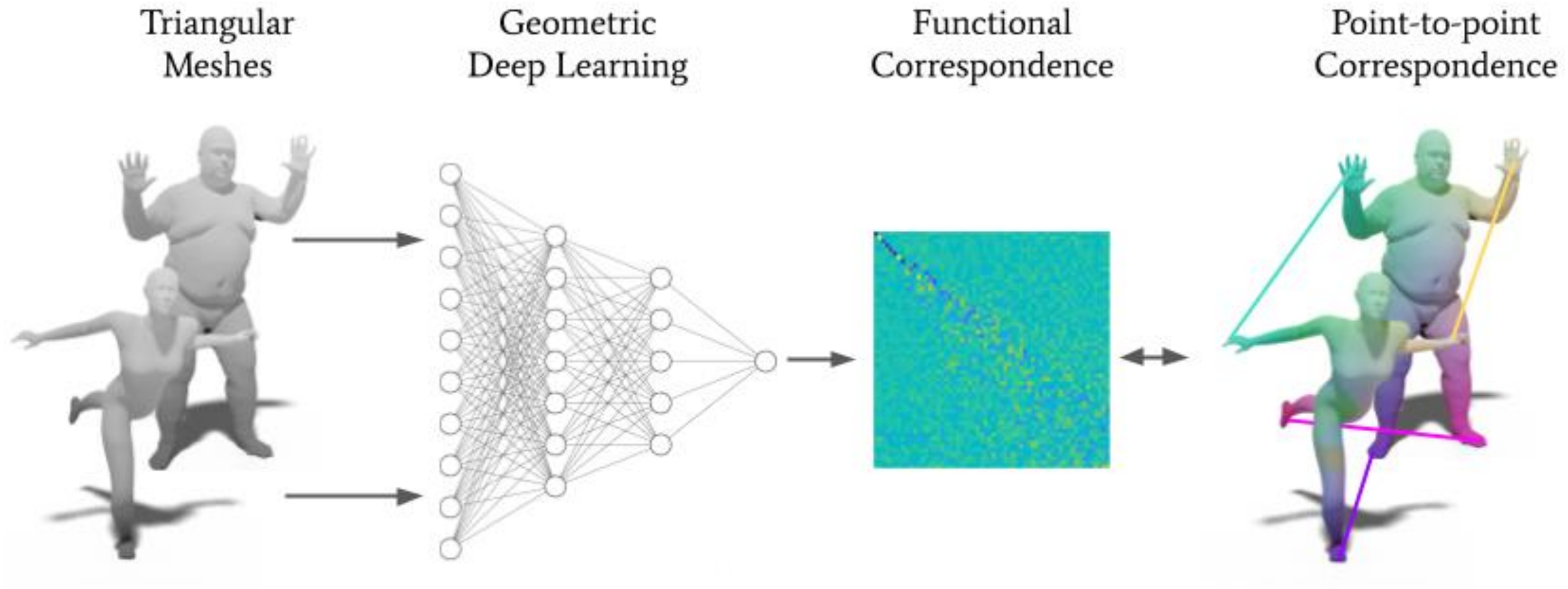
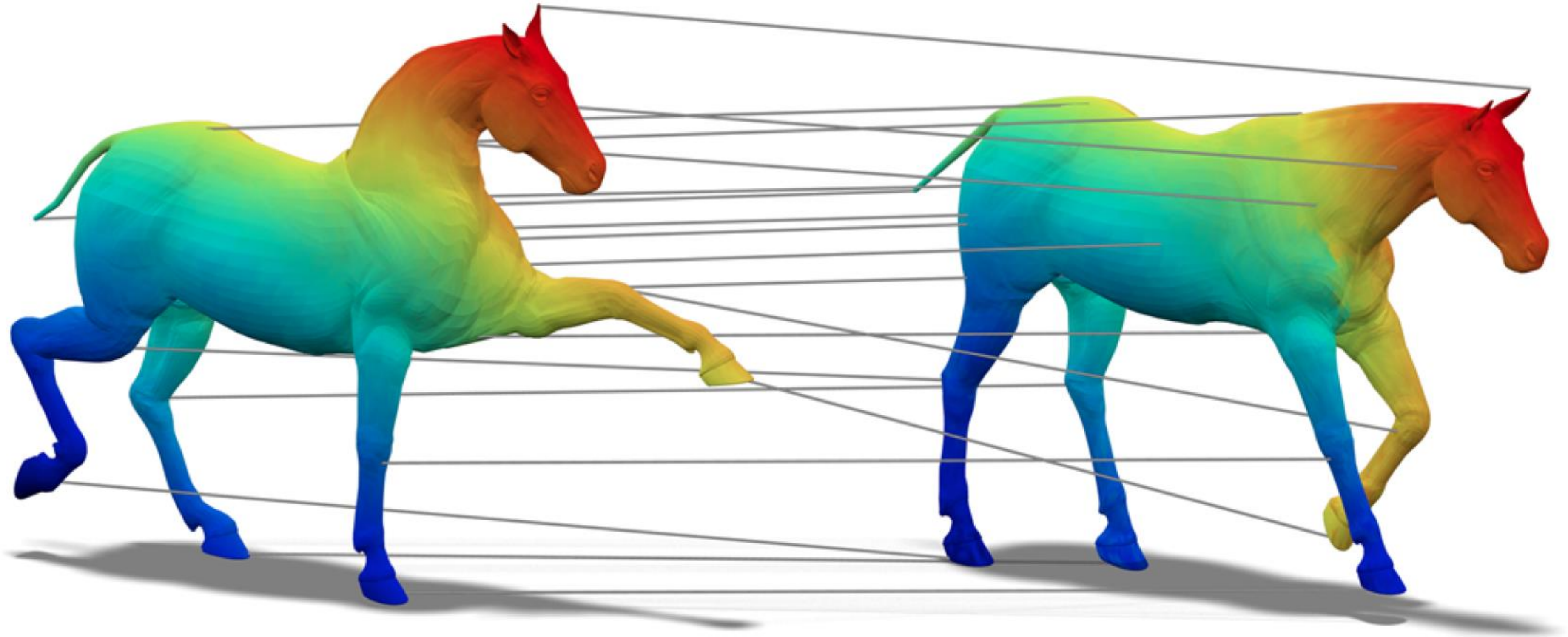
Functional Correspondence from Discrete Geometry to Learning

Riccardo Marin, Emanuele Rodolà



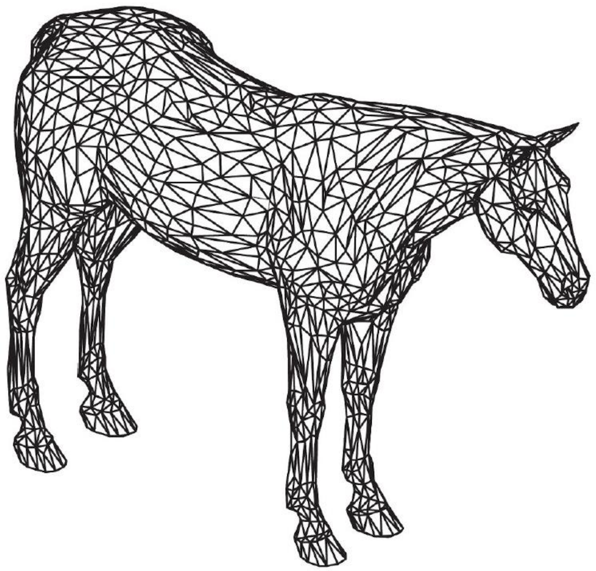
Genoa, 1st July 2023, Symposium on Geometry Processing

Task: Given a set of objects, find a correspondence between them



Huge permutation matrix

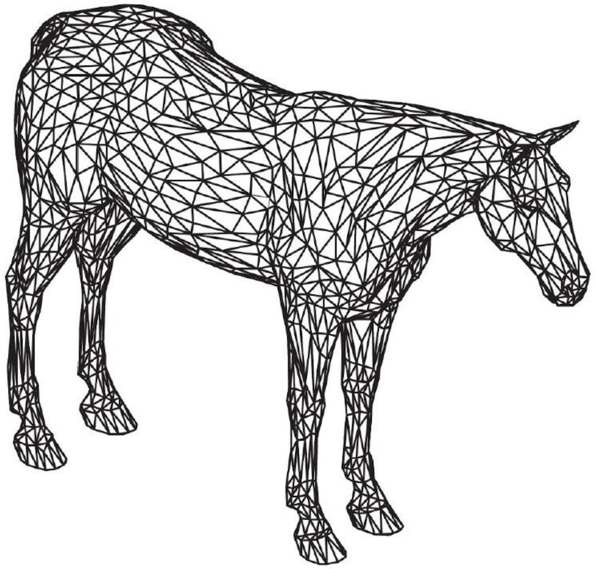
Can we make it more **compact**?



graph

Vertices $\mathcal{V} = \{1, \ldots, n\}$

Edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$



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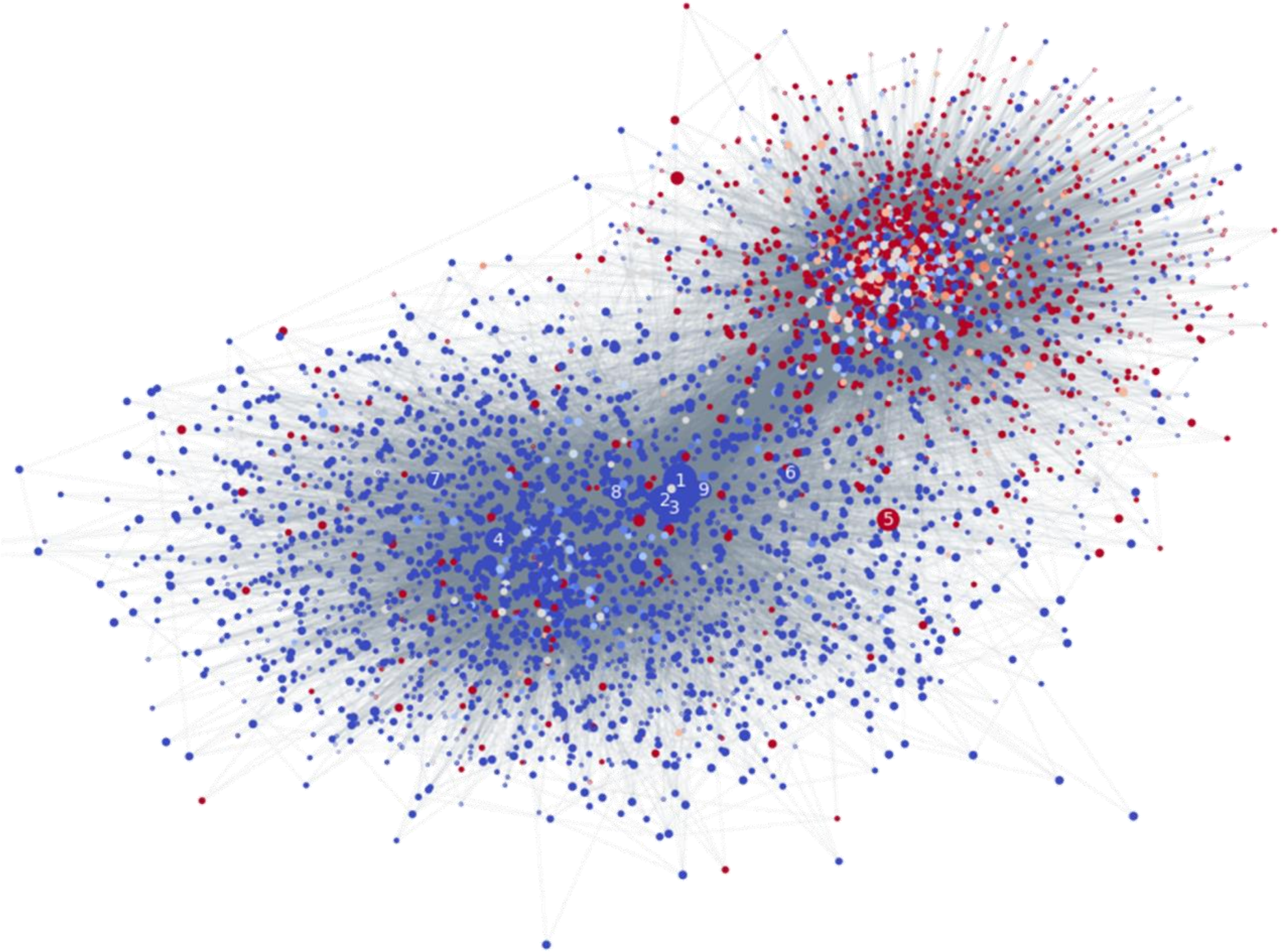
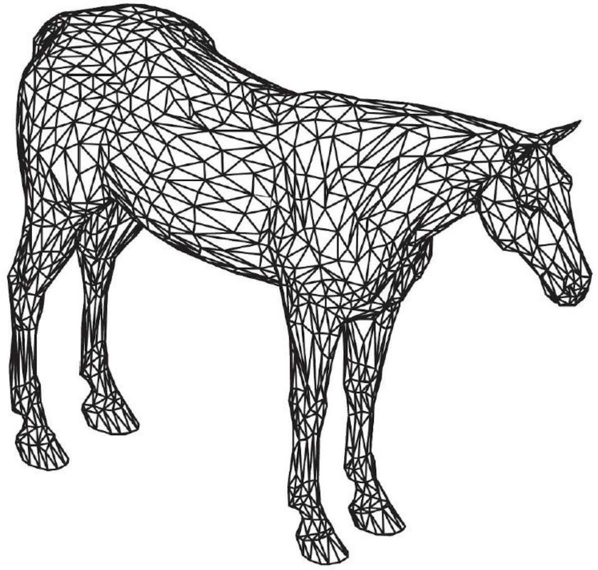
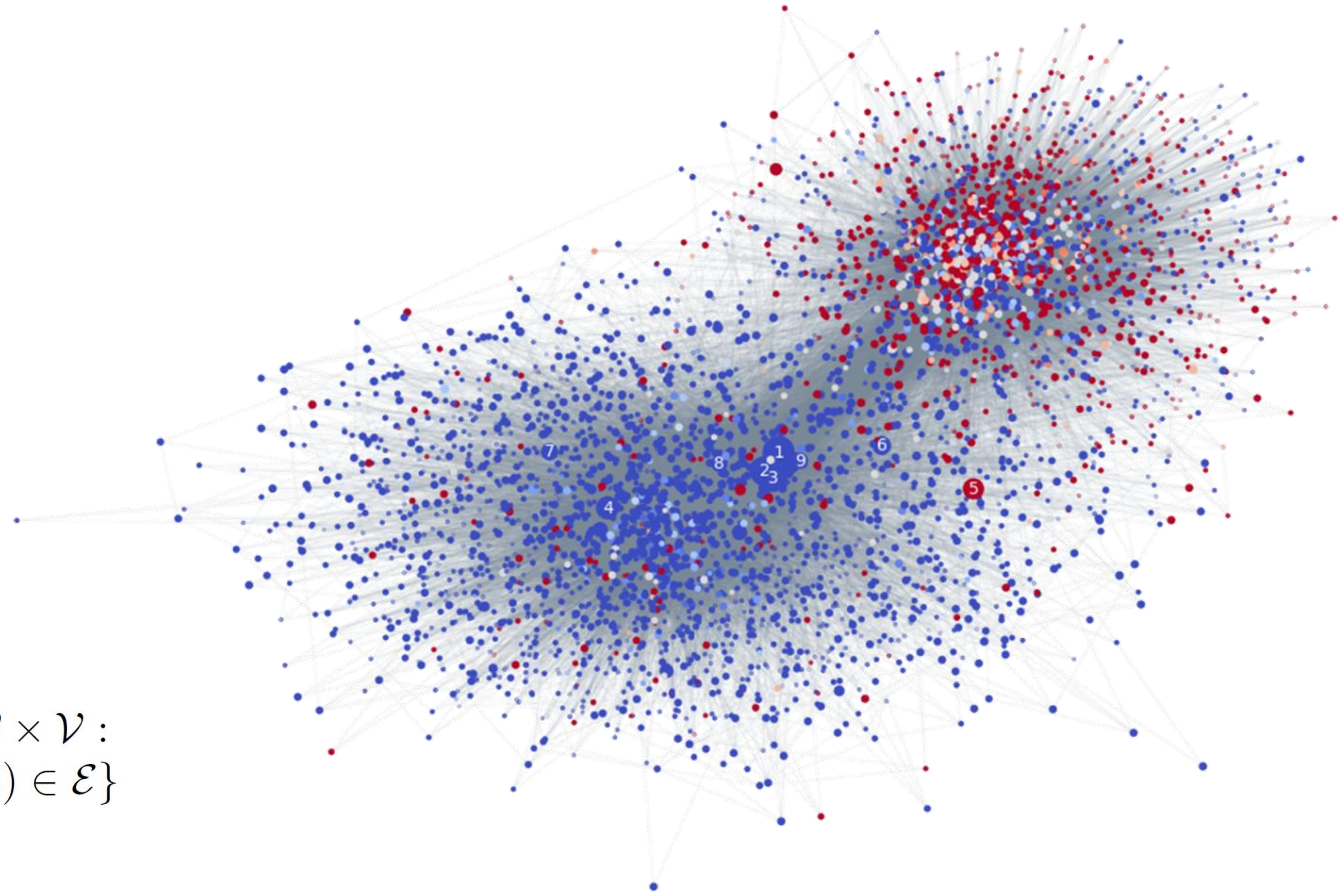




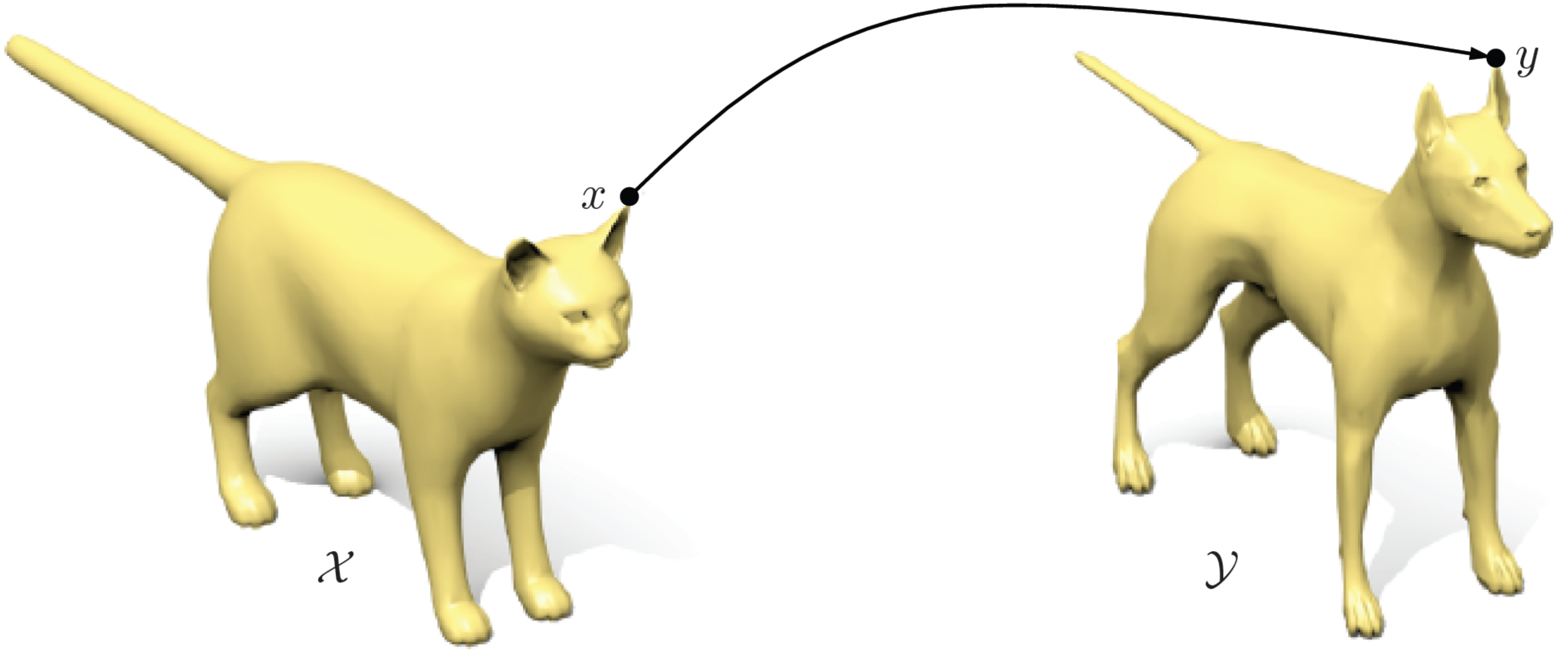
Triangular mesh



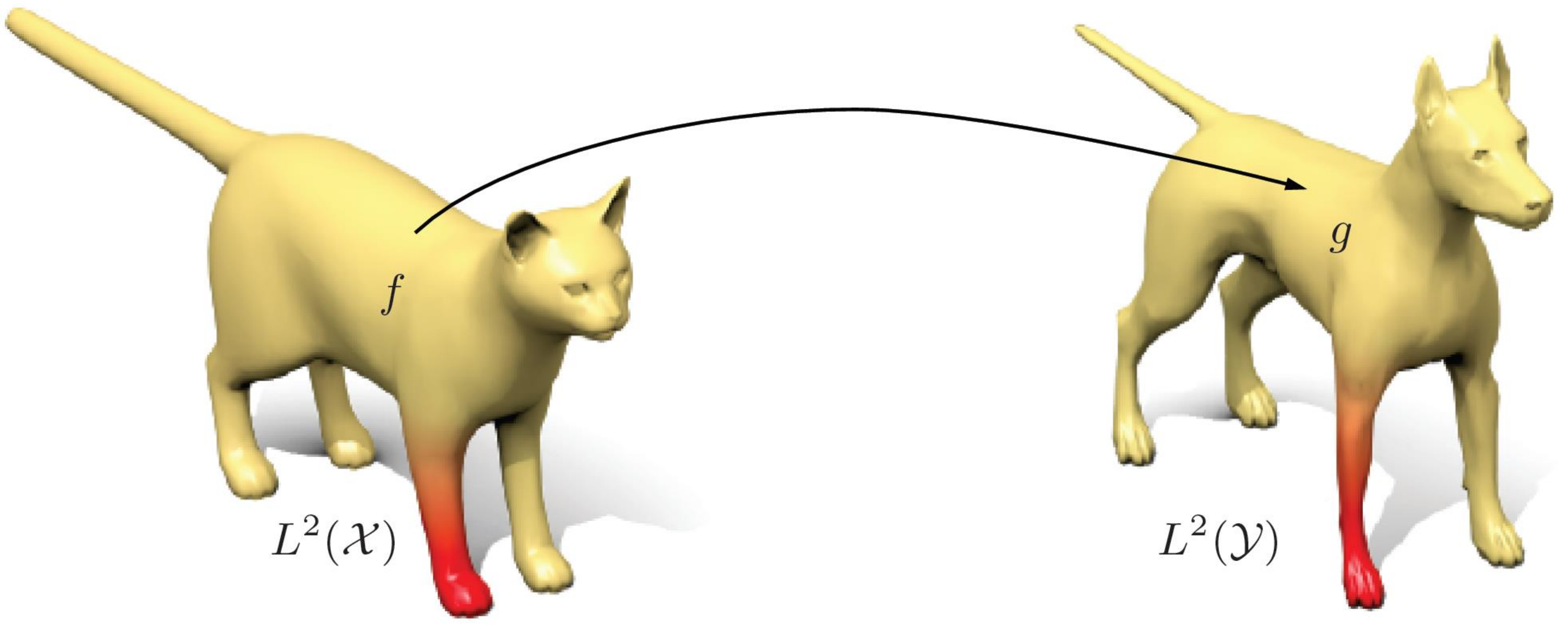
Vertices $\mathcal{V} = \{1, \ldots, n\}$

Edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$

Faces $\mathcal{F} = \{(i,j,k) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V} : (i,j), (j,k), (k,i) \in \mathcal{E}\}$

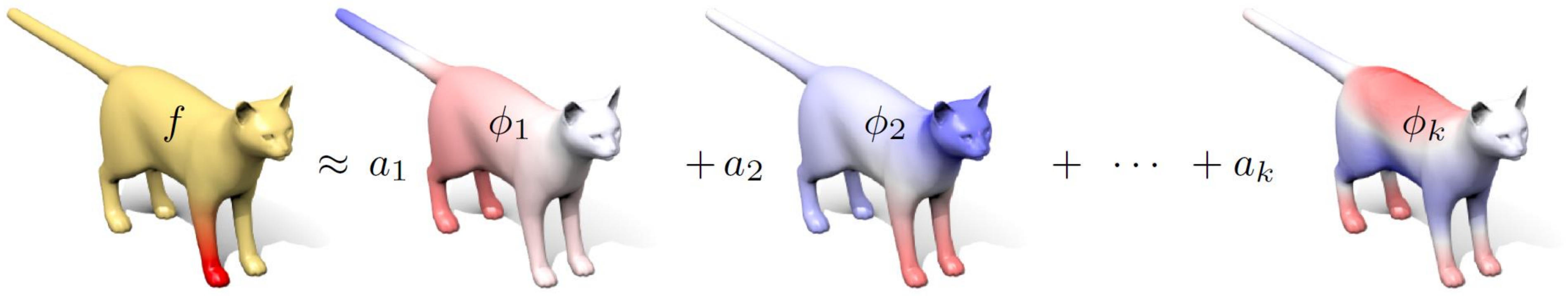


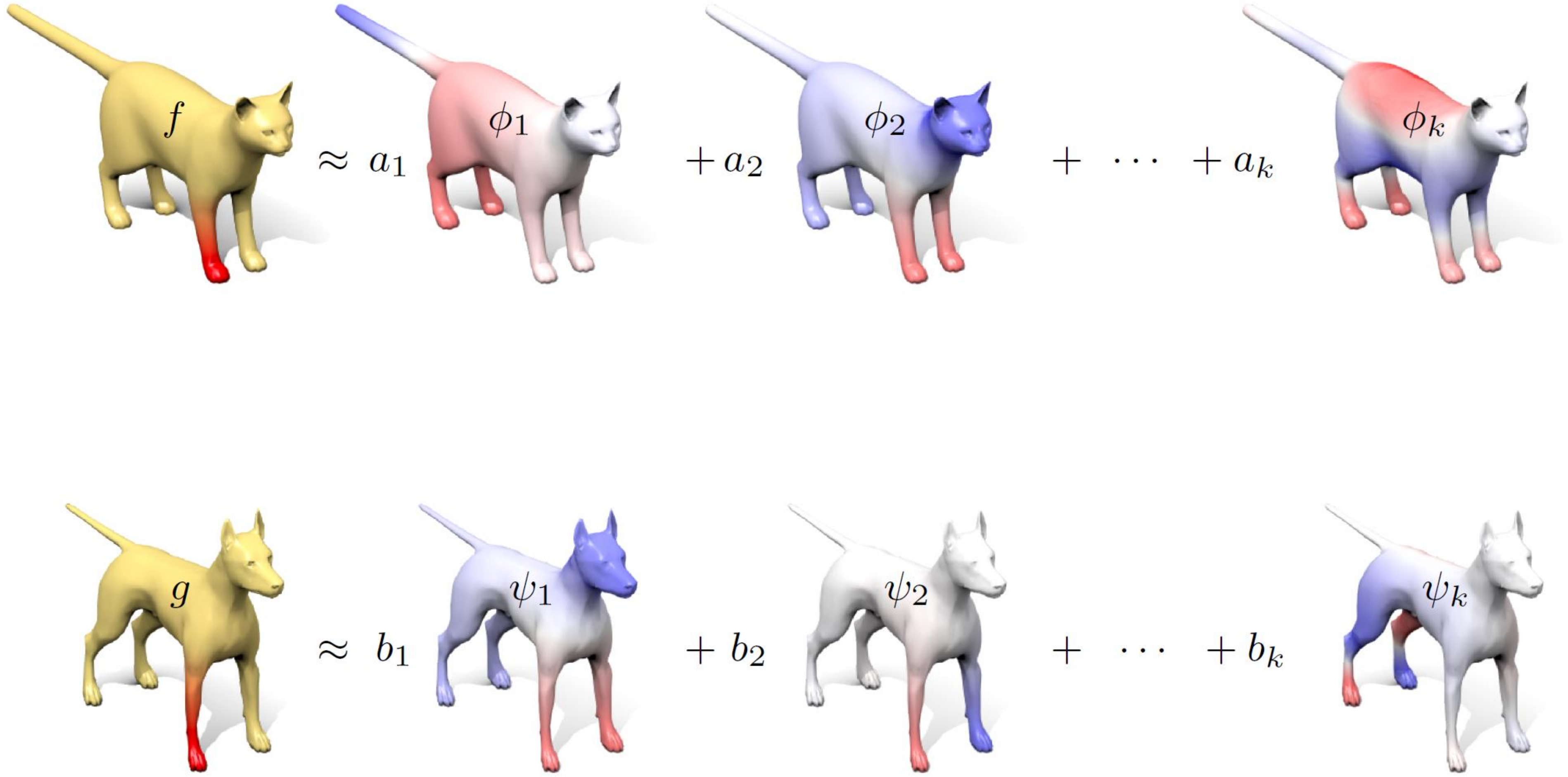
Point-wise map $t \colon \mathcal{X} \to \mathcal{Y}$



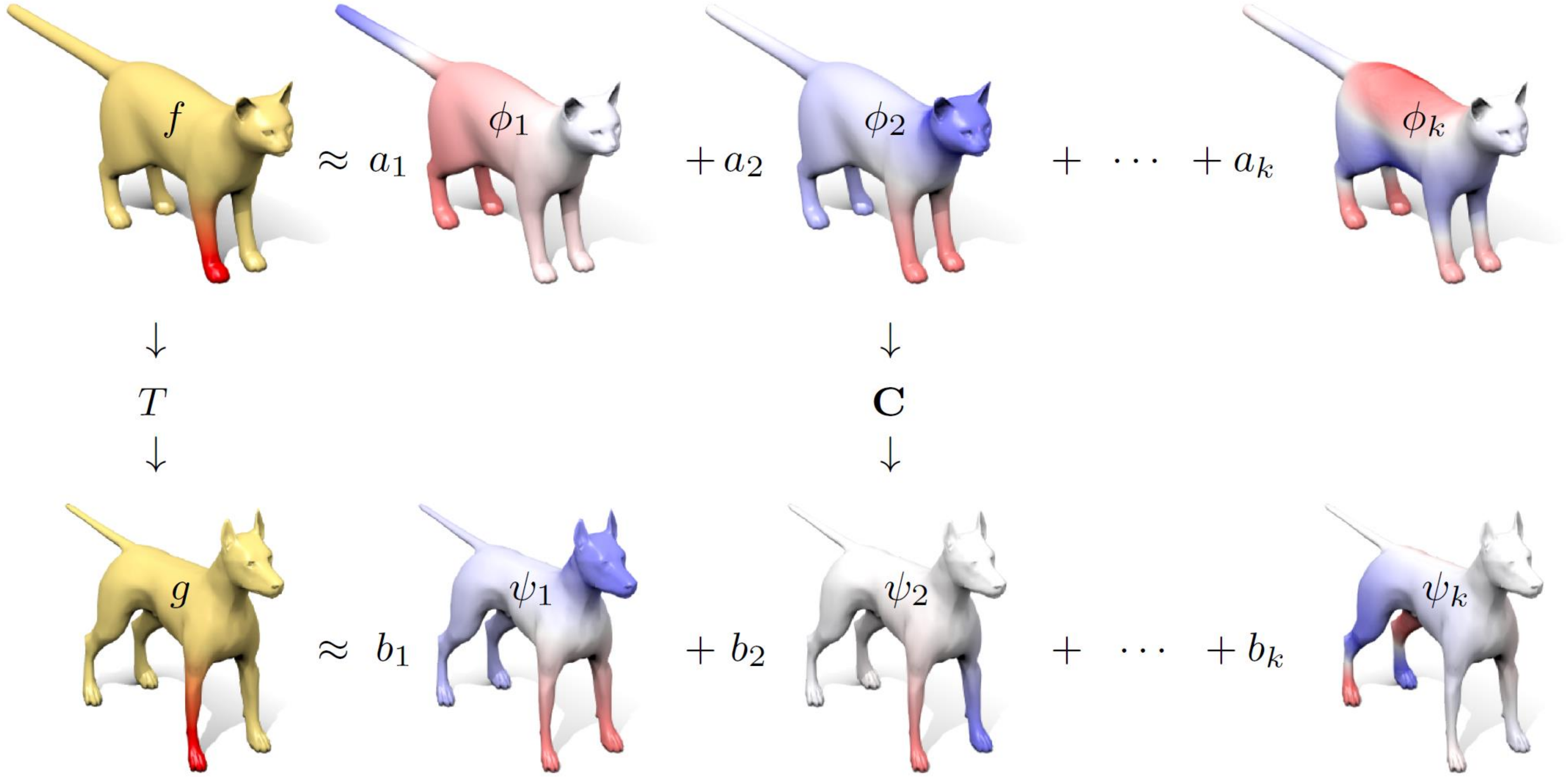
Functional map $T\colon L^2(\mathcal{X}) \to L^2(\mathcal{Y})$

$$Tf = f \circ t^{-1}$$

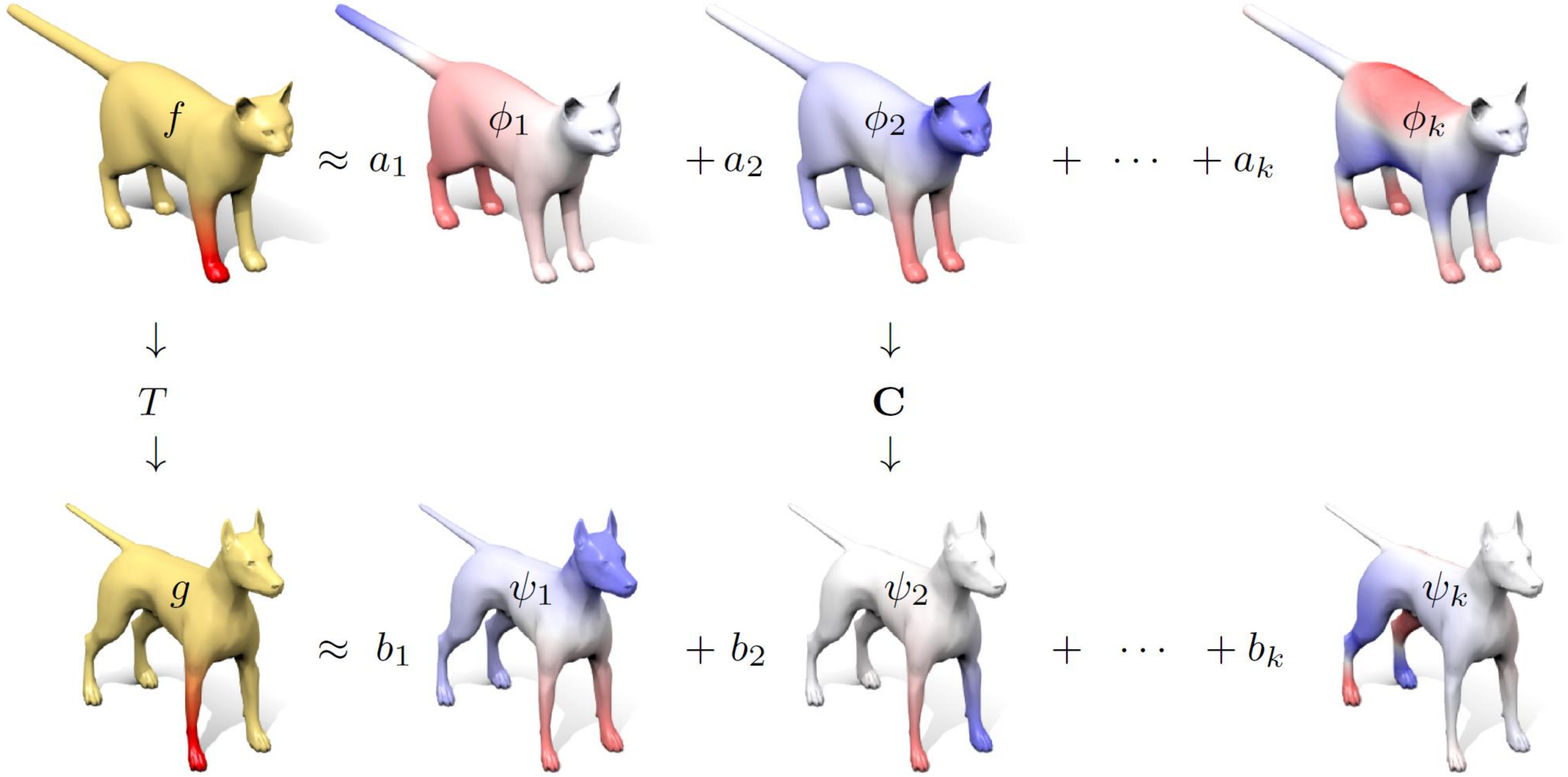

f
$\approx a_1$
ϕ_1
$+ a_2$
ϕ_2
$+ \cdots + a_k$
ϕ_k



$f \approx a_1 \phi_1 + a_2 \phi_2 + \cdots + a_k \phi_k$
$g \approx b_1 \psi_1 + b_2 \psi_2 + \cdots + b_k \psi_k$

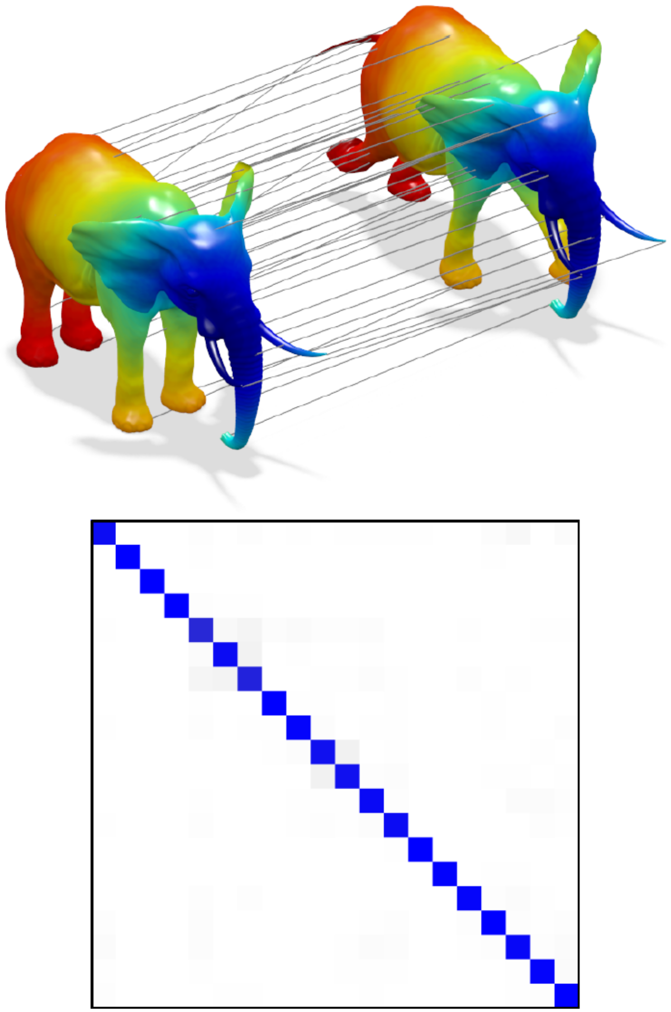


$f \approx a_1 \phi_1 + a_2 \phi_2 + \cdots + a_k \phi_k$
T
$\mathbf{C}$
$g \approx b_1 \psi_1 + b_2 \psi_2 + \cdots + b_k \psi_k$



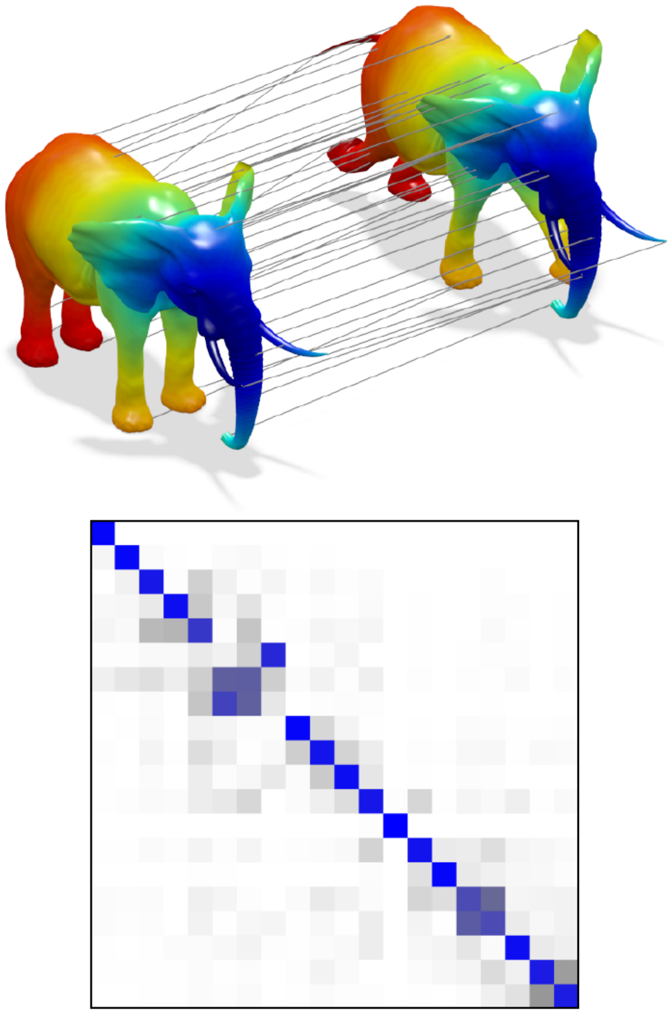
Functional correspondence boils down to a linear equation w.r.t. $\mathbf{C}$

$$g = Tf \qquad \Longleftrightarrow \qquad \mathbf{b} = \mathbf{C}\mathbf{a}$$



Isometric

$\mathbf{C} \approx \mathbf{I}$



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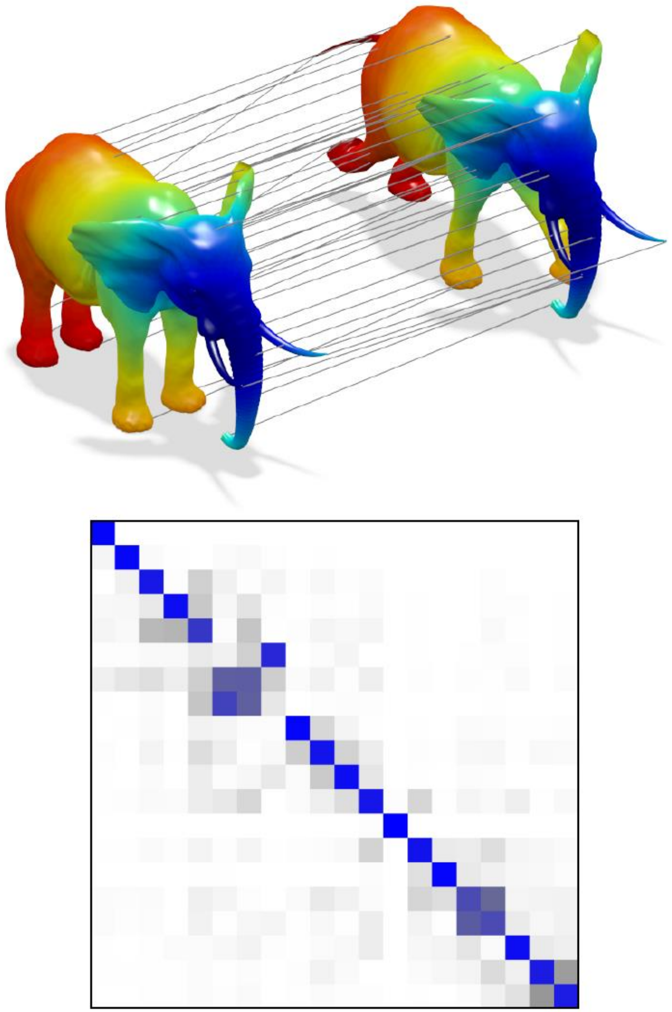
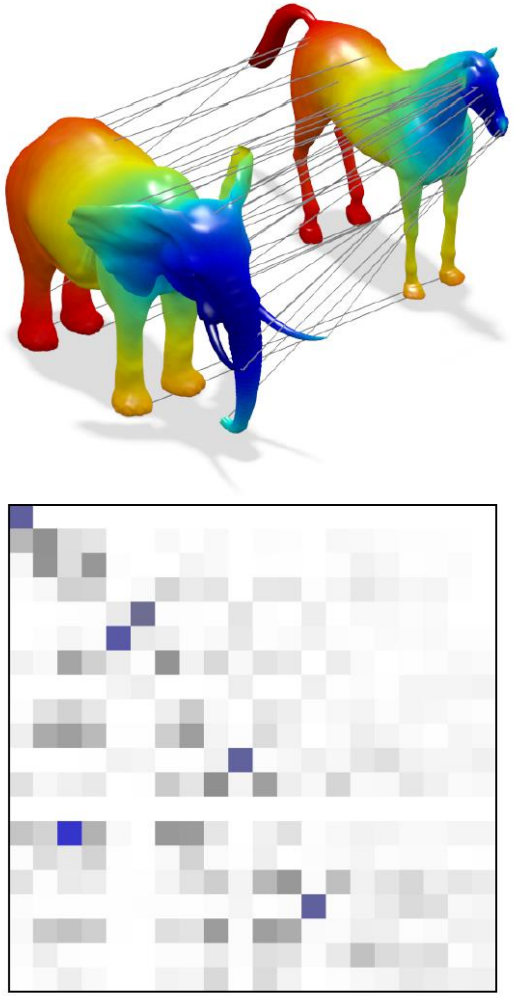


Isometric
$\mathbf{C} \approx \mathbf{I}$

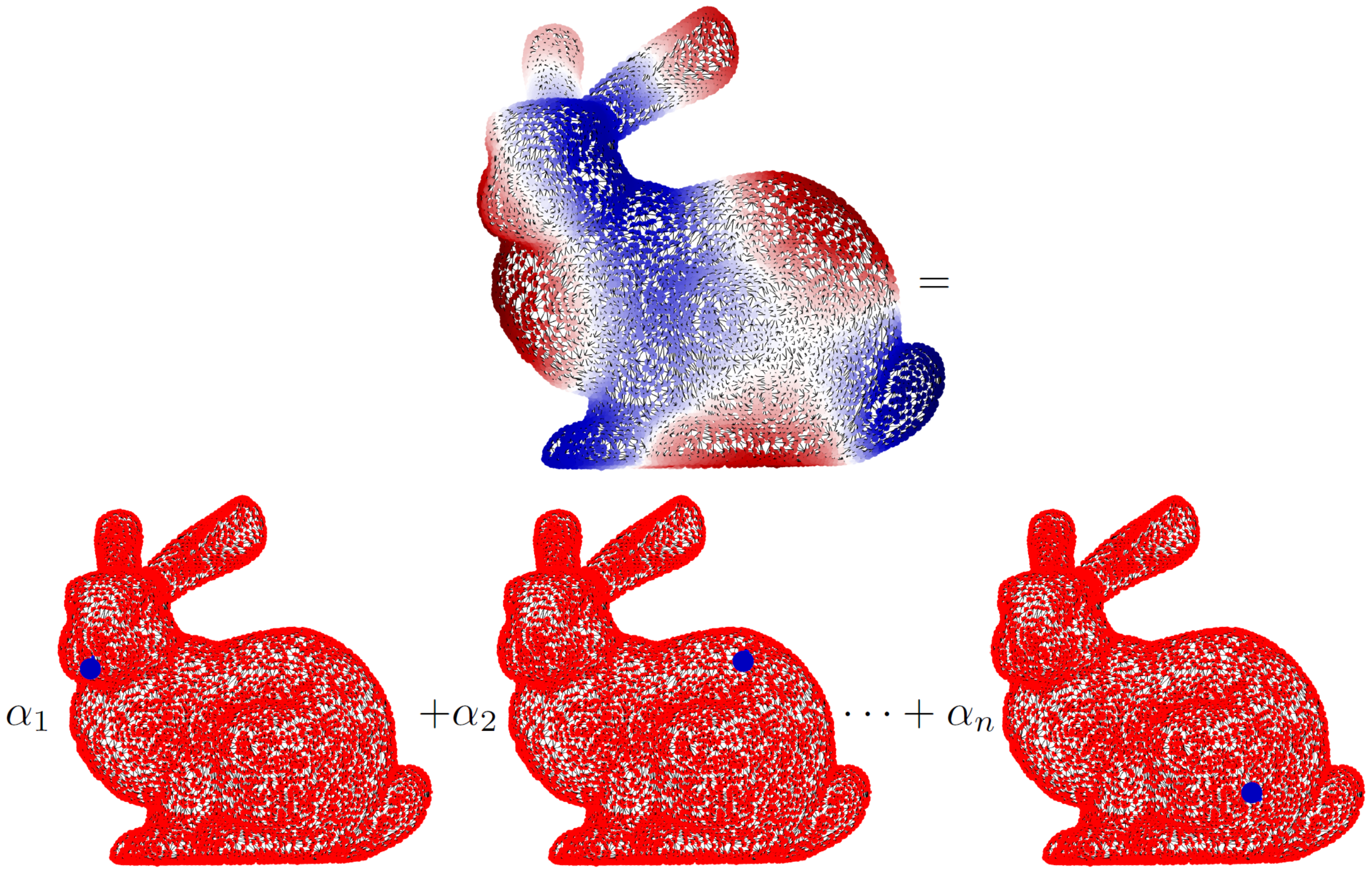


Non-isometric

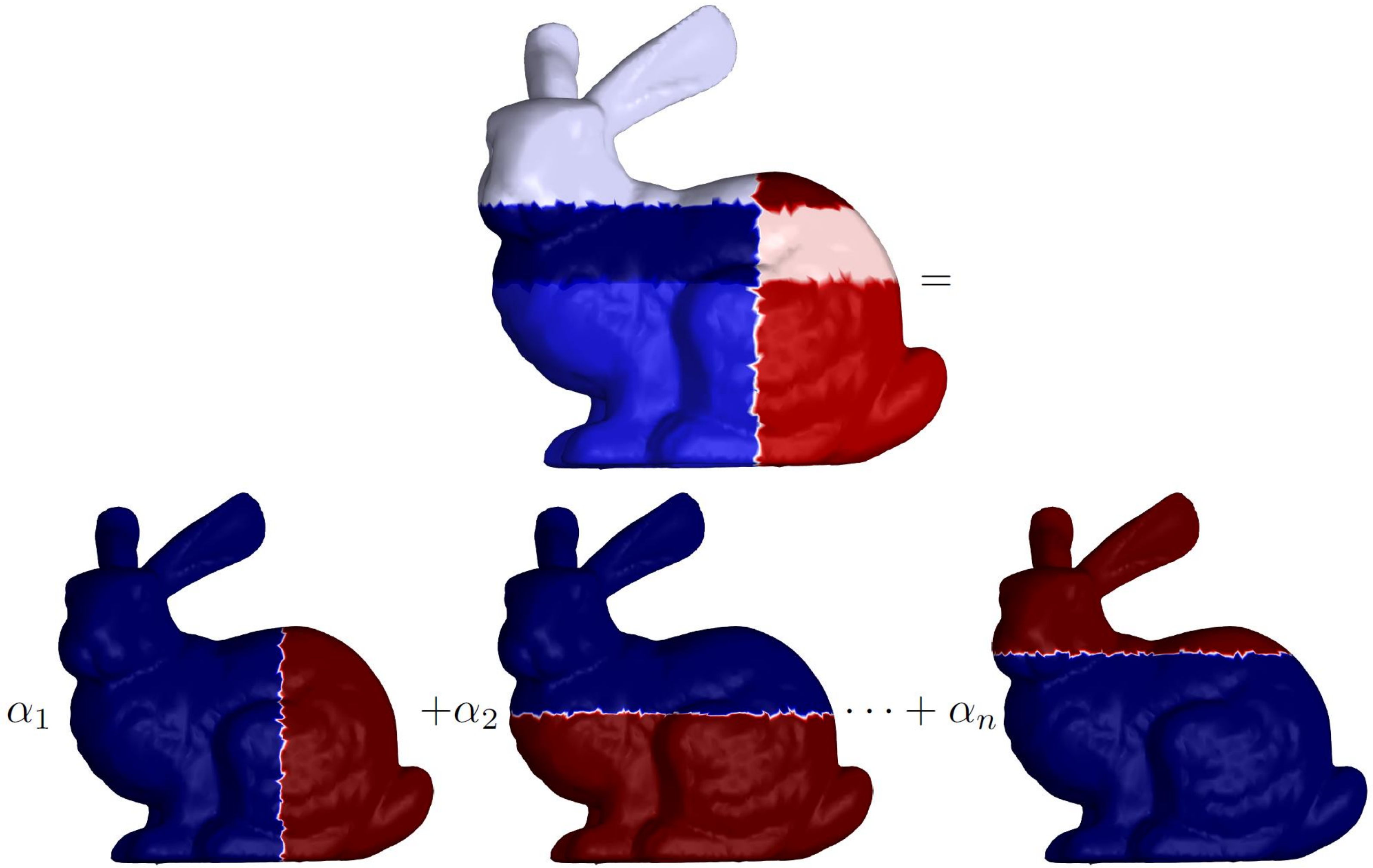
Where do we get a basis?

- Can be computed directly on a given shape
- Can be optimized on a given set of shapes
- Can be learned
- Combination of the above

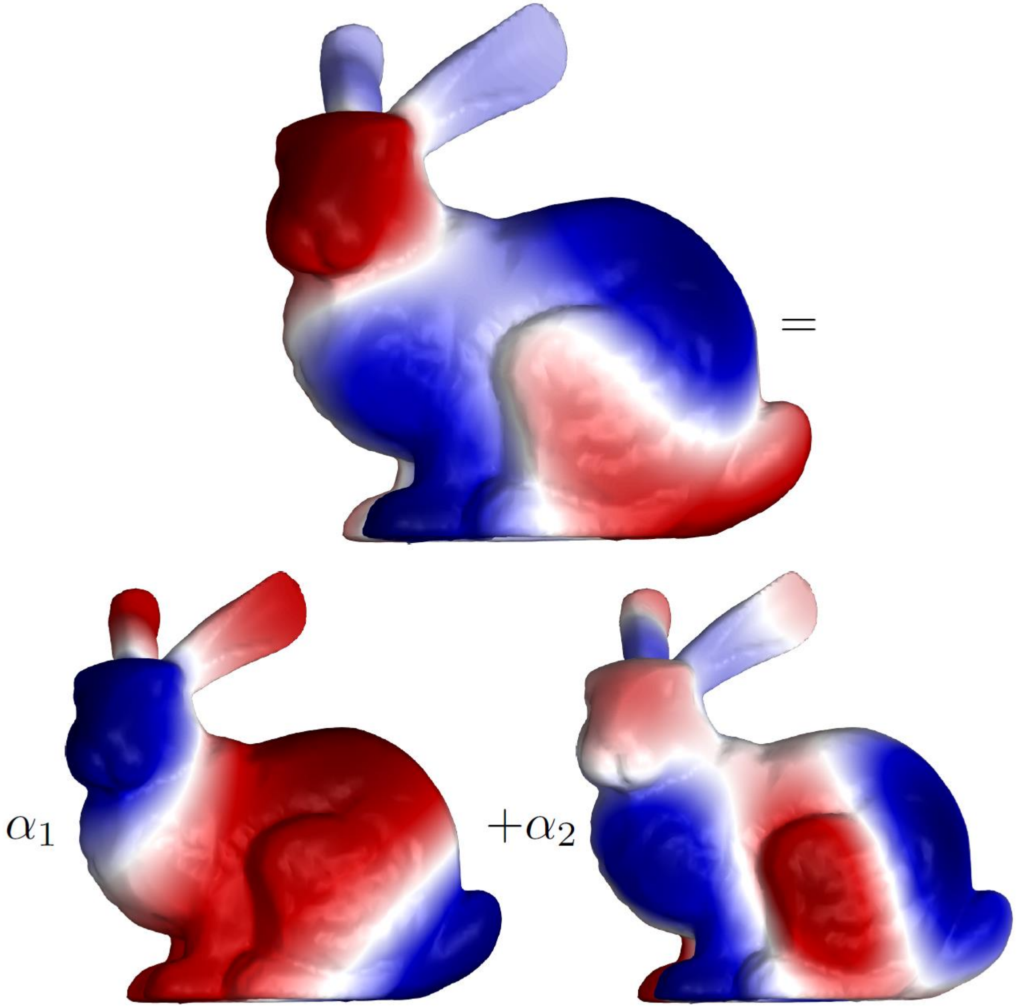
Basis vectors are indicator functions at all vertices in the mesh



Basis vectors are some piecewise-constant functions on the mesh



Basis vectors are two random smooth functions



$$g(v_i) = f(v_i) - \frac{1}{d_i} \sum_{j:(i,j)\in E} f(v_j)$$

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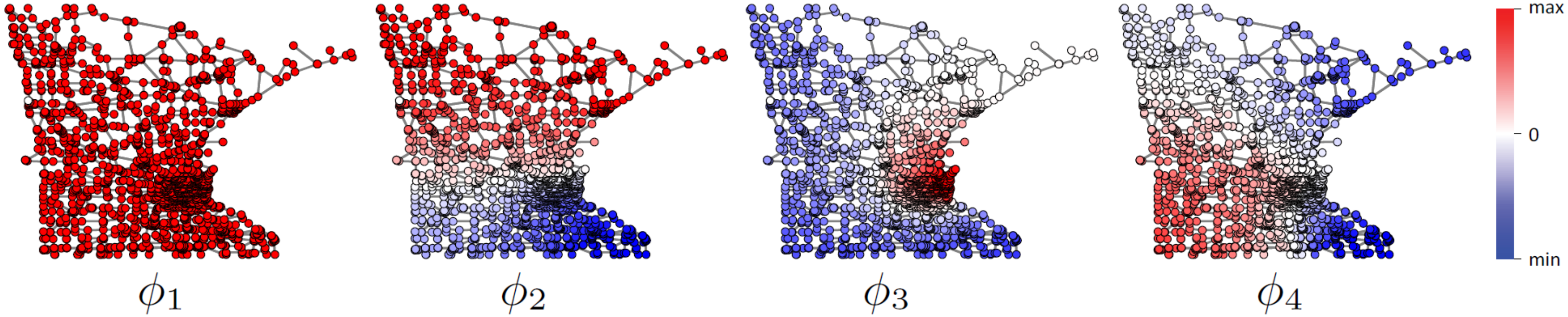
In matrix notation, we define the $n \times n$ matrix $\mathbf{L}$ as:

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{d_i} & \text{if } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases}$$

also known as the graph Laplacian of G.

Variants with different properties exist (e.g. normalized Laplacian, random walk Laplacian, etc.).

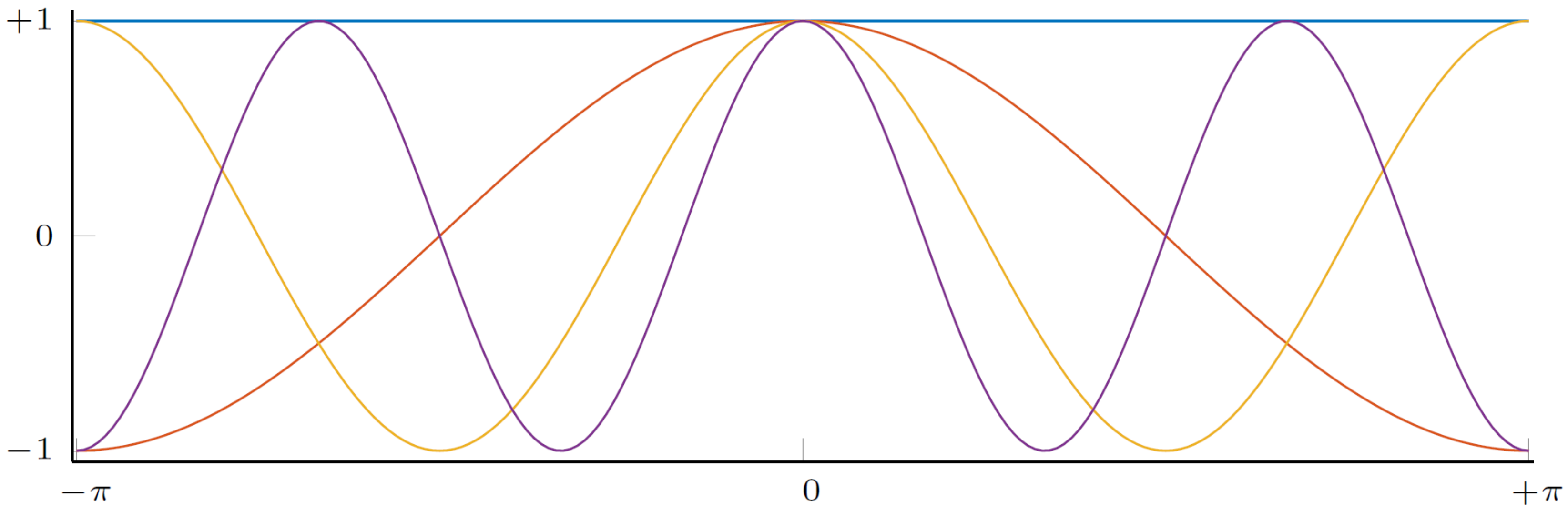
Theorem: The eigenfunctions $\{\phi_i\}$ of $\mathbf{L}$ form an orthonormal basis of $\mathcal{F}(G)$



First eigenfunctions of a graph Laplacian

(ordered by corresponding eigenvalue)

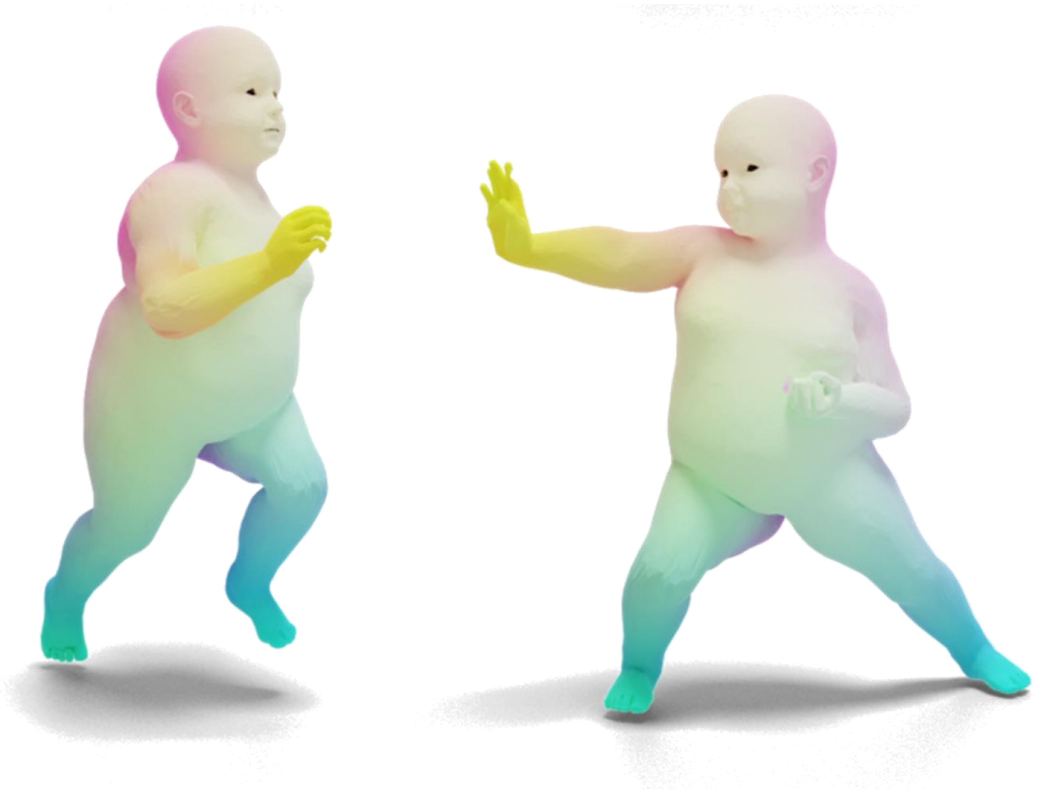
Theorem: The eigenfunctions $\{\phi_i\}$ of $\mathbf{L}$ form an orthonormal basis of $\mathcal{F}(G)$



First eigenfunctions of 1D Euclidean Laplacian = standard Fourier basis

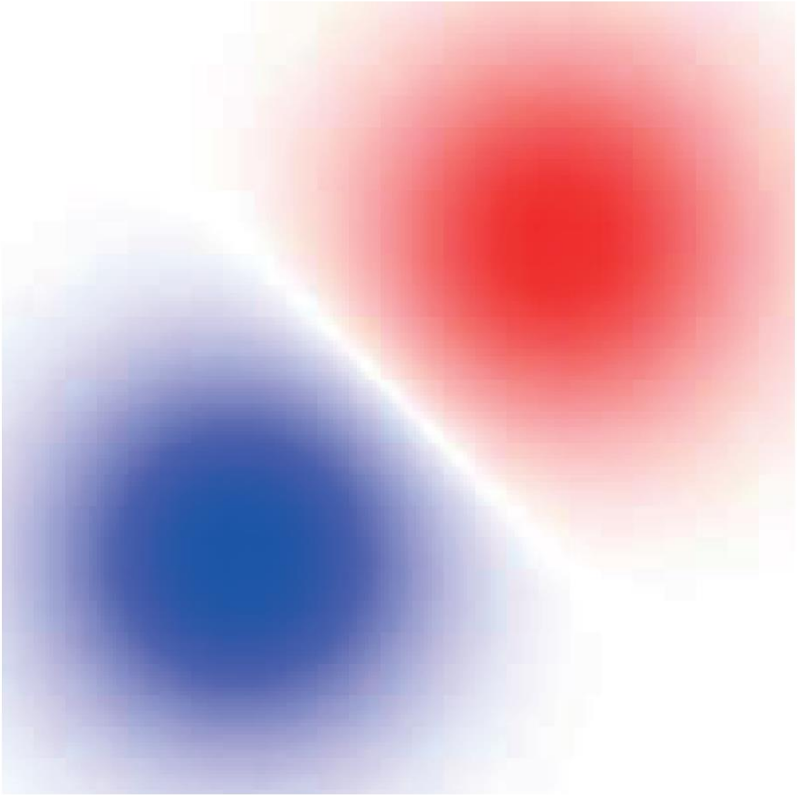
1. Compute bases $\mathbf{\Phi}$, $\mathbf{\Psi}$
2. Compute corresponding functions **F**, **G**
3. Represent **F** in $\mathbf{\Phi}$, and **G** in $\mathbf{\Psi}$, obtaining coefficients ***A*** and ***B***
4. Solve:

$$min_{\boldsymbol{C}} \, \|\boldsymbol{C}\boldsymbol{A} - \boldsymbol{B}\| + \rho(\boldsymbol{C})$$



5. Use the optimized ***C*** *...or...* Convert ***C*** to point-to-point

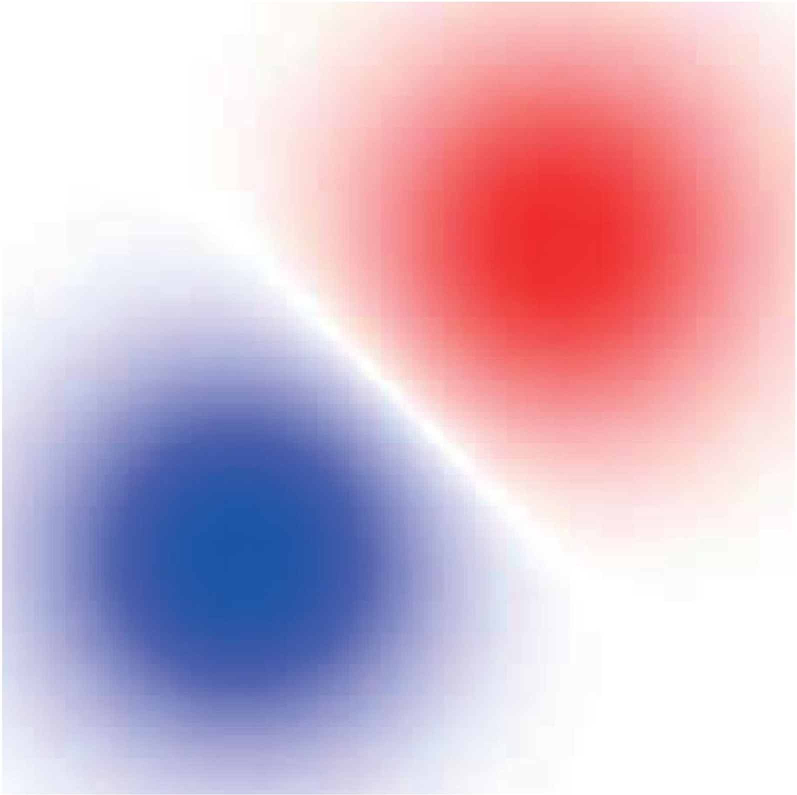
$$f : \mathcal{M} \to \mathbb{R}^3$$



Smooth scalar field f

Gradient $\nabla f(x)$

'direction of the steepest increase of f at x'



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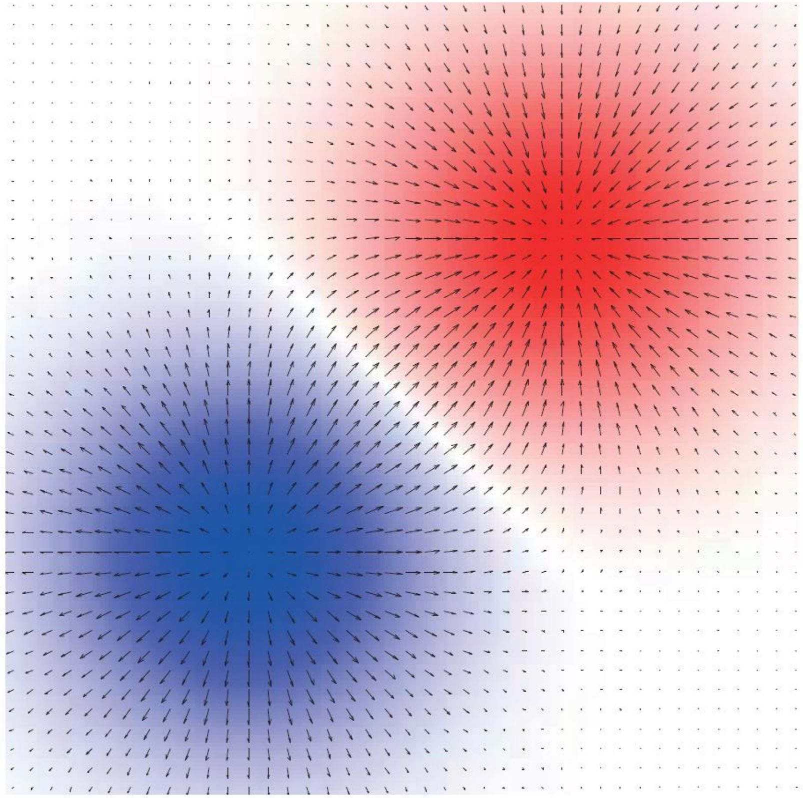


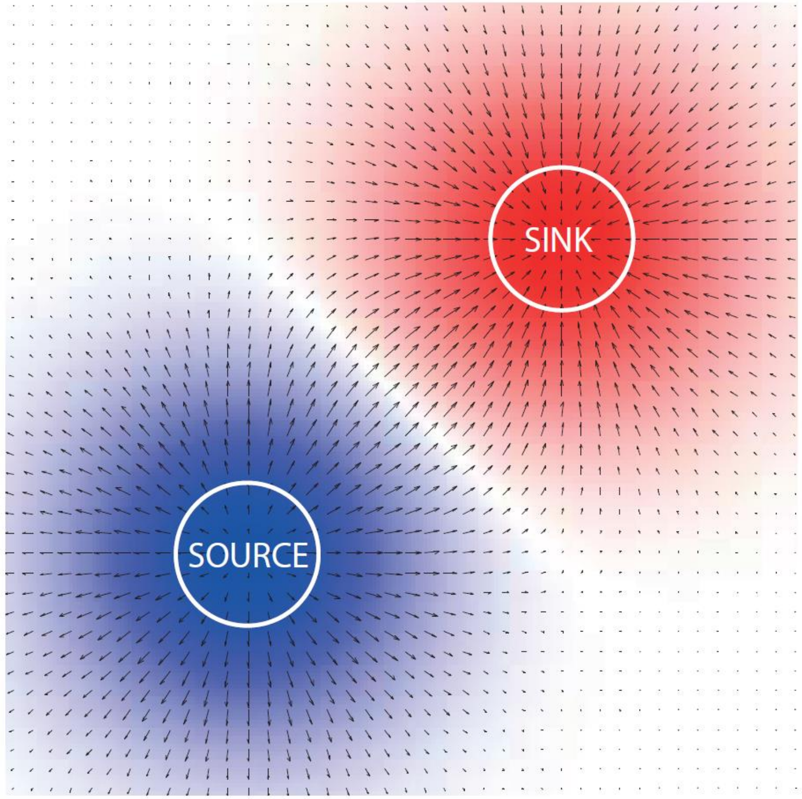
Smooth vector field F

Gradient $\nabla f(x)$

'direction of the steepest increase of f at x'

Divergence $\operatorname{div}(F(x))$

'scalar density of an outward flux of F from an infinitesimal volume around x'



Smooth vector field F

Gradient $\nabla f(x)$

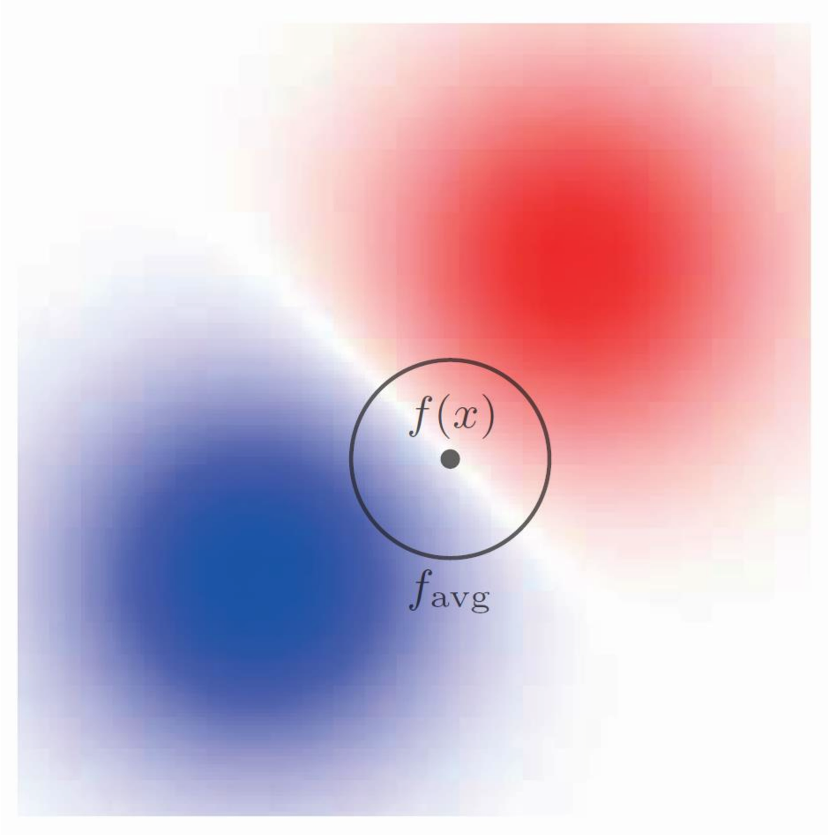
'direction of the steepest increase of f at x'

Divergence $\mathrm{div}(F(x))$

'scalar density of an outward flux of F from an infinitesimal volume around x'

Laplacian $\Delta f(x) = -\mathrm{div}(\nabla f(x))$

'scalar difference between $f(x)$ and the average of f on an infinitesimal sphere around x'



Smooth scalar field f

$$L_{ij} = \begin{cases} 1 & \text{if } i = j \\ -\frac{1}{d_i} & \text{if } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases}$$ graph Laplacian of G

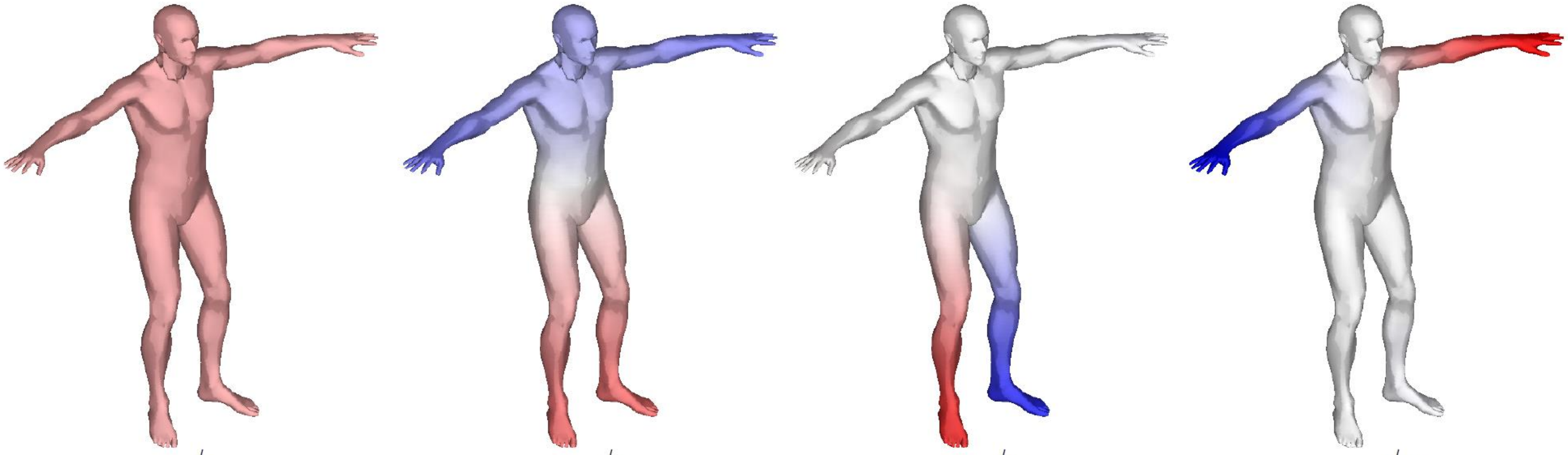
The discrete Laplace operator on a mesh is the $n \times n$ matrix:

$$\mathbf{L} = \mathbf{A}^{-1}\mathbf{S}$$

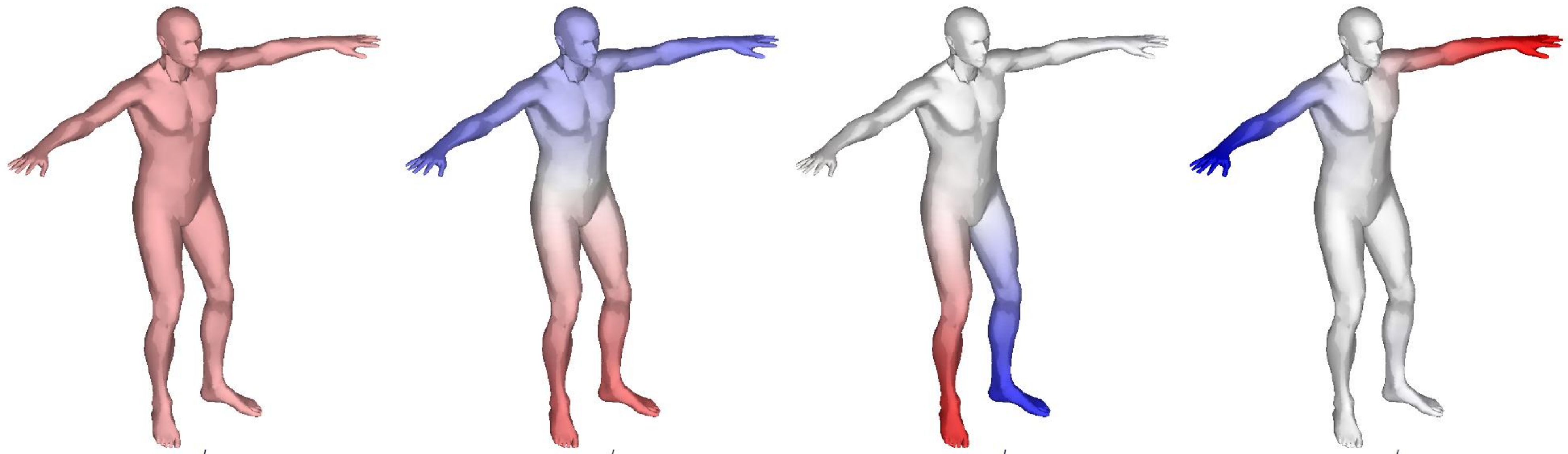
where

$$s_{ij} = \begin{cases} -\frac{1}{2}(\cot\alpha_{ij} + \cot\beta_{ij}) & \text{if } e_{ij} \in E \\ -\sum_{k \neq i} s_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$$a_{ij} = \begin{cases} \frac{1}{12}(A(T_{jii'}) + A(T_{ji''i})) & \text{if } e_{ij} \in E \\ \frac{1}{6}\sum_{k \in \mathcal{N}(i)} A(T_k) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$



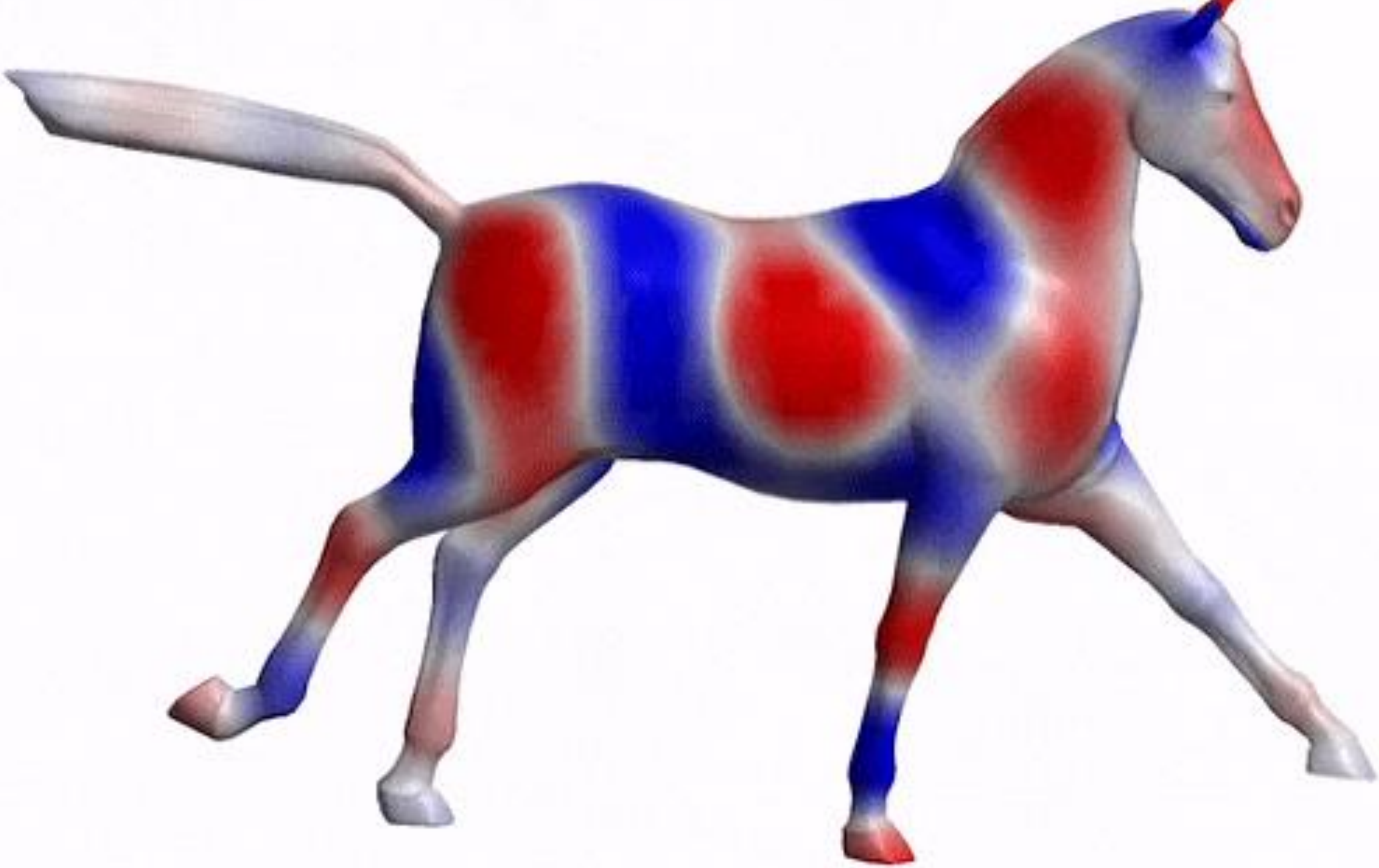
ϕ_1 ϕ_2 ϕ_3 ϕ_4

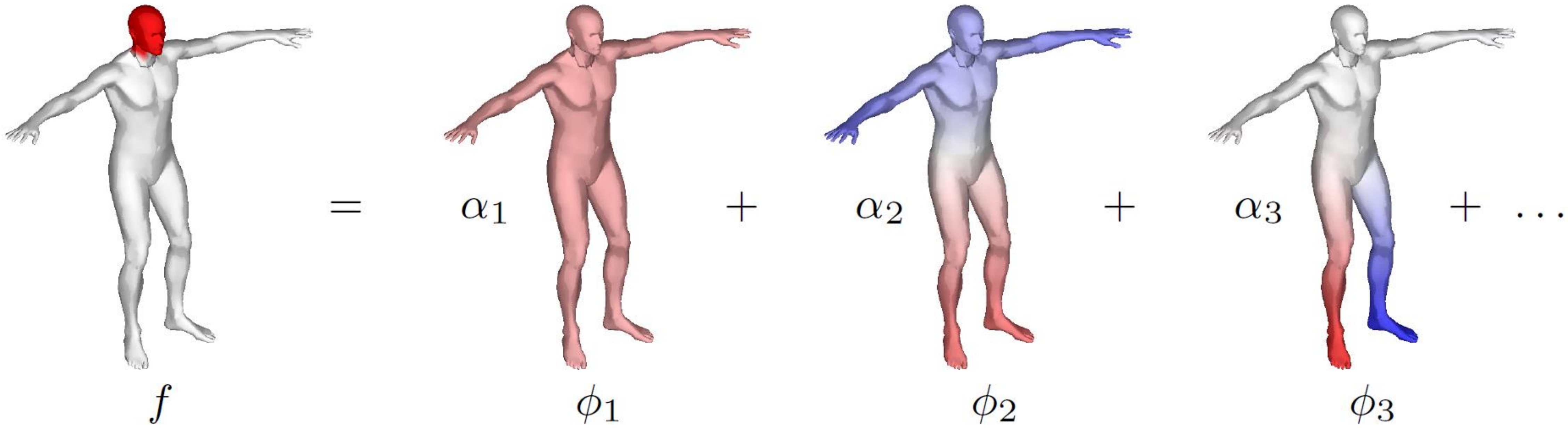


ϕ_1 ϕ_2 ϕ_3 ϕ_4

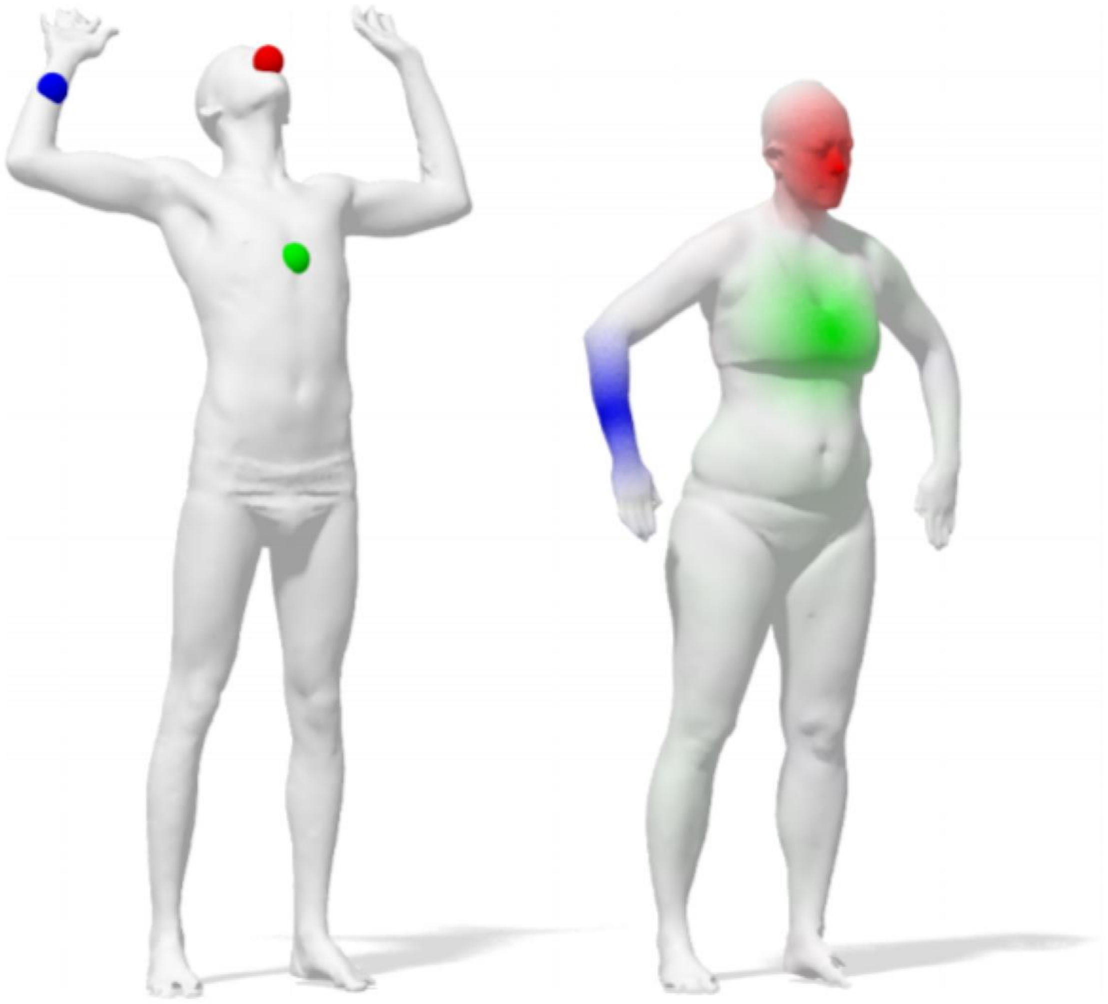
For shapes with simple spectrum, Laplacian eigenfunctions are invariant (up to sign) to isometric deformations, $\psi_i = \pm T\phi_i$

ψ_1 ψ_2 ψ_3 ψ_4

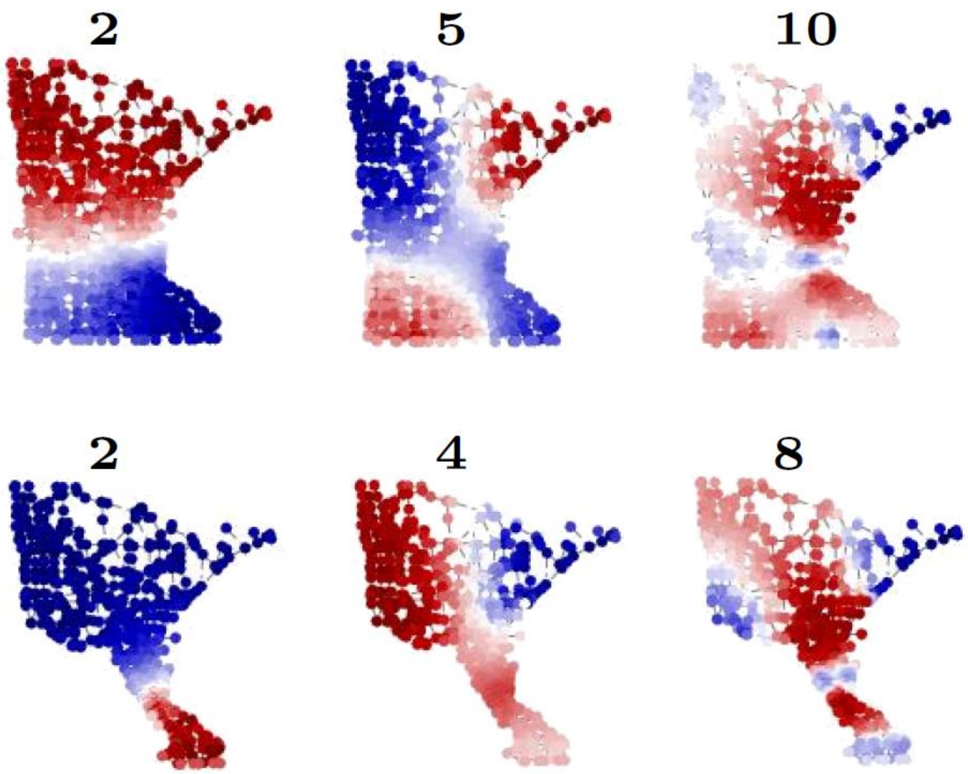




With coefficients $\langle f, \phi_k \rangle$

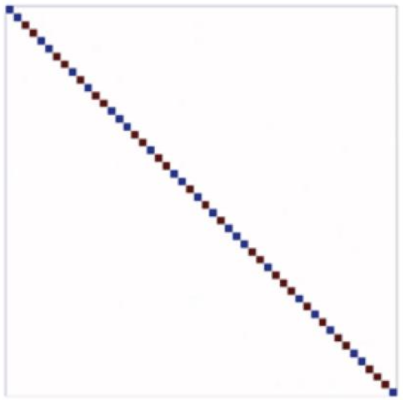


Truncating the sum to the first few terms has a smoothing effect

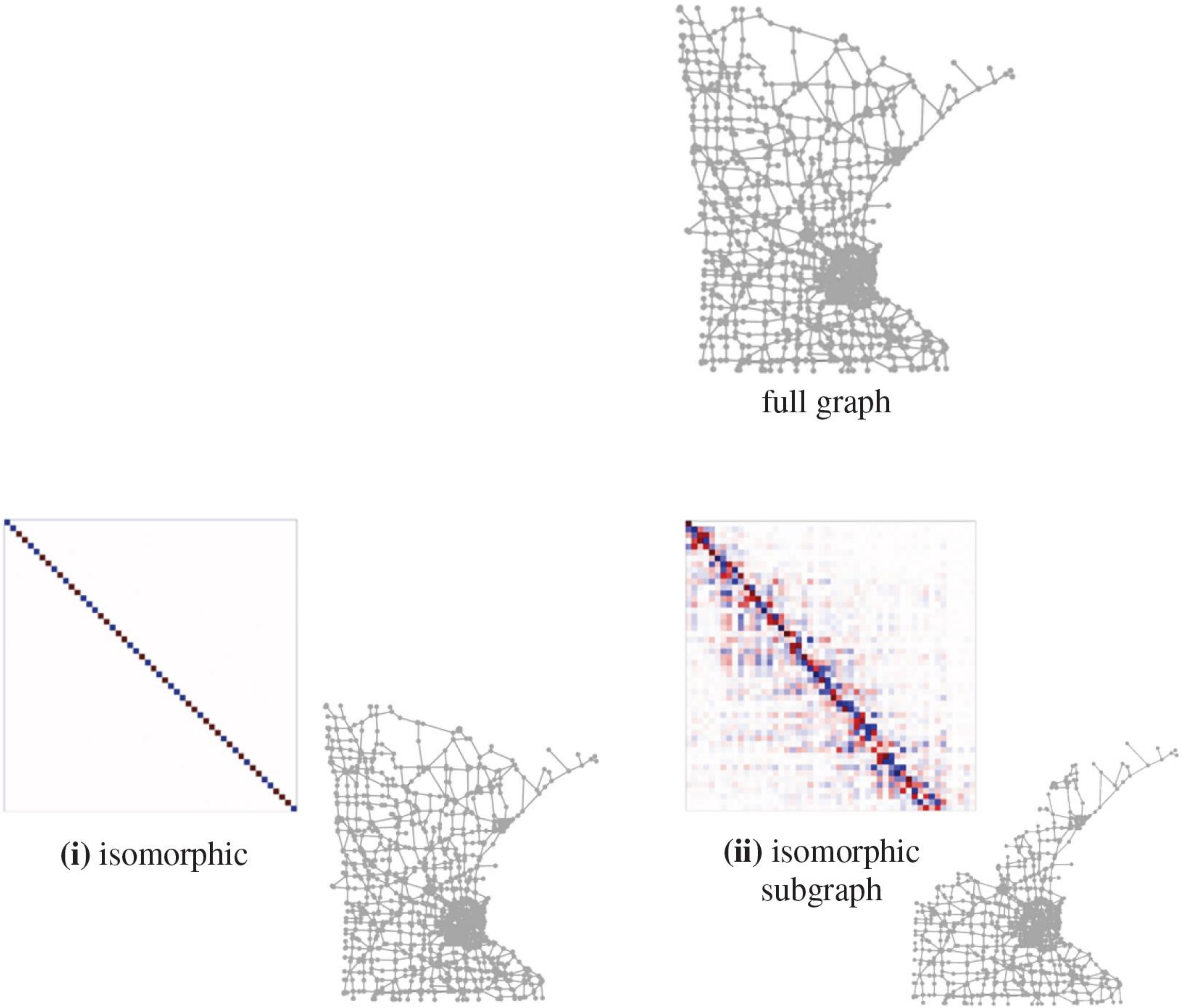


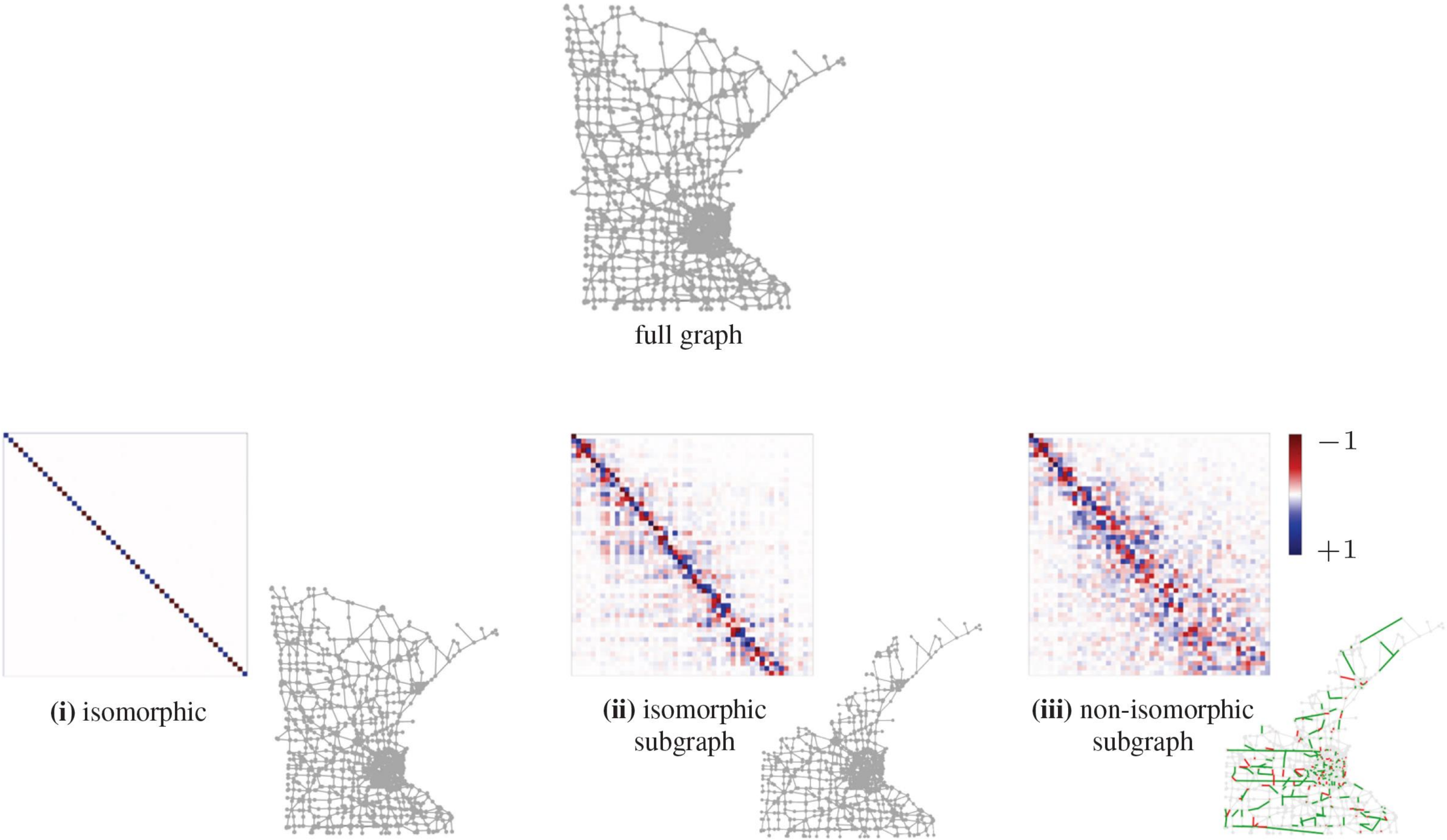
Eigenvectors of the graph Laplacian

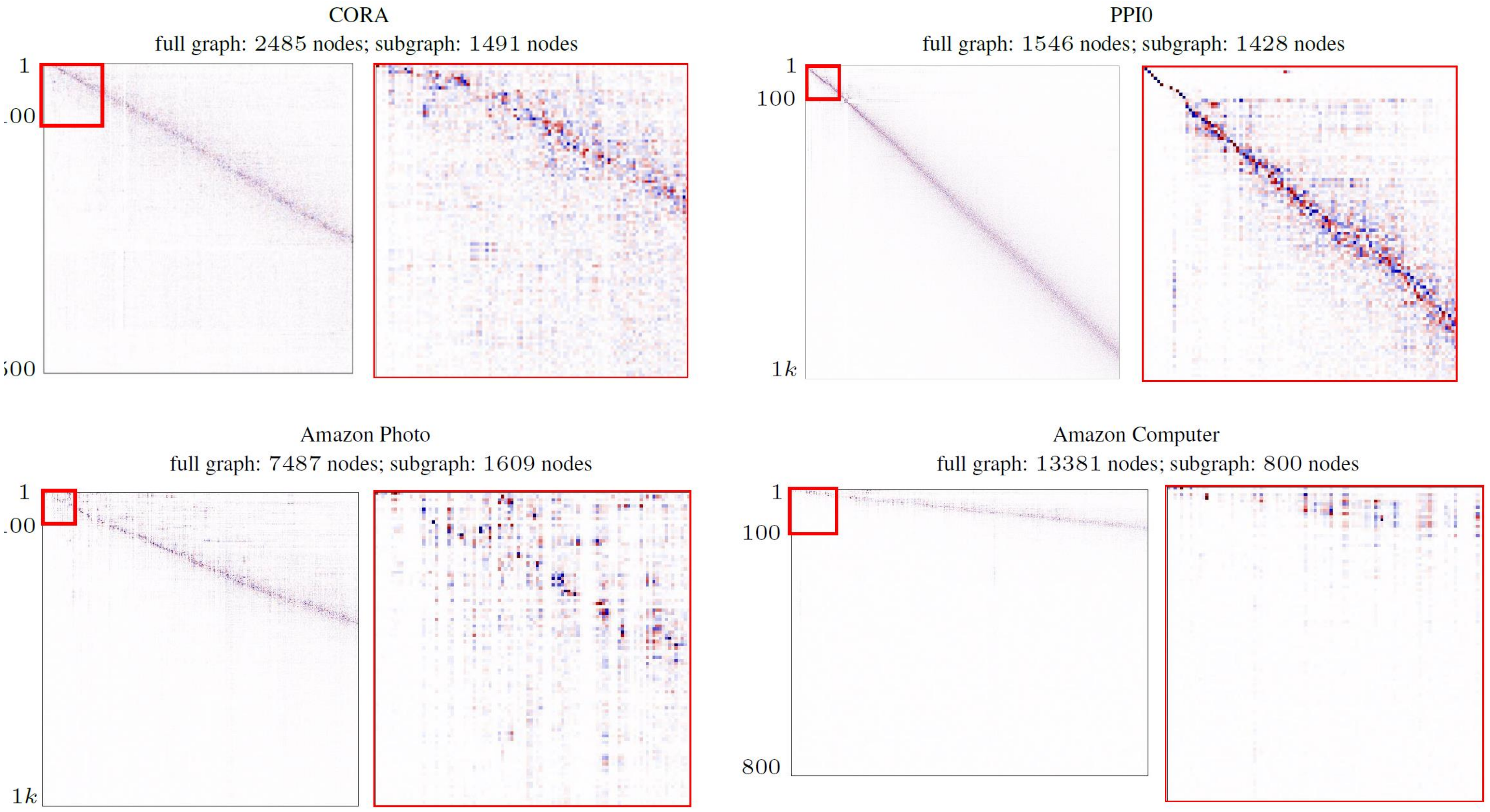
full graph



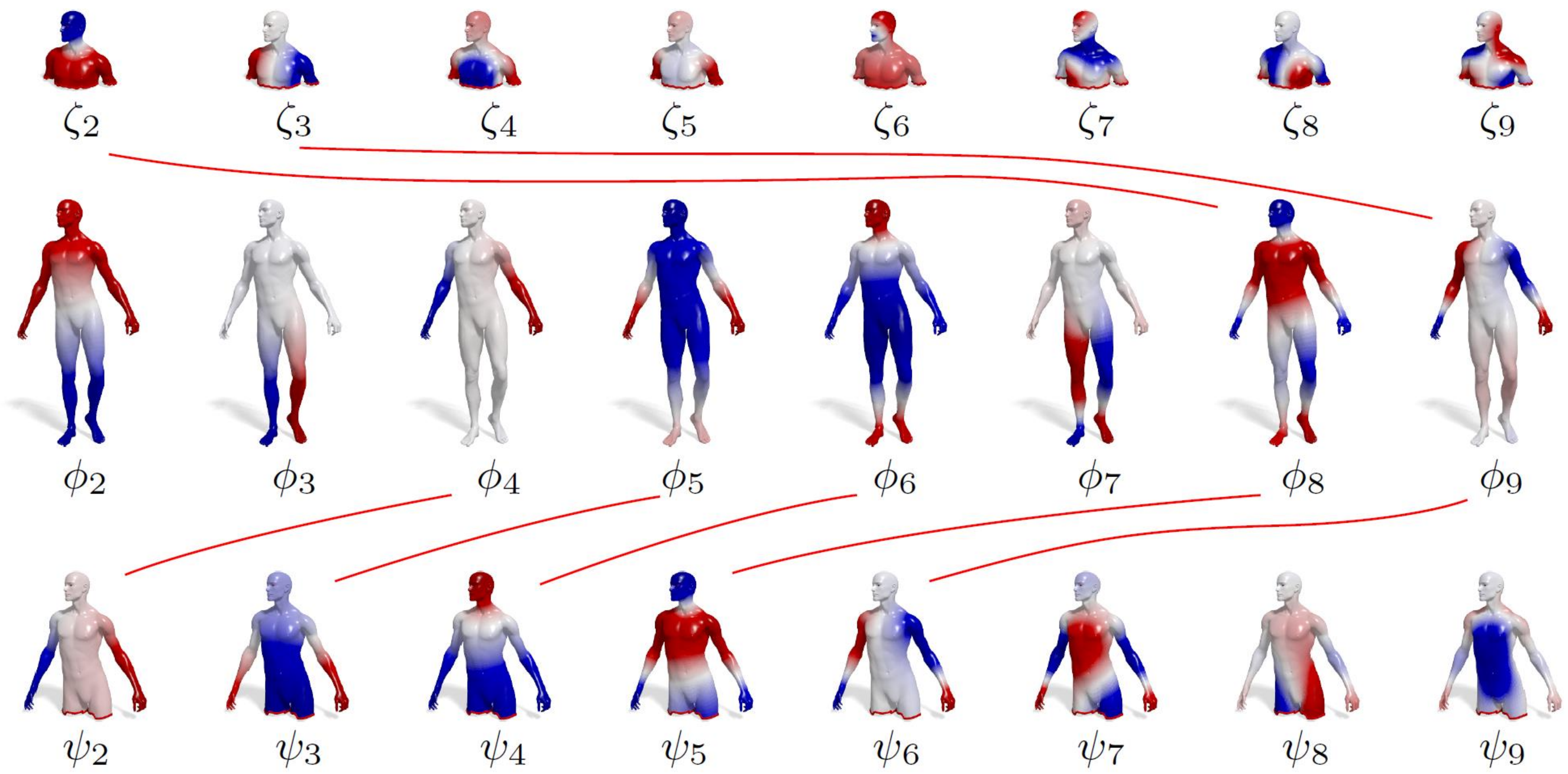
(i) isomorphic


full graph
(i) isomorphic
(ii) isomorphic subgraph

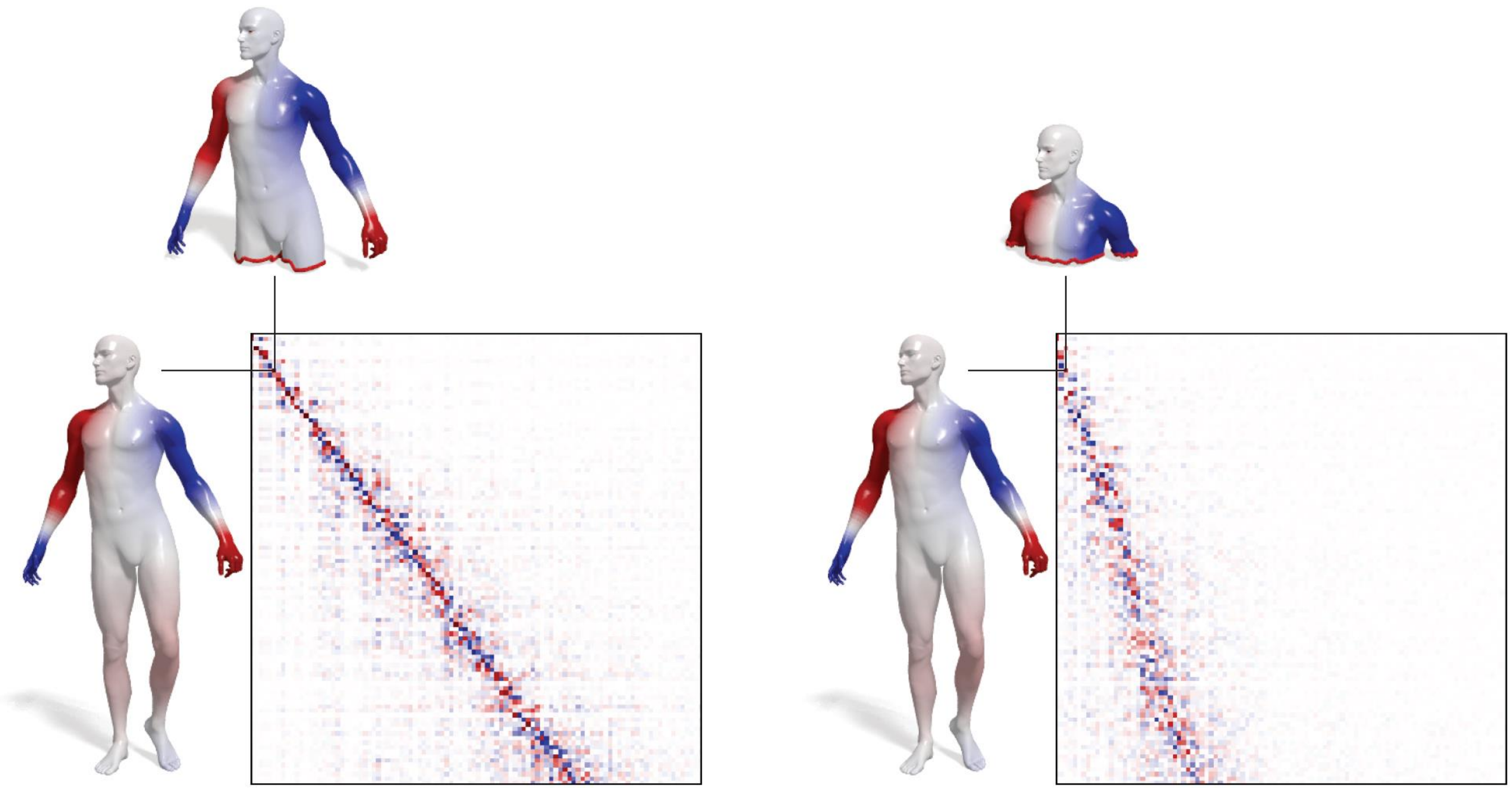

full graph
(i) isomorphic
(ii) isomorphic subgraph
(iii) non-isomorphic subgraph
−1
+1



CORA
full graph: 2485 nodes; subgraph: 1491 nodes
1
PPI0
full graph: 1546 nodes; subgraph: 1428 nodes
1
100
1k
Amazon Photo
full graph: 7487 nodes; subgraph: 1609 nodes
1
1k
Amazon Computer
full graph: 13381 nodes; subgraph: 800 nodes
1
100
800



ζ_2 ζ_3 ζ_4 ζ_5 ζ_6 ζ_7 ζ_8 ζ_9
ϕ_2 ϕ_3 ϕ_4 ϕ_5 ϕ_6 ϕ_7 ϕ_8 ϕ_9
ψ_2 ψ_3 ψ_4 ψ_5 ψ_6 ψ_7 ψ_8 ψ_9



Functional correspondence matrix $\mathbf{C}$

Slope angle depends on the areas ratio $\dfrac{|\mathcal{X}|}{|\mathcal{Y}|}$