Moduli spaces of elliptic curves

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Abstract

In this short article we summarize the theory of moduli spaces and we use it to give an algebraic definition of (affine) modular curves.

We want to give another interpretation of modular curves as *moduli spaces* of the family of elliptic curves over $\mathbb{C}$ endowed with some extra structure. This interpretation will allow us to define these modular curves in a purely algebraic way, as schemes of finite type over $\mathbb{Z}\left[{}^{1}/_{n}\right]$. We will see in the third chapter how this interpretation can be obtained also from a completely adèlic description of these modular curves, which is the main objective of this thesis.

We begin by recalling some definitions and theorems from the theory of elliptic curves.

Definition 1. An *elliptic curve* $E = {}^{\mathbb{C}}/_{\Lambda}$ over the complex numbers is a group quotient of the additive group $(\mathbb{C},+)$ by a *lattice* $\Lambda \subseteq \mathbb{C}$, i.e. $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ where $\{\omega_1, \omega_2\} \subseteq \mathbb{C}$ is a basis of $\mathbb{C}$ as a vector space over the real numbers.

The action $\Lambda \circlearrowleft \mathbb{C}$ defined as $\lambda * z = z + \lambda$ for every $\lambda \in \Lambda$ and every $z \in \mathbb{C}$ is clearly properly discontinuous and free and thus the elliptic curve $E = {}^{\mathbb{C}}/_{\Lambda}$ has a natural structure of a compact Riemann surface such that the quotient map $\mathbb{C} \twoheadrightarrow E$ is holomorphic.

Definition 2. A map of sets $E \to E'$ between two elliptic curves is a *morphism of elliptic curves* if it is holomorphic and $O \mapsto O'$, where $O \in E$ and $O' \in E'$ are the images of $\mathbf{0} \in \mathbb{C}$ through the quotient maps $\mathbb{C} \to E$ and $\mathbb{C} \to E'$.

We have thus defined the category $\mathbf{Ell}(\mathbb{C})$ of elliptic curves over the complex numbers. The morphisms and isomorphisms in this category can be characterized in a very immediate and explicit way, as it is shown in the following theorem.

Theorem 3. *For every couple of elliptic curves $E = {}^{\mathbb{C}}/_{\Lambda}$ and $E' = {}^{\mathbb{C}}/_{\Lambda'}$ the map*

$$\rho \colon \left(\{\alpha \in \mathbb{C} \mid \alpha \cdot \Lambda \subseteq \Lambda'\}, +\right) \to \left(\mathrm{Hom}_{\mathbf{Ell}(\mathbb{C})}(E, E'), +\right) \qquad \textit{such that} \qquad \begin{aligned} \rho(\alpha) \colon E &\to E' \\ [z] &\mapsto [\alpha \cdot z] \end{aligned}$$

is a group homomorphism.

Proof. it is clear that ρ is a group homomorphism. Suppose now that $\rho(\alpha) = \rho(\beta)$ for some $\alpha, \beta \in \mathbb{C}$. This implies that $(\alpha - \beta)z \in \Lambda'$ for every $z \in \mathbb{C}$, which can happen if and only if $\alpha = \beta$, because Λ' is discrete and the function $z \mapsto (\alpha - \beta)z$ is continuous. To conclude we have to show that ρ is surjective. Let $f \colon E \to E'$ be a morphism of elliptic curves and let $\pi \colon \mathbb{C} \to E$ and $\pi' \colon \mathbb{C} \to E'$ be the quotient maps. Since $\mathbb{C}$ is simply connected and π' is a covering map there exists a map $F:\mathbb{C}\to\mathbb{C}$ such that $f\circ\pi=\pi'\circ F$. This implies that $F(z+\lambda)-F(z)\in\Lambda'$ for every $\lambda\in\Lambda$ and $z\in\mathbb{C}$ and thus we can assume without loss of generality that $F(0)=0$ because by hypothesis $f(O)=O'$. The map F is also holomorphic since π and π' are local isomorphisms and f is holomorphic.

Observe now that for every $\lambda\in\Lambda$ the function $F_\lambda\overset{\mathsf{def}}{=}F(z+\lambda)-F(z)$ is a continuous function whose image is contained in the discrete set Λ', and thus it is constant. This implies that $F'(z)=F'(z+\lambda)$ for every $\lambda\in\Lambda$ and $z\in\mathbb{C}$, and this implies that F' is bounded. We get hence that F' is constant by applying Liouville's theorem as stated and proved in **Theorem 10.23** of [2]. Thus $F(z)=\alpha z+\beta$ for some $\alpha,\beta\in\mathbb{C}$. Since we have supposed that $F(0)=0$ we see that $\beta=0$ which implies that $f(\pi(z))=\pi'(F(z))=\pi'(\alpha z)$ for every $z\in\mathbb{C}$. Moreover since $F(\Lambda)\subseteq\Lambda'$ we have that $\alpha\Lambda\subseteq\Lambda'$, and thus that $f=\rho(\alpha)$. This implies that ρ is surjective, as we wanted to prove. □

Corollary 4. *There exists a bijection between the set of isomorphism classes of complex elliptic curves and the Riemann surface $Y(1)=\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$.*

Proof. For every $\tau\in\mathfrak{h}$ the set $\{1,\tau\}\subseteq\mathbb{C}$ is a basis of $\mathbb{C}$ as a real vector space. Thus we can define the elliptic curve $E_\tau\overset{\mathsf{def}}{=}\mathbb{C}/_{\mathbb{Z}\oplus\mathbb{Z}\tau}$. Observe now that for every lattice $\Lambda=\mathbb{Z}\omega_1\oplus\mathbb{Z}\omega_2$ we have that

$$\frac{1}{\omega_1}\Lambda=\mathbb{Z}\oplus\mathbb{Z}\left(\frac{\omega_2}{\omega_1}\right)=\mathbb{Z}\oplus\mathbb{Z}\left(-\frac{\omega_2}{\omega_1}\right)$$

and clearly either ${}^{\omega_2}/_{\omega_1}\in\mathfrak{h}$ or $-{}^{\omega_2}/_{\omega_1}\in\mathfrak{h}$. This implies, using the previous theorem, that for every elliptic curve $E\in\mathrm{Ell}(\mathbb{C})$ there exists $\tau\in\mathfrak{h}$ such that $E\cong E_\tau$.

To conclude we have only to prove that for every $\tau,\tau'\in\mathbb{C}$ we have $E_\tau\cong E_{\tau'}$ if and only if there exists $A\in\mathrm{SL}_2(\mathbb{Z})$ such that $\tau'=A*\tau$. To do so we use again the previous theorem to see that $E_\tau\cong E_{\tau'}$ if and only if there exists $\alpha\in\mathbb{C}^\times$ such that $\mathbb{Z}\oplus\mathbb{Z}\tau=\alpha(\mathbb{Z}\oplus\mathbb{Z}\tau')$. This is true if and only if we can find a matrix $B=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Z})$ such that $\left(\begin{smallmatrix}\alpha\tau'\\\alpha\end{smallmatrix}\right)=B\cdot\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$, and this is true if and only if $\tau'={}^{a\tau+b}/_{c\tau+d}$. Since $\tau,\tau'\in\mathfrak{h}$ we have that $\det(B)>0$ and since $B\in\mathrm{GL}_2(\mathbb{Z})$ this implies that $\det(B)=1$, which is equivalent to say that $B\in\mathrm{SL}_2(\mathbb{Z})$. □

What we have shown in the previous corollary is a first example of the interpretation of the modular curves as *moduli spaces* for the elliptic curves. Roughly speaking a moduli space is a geometric object whose points represent the solutions of a geometric classification problem. What we have proved above shows that the classification problem of finding all the isomorphism classes of elliptic curves over $\mathbb{C}$ admits $Y(1)$ as a moduli space. This is still very vague, and in particular it doesn't involve the algebraic description of elliptic curves. To make this all more precise we will thus need to use the language of schemes and functors, which we outline in the following pages.

Recall first of all that for every locally small category $\mathscr{C}$ and every object $X\in\mathscr{C}$ we can define a functor

$$h_X:\mathscr{C}^{\mathrm{op}}\to\mathbf{Sets}\qquad\text{by setting}\qquad h_X(T)=\mathrm{Hom}_{\mathscr{C}}(T,X)$$

where $\mathrm{Hom}_{\mathscr{C}}(T,X)$ denotes the set of all arrows $T\to X$ from T to X in the category $\mathscr{C}$. Denote with $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$ the category of functors $\mathscr{C}^{\mathrm{op}}\to\mathbf{Sets}$ whose morphisms are natural transformations. The fundamental result called *Yoneda's lemma* shows that the covariant functor

$$h_-:\mathscr{C}\to\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}\qquad\text{such that}\qquad h_-(X)=h_X$$

is full and faithful, and thus gives us an embedding of the category $\mathscr{C}$ inside the category of functors $\mathbf{Sets}^{\mathscr{C}^{\mathrm{op}}}$. Any functor $F:\mathscr{C}^{\mathrm{op}}\to\mathbf{Sets}$ is said to be *representable* if there exists an object $X\in\mathscr{C}$ such that $F\cong h_X$. The theory of moduli spaces is strictly related to the representability of certain functors $\mathscr{C}^{\mathrm{op}} \to \mathrm{Sets}$, where $\mathscr{C}$ is a category of schemes.

Definition 5. A *scheme* is a couple $(X, \mathscr{O}_X)$ where X is a topological space and $\mathscr{O}_X$ is a *sheaf* of rings on X, such that for every $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$, a commutative ring A, an homeomorphism $f: U \to \mathrm{Spec}(A)$ and an isomorphism of sheaves $\mathscr{O}_{\mathrm{Spec}(A)} \xrightarrow{\sim} f^*(\mathscr{O}_X|_U)$.

We are now ready to define the concept of a *moduli problem* in the context of schemes, which is simply a functor $\mathscr{P}: \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ where $\mathscr{C}$ is a category obtained by the category of schemes $\mathbf{Sch}$. The simplest example is to take $\mathscr{C} = \mathbf{Sch}/S$ whose objects are arrows $X \to S$ in the category $\mathbf{Sch}$ and whose morphisms are arrows $X \to Y$ such that the triangle on the right commutes.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We say that a moduli problem $\mathscr{P}: \mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ is representable if the functor $\mathscr{P}$ is representable. In this case every object $X \in \mathscr{C}$ such that $\mathscr{P} \cong h_X$ is called a *fine moduli space* for $\mathscr{P}$. A consequence of Yoneda's lemma is that two fine moduli spaces for the same moduli problem $\mathscr{P}$ are always canonically isomorphic. Unfortunately fine moduli spaces do not always exist because many moduli problems are not representable. Take for instance $\mathscr{C} = \mathbf{Sch}/\mathrm{Spec}(\mathbb{C})$ and take the functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Sets}$ which sends a $\mathbb{C}$-scheme S to the set of all isomorphism classes of complex vector bundles $E \to S$ of fixed rank $r \in \mathbb{N}$. Suppose by contradiction that $X \in \mathscr{C}$ is a fine moduli space for this moduli problem and observe that the isomorphism class of the vector bundle $\mathbb{C}^r \to \mathrm{Spec}(\mathbb{C})$ gives us a morphism $\mathrm{Spec}(\mathbb{C}) \to X$, i.e. a $\mathbb{C}$-rational point $x \in X$. Now for every complex vector bundle $E \to S$ we have a map $f_E: S \to X$ corresponding to its isomorphism class, and it is immediate to see that $E \to S$ is trivial (i.e. isomorphic to $\mathbb{C}^r \times_{\mathrm{Spec}(\mathbb{C})} S \to S$) if and only if $f_E(S) = \{x\}$. Unfortunately for every vector bundle $\pi: E \to S$ we can find an open cover $\mathscr{U}$ of S such that for every $U \in \mathscr{U}$ the vector bundle $\pi^{-1}(U) \to S$ is trivial, and thus $f_E\big|_U(U) = \{x\}$. But this implies that $f_E(S) = \{x\}$ and thus that $\pi: E \to S$ is trivial. This is a contradiction because not every complex vector bundle is trivial, as it is seen by taking the tangent bundle $T(\mathbb{P}^1_{\mathbb{C}}) \to \mathbb{P}^1_{\mathbb{C}}$.

The problem of the non existence of such fine moduli spaces can be solved by defining the concept of *coarse moduli space* for a moduli problem. Let $S \in \mathbf{Sch}$ be a scheme and let

$$\mathscr{P}: (\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Sets}$$

be a functor. Then a *coarse moduli space* for $\mathscr{P}$ is an S-scheme $X \in \mathbf{Sch}/S$ together with a natural transformation $\Phi: \mathscr{P} \to h_X$ such that for every affine S-scheme $\mathrm{Spec}(\mathbb{K}) \in \mathbf{Sch}/S$ where $\mathbb{K}$ is an algebraically closed field we have that the map

$$\Phi(\mathrm{Spec}(\mathbb{K})): \mathscr{P}(\mathrm{Spec}(\mathbb{K})) \to \mathrm{Hom}(\mathrm{Spec}(\mathbb{K}), X)$$

is bijective and for every $Y \in \mathbf{Sch}/S$ and every natural transformation $\Psi: \mathscr{P} \to h_Y$ there exists a morphism of schemes $f: X \to Y$ such that $\Psi = h_{\_}(f) \circ \Phi$.

How does this discussion relate to modular curves? As we have already said, modular curves are exactly coarse (and sometimes fine) moduli spaces for moduli problems starting from the category $\mathbf{Ell}_S$ of elliptic curves over a scheme $S \in (\mathbf{Sch})$. To give a precise definition of this fact we will first of all review why all the complex elliptic curves are actually algebraic objects, and then we will generalize this new definition of elliptic curve to schemes different from $\mathrm{Spec}(\mathbb{C})$.

Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \leq \mathbb{C}$ be a lattice and let $E_\Lambda = {}^{\mathbb{C}}/_{\Lambda}$ be the associated elliptic curve. We define the *Weierstrass* $\wp$ *function* associated to Λ as the meromorphic function

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\substack{\lambda\in\Lambda \\ \lambda\neq 0}} \frac{1}{(z-\lambda)^2} + \frac{1}{\lambda^2}$$

and we observe that if we suppose that $\tau = {}^{\omega_2}/_{\omega_1} \in \mathfrak{h}$ we have that

$$(\wp'_\Lambda(z))^2 = 4\wp^3_\Lambda(z) - g_2(\tau)\wp_\Lambda(z) - g_3(\tau) \tag{1}$$

for every $z \in \mathbb{C}$ as is not difficult to prove.

Using Equation 1 we can show that the map

$$E_\Lambda \to \mathbb{P}^2_{\mathbb{C}} \qquad \text{defined as} \qquad \pi(z) \mapsto \begin{cases} (\wp_\Lambda(z) : \wp'_\Lambda(z) : 1), \text{ if } z \notin \Lambda \\ (0:1:0), \text{ if } z \in \Lambda \end{cases}$$

is an isomorphism between the elliptic curve E_Λ and the algebraic curve

$$y^2 z = 4x^3 - g_2(\tau) x z^2 - g_3(\tau) z^3$$

defined inside the projective space $\mathbb{P}^2_{\mathbb{C}}$. We say that an equation of this kind is a *Weierstrass equation* for the elliptic curve E_Λ.

Generalizing this fact to arbitrary fields we say that an elliptic curve E over a field $\mathbb{F}$ is the algebraic curve defined inside the projective space $\mathbb{P}^2_{\mathbb{F}}$ given by an equation of the form

$$y^2 z + axyz + byz^2 = x^3 + cx^2 z + dxz^2 + ez^3 \qquad \text{for} \qquad a,b,c,d,e \in \mathbb{F} \tag{2}$$

such that $\Delta = 9\alpha\beta\gamma - \alpha^2\delta - 8\beta^3 - 27\gamma^2 \neq 0$, where $\alpha = a^2 + 4c$, $\beta = ab + 2d$, $\gamma = b^2 + 4e$ and $\delta = a^2 e + 4ce - abd + cb^2 - d^2$. It is not hard to see that the condition $\Delta \neq 0$ is equivalent to the fact that the curve defined in Equation 2 is a smooth curve of genus 1. Moreover each curve defined by a Weierstrass equation of the form (2) contains the point $(0:1:0) \in \mathbb{P}^2_{\mathbb{F}}$ which can be chosen as a special point of our curve. Thus every elliptic curve is a smooth, projective curve of genus 1 defined over $\mathbb{F}$ with a specific point chosen on it. The converse is also true, because for every smooth, projective curve C over a field $\mathbb{F}$ of genus 1 we can choose a point $O \in C$ and we can define an embedding $C \hookrightarrow \mathbb{P}^2_{\mathbb{F}}$ such that $O \mapsto (0:1:0)$ and C is identified with a curve defined by a Weierstrass equation.

The previous discussion allows us to define the more general notion of an elliptic curve over a scheme $S \in \mathbf{Sch}$ as a scheme $E \in \mathbf{Sch}/S$ such that the morphism $E \to S$ is smooth and proper and its fibers are all connected curves of genus one, together with a section $O: S \to E$ of the morphism $E \to S$. We have the naturally defined category $\mathbf{Ell}_S$ of elliptic curves over a scheme $S \in \mathbf{Sch}$ which will be the crucial category for the definition of the modular curves $Y(N)$, $Y_0(N)$ and $Y_1(N)$ as solutions to moduli problems. In order to define these moduli problems we have to define a group structure on the elliptic curves $E \in \mathbf{Ell}_S$, which will give us another new definition of elliptic curve.

We look first of all at an elliptic curve $E/\mathrm{Spec}(\mathbb{F})$ defined by a Weierstrass equation of the form (2) inside the projective plane $\mathbb{P}^2_{\mathbb{F}}$ and we observe that every line $\mathfrak{l} \subseteq \mathbb{P}^2_{\mathbb{F}}$ intersects E in at most three points. This implies that we can define a commutative and associative operation

$$+ : E(\mathbb{F}) \times E(\mathbb{F}) \to E(\mathbb{F})$$

such that for every line $\mathfrak{l} \subseteq \mathbb{P}^2_{\mathbb{F}}$ we have $P+Q+R=O$ if $\mathfrak{l} \cap E(\mathbb{F}) = \{P,Q,R\}$ and $P+(0:1:0)=(0:1:0)+P=P$ for every $P \in E(\mathbb{F})$. This group law, which can be defined very explicitly using some homogeneous polynomials on $E(\mathbb{F})$, can be also described in a more abstract way using the Picard group of our elliptic curve E, and this construction can be done also for elliptic curves in the category $\mathbf{Ell}_S$.

Recall first of all that a scheme $X \in \mathbf{Sch}$ admits the structure of a group scheme if and only if the functor $h_X : \mathbf{Sch} \to \mathbf{Sets}$ admits a factorization through the forgetful functor $\mathbf{Grp} \to \mathbf{Sch}$, where $\mathbf{Grp}$ is the category of groups. Recall moreover that for every scheme $X \in \mathbf{Sch}$ the Picard group $\mathrm{Pic}(X)$ is defined as the group of all the invertible sheaves of $\mathcal{O}_X$ modules on X. If $E \in \mathbf{Ell}_S$ we can define a subgroup $\mathrm{Pic}^0(E) \le \mathrm{Pic}(E)$ as the set of those sheaves $\mathscr{L} \in \mathrm{Pic}(E)$ such that the restriction $\mathscr{L}|_F$ of any of these sheaves to a fiber $F \subseteq E$ of the map $E \to S$ has degree zero. Then we can define

$$\mathrm{Pic}^0(E/S) \stackrel{\mathsf{def}}{=} \frac{\mathrm{Pic}^0(E)}{f^*(\mathrm{Pic}(S))} \qquad \text{where} \qquad f : E \to S$$

and we have a functor $(\mathbf{Sch}/S)^{\mathrm{op}} \to \mathbf{Grp}$ which sends every S-scheme T to $\mathrm{Pic}^0(E_T/T)$ where the elliptic curve $E_T \to T$ is obtained as the pullback of $E \to S$ along the map $T \to S$. it is not difficult to see that the composition of this functor with the forgetful functor $\mathbf{Grp} \to \mathbf{Sets}$ is isomorphic to h_E, and thus that E has the natural structure of an abelian group scheme over S.

Finally we are ready to define the moduli problems associated to the modular curves $Y(N)$, $Y_0(N)$ and $Y_1(N)$. Observe first of all that for every scheme S, every elliptic curve $E \in \mathbf{Ell}_S$ and every $n \in \mathbb{Z} \setminus \{0\}$ the map $[n] : E \to E$ which sends any point P to nP is a *finite* and *locally free* morphism of rank n^2, meaning that for every $P \in E$ there exists an affine open neighbourhood $P \in \mathrm{Spec}(A) \subseteq E$ such that $[n]^{-1}(\mathrm{Spec}(A)) = \mathrm{Spec}(B)$ with $B \cong A^{n^2}$ as a module over A. Moreover

$$\begin{tikzcd} E[n] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "{[n]}"] \\ S \arrow[r, "O"] & E \end{tikzcd}$$

if $E[n] = \ker([n])$ is defined as the pullback on the right and if $S \in \mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])$ then the morphism $E[n] \to S$ is finite and étale and there exists a surjective, finite and étale morphism $T \to S$ such that $E_T[n] \cong \left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^2_T$.

Using the facts proved above we can associate to every scheme $S \in \mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])$ the set of pairs (E, φ) where $E \in \mathbf{Ell}_S$ and $\varphi : ({}^{\mathbb{Z}}/_{n\mathbb{Z}})^2_T \xrightarrow{\sim} E[n]$ is an isomorphism. This association defines a functor

$$[\Gamma(n)] : \left(\mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])\right)^{\mathrm{op}} \to \mathbf{Sets}$$

which is a moduli problem for the category of $\mathbb{Z}[{}^1/_n]$-schemes. In a similar way we can define the moduli problems $[\Gamma_1(n)]$ and $[\Gamma_0(n)]$. The first one is given by a functor $[\Gamma_1(n)] : \left(\mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])\right)^{\mathrm{op}} \to \mathbf{Sets}$ which associates to every $\mathbb{Z}[{}^1/_n]$-scheme S the set of isomorphism classes of pairs (E, ι) where $E \in \mathbf{Ell}_S$ and $\iota : \left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)_S \hookrightarrow E[n]$ is an inclusion of group schemes. The second one is given by the functor $[\Gamma_0(n)]$ sending $S \in \mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])$ to the set of isomorphism classes of couples (E, H) where $E \in \mathbf{Ell}_S$ and $H \hookrightarrow E$ is a subgroup scheme such that there exists a surjective, finite étale morphism $T \to S$ with respect to which we have an isomorphism $H \times_S T \cong \left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)_T$. The names $[\Gamma(n)]$, $[\Gamma_0(n)]$ and $[\Gamma_1(n)]$ given to these moduli problems are explained by the following theorem.

Theorem 6. *For every $n \in \mathbb{N}_{\ge 1}$ there exist three smooth, affine curves over* $\mathrm{Spec}(\mathbb{Z}[{}^1/_n])$ *called $Y(n)$, $Y_0(n)$ and $Y_1(n)$ which are coarse moduli spaces for the moduli problems $[\Gamma(n)]$, $[\Gamma_0(n)]$ and $[\Gamma_1(n)]$ respectively. Moreover for every* $S \in \mathbf{Sch}/\mathrm{Spec}(\mathbb{Z}[{}^1/_n])$ *the induced morphisms $Y(n)_S \to S$, $Y_0(n)_S \to S$ and $Y_1(n)_S \to S$ are finite and étale, $Y(n)$ is a fine moduli space for every $n \ge 3$ and $Y_1(n)$ is a fine moduli space for every $n \ge 4$. Finally we have*

$$Y(n)_{\mathrm{Spec}(\mathbb{C})} \cong \Gamma(n)\backslash\mathfrak{h} \qquad Y_0(n)_{\mathrm{Spec}(\mathbb{C})} \cong \Gamma_0(n)\backslash\mathfrak{h} \qquad Y_1(n)_{\mathrm{Spec}(\mathbb{C})} \cong \Gamma_1(n)\backslash\mathfrak{h}$$

where the isomorphisms are isomorphisms of complex affine curves.

Proof. See **Corollary 2.7.3**, **Theorem 3.7.1** and **Corollary 4.7.1** of [1] □

As a corollary to this theorem we get a different set description of the classical complex modular curves $\Gamma(n)\backslash\mathfrak{h}$, $\Gamma_0(n)\backslash\mathfrak{h}$ and $\Gamma_1(n)\backslash\mathfrak{h}$. Observe first of all that the association

$$\tau \mapsto \left(\frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\tau}, \frac{\frac{1}{n}\mathbb{Z}}{\mathbb{Z}}\right)$$

induces a bijection between the set $\Gamma_0(n)\backslash\mathfrak{h}$ and the set of isomorphism classes of couples (E,C) where $E\in\mathbf{Ell}_{\mathbb{C}}$ and $C\leq E$ is a cyclic subgroup of E of order n. This bijection is the same that we obtain from the isomorphism $\Gamma_0(n)\backslash\mathfrak{h}\cong Y_0(n)_{\mathrm{Spec}(\mathbb{C})}$ using the definition of $Y_0(n)_{\mathrm{Spec}(\mathbb{C})}$ as a coarse moduli space. We can also obtain similar bijections for $\Gamma_1(n)\backslash\mathfrak{h}$ and $\Gamma(n)\backslash\mathfrak{h}$. Indeed the association

$$\tau \mapsto \left(\frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\tau}, \frac{1}{n} \bmod \mathbb{Z}\oplus\mathbb{Z}\tau\right)$$

induces a bijection between $\Gamma_1(n)\backslash\mathfrak{h}$ and the set of isomorphism classes of couples (E,P) where $E\in\mathbf{Ell}_{\mathbb{C}}$ and $P\in E$ is a point of order n. Finally the association

$$\tau \mapsto \left(\frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\tau}, \phi_\tau\right) \qquad \text{where} \qquad \begin{aligned} \phi_\tau \colon \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^2 &\xrightarrow{\sim} \left(\frac{\mathbb{C}}{\mathbb{Z}\oplus\mathbb{Z}\tau}\right)[n] \\ (1,0) &\mapsto {}^1/_n \bmod \mathbb{Z}\oplus\mathbb{Z}\tau \\ (0,1) &\mapsto {}^\tau/_n \bmod \mathbb{Z}\oplus\mathbb{Z}\tau \end{aligned}$$

induces a bijection between $\Gamma(n)\backslash\mathfrak{h}$ and the set of isomorphism classes of couples (E,ϕ) where $E\in\mathbf{Ell}_{\mathbb{C}}$ and ϕ is an isomorphism $\phi\colon\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^2 \xrightarrow{\sim} E[n]$. Using these descriptions of the modular curves as moduli spaces of the elliptic curves we are now able to give a new and third description of these modular curves, which uses the adèle ring $\mathbb{A}_{\mathbb{Q}}$ which we will define in the following section.

References

[1] Nicholas Michael Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR 772569

[2] Walter Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987. MR 924157