$$R(P_{12}) = \|(Id - \Phi_1 \Phi_1^\dagger) P_{12} \Phi_2\|_{M_1}^2$$

$$P_{12} \Phi_2 \in \text{span}(\Phi_1)$$

$$\begin{aligned} & \underset{P_{12}}{\text{minimize}} && R(P_{12}) + \|C_{12} - \Phi_1^\dagger P_{12} \Phi_2\|_F^2 \\ & \text{subject to} && P_{12} \in \mathcal{P}_{12} \end{aligned}$$



Progressively registering the eigenfunctions  
Exploiting the connection between functional and point-to-point map

## ZoomOut

- 5 lines of code
- Similar complexity to ICP

```
1 function [C,P]=ZoomOut(M,N,C,k_final)
2
3 for k=size(C,1):k_final-1
4 x = knnsearch(N.Phi(:,1:k)*C',M.Phi(:,1:k));
5 P = sparse(1:M.n,x,1,M.n,N.n);
6 C = M.Phi(:,1:k+1)'*M.A*P*N.Phi(:,1:k+1);
7 end
```

# ZoomOut Algorithm

Slide credit J. Ren

## Algorithm

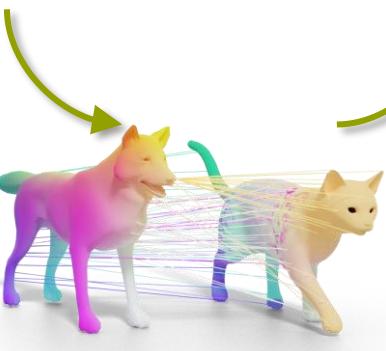
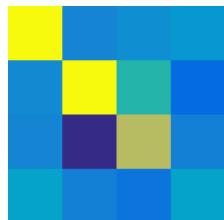
$$\min_{C, \Pi} \|\Phi_1 C - \Phi_2(\Pi, :) \|_F^2$$

1. Input: an initial map  $\Pi$  and an integer  $k$
2. Solve  $C^k = \operatorname{argmin}_C \|\Phi_1^k C - \Phi_2^k(\Pi, :) \|_F^2$   
 $\Phi_i^k$  are the first  $k$  eigenfunctions of  $S_i$
3. Update  $\Pi = \operatorname{argmin}_{\Pi} \|\Phi_1^k C^k - \Phi_2^k(\Pi, :) \|_F^2$
4. Update  $k = k + 1$
5. Return to step 2.

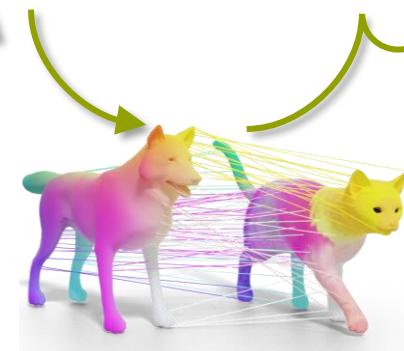
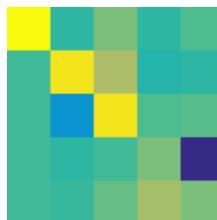
# ZoomOut: a visualization

Slide credit J. Ren

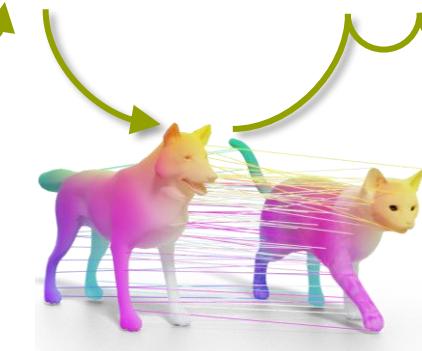
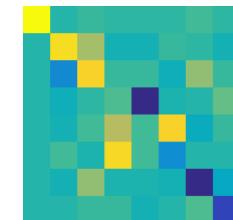
$C$ : dim = 4



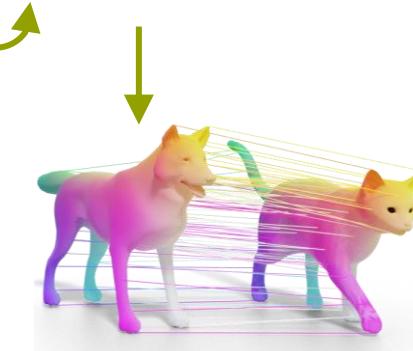
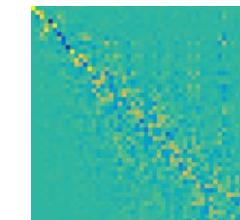
$C$ : dim = 5



$C$ : dim = 8



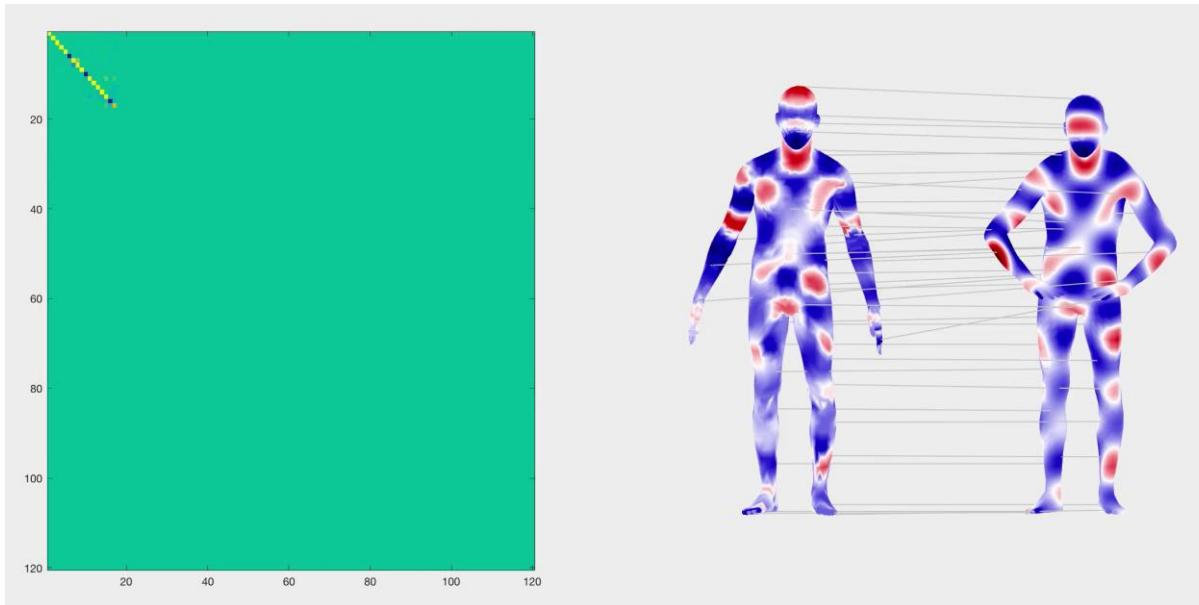
$C$ : dim = 50



# Increasing the size of the functional map

Slide credit M. Ovsjanikov

From 20x20 to 120x120

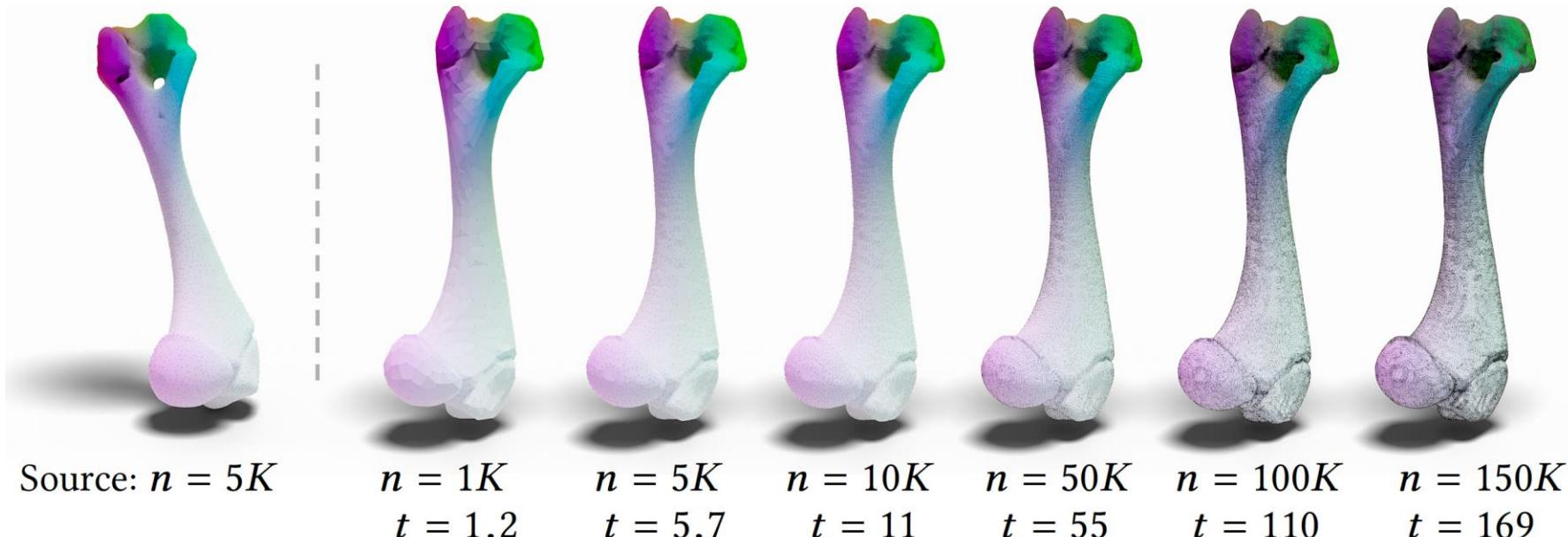


ZoomOut: Spectral Upsampling for Efficient Shape Correspondence, Melzi, Ren, Sharma, Rodolà, Wonka, Ovsjanikov, SIGGRAPH Asia 2019

# Scalability

Slide credit J. Ren

Computation time dominated by:  
eigenfunctions computation and nearest-neighbor search

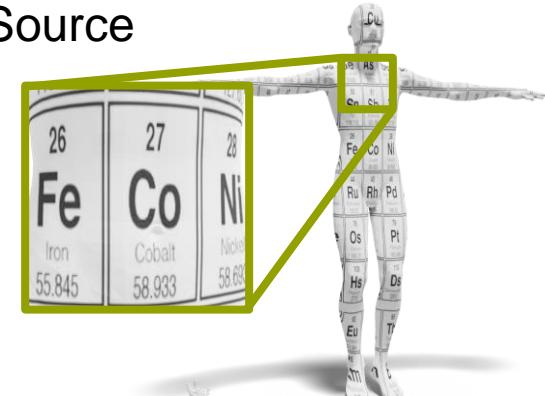


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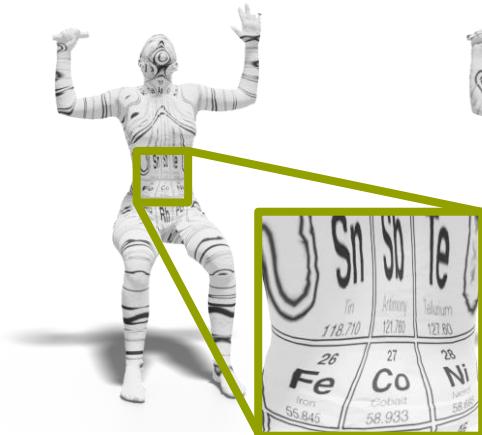
# Qualitative results

Slide credit J. Ren

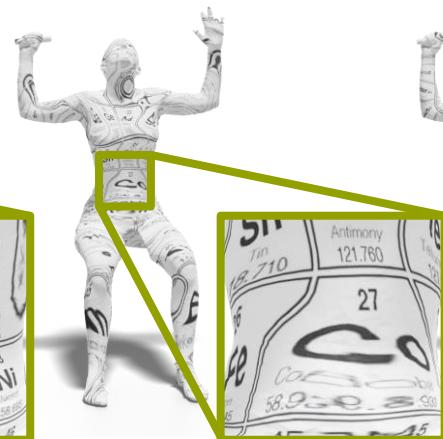
Source



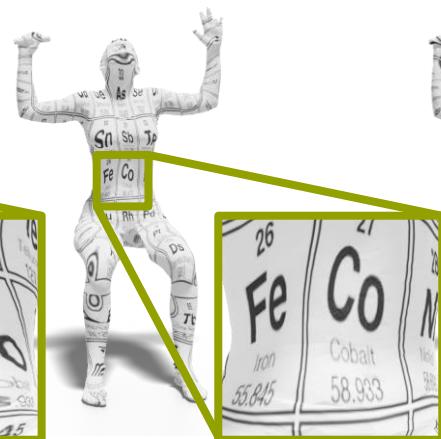
Initialization



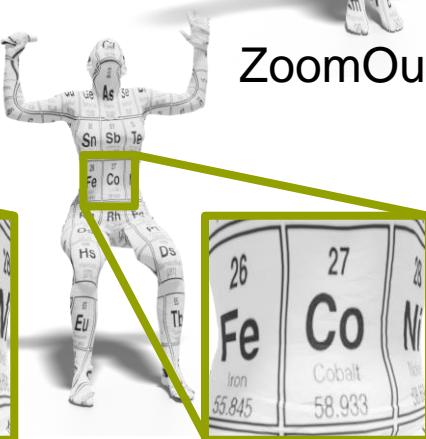
PMF



BCICP



ZoomOut

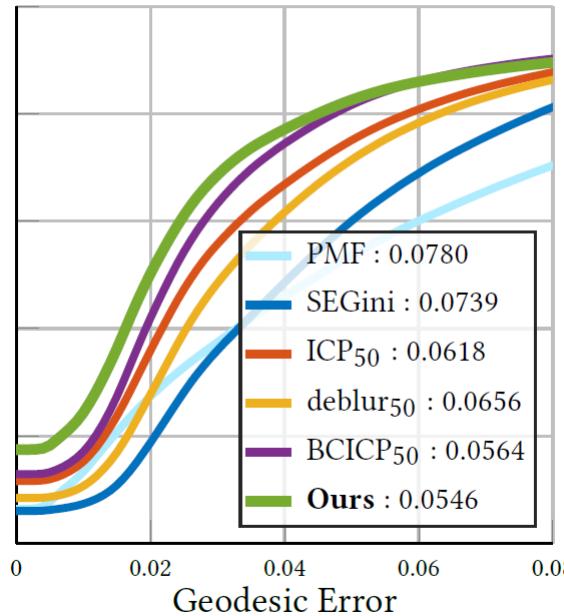


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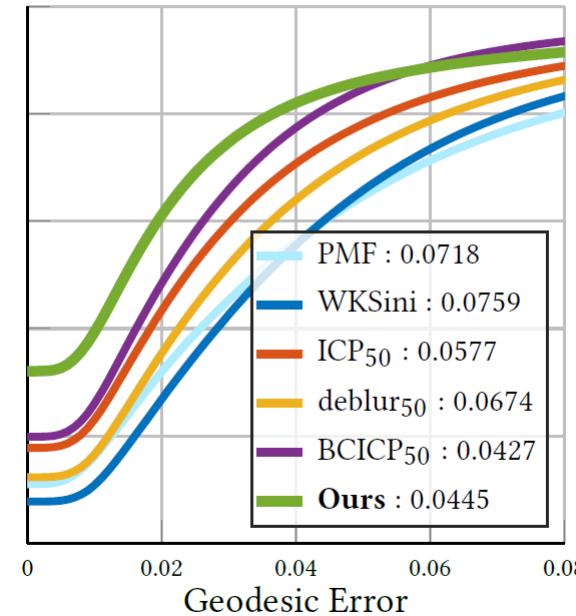
# Quantitative results

Slide credit J. Ren

**FAUST: direct error**  
different person, different pose



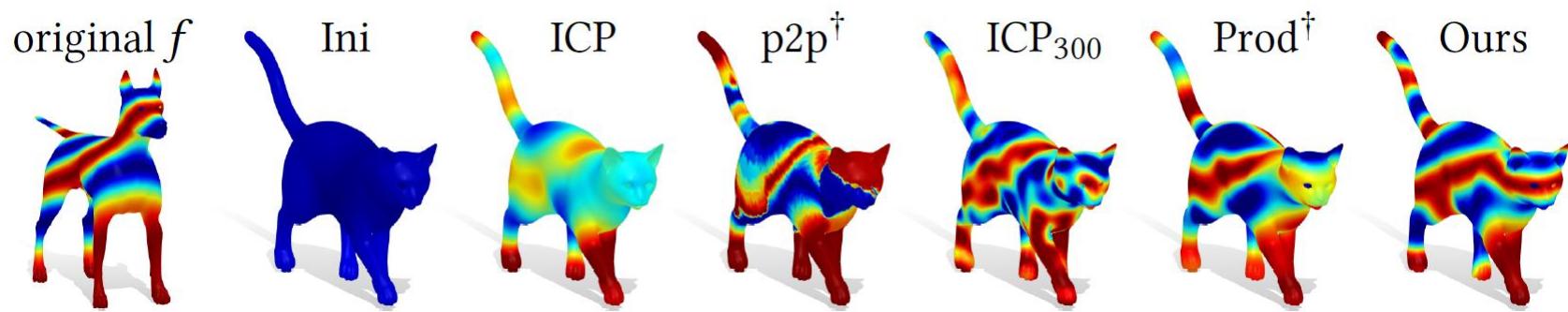
**TOSCA Isometric**  
direct error



ZoomOut: Spectral Upsampling for Efficient Shape Correspondence, Melzi, Ren, Sharma, Rodolà, Wonka, Ovsjanikov, SIGGRAPH Asia 2019

# Information transfer results

Slide credit J. Ren

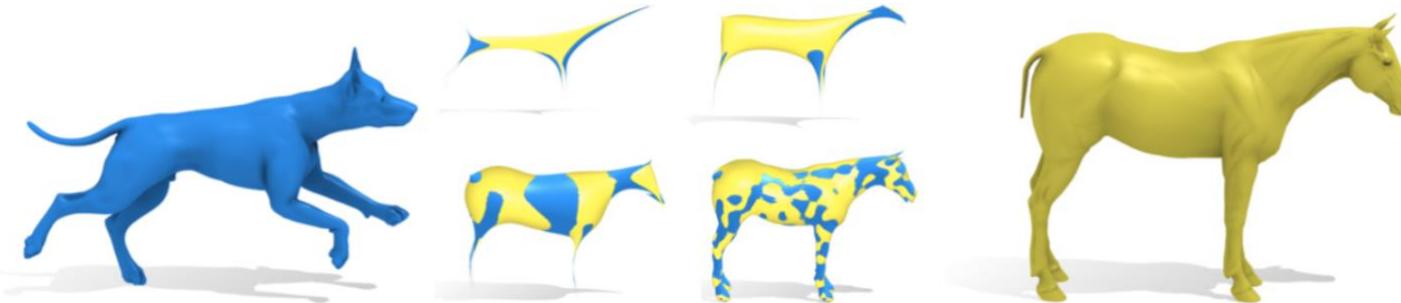


# Smooth Shells

A fully automatic initialization is proposed

A deformation and extrinsic registration energy are involved

Additional extrinsic information (coordinates and normals)



# Smooth Shells: iterative scheme

Correspondence is calculated by nearest neighbors between:

$$\bar{Y}_K = (\Psi_K, Y_K, n_K^Y) \in \mathbb{R}^{M \times (K+6)}$$

Laplace-Beltrami  
Eigenfunctions

K-th Shell  
Coordinates

K-th Shell  
Normals

$$\bar{X}_K^* = (\Phi_K C^\top, X_K + \Phi_k \tau, n_K^*) \in \mathbb{R}^{N \times (K+6)}$$

# **Open problems**

# Why not and why yes to high-frequencies

**NOT:** Avoid to make the functional map estimation more difficult

**NOT:** The alignment of high-frequencies is not easy (too unstable  
for different shapes)

**YES:** the point to point maps contain high-frequencies

**YES:** high-frequencies are necessary for real application  
(information transfer)

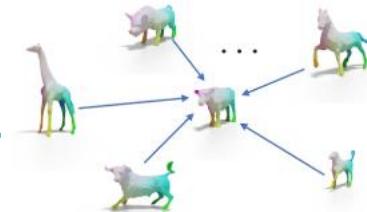
# How to recover high frequencies

1. The **algebra structure** of the functional space is useful
2. More information can be recovered from the fact that the functional map encodes a **point-to-point map**
3. Define **New basis** for the functional space

# ZoomOut is a trend in shape matching

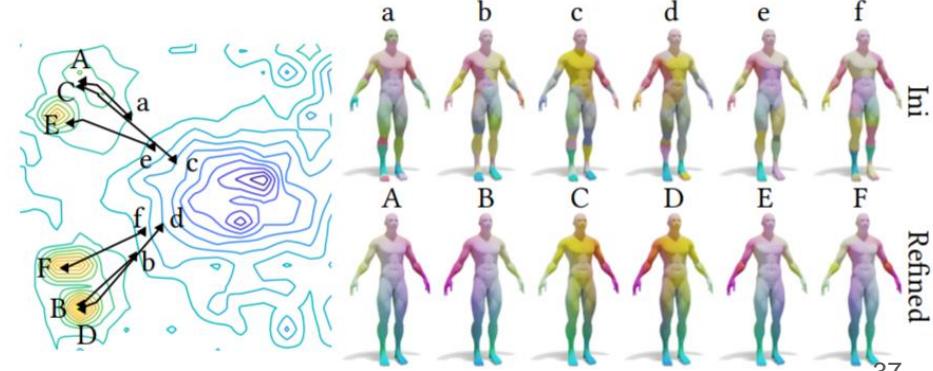
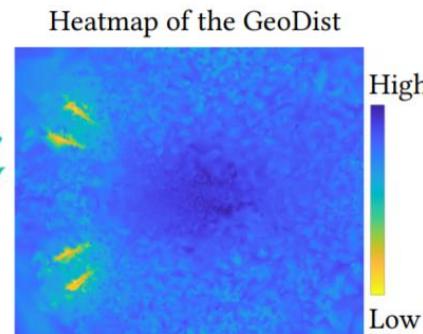
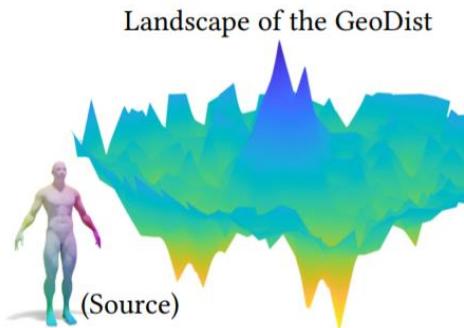
Consistent ZoomOut proposed at SGP 2020 one month ago:

## Consistent ZoomOut: Efficient Spectral Map Synchronization



We have a recently accepted at SIGGRAPH ASIA (yesterday) paper based on ZoomOut:

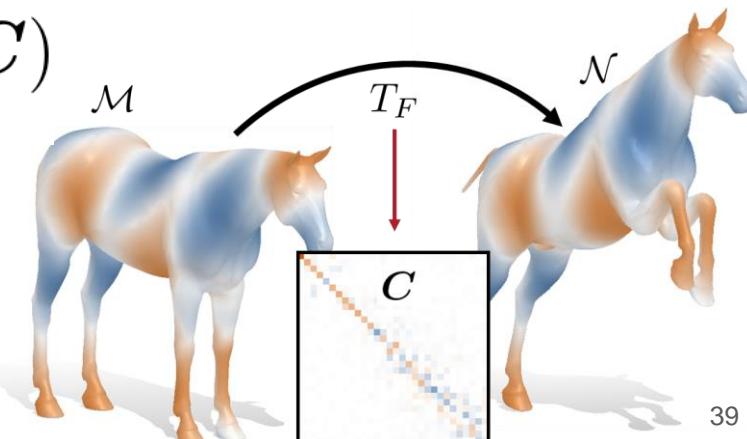
## MapTree: Recovering Multiple Solutions in the Space of Maps



# **Data-driven approaches for functional maps**

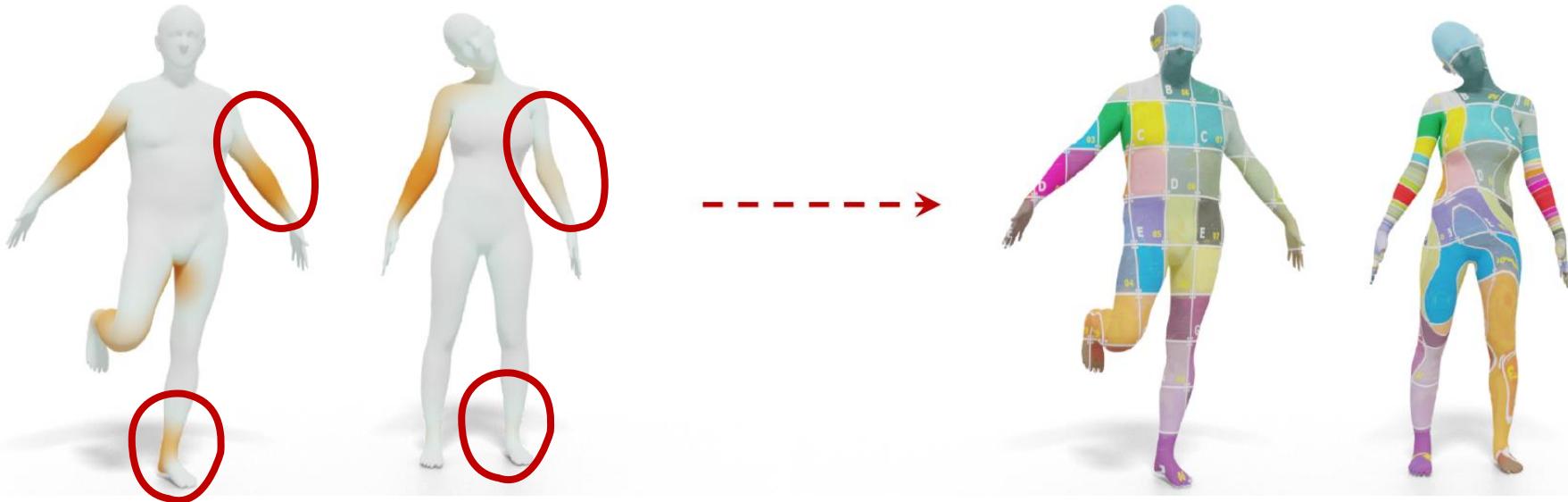
# FMAP pipeline again

1. Compute the first  $k$  ( $\sim 30-100$ ) eigenfunctions of the LBO.  
Store them in matrices:  $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$
2. Compute probe functions (e.g., landmarks or descriptors) on  $\mathcal{M}, \mathcal{N}$ . Express them in  $\Phi_{\mathcal{M}}, \Phi_{\mathcal{N}}$ , as columns of  $A$  and  $B$
3. Solve  $\underset{C}{argmin} \|Ca - b\|_F^2 + \mathcal{R}(C)$
4. Convert the functional map  
to a point to point map  $T$ .



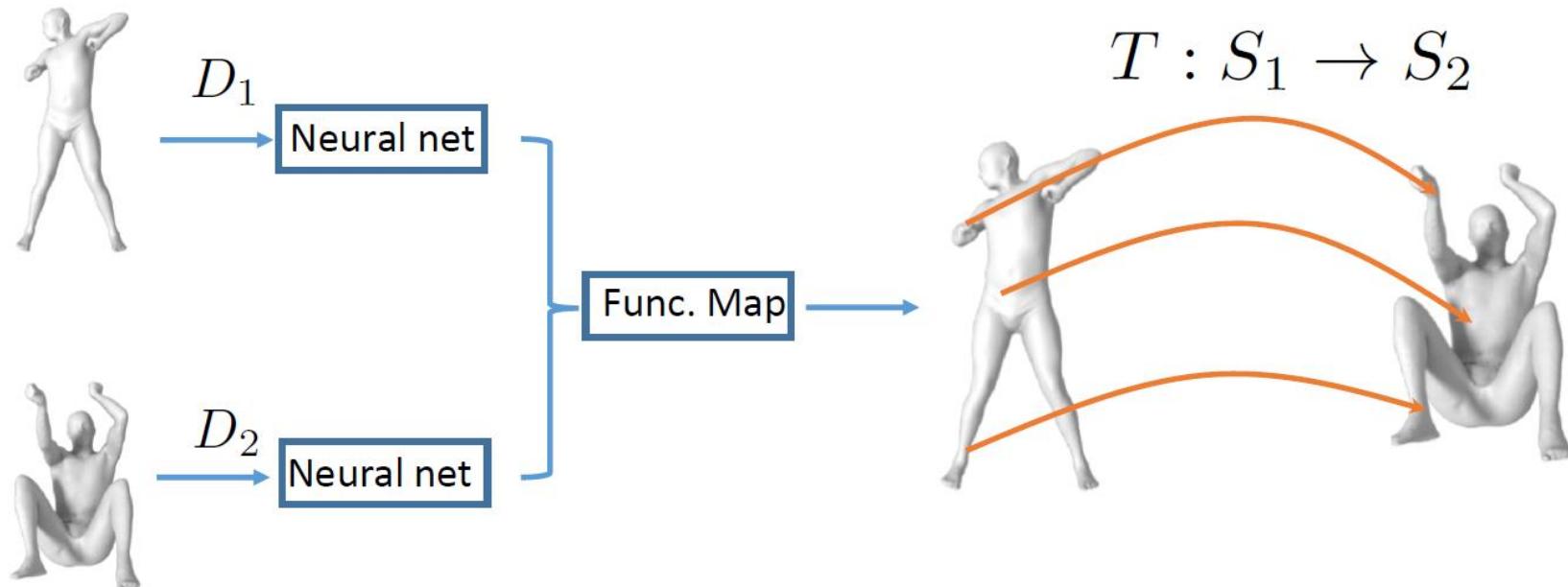
# Bad descriptors

Slide credit M. Ovsjanikov



# FMNET idea

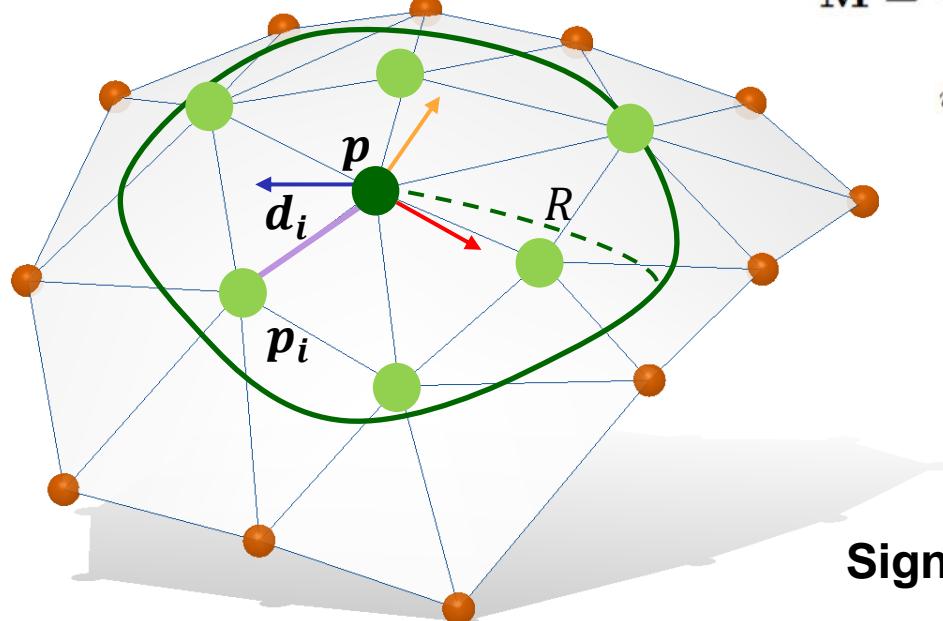
Slide credit M. Ovsjanikov



Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# Do you remember SHOT?

For all  $p$  we define the covariance matrix:



$$\mathbf{M} = \frac{1}{\sum_{i:d_i \leq R} (R - d_i)} \sum_{i:d_i \leq R} (R - d_i)(\mathbf{p}_i - \mathbf{p})(\mathbf{p}_i - \mathbf{p})^T$$

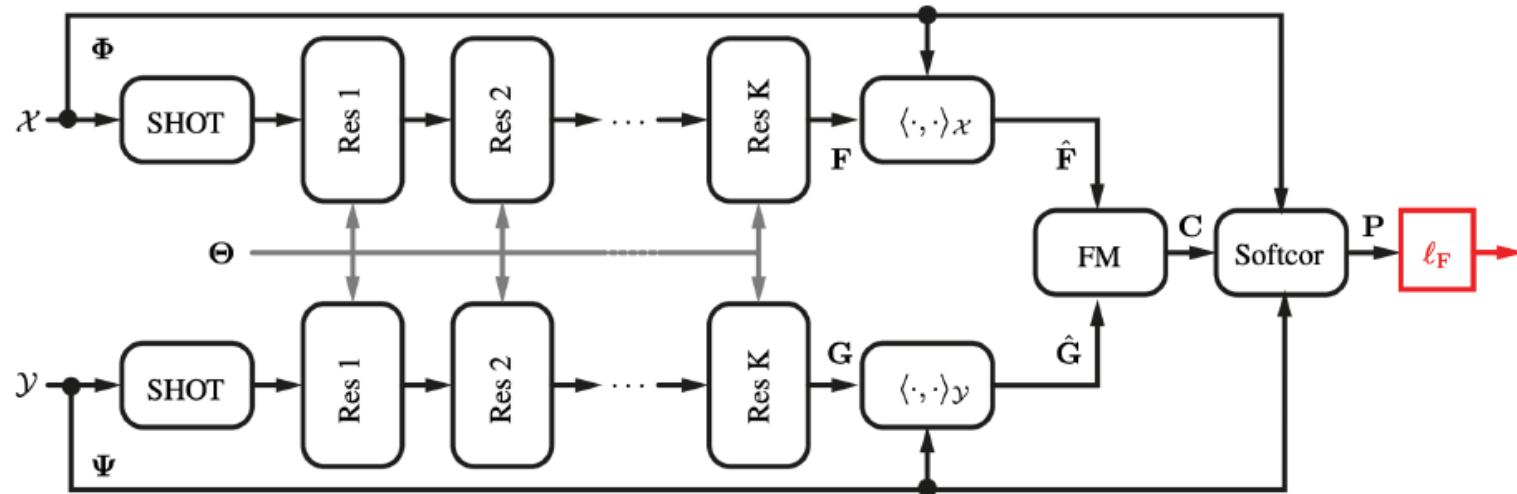
From the eigenvectors of  $M$  we obtain a LRF ( $x, y, z$ ) that is then used to define:

**Signature of Histograms of Orientations**

# FMNET network

Slide credit M. Ovsjanikov

A learning-based strategy to compute correspondences

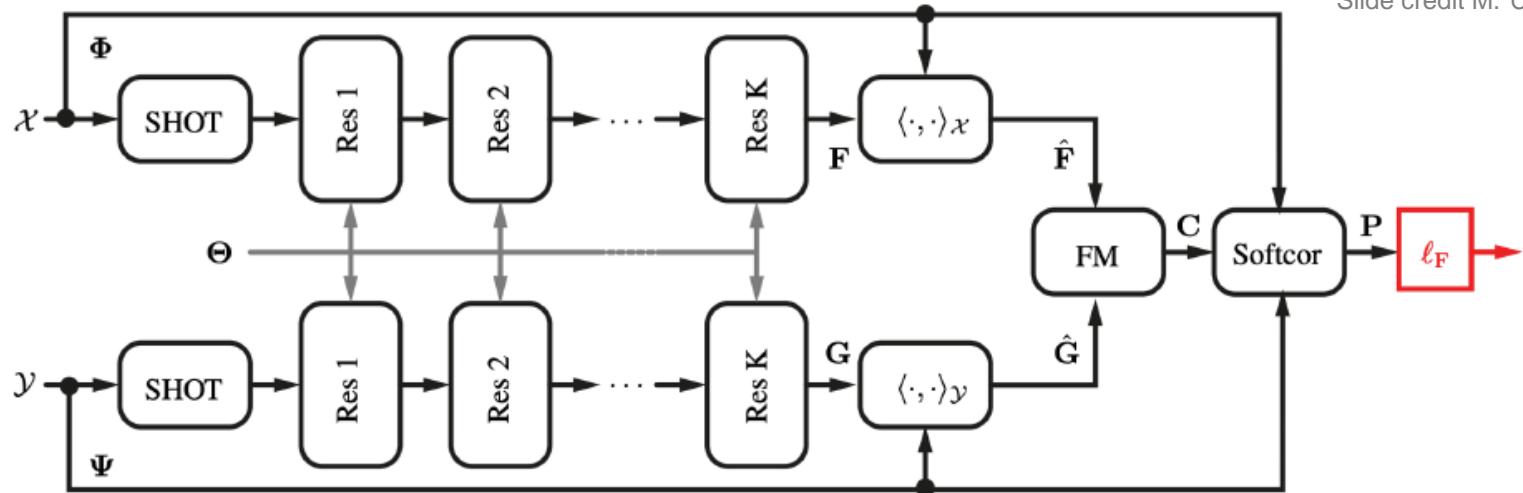


The *FM* layer corresponds to: 
$$C = \underset{C}{\operatorname{argmin}} \| C \hat{F} - \hat{G} \|$$

Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# FMNET loss

Slide credit M. Ovsjanikov

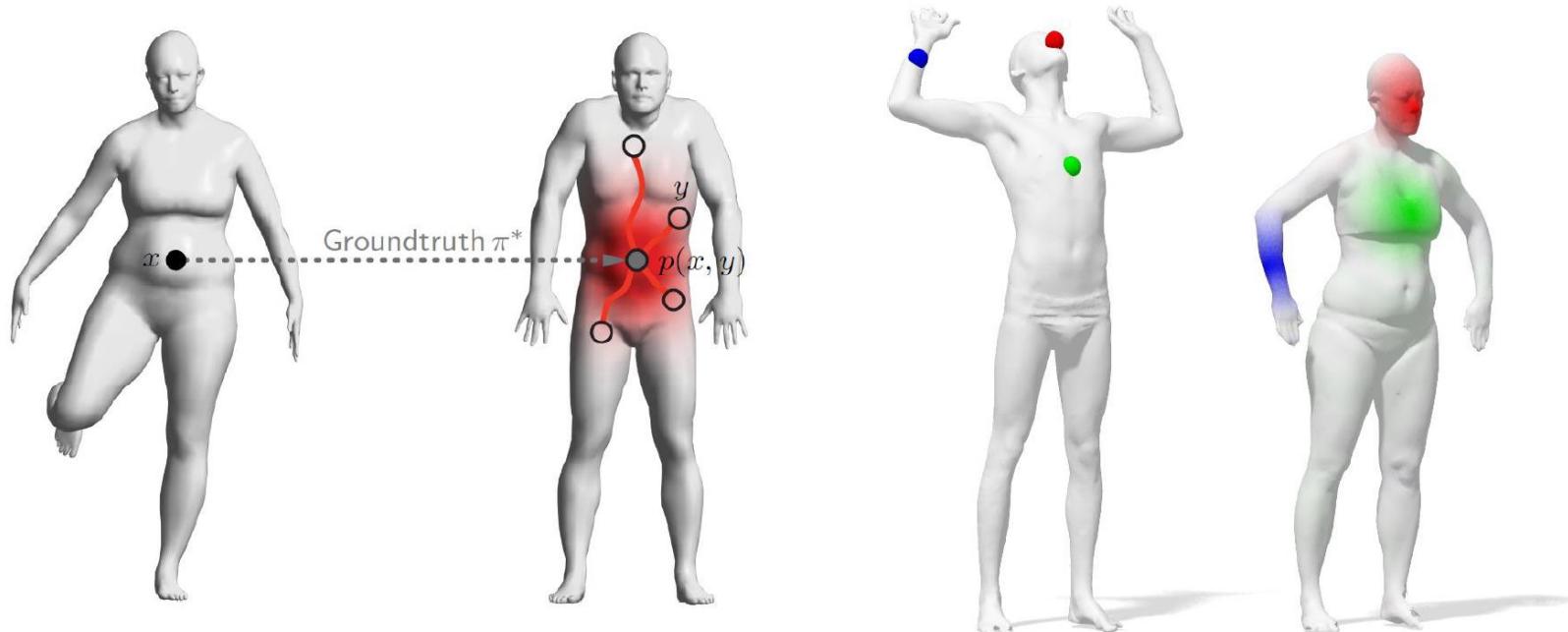


Training loss: 
$$\ell_F = \sum_{(x,y) \in (\mathcal{X},\mathcal{Y})} P(x,y) d_Y(y, \pi^*(x))$$

Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# **P = soft correspondence error**

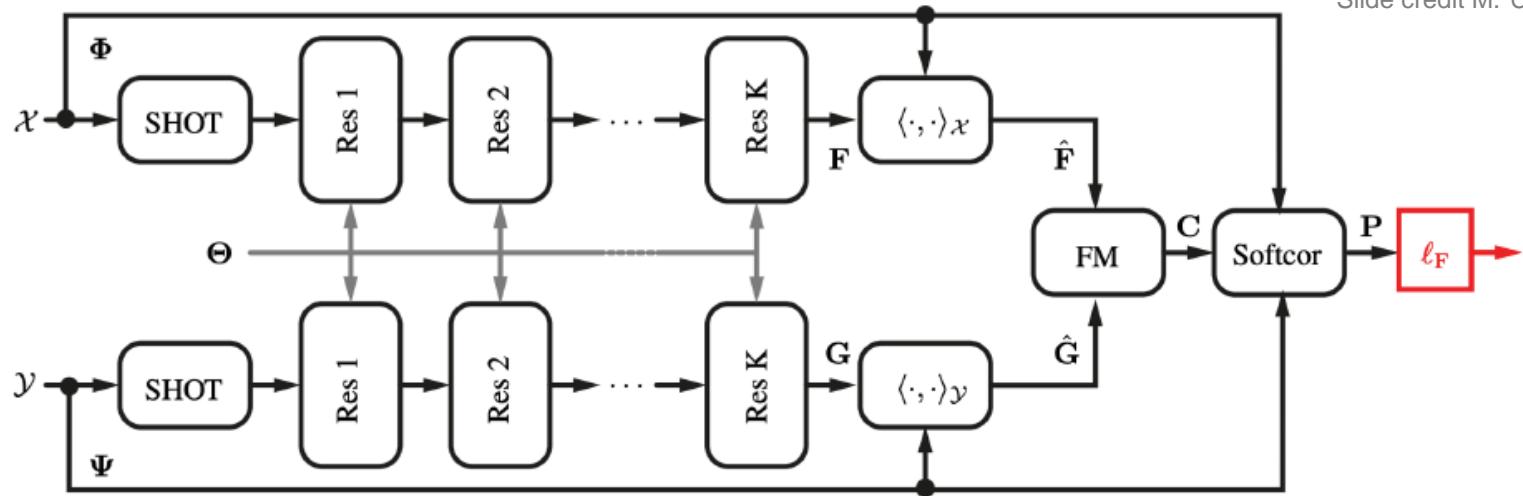
$$P = |\Psi C \Phi^\top \Omega_{\mathcal{X}}| \longrightarrow P(x, y) = |\Psi C \Phi^\top \Omega_{\mathcal{X}} \delta_x|_y$$



Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# Positive aspect

Slide credit M. Ovsjanikov

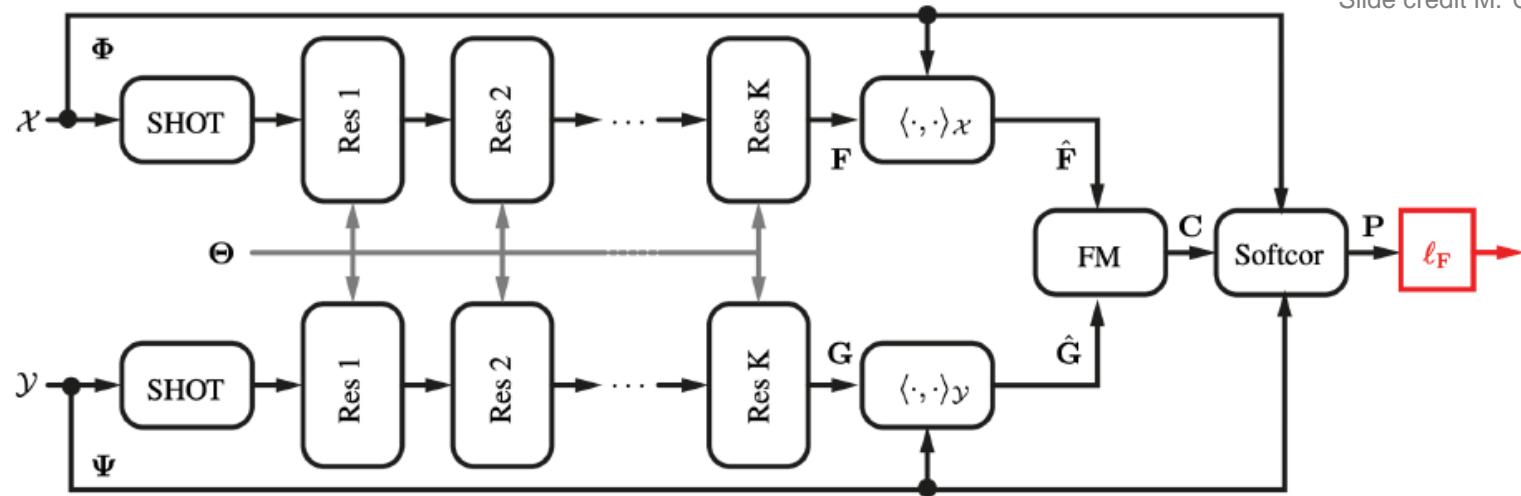


**Positive aspect:** evaluating the fmaps representation all the correspondences are evaluated together

Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# Negative aspect

Slide credit M. Ovsjanikov

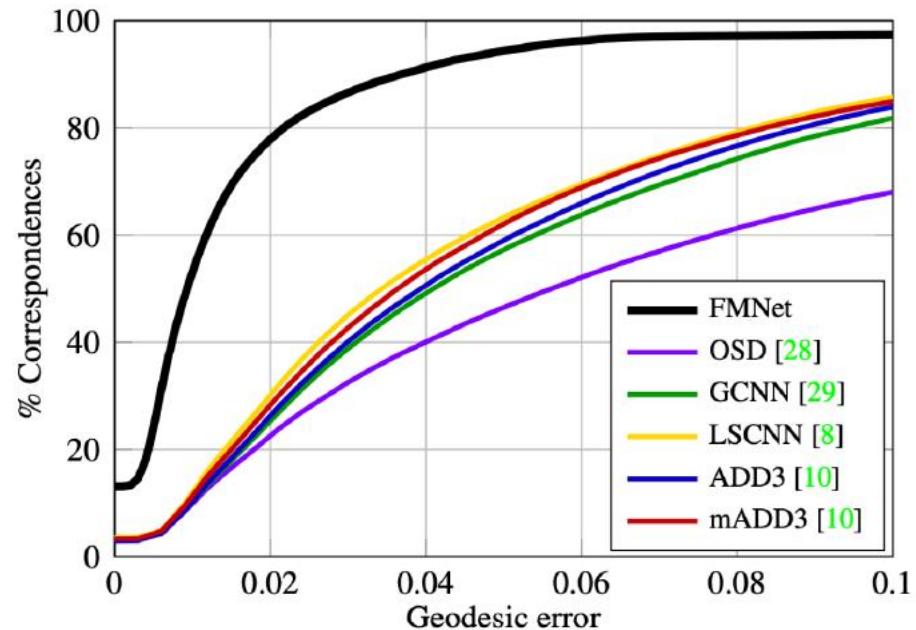


**Negative aspect:** the ground-truth correspondences  
between training data are required

Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# State of the art results in 2017

Slide credit M. Ovsjanikov



Deep functional maps: Structured prediction for dense shape correspondence, O. Litany, et al, ICCV (2017).

# Learning and shape matching questions (1)

Problem: the need for ground-truth correspondences between pairs of shapes:

- Not always available
- Not available for all the classes or type of data
- Not easy to encode for different discretizations

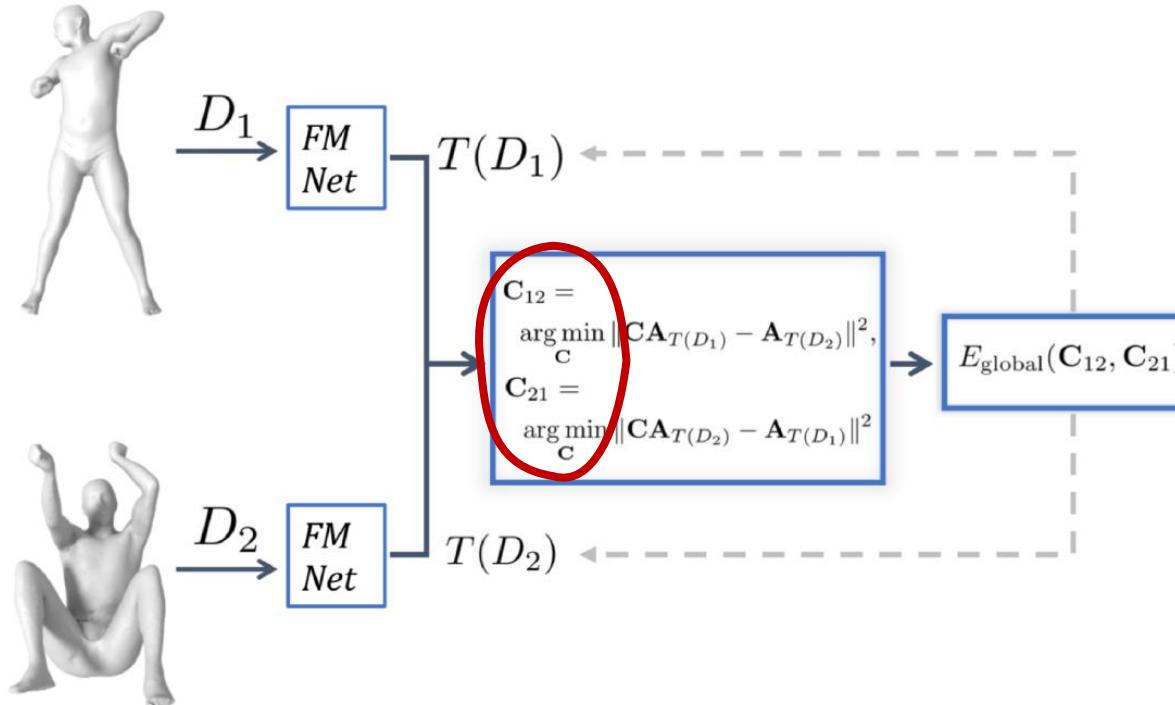
1. Can we avoid ground-truth correspondences and compute a globally coherent correspondence efficiently?

Note: related concurrent paper: Unsupervised learning of dense shape correspondence. Halimi et al. , CVPR, 2019

# Unsupervised learning

Slide credit M. Ovsjanikov

Replace supervised loss with **unsupervised** loss



Unsupervised Deep Learning for Structured Shape Matching, J.-M. Rouffosse et al., ICCV 2019

$$\text{loss}_{\text{unsupervised}} = \sum_{i \in \text{penalties}} w_i E_i(C_{1-2}, C_{2-1})$$

$$\begin{cases} E_1(C_{12}, C_{21}) = \|C_{12}C_{21} - Id\|^2 & \text{Bijectivity} \\ E_1(C_{12}, C_{21}) = \|C_{21}C_{12} - Id\|^2 \end{cases}$$

$$E_2(C) = \|C^T C - Id\|^2 \quad \text{Area-preservation}$$

$$E_3(C) = \|\Lambda_2 C - C \Lambda_1\|^2 \quad \text{Near-isometry}$$

$$E_4(C) = \sum_i \|CX_{f_i} - Y_{g_i}C\|^2 \quad \text{Functional map close to pointwise one.}$$

50 times faster  
than FMNET

All these penalties are represented in the reduced basis (50 Fourier basis)

# Training dataset

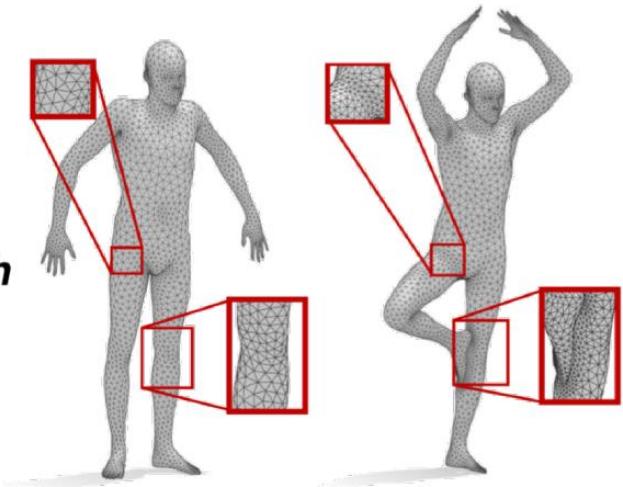
Slide credit M. Ovsjanikov

FAUST :

- ▶ Subset: train on 80 and test on 20
- ▶ Whole set : train on 100 shapes, ***without ground truth***

SCAPE :

- ▶ Subset: train on 50 and test on 10
- ▶ Whole set : train on 60 shapes, ***without ground truth***



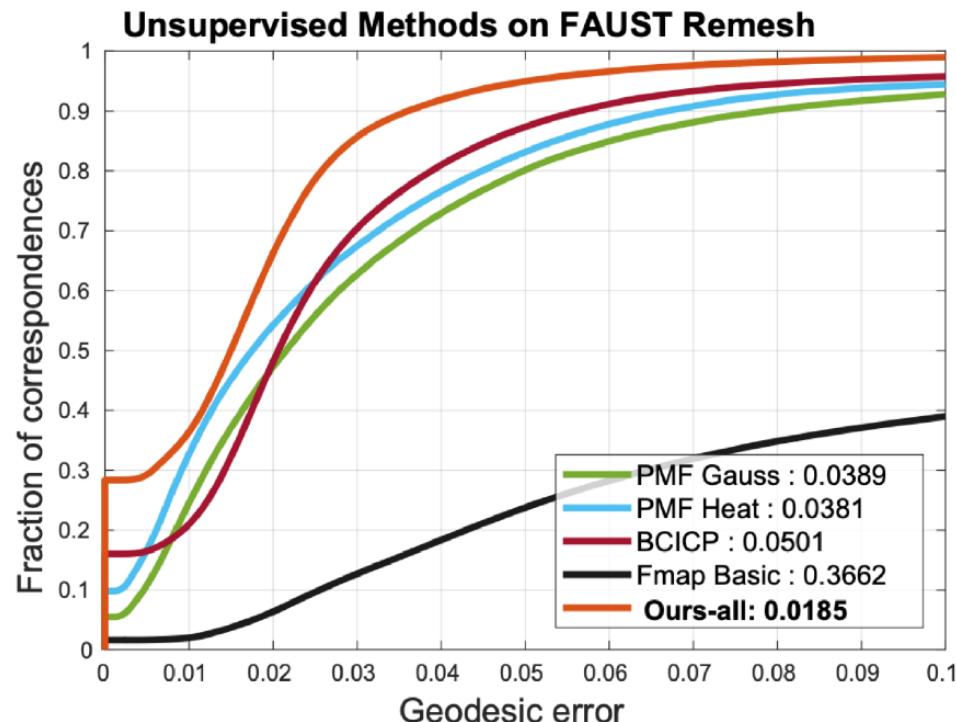
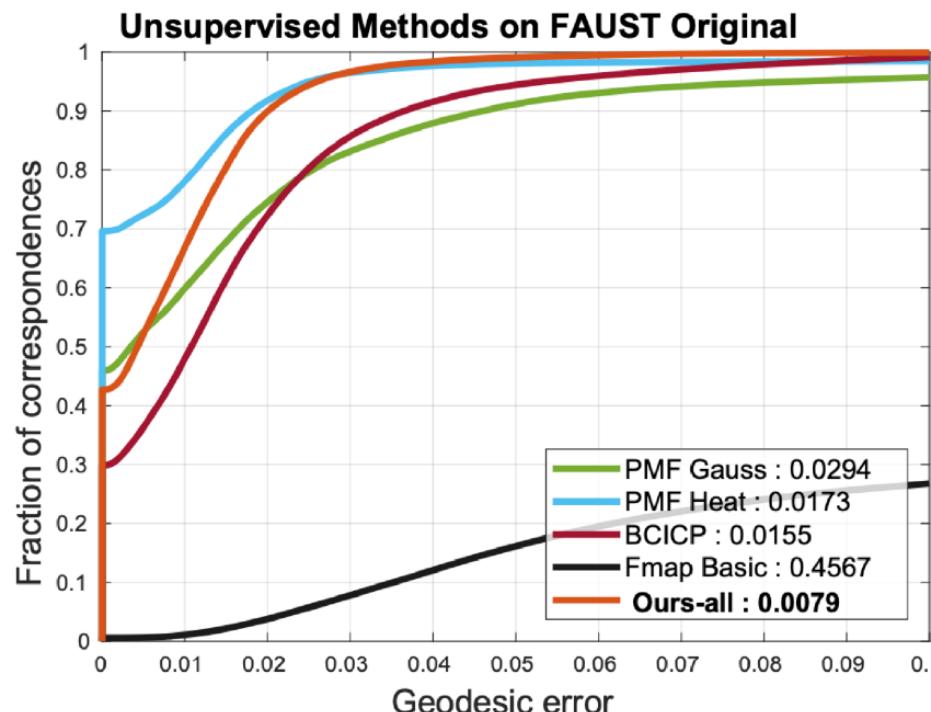
Remeshed FAUST - 5000 vertices

Datasets released as part of: *Continuous and Orientation-preserving Correspondences via Functional Maps*, J. Ren, A. Poulenard, P. Wonka, M. O, SIGGRAPH Asia 2018

Unsupervised Deep Learning for Structured Shape Matching, J.-M. Roufosse et al., ICCV 2019

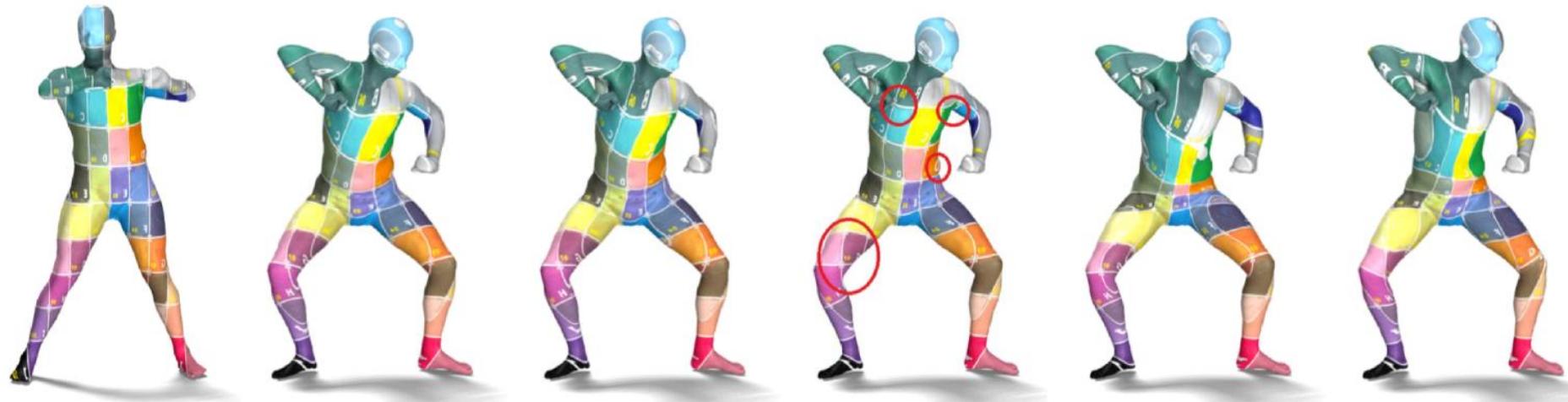
# Comparison to unsupervised methods

Slide credit M. Ovsjanikov



# Qualitative comparison

Slide credit M. Ovsjanikov



Source

Ground-Truth

SURFMNet

BCICP

PMF (heat)

PMF (gauss)

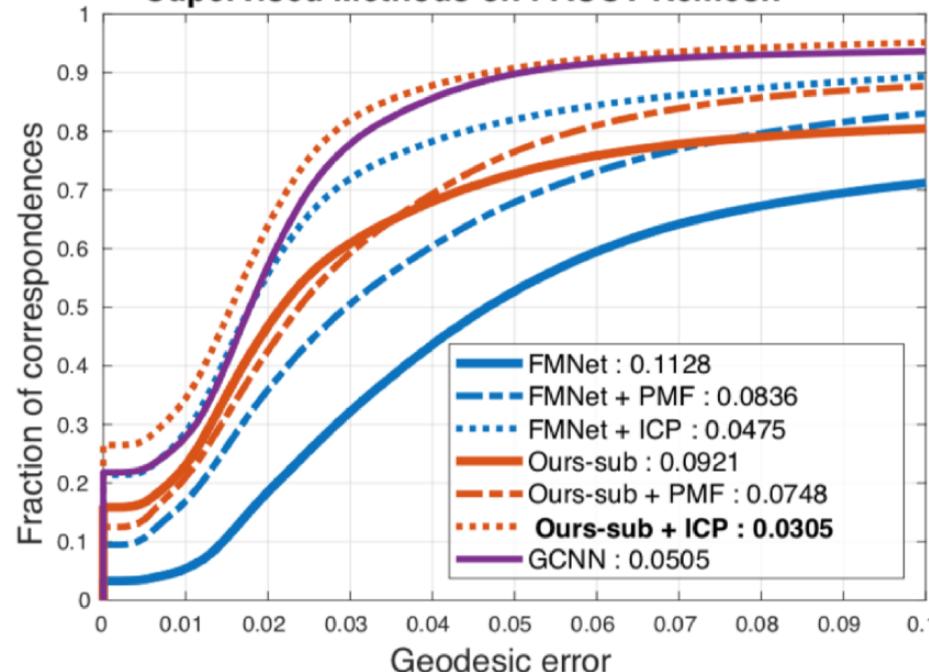
State-of-the-art for unsupervised methods

Unsupervised Deep Learning for Structured Shape Matching, J.-M. Rouffosse et al., ICCV 2019

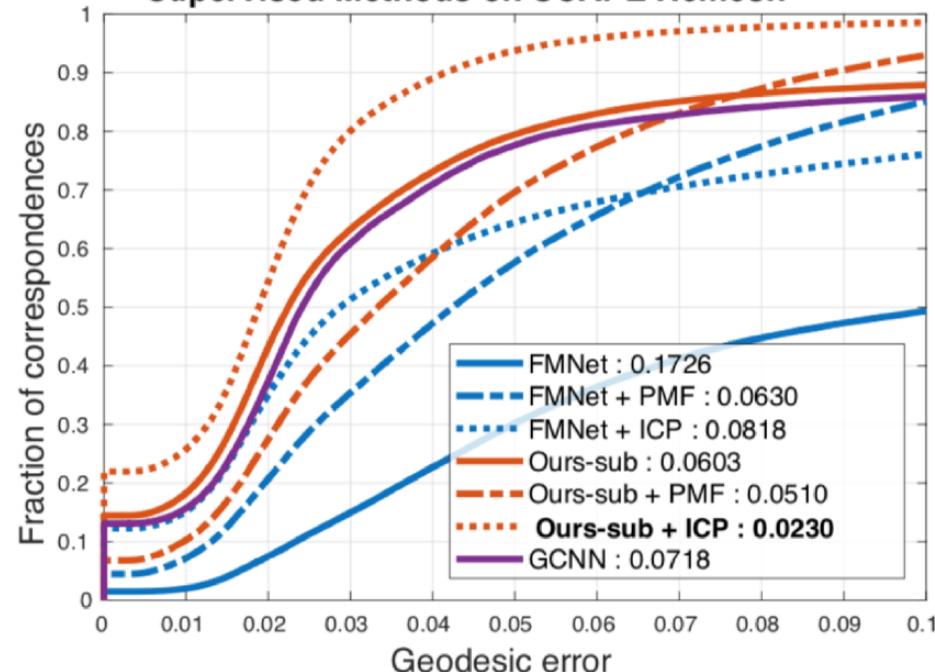
# Comparison to supervised methods

Slide credit M. Ovsjanikov

Supervised Methods on FAUST Remesh



Supervised Methods on SCAPE Remesh



# Qualitative comparison to supervised methods

Slide credit M. Ovsjanikov



Source	Ground-Truth	Ours-sub + ICP	Ours-sub	FMNet	FMNet + PMF	GCNN
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Comparable to supervised methods

Unsupervised Deep Learning for Structured Shape Matching, J.-M. Roufosse et al., ICCV 2019

# Original vs learned descriptors

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Source descriptor before



Target descriptor before



Source descriptor after



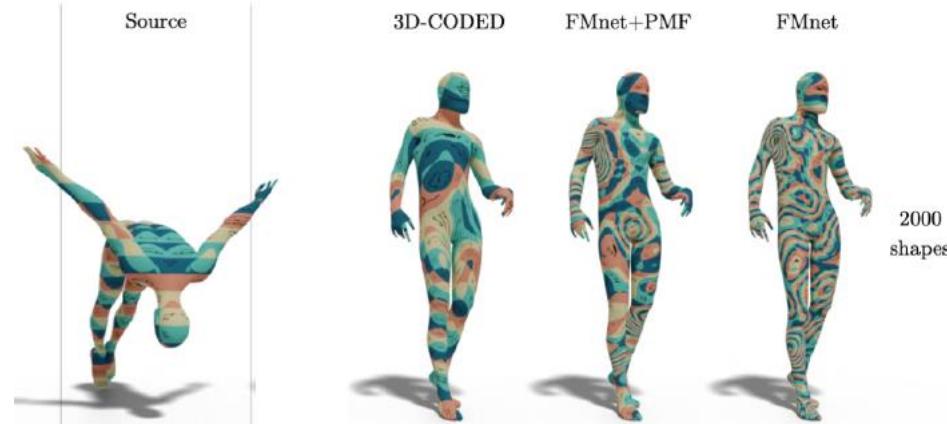
Target descriptor after

Unsupervised Deep Learning for Structured Shape Matching, J.-M. Roufosse et al., ICCV 2019

# Learning and shape matching questions (2)

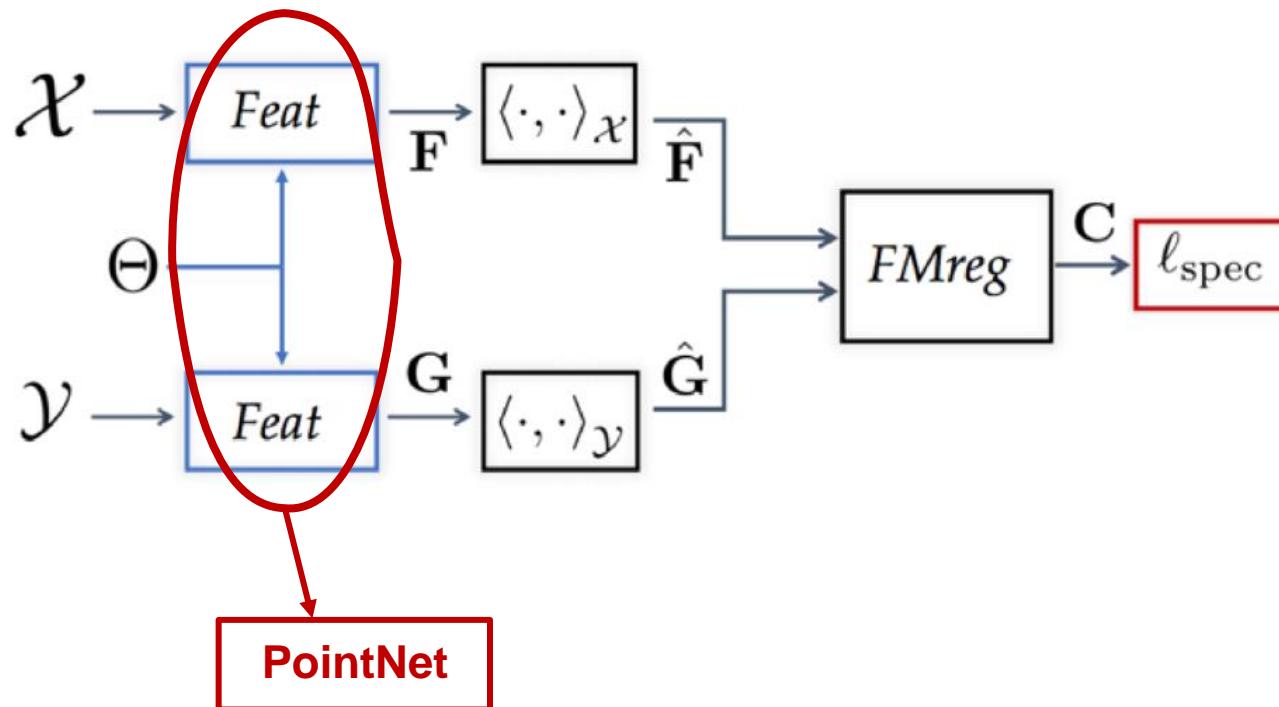
Slide credit M. Ovsjanikov

1. Can we directly learn the descriptors from raw data without using pre-computed descriptors like SHOT?
2. Can we be invariant to different connectivity or discrete representations?



# Descriptors from raw geometry

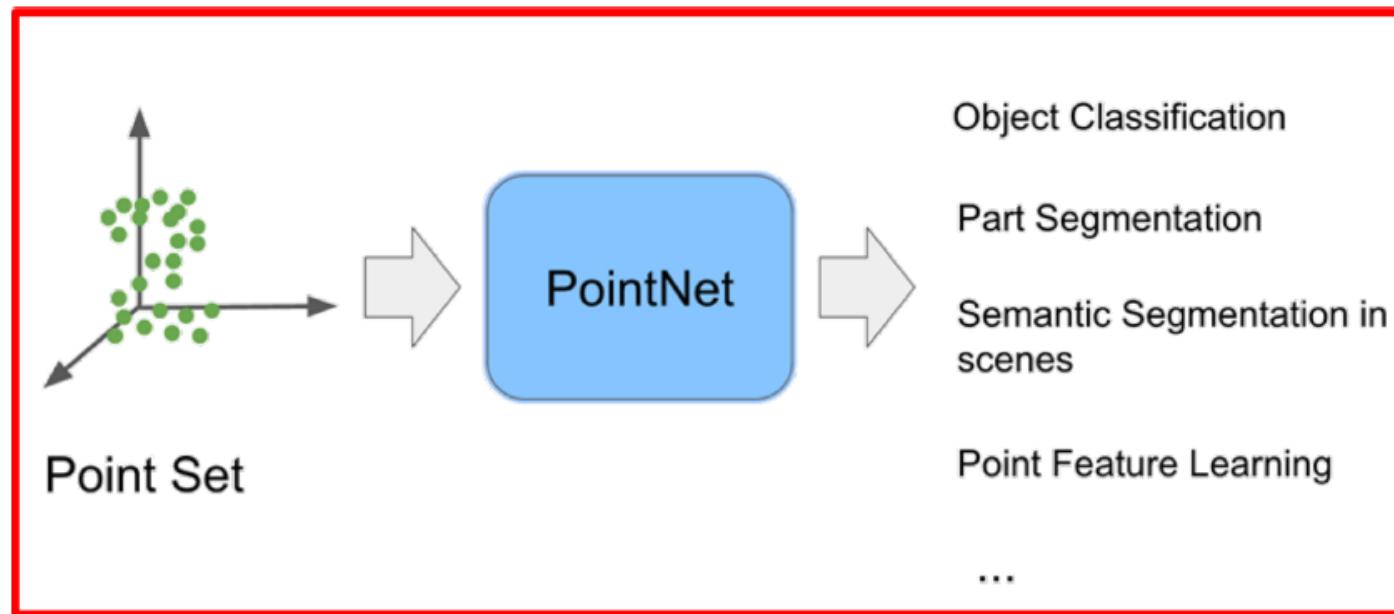
Slide credit M. Ovsjanikov



Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence, Donati et al., CVPR 2020

# PointNet

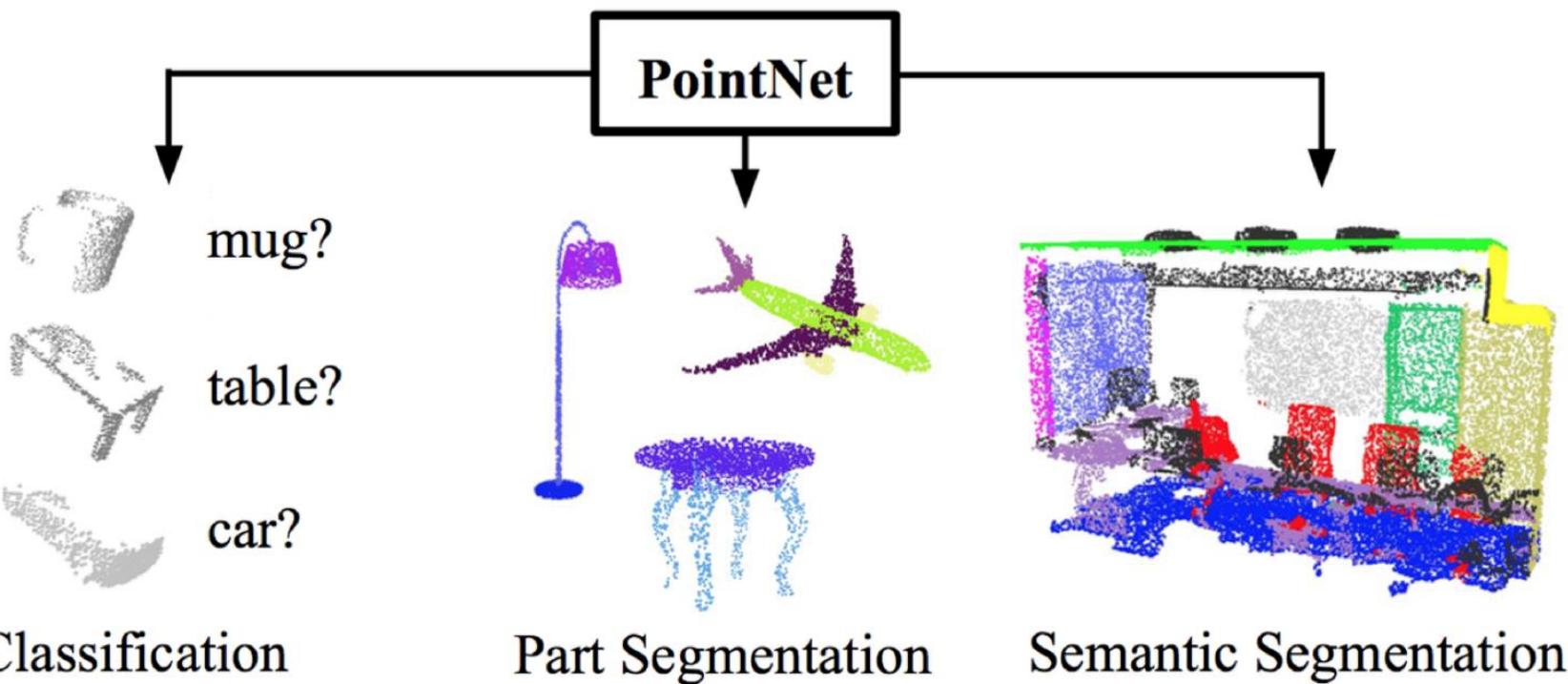
Slide credit M. Ovsjanikov



Pointnet: Deep learning on point sets for 3d classification and segmentation, Qi et al., CVPR 2017

# PointNet applications

Slide credit M. Ovsjanikov



Classification

Part Segmentation

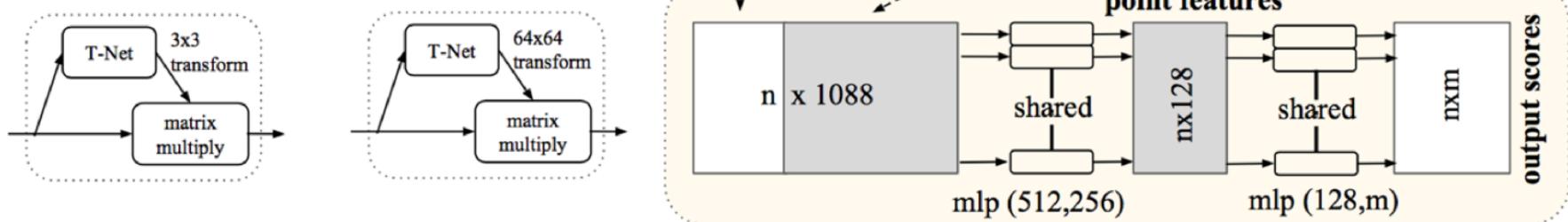
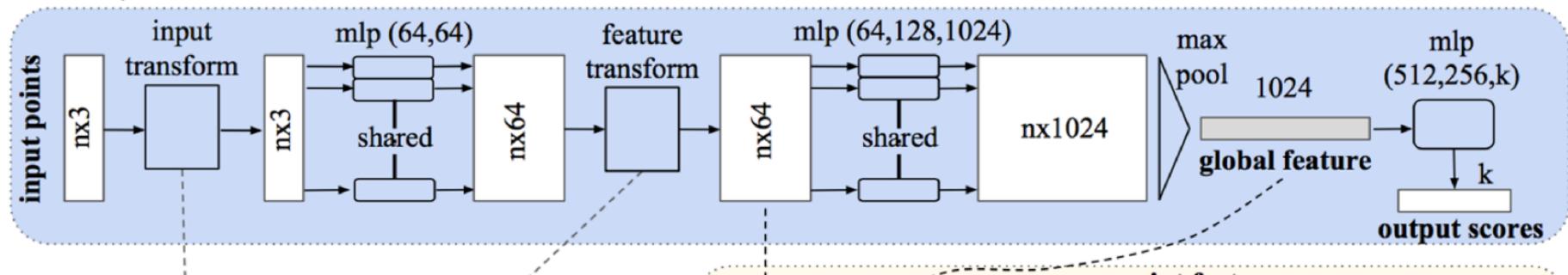
Semantic Segmentation

Pointnet: Deep learning on point sets for 3d classification and segmentation, Qi et al., CVPR 2017

# PointNet-architecture (in brief)

Slide credit M. Ovsjanikov

*Classification Network*

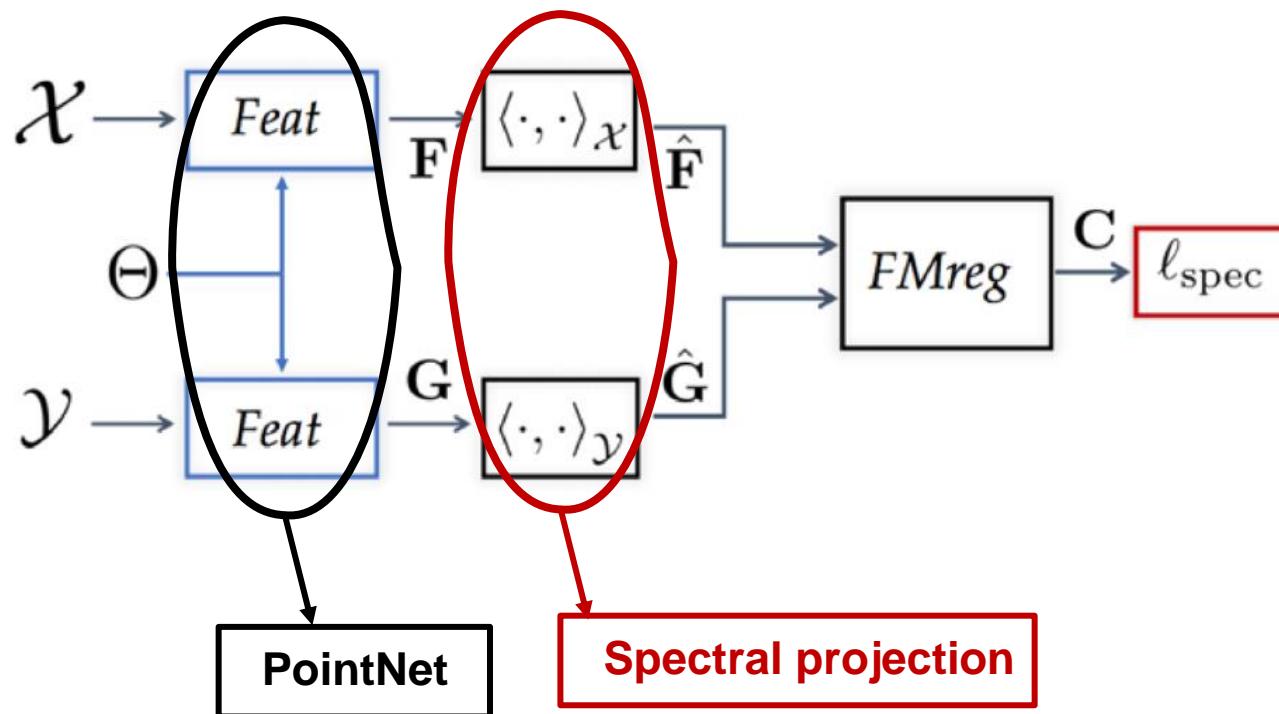


*Segmentation Network*

Pointnet: Deep learning on point sets for 3d classification and segmentation, Qi et al., CVPR 2017

# Deep-Geometry FMaps

Slide credit M. Ovsjanikov



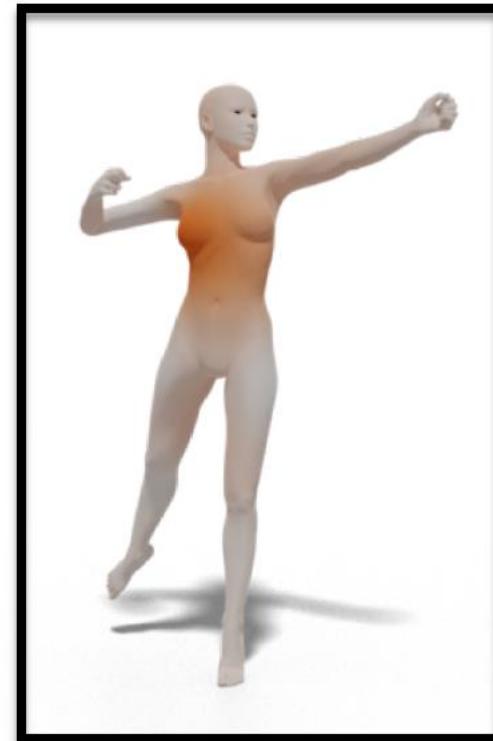
Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence, Donati et al., CVPR 2020

# Descriptors from raw geometry

Slide credit M. Ovsjanikov



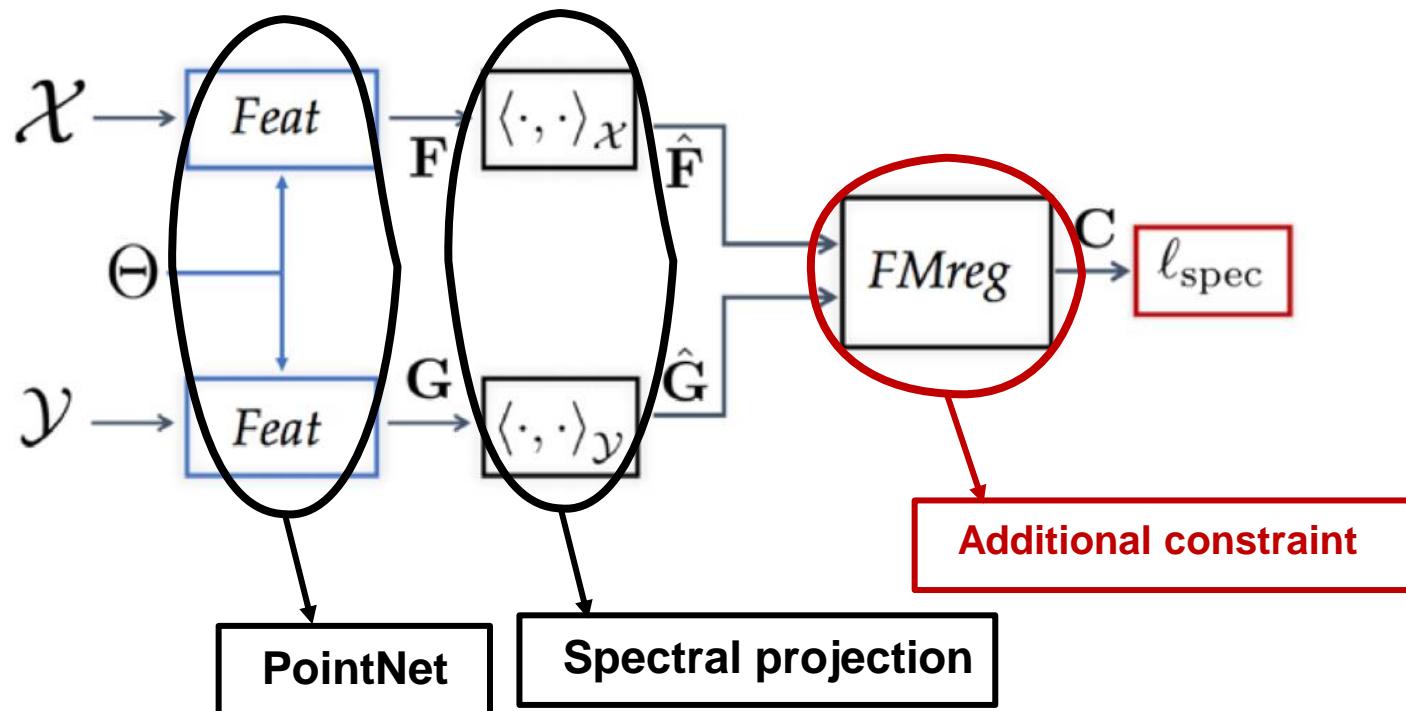
spectral projection  
→  
(on Laplace Basis)



Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence, Donati et al., CVPR 2020

# Additional constraint in the FM layer

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Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence, Donati et al., CVPR 2020

# Commutativity with the Laplacian

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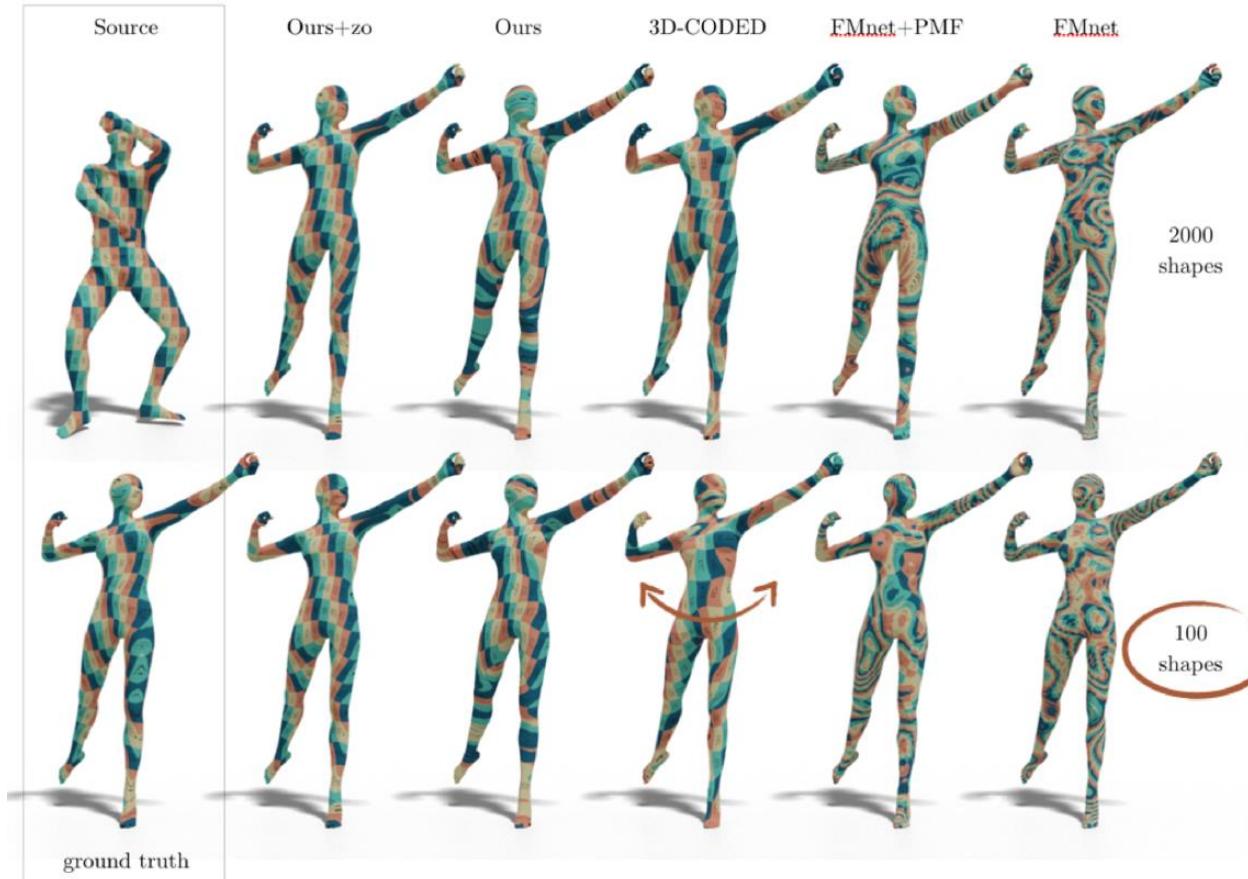
$$\min_{\mathbf{C}} \|\mathbf{CA} - \mathbf{B}\|^2 + \lambda \|\mathbf{C}\Delta_{\mathcal{M}} - \Delta_{\mathcal{N}}\mathbf{C}\|^2$$

Linear system for  
every row in  $\mathbf{C}$ !

- Fully differentiable
- gives better maps

Already used for the standard FMAPS methods and for  
the first time adopted in a data-driven approach

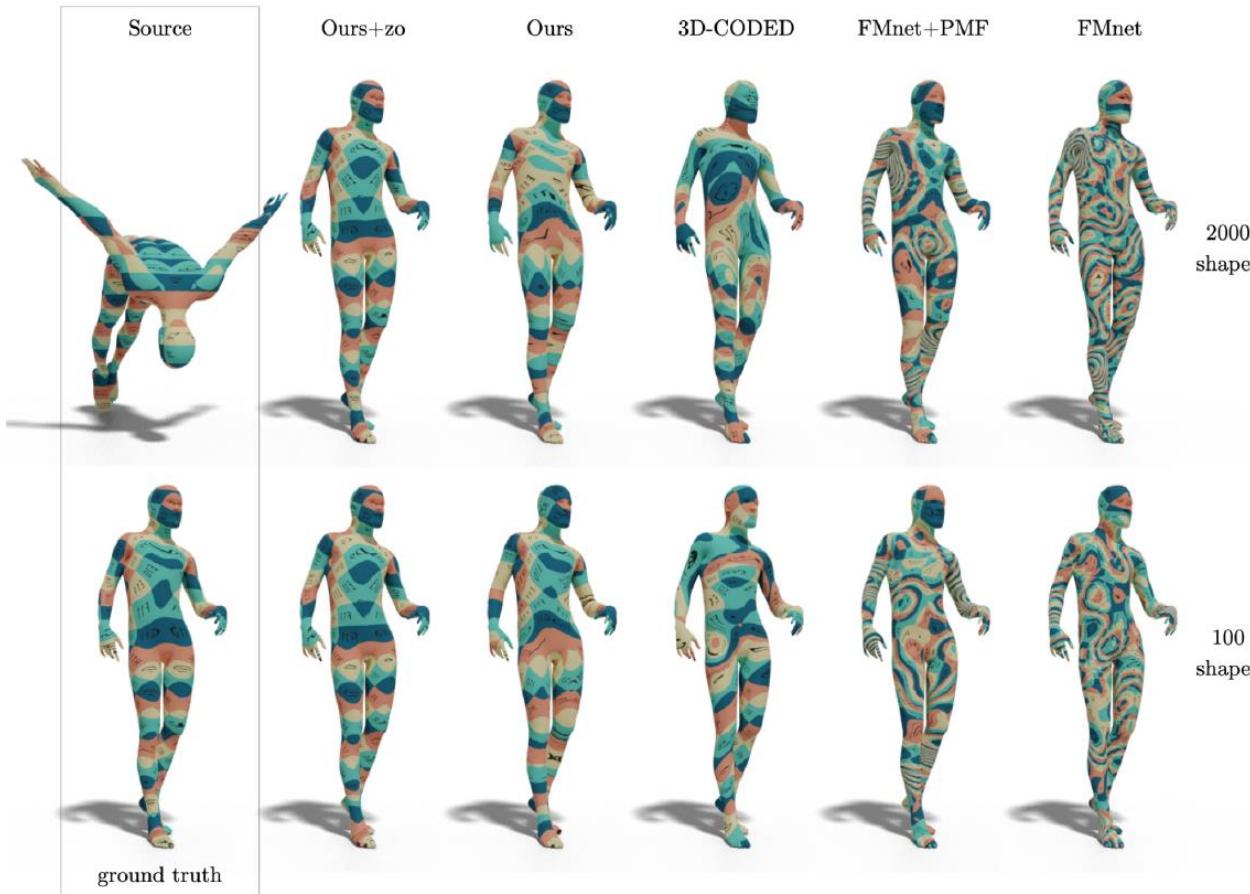
# Qualitative comparison



Slide credit M. Ovsjanikov

Deep Geometric Functional  
Maps: Robust Feature Learning  
for Shape Correspondence,  
Donati et al., CVPR 2020

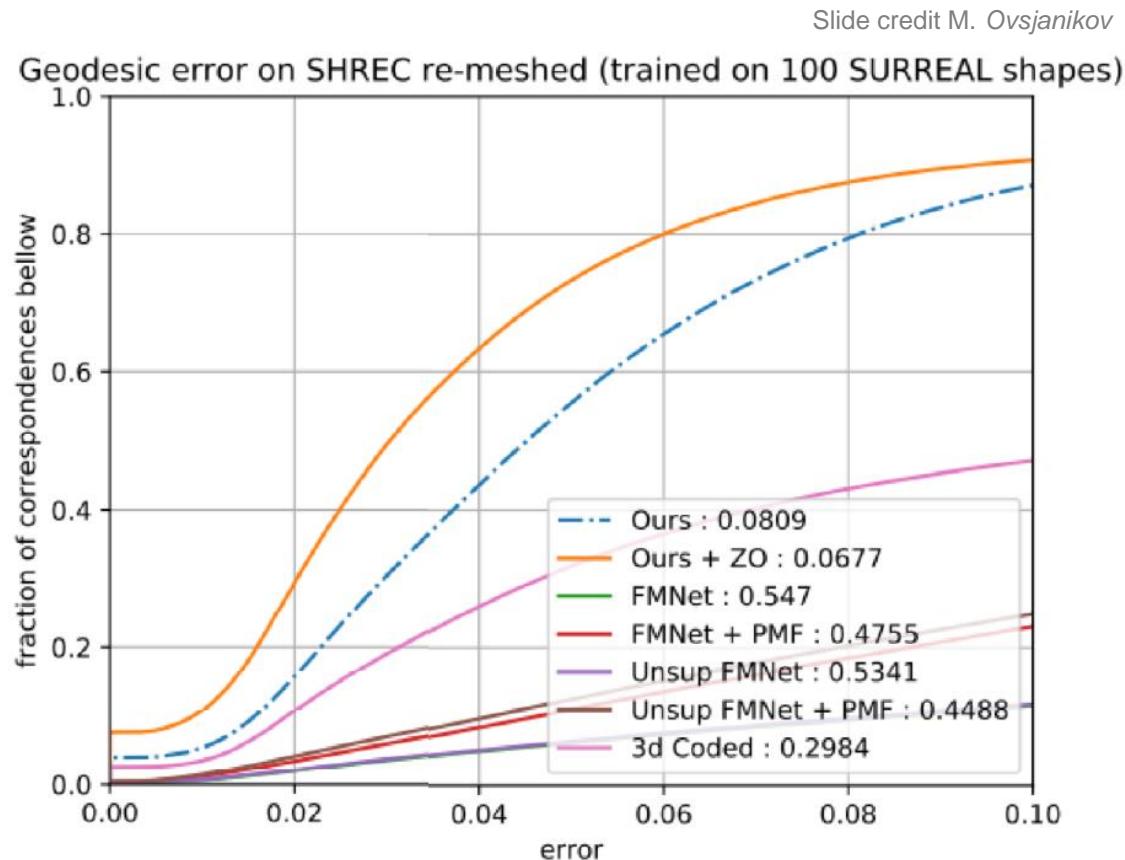
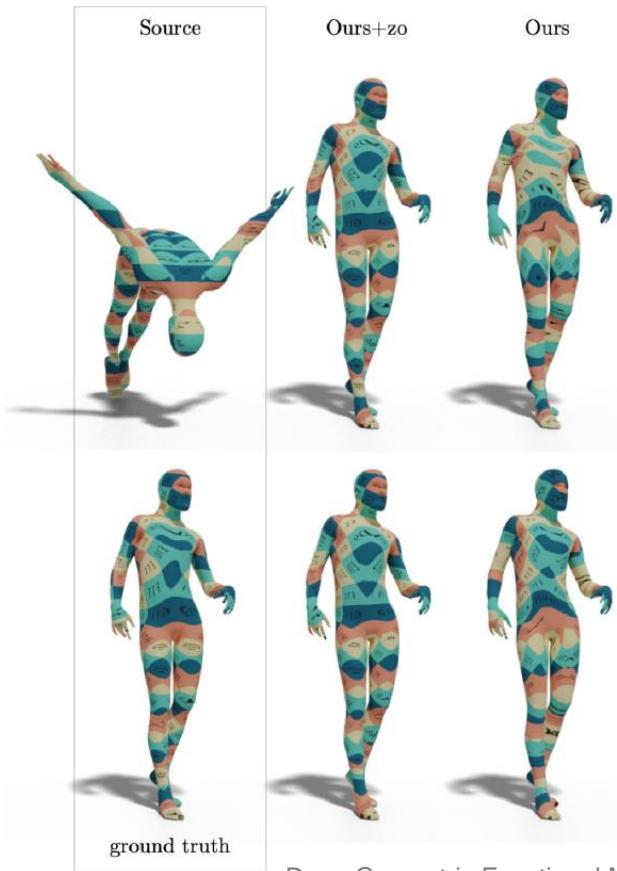
# Generalization across datasets



Deep Geometric Functional  
Maps: Robust Feature Learning  
for Shape Correspondence,  
Donati et al., CVPR 2020

Slide credit M. Ovsjanikov

# Quantitative evaluation



Deep Geometric Functional Maps: Robust Feature Learning for Shape Correspondence, Donati et al., CVPR 2020

# Conclusion on data-driven approaches

1. It is useful to compute better descriptors to generate constraints for FMAPS
2. In the context of FMAPS different unsupervised constraints can be defined
3. The dimensionality of the learned variables and the problem are smaller
4. The descriptors can be learned directly from the raw geometry
5. Refinement techniques can be applied to the estimated correspondence

## Possible future directions:

- Can we learn a new basis for FMAPS?
- Can we learn new constraints for FMAPS?
- Can we learn a refinement process for the estimated correspondences?

# questions?

