Background material for the PhD project

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Abstract

In this paper we collect some outlines of the main initial papers needed to start our PhD project.

SECTION 1

Outline of the PhD project

The aim of the project is to understand more deeply the evidences of links between heights of polynomials and special values of L-functions. One of the simplest notions of height of a Laurent polynomial $P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is its **Mahler measure**

$$m(P) := \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}\right) \right| dt_1 \cdots dt_n = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(z_1, \dots, z_n)| \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n} \tag{1}$$

where $\mathbb{T}^n := \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_j| = 1, \forall j = 1, \dots, n\} = S^1 \times \cdots \times S^1$ is the real n-torus. We expect to find equalities of the following kind

$$m(P) = r \cdot L(X, s_0) \qquad \text{or} \qquad m(P) = r \cdot L'(X, s_0) \qquad \text{or} \qquad m(P) = r \cdot \frac{L'(X, s_0)}{L(X, s_0)} \tag{2}$$

for some “geometric object” X and some $r \in \mathbb{Q}^\times$. It is natural now to ask the following question.

Question 1.1. *What is the relationship between P and X? Given the polynomial P can we always find an object X such that a relation as above holds? Vice-versa, given a geometric object X can we find a polynomial P such that a relation as the ones displayed above holds?*

One first guess would be that X is the affine variety defined by $P(x_1, \dots, x_n) = 0$ inside $\mathbb{A}^n$, or its projective closure inside $\mathbb{P}^n$. This seems to be the case for example if $P(x, y) \in \mathbb{Z}[x, y]$ defines an elliptic curve E and P satisfies some further technical conditions which are explained in full detail in [4]. In this situation Boyd conjectured that $m(P(x, y)) = r \cdot L'(E, 0)$ for some $r \in \mathbb{Q}^\times$. On the other hand if the projective closure of $P(x, y) = 0$ has genus zero Boyd conjectured that $m(P(x, y)) = r \cdot L'(\chi_f, -1)$, where $\chi_f(n) := \left(\frac{f}{n}\right)$ is the real odd Dirichlet character of conductor $f \in \mathbb{Z}$. How the Dirichlet character χ_f is related to $P(x, y)$ is still somehow misterious, and deserves further attention.

If we suppose that taking X to be (the projective closure of) $P = 0$ is the right choice to obtain equalities of the form (2) then another question come naturally to the mind.

Question 1.2. *Let P_1 and P_2 be two polynomials defining the same variety X. What is the relation between $m(P_1)$ and $m(P_2)$?*

The answer to this question is not so clear, and we don't have even a conjectural answer so far.

Finally another extremely interesting problem is given by taking a polynomial $P(x, y)$ which defines a modular curve $X(\Gamma)$ for some congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$. In the paper [5] by Brunault it is proved that $m(P) = 2L'(f, 0)$ where $P(x, y) \in \mathbb{Z}[x, y]$ is a specific equation defining the modular curve $X_1(13) := X(\Gamma_1(13))$ and $f \in \mathscr{S}_2(\Gamma_0(13))$ is the trace of the newform 13.2.4.a of the LFMDB database. A natural question arises from this.

Question 1.3. *Given an integer $N \in \mathbb{N}$ and two affine polynomials $P(x,y), Q(x,y) \in \mathbb{Z}[x,y]$ which define the modular curves $X_0(N)$ and $X_1(n)$ can we find two modular forms $f \in \mathscr{M}_k(\Gamma_0(N))$ and $g \in \mathscr{M}_k(\Gamma_1(N))$ such that $m(P) = r \cdot L'(f,0)$ and $m(Q) = s \cdot L'(g,0)$ for some $r,s \in \mathbb{Q}^\times$?*

Finally, all these papers make a great use of the method portrayed in Deninger's paper [6], which uses motives and K-theory to give a very interesting and new interpretation of the links that appear in the field. In particular these links between heights and L-functions have their natural collocation in the frame of the conjectures of Beĭlinson and Bloch-Kato. To learn more about these see [8] and [1].

SECTION 2

Boyd's paper

In this paper Boyd provides some experimental evidence of the following conjecture. Let $P(x,y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij} x^i y^j \in \mathbb{Z}[x,y]$ and define the Newton polytope $N(P)$ associated to P as the convex hull in $\mathbb{R}^2$ of the finite set $S = \{(i,j) \in \mathbb{R}^2 \mid a_{ij} \neq 0\}$. Then we define a face F of $N(P)$ as the intersection of $N(P)$ with a support line of $N(P)$, and we define the corresponding face P_F of P as $P_F(x,y) := \sum_{(i,j)\in F\cap S} a_{ij} x^i y^j$. We are now ready to state Boyd's conjecture, which concerns a specific family of polynomials.

Conjecture 2.1. *Let $P(x,y) = A(x)y^2 + B(x)y + C(x) \in \mathbb{Z}[x,y]$ and let $D(x) := B(x)^2 - 4A(x)C(x)$. Suppose that:*

1. *the degree of $D(x)$ is 3 or 4;*
2. *$m(P_F(x,y)) = 0$ for all faces F of $N(P)$;*
3. *$m(P(x,y)) = r \cdot \oint_\gamma \omega$ for some $r \in \mathbb{Q}^\times$ where*

$$\omega := \frac{1}{2\pi}(\log|y|\, d\log|x| - \log|x|\, d\log|y|)$$

   and γ is a branch cut of the algebraic function $y(x)$ defined by $P(x,y) = 0$.

Then if $P(x,y) = 0$ defines an elliptic curve E we have $m(P(x,y)) = r \cdot L'(E,0)$ for some $r \in \mathbb{Q}^\times$, whereas if $P(x,y)$ has genus zero we have $m(P(x,y)) = \sum_{j=1}^m r_j L'(\chi_{f_j}, -1)$ for some $r_j \in \mathbb{Q}^\times$ and $f_j \in \mathbb{Z} \setminus \{0\}$.

F. Rodriguez Villegas and H. Bornhorn proved in [15] and [3] that this conjecture follows from the Bloch-Beĭlinson's conjectures when $P(x,y) \neq 0$ for every $(x,y) \in \mathbb{T}^2$. In Boyd's paper, on the other hand, he concentrated on families of polynomials of type

$$P_k(x,y) = A(x)y^2 + (B(x) + kx)y + C(x) \tag{3}$$

where $A(x), B(x), C(x) \in \mathbb{Z}[x]$ and $k \in S \subseteq \mathbb{C}$. Let now K be the set of all $k \in S$ such that $P_k(x,y) = 0$ for some $(x,y) \in \mathbb{T}^2$. It is easy to see that K is bounded. Let thus G_∞ be the unbounded component of $G := \mathbb{C} \setminus K$. Then Boyd makes another conjecture.

Conjecture 2.2. *Let $S \subseteq \mathbb{C}$ and let $\{P_k\}_{k\in S}$ be the family of polynomials defined by $P_k(x,y) = A(x)y^2 + (B(x) + kx)y + C(x)$. Suppose that the Mahler measure of every face of $P_k(x,y)$ is zero. Then we conjecture that:*

- *if $k \in G_\infty$ we have that $m(P_k(x,y)) = r \cdot L'(E,0)$ for some $r \in \mathbb{Q}^\times$ and some elliptic curve E related to $P_k(x,y)$;*
- *if $k \in \partial G_\infty$ we have that $m(P_k(x,y)) = r \cdot L'(E,0)$ or $m(P_k(x,y)) = r \cdot L'(\chi,-1)$ (if the discriminant of P_k vanishes) for some $r \in \mathbb{Q}^\times$ and some elliptic curve E and Dirichlet character χ related to $P_k(x,y)$;*
- *if $k \in \mathring{K}$ we don't have any formula of this kind.*

Then Boyd started to use PARI and Magma to compute some tables of Mahler measures and L-function values. He started considering families of polynomials $\{P_k\}_{k\in\mathbb{Z}}$ which have genus one for almost all k. Thus either P_k is a model of some elliptic curve E_k or the Jacobian J_k of the curve defined by $P_k(x,y)=0$ is an elliptic curve. In both cases we provide a Weierstrass equation for these elliptic curves.

$$P_k(x,y)=y^2+(x^2+kx+1)y+x^2 \qquad E_k : y^2=x^3+(k^2-8)x^2+16x \tag{F1}$$
$$P_k(x,y)=(x+1)y^2+(x^2+kx+1)y+(x^2+x) \qquad E_k : y^2=x^3+(k^2-12)x^2-16(k-3)x \tag{F2}$$
$$P_k(x,y)=(x^2+x-1)y^2+(kx)y+(-x^2+x+1) \qquad J_k : y^2=x(x-(k^2-4))(x-(k^2-20)) \tag{F3}$$
$$P_k(x,y)=(x^2+x-1)y^2+(x^2+kx+1)y+(-x^2+x+1) \qquad J_k : y^2=x^3+(k^2-40)x^2-16(k^2-25)x \tag{F4}$$
$$P_k(x,y)=(x+1)y^2+(x^2+(k+3)x+2)y+(x+1)^2 \qquad E_k : y^2=x^3+(k^2-6k+1)x^2+(-8k^3+8k^2)x+16k^4 \tag{F5}$$
$$P_k(x,y)=y^3+(3x-2)y^2+(3x^2+kx+1)y+(x^3+2x^2+x) \qquad E_k : y^2=x(x+16)(x+k^2) \tag{F6}$$
$$P_k(x,y)=y^3+(kx)y+(x^3+1) \qquad E_k : y^2=x^3-27k^2x^2+216k(k^3+27)x-432(k^3+27)^2 \tag{F7}$$
$$P_k(x,y)=y^2+(x^2+kx)y+x \qquad E_k : y^2=x^3+k^2x^2-8kx+16. \tag{F8}$$

He also considered two parameter families $\{P_{k,b}\}$ of polynomials which are already in generalised Weierstrass equation

$$P_{k,b}(x,y)=y^2+(kx+b)y-x^3+1 \qquad E_{k,b} : y^2+(kx+b)y=x^3-1 \tag{F9}$$
$$P_{k,b}(x,y)=y^2+(kx+b)y-x^3 \qquad E_{k,b} : y^2+(kx+b)y=x^3 \tag{F10}$$
$$P_{k,b}(x,y)=y^2+kxy-x^3-bx \qquad E_{k,b} : y^2+kxy=x^3+bx \tag{F11}$$

and he worked with them by fixing the parameter $b\in\mathbb{Z}$ and letting $k\in\mathbb{Z}$ vary.

We would like now to make a couple of remarks about these families:

- the families (F6) and (F7) are not of type (3) but the same considerations about the sets K and G_∞ are valid also in this setting;
- the families (F1), (F2), (F3) and (F4) are made of reciprocal polynomials. Recall that if R is a ring, a polynomial $P\in R[x_1,\dots,x_n]$ is said to be reciprocal if

$$\frac{P(x_1,\dots,x_n)}{P\left(\frac{1}{x_1},\dots,\frac{1}{x_n}\right)}=x_1^{b_1}\cdots x_n^{b_n}$$

  for some $b_1,\dots,b_n\in\mathbb{N}$;

Observe that if all the polynomials $P_k(x,y)$ of a family are reciprocal then $K\subseteq\mathbb{R}$. This implies in particular that $\overline{G}_\infty=\mathbb{C}$ and thus that Conjecture 2.2 predicts that we should get an identity of the types mentioned above for all $k\in\mathbb{C}$. For the reciprocal families (F1), (F2), (F3) and (F4) Boyd proved some identities and found numerically some other conjectural identities, verified up to 28 decimal places (the default precision of PARI back in 1998).

Family (F1) $m(P_k(x,y))\stackrel{?}{=}r_kL'(E_k,0)$ for some $r_k\in\mathbb{Z}\cup{}^1/_{\mathbb{Z}}$ and for $k\leq 100$ with $k\neq 0,4$. In these cases $m(P_0(x,y))=0$ and $m(P_4(x,y))=2L'(\chi_{-4},-1)$.

Family (F2) $m(P_k(x,y))\stackrel{?}{=}r_kL'(E_k,0)$ for some $r_k\in\mathbb{Z}\cup{}^1/_{\mathbb{Z}}$ and for $|k|\leq 100$ with $k\neq 2,3,-6$. Moreover $m(P_2(x,y))=0$, $m(P_3(x,y))=2L'(\chi_{-3},-1)$ and $m(P_{-6}(x,y))\stackrel{?}{=}5L'(\chi_{-3},-1)$ (now proved by Rodriguez Villegas).

Family (F3) $m(P_k(x,y))=\log\left({}^{1+\sqrt{5}}/_2\right)$ for $k=0,1,2$, whereas for $3\leq k\leq 100$ no identities of the types mentioned above seem to appear; this could be because some faces of $P_k(x,y)$ have Mahler measure different from zero;

Family (F4) $m(P_0(x,y)) = \log\left({}^{1+\sqrt{5}}/_{2}\right)$, $m(P_4(x,y)) \stackrel{?}{=} 2\log\left({}^{1+\sqrt{5}}/_{2}\right)$ and $m(P_5(x,y)) \stackrel{?}{=} \frac{2}{3}\log\left({}^{1+\sqrt{5}}/_{2}\right) + \frac{1}{6}L'(\chi_{-15},1)$. Again for $k \leq 20$ and $k \neq 0,4,5$ it seems that no relation of the types introduced above exists. This agrees again with Conjecture 2.1 because some faces of $\{P_k(x,y)\}$ have Mahler measure different from zero when $k \notin \{0,\pm 4,\pm 5\}$.

Moreover for the one-parameter non-reciprocal families (F5), (F6), (F7) and (F8) he proved and verified experimentally the following identities:

Family (F5) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z} \cup {}^1/_{\mathbb{Z}}$ and for $|k| \leq 50$ with $k \notin \{-11,-10,\dots,0\} = \mathbb{Z} \cap \mathring{K}$. When $k \in \mathring{K}$ PARI didn't recognise any suitable identity, and this is coherent with Conjecture 2.2.

Family (F6) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z} \cup {}^1/_{\mathbb{Z}}$ and for $10 \leq k \leq 200$. This range for k has been chosen because Conjecture 2.2 here predicts that we should get a suitable identity for $|k| \geq 10$ since $\mathbb{Z} \cap \mathring{K} = \{-9,-8,\dots,9\}$;

Family (F7) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z} \cup {}^1/_{\mathbb{Z}}$ and for $|k| \leq 40$ and $k \notin \{-3,-2,-1,0\} = \mathbb{Z} \cap \mathring{K}$. Moreover $m(P_{-3}(x,y)) = 3L'(\chi_{-3},-1)$ and $m(P_0(x,y)) = L'(\chi_{-3},-1)$ as it's easy to prove using the fact that

$$m(x+y+1) = L'(\chi_{-3},-1)$$

as it's shown in C.J. Smyth's paper [13].

Family (F8) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z} \cup {}^1/_{\mathbb{Z}}$ and for $|k| \leq 40$ and $k \notin \{-3,-2,-1,0\} = \mathbb{Z} \cap \mathring{K}$.

Finally for the two-parameter families (F9), (F10) and (F11) Boyd found out that

Family (F9) $m(P_{k,b}(x,y)) \stackrel{?}{=} r_{k,b} L'(E_{k,b},0)$ for some $r_{k,b} \in \mathbb{Z} \cup {}^1/_{\mathbb{Z}}$ and for $b \in \{0,1,2\}$ and $b+3 < |k| \leq 18$. This agrees with Conjecture 2.1 since for $b \notin \{0,1,2\}$ the face y^2+by+1 of the polynomial $P_{k,b}$ has Mahler measure different from zero, and it agrees also with Conjecture 2.2 since for every fixed $b \in \{0,1,2\}$ we have that $G_{b,\infty} \cap \mathbb{Z} = \{k \in \mathbb{Z} \mid |k| > b+3\}$. In fact for $b=3$ PARI wasn't able to find any suitable identity for $6 \leq |k| \leq 19$. Finally for $|b| > |k|+3$ we have that

$$m(P_{k,b}(x,y)) = m(y^2+by+1) = \log^+\left(\frac{-b+\sqrt{b^2-4}}{2}\right) + \log^+\left(\frac{-b-\sqrt{b^2-4}}{2}\right)$$

where $\log^+(v) := \max(\log(v),0)$ if $v > 0$ and $\log^+(0) := 0$;

Family (F10) We have that the two faces y^2+by and $by-x^3$ of the polynomial $P_{k,b}(x,y)$ have Mahler measure

$$m(y^2+by) = m(by-x^3) = \log(|b|)$$

and so the hypotheses of Conjecture 2.1 hold only if $|b| = 1$. Studying the one-parameter family $\{P_{k,1}\}$ is completely equivalent to study the family (F8), and in particular yields to the identity $m(P_{-6,1}(x,y)) \stackrel{?}{=} 3L'(E,0)$ where E is the unique elliptic curve of conductor 27, up to isogeny. This identity has now been proved in [15]. Moreover if $2 \leq b \leq 8$ we still obtain numerically the identity $m(P_{k,b}(x,y)) \stackrel{?}{=} \frac{1}{3}\log(b) + r_{k,b}L'(E_{k,b},0)$ but only if for every prime $p \mid b$ we have that $p \mid k$ too. This has been verified for $b=2$ and $2 \leq |k| \leq 40$ with $k \neq 2,3$. If $b > 2$ then for $k \in G_{b,\infty}$ we have that $m(P_{k,b}) = \log b$;

Family (F11) Again we have that the two faces y^2-bx and x^3+bx of $P_{k,b}(x,y)$ have Mahler measure

$$m(y^2-bx) = m(x^3+bx) = \log(|b|)$$

and thus the hypotheses of Conjecture 2.1 hold only if $|b| = 1$. It has been verified that $m(P_{k,1}(x,y)) \stackrel{?}{=} r_{k,1}L'(E_{k,1},0)$ for $3 \leq k \leq 40$, which agrees with Conjecture 2.2 because $\mathbb{R} \cap K_b = [-(b+2), b+2]$ if $b > 0$. It has also been verified that $m(P_{k,-1}(x,y)) \stackrel{?}{=} r_{k,-1}L'(E_{k,-1},0)$ for $2 \leq k \leq 40$, which agrees with Conjecture 2.2 because $\mathbb{Z} \cap K_{-1} = \{-1,0,1\}$. We have also another mixed-type formula, namely $m(P_{k,b}(x,y)) \stackrel{?}{=} \frac{1}{4}\log(|b|) + r_{k,b}L'(E_{k,b},0)$ which it's thought to hold for $|b| > 1$ when for every prime $p \mid b$ we also have that $p \mid k$. This has been verified for $b \in \{-2,-1,\dots,6\} \setminus \{0\}$ and $|b|+2 \leq k \leq 40$.

Now Boyd turns his attention to some families of polynomials of genus two. In particular he restricts to families $\{P_k(x,y)\}$ which satisfy the hypotheses of Conjecture 2.1 and such that the Jacobian of the curve defined by $P_k(x,y)=0$ splits as $E_k \times F_k$ where E_k and F_k are elliptic curves. We will write here the equations of the families $\{P_k(x,y)\}$ with corresponding Weierstrass equations for E_k and F_k:

$$\boxed{\begin{aligned} &P_k(x,y)=(x^2+x+1)y^2+(kx(x+1))y+x(x^2+x+1)\\ &E_k: y^2=x^3+(k^2-24)x^2-16(k^2-9)x\\ &F_k: y^2=x^3+(k^2+8)x^2+16x \end{aligned}} \tag{F12}$$

$$\boxed{\begin{aligned} &P_k(x,y)=(x-1)^2y^2+(x^3+kx^2+kx+1)y+x(x-1)^2\\ &E_k: y^2=x^3-2(k+1)(k-3)x^2+(k+1)^3(k-7)x+16(k+1)^4\\ &F_k: y^2=x^3+(k+1)(k-7)x^2-32(k+1)(k-3)x+256(k+1)^2 \end{aligned}} \tag{F13}$$

$$\boxed{\begin{aligned} &P_k(x,y)=(x^2+x+1)y^2+(x^4+kx^3+(2k-4)x^2+kx+1)y+x^2(x^2+x+1)\\ &E_k: y^2=x^3+(k^2-4k-20)x^2-16(k-5)(k+1)x\\ &F_k: y^2=x^3+(k^2-8k+20)x^2+16(k-5)x \end{aligned}} \tag{F14}$$

$$\boxed{\begin{aligned} &P_k(x,y)=y^2+(x^4+kx^3+2kx^2+kx+1)y+x^4\\ &E_k: y^2=x^3+(k^2-4k-8)x^2+16(k+1)x\\ &F_k: y^2=x^3+(k^2-8k+8)x^2+16x \end{aligned}} \tag{F15}$$

In these cases Boyd reports the following discoveries:

Family (F12) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z}\cup {}^1/_{\mathbb{Z}}$ and for $1 \le k \le 33$ with $k \neq 3$. Moreover for $k=0,3$ the curve E_k is not an elliptic curve and we have $m(P_0(x,y))=0$ whereas $m(P_k(x,y)) \stackrel{?}{=} \frac{1}{6}L'(\chi_{-15},-1)$;

Family (F13) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Q}^\times$ and for $|k|\le 20$ with $k\neq -1$. Moreover $m(P_{-1}(x,y))=0$;

Family (F14) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z}$ and for $6\le k \le 35$. Moreover $m(P_2(x,y)) = \frac{4}{3}L'(\chi_{-4},-1)$, $m(P_5(x,y))=2L'(\chi_{-4},-1)$ and $m(P_{-1}(x,y)) \stackrel{?}{=} \frac{1}{3}L'(\chi_{-7},-1)+\frac{1}{6}L'(\chi_{-15},-1)$. Finally PARI didn't find any suitable relation for $k<6$ with $k \notin \{-1,2,5\}$. This partially agrees with Conjecture 2.1 since the third hypothesis of the conjecture does not hold for $k\le -1$. The cases $k\in\{0,3,4\}$ deserve a special treatment in Boyd's paper;

Family (F15) $m(P_k(x,y)) \stackrel{?}{=} r_k L'(E_k,0)$ for some $r_k \in \mathbb{Z}\cup {}^1/_{\mathbb{Z}}$ and for $-50\le k\le 4$ with $k\notin\{-1,8\}$. Moreover $m(P_{-1}(x,y)) \stackrel{?}{=} 2L'(\chi_{-3},-1)$ and $m(P_8(x,y)) \stackrel{?}{=} 4L'(\chi_{-4},-1)$. Finally PARI didn't find any suitable relation for $k>4$, which agrees with Conjecture 2.1 since the third hypothesis of the conjecture does not hold in these cases.

Finally Boyd provides some experimental evidences of equalities of the kind $m(P(x,y))=rL'(\chi,-1)$ where χ is a suitable Dirichlet character related to P. Boyd's attempt was to generalise the equalities of this kind proved in G.A. Ray's paper [9]. In particular Ray proved that $m(P_f(x,y))=r_f L'(\chi_{-f},-1)$ where $f\in\{3,4,7,8,20,24\}$ and $P_f(x,y)$ and r_f are given by

$$\begin{aligned} P_3(x,y) &= ?\\ P_4(x,y) &= ?\\ P_7(x,y) &= \Phi_7(x)(y-1)^2+7x^2(x+1)^2y\\ P_8(x,y) &= \Phi_8(x)(y-1)^2+8x^2y\\ P_{20}(x,y) &= \Phi_{20}(x)(y-1)^2+20x^2(x^2-1)^2\\ P_{24}(x,y) &= \Phi_{24}(x)(y-1)^2+24x^2(x^2-1)^2y \end{aligned} \quad \text{where} \quad \begin{aligned} r_f &= \begin{cases} \frac{8}{7}, & \text{if } f=7\\ \frac{8-\chi_{-f}(2)}{f}, & \text{if } f\neq 7 \end{cases}\\ \Phi_n(x) &= \prod_{\substack{1\le k\le n\\ \gcd(k,n)=1}} \left(x-e^{2i\pi\frac{k}{n}}\right) \end{aligned}.$$

Observe that every polynomial $P_f(x,y)$ splits completely into linear factors over the field $\mathbb{Q}(\sqrt{f})$. Thus Boyd looked for examples of identities of the form $m(P(x,y)) = rL'(\chi_f,-1)$ where $P(x,y)$ does not split completely over $\mathbb{Q}(\sqrt{f})$. He verified in particular the following identities:

$$\begin{aligned} m(y^2+(x^3-4x^2-4x+1)y+x^3) &\overset{?}{=} L'(\chi_{-7},-1) \\ m((x^2+x+1)(y^2+1)+2xy) &\overset{?}{=} \frac{1}{3}L'(\chi_{-8},-1) \\ m((x^2+x+1)(y^2+x)+3x(x+1)y) &\overset{?}{=} \frac{1}{6}L'(\chi_{-15},-1) \\ m((x^2+x+1)(y^2+x^2)+(x^4-x^3-6x^2-x+1)y) &\overset{?}{=} \frac{1}{3}L'(\chi_{-7},-1)+\frac{1}{6}L'(\chi_{-15},-1) \\ m((x^2+x+1)(y^2+1)+6xy) &\overset{?}{=} \frac{1}{6}L'(\chi_{-24},-1) \\ m((x^2+x+1)(y^2+x)+(x^3-4x^2-4x+1)y) &\overset{?}{=} \frac{1}{18}L'(\chi_{-39},-1) \\ m((x^4+x^3+x^2+x+1)(y^2+1)+(x^4-3x^3-6x^2-3x-1)y) &\overset{?}{=} \frac{1}{30}L'(\chi_{-55},-1) \end{aligned}$$

where all the polynomials do not split completely over the corresponding quadratic field $\mathbb{Q}(\sqrt{f})$. In the end Boyd pointed out that the study of the family (F4) led him to verify the identity

$$m((x^2+x-1)y^2+(x^2+5x+1)y+(-x^2+x+1)) \overset{?}{=} \frac{2}{3}\log\left({}^{1+\sqrt{5}}/_{2}\right)+\frac{1}{6}L'(\chi_{-15},1)$$

in which the polynomial splits completely over $\mathbb{Q}(\sqrt{5})$. Hence perhaps this identity could be proved using methods similar to the ones described in [9].

SECTION 3

Deninger's paper

In this paper C. Deninger interpreted the Mahler measure of a polynomial $0 \neq P(x_1,\dots,x_n) \in \mathbb{Q}[x_1,\dots,x_n]$ in several different ways, using dynamical systems, homology, K-theory and motives.

3.1 Mahler measure and dynamical systems

First of all he noticed that if A is a commutative ring and $\mathbb{G}^n_{m,A} := \operatorname{Spec}(A[\mathbb{Z}^n])$ is the split n-torus over A then for every coherent sheaf $\mathscr{M}$ over $\mathbb{G}^n_{m,A}$ the set of global sections $\mathscr{M}(\mathbb{G}^n_{m,A}) = \Gamma(\mathbb{G}^n_{m,A},\mathscr{M}) = H^0(\mathbb{G}^n_{m,A},\mathscr{M})$ is a finitely generated module over $A[\mathbb{Z}^n]$ and thus we have a canonical $\mathbb{Z}^n$-action on $\mathscr{M}(\mathbb{G}^n_{m,A})$ given by $\mathbf{v} * m := \mathbf{v}\cdot m$ for every $\mathbf{v} \in \mathbb{Z}^n \subseteq A[\mathbb{Z}^n]$ and $m \in \mathscr{M}(\mathbb{G}^n_{m,A})$. This induces an action on $\widehat{\mathscr{M}(\mathbb{G}^n_{m,A})}$ which is by definition the Pontryagin dual of $\mathscr{M}(\mathbb{G}^n_{m,A})$ seen as a discrete topological group. Then he reports from **Chapter V** of [11] that when $A=\mathbb{Z}$ the entropy $h(\mathscr{M})$ of the action of $\mathbb{Z}^n$ on $\widehat{\mathscr{M}(\mathbb{G}^n_{m,A})}$ is finite if and only if $\mathscr{M} \in \operatorname{Obj}(\mathscr{T})$, where $\mathscr{T}$ is the category of coherent torsion sheaves on $\mathbb{G}^n_{m,\mathbb{Z}}$. Since moreover $h(\mathscr{M}\oplus\mathscr{N}) = h(\mathscr{M})+h(\mathscr{N})$ as it is proved in **Theorem 14.1** of [11] we have an induced group morphism $h\colon K_0(\mathscr{T}) \to \mathbb{R}$ where $K_0(\mathscr{T})$ is the "smallest" group containing the abelian monoid $(\operatorname{Obj}(\mathscr{T}),\oplus,0)$. For a more rigorous definition of $K_0(\mathscr{C})$ where $\mathscr{C}$ is any symmetric monoidal category see **§5** of [16]. It is not difficult to see now that $K_0(\mathscr{T})$ is generated by the subset $\{\mathscr{O}_Z\} \subseteq K_0(\mathscr{T})$ where Z runs over all the irreducible closed sub-schemes $Z \subset \mathbb{G}^n_{m,\mathbb{Z}}$. Thus the morphism h is completely determined by the values $h(\mathscr{O}_Z)$. In particular if $0 \neq P \in \mathbb{Z}[\mathbb{Z}^n] = \mathbb{Z}[x_1^{\pm1},\dots,x_n^{\pm1}]$ is an irreducible Laurent polynomial and $Z(P) \subseteq \mathbb{G}^n_{m,\mathbb{Z}}$ is defined by $P=0$ we have that $h(\mathscr{O}_{Z(P)}) = m(P)$ where $m(P)$ is the Mahler measure of the polynomial P defined in (1). This gives us an interpretation of the Mahler measure of a polynomial as the entropy of a dynamical system.

3.2 Mahler measure and (co)homology

In this section we will explain how Deninger links the Mahler measure of a polynomial P to the evaluation of the cup product $\log(|P|)\cup\log(|z_1|)\cup\cdots\cup\log(|z_n|)$ on a "topological" homology class, as explained in [6]. To do so we need to recall the definition of the Deligne cohomology of a complex manifold X, which is clearly given in **§2** of [12].

Definition 3.1. Let $A \subseteq \mathbb{C}$ be a sub-group and for every $j \in \mathbb{N}$ define $A(j) := (2\pi i)^j A \subseteq \mathbb{C}$ where $i = \sqrt{-1}$. Then for every smooth complex manifold X we define the complex Deligne cohomology groups of X with coefficients in $A(j)$ as the cohomology $\{H^n_{\mathscr{D}}(X_{/\mathbb{C}}, A(j))\}_{n\in\mathbb{N}}$ of the complex

$$\mathbf{0} \to A(j) \to \mathscr{O}_X \to \Omega^1_X \to \ldots \to \Omega^{j-1}_X \to \mathbf{0}$$

where the inclusion $A(j) \hookrightarrow \mathscr{O}_X$ is given by the composition $A(j) \subseteq \mathbb{C} \subseteq \mathscr{O}_X$.

Observe now that if X is a variety over $\mathbb{R}$ then the complex conjugation $z \mapsto \overline{z}$ induces a canonical anti-holomorphic map $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ such that $F_\infty^2 = \mathrm{Id}_{X(\mathbb{C})}$.

Definition 3.2. Let $A \subseteq \mathbb{C}$ be a sub-ring and let $j \in \mathbb{N}$. Let moreover X be a variety over $\mathbb{R}$ and define the real Deligne cohomology groups with coefficients in $A(j)$ by setting $H^n_{\mathscr{D}}(X_{/\mathbb{R}}, A(j))_{n\in\mathbb{N}} := H^n_{\mathscr{D}}(X(\mathbb{C})_{/\mathbb{C}}, A(j))^+ := \{\omega \in H^n_{\mathscr{D}}(X(\mathbb{C})_{/\mathbb{C}}, A(j)) \mid \overline{F^*_\infty}(\omega) = \omega\}$ where $\overline{F^*_\infty}(\omega) := F^*_\infty(\overline{\omega}) = \overline{F^*_\infty(\omega)}$.

Recall now that we have a cup product operation

$$\cup : H^n_{\mathscr{D}}(X/\mathbb{K}, A(n)) \times H^m_{\mathscr{D}}(X/\mathbb{K}, A(m)) \to H^{n+m}_{\mathscr{D}}(X/\mathbb{K}, A(n+m))$$

which makes the abelian group $H^\bullet_{\mathscr{D}}(X/\mathbb{K}, A) := \bigoplus_{n\in\mathbb{N}} H^n_{\mathscr{D}}(X/\mathbb{K}, A(n))$ into a graded ring.

We fix also some notation for singular (co)homology groups. In particular if $K \in \{\mathbb{R}, \mathbb{C}\}$ and $\Lambda \subseteq \mathbb{C}$ is a subgroup which if $K = \mathbb{R}$ satisfies also $\overline{\Lambda} = \Lambda$ we set

$$H_n(X_{/\mathbb{C}}, \Lambda) := H^{\mathrm{sing}}_n(X(\mathbb{C}), \Lambda) \qquad \text{and} \qquad H^n(X_{/\mathbb{C}}, \Lambda) := H^n_{\mathrm{sing}}(X(\mathbb{C}), \Lambda)$$
$$H_n(X_{/\mathbb{R}}, \Lambda) := H^{\mathrm{sing}}_n(X(\mathbb{C}), \Lambda)^+ \qquad \qquad H^n(X_{/\mathbb{R}}, \Lambda) := H^n_{\mathrm{sing}}(X(\mathbb{C}), \Lambda)^+$$

where again when we write $H^{\mathrm{sing}}_n(X(\mathbb{C}), \Lambda)^+$ and $H^n_{\mathrm{sing}}(X(\mathbb{C}), \Lambda)^+$ we mean the elements fixed by $\overline{F^*_\infty}$.

Observe now that if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and X is an n-dimensional manifold over $\mathbb{K}$ then $H^{n+1}_{\mathscr{D}}(X/\mathbb{K}, \mathbb{R}(n+1)) \cong H^n(X/\mathbb{K}, \mathbb{R}(n))$. Recall moreover that we can associate to every orientable n-dimensional compact manifold X a homology class $[X] \in H^{\mathrm{sing}}_n(X, \mathbb{Z})$ such that under the pairing $<,> : H^{\mathrm{sing}}_n(X, \mathbb{R}) \times H^n_{\mathrm{sing}}(X, \mathbb{R}) \to \mathbb{R}$ we have $< \omega, [X] >= \int_X \omega$. Thus if $\iota : Y \to X$ is an embedding of a closed m-dimensional orientable sub-manifold Y inside a compact manifold X we have a naturally defined homology class $[Y] \in H^{\mathrm{sing}}_m(X, \mathbb{Z})$. In particular if $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $P \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$ is a polynomial such that $P(\mathbf{z}) \neq 0$ for every $\mathbf{z} \in \mathbb{T}^n$ we can consider the homology class $(2\pi i)^{-n} \cdot [\mathbb{T}^n] \in H_n(X_P/\mathbb{K}, \mathbb{Z}(-n))$ where $X_P := \mathbb{G}^n_{m,\mathbb{K}} \setminus \mathrm{Spec}(\mathbb{K}[\mathbb{Z}^n]/(P))$. We are now ready to give the cohomological interpretation of the Mahler measure $m(P)$.

Theorem 3.3. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $P \in \mathbb{K}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$ be a polynomial such that $P(\mathbf{z}) \neq 0$ for every $\mathbf{z} \in \mathbb{T}^n$. Then if $X_P := \mathbb{G}^n_{m,\mathbb{K}} \setminus \mathrm{Spec}(\mathbb{K}[\mathbb{Z}^n]/(P))$ we have that*

$$m(P) =< [\log(|P|)] \cup [\log(|z_1|)] \cup \ldots \cup [\log(|z_n|)], (2\pi i)^{-n} \cdot [\mathbb{T}^n] >$$

where the pairing $<,> : H^{n+1}_{\mathscr{D}}(X_P/\mathbb{K}, \mathbb{R}(n)) \times H_n(X_P/\mathbb{K}, \mathbb{R}(-n)) \to \mathbb{R}$ is obtained by combining the cap product pairing

$$\frown : H^n(X_P/\mathbb{K}, \mathbb{R}(n)) \times H_n(X_P/\mathbb{K}, \mathbb{R}(-n)) \to \mathbb{R}$$

with the isomorphism $H^{n+1}_{\mathscr{D}}(X_P/\mathbb{K}, \mathbb{R}(n+1)) \cong H^n(X_P/\mathbb{K}, \mathbb{R}(n))$.

Then Deninger derives also formulas for the Mahler measure of a polynomial $P \in \mathbb{C}[x_1,\dots,x_n] \setminus \{0\}$ which vanishes on the torus $\mathbb{T}^n$. In particular if $P(x_1,\dots,x_n) = \sum_{j \in \mathbb{N}} a_j(x_1,\dots,x_{n-1}) \cdot x_n^j$ we define $j_0 := \min\{j \in \mathbb{N} \mid a_j \neq 0\}$ and $P^*(x_1,\dots,x_{n-1}) := a_{j_0}(x_1,\dots,x_{n-1})$. Then Deninger proves using Jensen's formula (which is stated and proved in **Theorem 1.1** of **Chapter 5** of [14]) that

$$m(P^*) - m(P) = \int_{\mathbb{T}^{n-1}} \eta \qquad \text{where} \qquad \eta := (2\pi i)^{1-n} \cdot \sum_{\substack{0<|b|<1 \\ P(z',b)=0}} \log(|b|) \cdot \frac{dz_1}{z_1} \wedge \dots \wedge \frac{dz_{n-1}}{z_{n-1}}$$

and he notes in particular that the integral $\int_{\mathbb{T}^{n-1}} \eta$ is well defined because η is a $(n-1)$-form defined over

$$\mathbb{T}^{n-1} \setminus S_P \qquad \text{where} \qquad S_P := \{\mathbf{z} \in \mathbb{T}^{n-1} \mid P(\mathbf{z}, \cdot) = 0\}$$

and the set S_P has measure zero with respect to the Haar measure of $\mathbb{T}^{n-1}$. Deninger then imposes the following conditions on the polynomial $P(x_1,\dots,x_n)$:

C1 he requires that the divisor $\operatorname{div}(P)$ of P in $\mathbb{G}_{m,\mathbb{C}}^{n-1} \times \mathbb{A}_{\mathbb{C}}^1$ has no multiple components and $\operatorname{supp}(\operatorname{div}(P)) \cap (\mathbb{T}^{n-1} \times \{0\}) = \emptyset$, which is equivalent to say that $P^*(x_1,\dots,x_{n-1}) = P(x_1,\dots,x_{n-1},0)$ and $P^*(\mathbf{z}') \neq 0$ for every $\mathbf{z}' \in \mathbb{T}^{n-1}$;

C2 if A is the union of the $(n-1)$-dimensional connected components of

$$\overline{(\mathbb{T}^{n-1} \times \mathring{B}) \cap Z_P} \qquad \text{where} \qquad Z_P := \{\mathbf{z} \in (\mathbb{C}^\times)^n \mid P(z) = 0\} \qquad \text{and} \qquad B := \{z \in \mathbb{C} \mid 0 \leq |z| \leq 1\}$$

then Deninger requires that $A \subseteq Z_P^{\mathrm{reg}}$ and that $A \subseteq (\mathbb{C}^\times)^n$ is a real sub-manifold with boundary. Moreover he requires that if $(z_1,\dots,z_n) \in \partial A$ then $z_n \in \{\pm 1\}$, which is equivalent to say that the action $\{\pm 1\} \circlearrowright \mathbb{G}_{m,\mathbb{C}}^n$ given by $\varepsilon * (z_1,\dots,z_n) := (z_1,\dots,z_{n-1},z_n^\varepsilon)$ is trivial when restricted to ∂A.

The last part of this second condition allows us to say that $H_\bullet(\partial A;\mathbb{Q})(\delta) = 0$ where $\delta\colon \{\pm 1\} \to \mathbb{Q}^\times$ is the non-trivial character. Recall in particular that for every object X endowed with a $\mathbb{Q}$-linear action of a finite group G and for every character $\varepsilon\colon G \to \mathbb{Q}^\times$ we define $X(\varepsilon)$ by setting $X(\varepsilon) := \left(|G|^{-1} \cdot \sum_{g \in G} \varepsilon(g) \cdot g^{-1}\right) X$. Thus we obtain a homomorphism

$$H_{n-1}(Z^{\mathrm{reg}};\mathbb{Q}(1-n))(\delta) \xrightarrow{\sim} H_{n-1}(Z^{\mathrm{reg}},\partial A;\mathbb{Q}(1-n))(\delta) \tag{4}$$

given by the exact homology sequence of the pair $(Z^{\mathrm{reg}},\partial A)$ shifted by δ. Observe moreover that $A \hookrightarrow Z^{\mathrm{reg}}$ is a sub-manifold which is endowed with a canonical orientation given by $\mathbb{T}^{n-1} \times \mathring{B}$, and thus it defines a homology class $[A] \in H_{n-1}(Z^{\mathrm{reg}},\partial A;\mathbb{Z})$. This finally allows us to give a homological description of the Mahler measure $m(P)$.

Theorem 3.4. *Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and let $P \in \mathbb{K}[x_1,\dots,x_n] \setminus \{0\}$ be a polynomial which satisfies the conditions C1 and C2. Then we have*

$$m(P^*) - m(P) = < [\log(|z_1|)] \cup \ldots \cup [\log(|z_n|)], \gamma >$$

where $\gamma \in H_{n-1}(Z^{reg};\mathbb{Q}(1-n))(\delta)$ is the pre-image of the class $(2\pi i)^{1-n} \cdot [A](\delta) \in H_{n-1}(Z^{reg},\partial A;\mathbb{Q}(1-n))(\delta)$ under the isomorphism (4) and the pairing

$$<,>\colon H^n_{\mathscr{D}}(Z^{reg}/\mathbb{K},\mathbb{R}(n))(\delta) \times H_{n-1}(Z^{reg};\mathbb{R}(1-n))(\delta) \to \mathbb{R}$$

is obtained by combining the cap product pairing $\frown\colon H^{n-1}(Z^{reg}/\mathbb{K},\mathbb{R}(n-1))(\delta) \times H_{n-1}(Z^{reg}/\mathbb{K},\mathbb{R}(1-n))(\delta) \to \mathbb{R}$ with the isomorphism $H^n_{\mathscr{D}}(Z^{reg}/\mathbb{K},\mathbb{R}(n))(\delta) \cong H^{n-1}(Z^{reg}/\mathbb{K},\mathbb{R}(n-1))(\delta)$.

3.3 Mahler measure and K-theory

Here I try to summarize how Deninger interprets the Mahler measure $m(P)$ of a polynomial using K-theory. For a very brief recall of the definitions needed to understand this section see Appendix A. We start with the following definitions of the absolute cohomology groups for a regular, quasi-projective scheme X.

Definition 3.5. Define $H^n_{\mathscr{M}}(X;\mathbb{Q}(m)) := K^{(m)}_{2m-n}(X)\otimes\mathbb{Q}$, where $K^{(m)}_{2m-n}(X)$ is the m-graded part of the γ-filtration on the Quillen K-group $K_{2m-n}(X)$ defined in Appendix A. Observe that $H^n_{\mathscr{M}}(X;\mathbb{Q}(m))$ is defined in §3 of [12] but it is called $H^n_{\mathscr{A}}(X;\mathbb{Q}(m))$.

It is easy to prove that $H^1_{\mathscr{M}}(X;\mathbb{Q}(1)) = K^{(1)}_1(X)\otimes\mathbb{Q}\cong\mathscr{O}^\times_X(X)\otimes\mathbb{Q}$. Recall now that $H^\bullet_{\mathscr{M}}(X;\mathbb{Q}) := \bigoplus_{n\in\mathbb{N}} H^n_{\mathscr{M}}(X;\mathbb{Q}(n))$ inherits a natural notion of cup product from K-theory, which makes it into a graded ring. In particular if $f_0,\dots,f_n\in H^1_{\mathscr{M}}(X;\mathbb{Q}(1))$ it is custom to define

$$\{f_0,\dots,f_n\} := f_0\cup\cdots\cup f_n \in H^{n+1}_{\mathscr{M}}(X;\mathbb{Q}(n+1))$$

which is called the symbol of $f_0,\dots,f_n$.

Recall finally that if $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ then for every variety X defined over $\mathbb{K}$ we have a morphism of graded rings

$$r_{\mathscr{D}}\colon H^\bullet_{\mathscr{M}}(X;\mathbb{Q})\to H^\bullet_{\mathscr{D}}(X/\mathbb{K};\mathbb{R})$$

which restricted to $H^1_{\mathscr{M}}(X;\mathbb{Q}(1))$ sends the function $f\in\mathscr{O}^\times_X(X)\otimes\mathbb{Q}$ to $[\log(|f|)]\in H^1_{\mathscr{D}}(X/\mathbb{K},\mathbb{R}(1))$.

If X is a regular and quasi-projective variety over $\mathbb{Q}$ we set $r_{\mathscr{D}}\colon H^\bullet_{\mathscr{M}}(X;\mathbb{Q})\to H^\bullet_{\mathscr{D}}((X\otimes\mathbb{R})/\mathbb{R};\mathbb{R})$ as the composition

$$H^\bullet_{\mathscr{M}}(X;\mathbb{Q})\to H^\bullet_{\mathscr{M}}(X\otimes\mathbb{R};\mathbb{Q})\xrightarrow{r_{\mathscr{D}}} H^\bullet_{\mathscr{D}}((X\otimes\mathbb{R})/\mathbb{R};\mathbb{R}).$$

Moreover if X is projective over $\mathbb{Q}$ and $\mathfrak{X}$ is a proper and regular model of X over $\mathbb{Z}$ we can define

$$H^n_{\mathscr{M}}(X;\mathbb{Q}(m))_{\mathbb{Z}} := \operatorname{Im}\left(K_{2m-n}(\mathfrak{X})\otimes\mathbb{Q}\to H^n_{\mathscr{M}}(X;\mathbb{Q}(m))\right)$$

and so we have the definition of an "integral K-theory" $H^\bullet_{\mathscr{M}}(X;\mathbb{Q})_{\mathbb{Z}} := \bigoplus_{n\in\mathbb{N}} H^n_{\mathscr{M}}(X;\mathbb{Q}(n))_{\mathbb{Z}}$.

Using this notation the previous Theorem 3.3 and Theorem 3.4 can be summarised as follows.

Theorem 3.6. *Let $P\in\mathbb{Q}[t_1,\dots,t_n]\setminus\{0\}$ and let $Z_P := \operatorname{Spec}(\mathbb{Q}[\mathbb{Z}^n]/(P))\subseteq\mathbb{G}^n_{m,\mathbb{Q}}$ and let $X_P := \mathbb{G}^n_{m,\mathbb{Q}}\setminus Z_P$. Then if $P(\mathbf{z})\neq 0$ for every $\mathbf{z}\in\mathbb{T}^n$ we have that*

$$m(P) = < r_{\mathscr{D}}(\{P,t_1,\dots,t_n\}), (2\pi i)^{-n}\cdot[\mathbb{T}^n] >$$

where $\{P,t_1,\dots,t_n\}\in H^{n+1}_{\mathscr{M}}(X_P;\mathbb{Q}(n+1))$. On the other hand if P satisfies the conditions C1 and C2 (and perhaps vanishes on $\mathbb{T}^n$) we have that

$$m(P) = m(P^*) - < r_{\mathscr{D}}(\{t_1,\dots,t_n\}),\gamma > = < r_{\mathscr{D}}(\{P^*,t_1,\dots,t_{n-1}\}),(2\pi i)^{1-n}\cdot[\mathbb{T}^{n-1}] > - < r_{\mathscr{D}}(\{t_1,\dots,t_n\}),\gamma >$$

where $\{t_1,\dots,t_n\}\in H^n_{\mathscr{M}}(Z^{reg}_P;\mathbb{Q}(n))(\delta)$.

Observe finally that if $P\in\mathbb{Z}[t_1,\dots,t_n]$ then $\{P,t_1,\dots,t_n\}\in H^{n+1}_{\mathscr{M}}(X_P;\mathbb{Q}(n+1))_{\mathbb{Z}}$ and analogously for all the other symbols in K-theory.

3.4 Mahler measure and motives

3.5 Examples

The first very interesting example of a polynomial to which Deninger applies all this work is the Laurent polynomial

$$\frac{1}{x}+\frac{1}{y}+1+x+y\in\mathbb{Z}[x^{\pm1},y^{\pm1}]$$

which can be replaced in our study by the polynomial $P(t_1,t_2) = t_1t_2^2+(t_1^2+t_1+1)t_2+t_1\in\mathbb{Z}[t_1,t_2]$ since in general for every Laurent polynomial $Q(t_1,\dots,t_n)\in\mathbb{C}[t_1^{\pm1},\dots,t_n^{\pm1}]$ and for every $(a_1,\dots,a_n)\in\mathbb{Z}^n$ we have that $m(Q) = m(t_1^{a_1}\cdots t_n^{a_n}\cdot Q)$. It can be shown that P satisfies the conditions C1 and C2 and since $P^*(t_1) = t_1$ and $m(t_1) = 0$ we can apply what we have outlined in subsection 3.3 to obtain $m(P) = - < r_{\mathscr{D}}(\{t_1,t_2\}),\gamma >$.

If we now consider the standard embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ given by $(t_1, t_2) \mapsto (1\colon t_1\colon t_2)$ we see that the projective completion E of the set $Z = \{(t_1, t_2) \in \mathbb{A}^2 \mid P(t_1, t_2) = 0\}$ is a smooth curve of genus one. If we take $\mathbf{0} := (0\colon 1\colon 0) \in E$ as a distinguished point we obtain an elliptic curve defined over $\mathbb{Q}$. Moreover if $Q := (0\colon 0\colon 1) \in E$ we can check that $4Q = \mathbf{0}$ and that $G := E \setminus Z = < Q >$. This gives us the following exact sequence

$$0 \to H^2_{\mathcal{M}}(E; \mathbb{Q}(2)) \xrightarrow{\iota} H^2_{\mathcal{M}}(Z; \mathbb{Q}(2)) \xrightarrow{\partial} \mathbb{Q}[G]^0 \otimes_{\mathbb{Z}} \mathbb{Q} \to 0$$

where $\mathbb{Q}[G]^0$ is by definition the vector space generated over $\mathbb{Q}$ by the set of divisors $\{(nQ) - (\mathbf{0}) \mid n \in \{1, 2, 3\}\}$. Using this exact sequence and an explicit description for the divisors $\mathrm{div}(t_1)$ and $\mathrm{div}(t_2)$ one can prove that $\{t_1, t_2\} \in \ker(\partial)$ and thus that there exists a unique element $[t_1, t_2]_E \in H^2_{\mathcal{M}}(E; \mathbb{Q}(2))$ such that $\iota([t_1, t_2]_E) = \{t_1, t_2\}$. This proves in particular that $m(P) = - < r_{\mathcal{D}}([t_1, t_2]_E), \gamma >$ where now we consider $\gamma \in H_1(E/\mathbb{R}; \mathbb{Q}(-1))$.

Using this fact and a couple of lemmas proved in [7] we can prove that the Mahler measure $m(P)$ is equal to a specific value of an Eisenstein-Kronecker series. In particular we have

$$m(P) = 4A(\Lambda)\left(\int_\gamma \omega\right) \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \frac{\lambda}{|\lambda|^4} \cdot \exp\left(A(\Lambda)^{-1}(\overline{Q}\lambda - Q\overline{\lambda})\right)$$

where $\omega := t_2 t_1^{-1}(t_2^2 - 1)^{-1} dt_1$ is the invariant differential of the first kind on E, $\Lambda \subseteq \mathbb{C}$ is the lattice of periods of ω and $A(\Lambda) := (2\pi i)^{-1}(\overline{u}v - u\overline{v})$ where $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$ with $\mathfrak{I}({}^{v}/_{u}) > 0$.

Finally, Deninger notes that the Bloch-Beĭlinson conjectures would imply that the Mahler measure $m(P)$ is a rational multiple of a special value of an L-function. In particular he notes that E has bad reduction only at the primes 3 and 5 and then he uses some results proved in **§3.5** of [10] to prove that $[t_1, t_2]_E \in H^2_{\mathcal{M}}(E; \mathbb{Q}(2))_{\mathbb{Z}}$. Then he observes that the Bloch-Beĭlinson conjectures would imply that $\dim_{\mathbb{Q}}(H^2_{\mathcal{M}}(E; \mathbb{Q}(2))_{\mathbb{Z}}) = 1$ and thus that if $\eta \in H^1(E/\mathbb{R}; \mathbb{Q}(1)) \setminus \{0\}$ we would have that

$$r_{\mathcal{D}}([t_1, t_2]_E) = s \cdot L'(E, 0) \cdot \eta$$

for some $s \in \mathbb{Q}$. Thus pairing with γ we would obtain that $m(P) = r \cdot L'(E, 0)$ for some $r \in \mathbb{Q}^\times$. This was the starting point for Boyd to formulate his conjectures and write his paper [4].

Deninger points also out that the two formulas

$$m((x+y)^2 \pm 3) = \frac{2}{3}\log(3) + L'(\chi_{-3}, -1) \qquad \text{and} \qquad m((x+y)^2 \pm 2) = \frac{1}{2}\log(2) + L'(\chi_{-4}, -1)$$

which are proved in Smyth's paper [13] can also be proved up to rational multiples in a more geometric fashion by looking at the relative cohomology exact sequence

$$H^1_{\mathcal{M}}(\partial A; \mathbb{Q}(2)) \to H^2_{\mathcal{M}}(Z^{\mathrm{reg}}, \partial A; \mathbb{Q}(2)) \to H^2_{\mathcal{M}}(Z^{\mathrm{reg}}; \mathbb{Q}(2)) \to 0$$

and using the theorems proved by Borel in [2] which provide a special case of the Bloch-Beĭlinson conjectures.

SECTION 4

Brunault's paper

APPENDIX A

A very brief introduction to algebraic K-theory

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