The classical, analytic definition of affine and compactified modular curves

Abstract

In this short article we summarize the main theorems (with proofs) concerning the analytic definition of modular curves.

We give in this section the first, analytic description of the non-compact Riemann surfaces $Y(N)$, $Y_0(N)$ and $Y_1(N)$ and of their compactification $X(N)$, $X_0(N)$ and $X_1(N)$. To do so recall that the group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(\mathbb{R}) \mid ad - bc > 0 \right\}$$

of invertible 2×2 matrices with positive determinant acts on the upper half plane

$$\mathfrak{h} = \{z \in \mathbb{C} \mid \Im(z) > 0\} \qquad \text{by setting} \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) * z = \frac{az+b}{cz+d}$$

which is a well defined action because $cz + d \neq 0$ for every $c, d \in \mathbb{R}$ and $z \in \mathfrak{h}$, and moreover

$$\Im\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2} \cdot \Im(z)$$

which implies that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) * z \in \mathfrak{h}$ if $z \in \mathfrak{h}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. Using this action we can describe the full group of bi-holomorphic transformations of $\mathfrak{h}$, as shown in the following lemma.

Theorem 1. *Let* $\mathrm{Aut}(\mathfrak{h})$ *be the group of all bi-holomorphic functions* $\mathfrak{h} \to \mathfrak{h}$ *and let*

$$\rho\colon \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{h}) \qquad \textit{such that} \qquad \rho\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(z) = \frac{az+b}{cz+d}$$

be the group homomorphism induced by the action $\mathrm{GL}_2^+(\mathbb{R}) \circlearrowright \mathfrak{h}$. *Then* ρ *is surjective and*

$$\ker(\rho) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\} \qquad \textit{which implies that} \qquad \mathrm{Aut}(\mathfrak{h}) \cong \mathrm{PSL}_2(\mathbb{R}) \overset{\mathsf{def}}{=} \mathrm{SL}_2(\mathbb{R})/\{\pm I_2\}$$

where $\mathrm{SL}_2(\mathbb{R}) \overset{\mathsf{def}}{=} \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(\mathbb{R}) \mid ad - bc = 1 \right\}$ *and* $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *is the identity matrix.*

Proof. Observe first of all that $\rho(A) = \mathrm{Id}_{\mathfrak{h}}$ if and only if

$$\frac{az+b}{cz+d} = z \qquad \text{for every} \qquad z \in \mathfrak{h} \qquad \text{where} \qquad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and thus if and only if $cz^2 + (d-a)z - b = 0$ for every $z \in \mathfrak{h}$, which is true if and only if $c = b = 0$ and $a = d$. This implies that $\ker(\rho) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\}$, as we wanted to prove.

We have now to prove that ρ is surjective. First of all we observe that for every $z = x + iy \in \mathfrak{h}$ we have a matrix $A_z \stackrel{\text{def}}{=} \frac{1}{\sqrt{y}} \cdot \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\rho(A_z)(i) = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} * i = \frac{y\,i + x}{0 \cdot i + 1} = z$$

which implies that for every bi-holomorphic map $\varphi\colon \mathfrak{h} \to \mathfrak{h}$ we have $\widetilde{\varphi}(i) = i$, where $\widetilde{\varphi}\colon \mathfrak{h} \to \mathfrak{h}$ is defined as $\widetilde{\varphi} \stackrel{\text{def}}{=} \rho(A_{\varphi(i)}^{-1}) \circ \varphi$.

Let now $\mathfrak{b} = \{z \in \mathbb{C} \mid |z| < 1\}$ and observe that we have a bi-holomorphic map

$$\eta\colon \mathfrak{h} \to \mathfrak{b} \qquad \text{such that} \qquad \eta(z) = \frac{z - i}{z + i}$$

which satisfies $\eta(i) = 0$. This implies that given a bi-holomorphic function $\varphi\colon \mathfrak{h} \to \mathfrak{h}$ we can define a function $\psi \stackrel{\text{def}}{=} \eta \circ \widetilde{\varphi} \circ \eta^{-1}$ which is a bi-holomorphic function $\psi\colon \mathfrak{b} \to \mathfrak{b}$ satisfying $\psi(0) = 0$. Thus we can apply *Schwarz theorem*, as stated and proved in **Theorem 12.2** of [2], to see that $|\psi(z)| = |z|$ for every $z \in \mathfrak{b}$. This implies that we can find a real number $\theta \in [0, \pi)$ such that $\psi(z) = e^{2i\theta} \cdot z$ for every $z \in \mathfrak{b}$. Observe now that

$$\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}^{-1} \cdot \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2} \cdot \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \cdot \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{e^{2i\theta}+1}{2} & \frac{e^{2i\theta}-1}{2i} \\ -\frac{e^{2i\theta}-1}{2i} & \frac{e^{2i\theta}+1}{2} \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

which implies that

$$\widetilde{\varphi}(z) = (\eta^{-1} \circ \psi \circ \eta)(z) = \frac{\cos(\theta)\, z + \sin(\theta)}{-\sin(\theta)\, z + \cos(\theta)} = \rho(r_\theta)(z)$$

where we define $r_\theta \stackrel{\text{def}}{=} \left(\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right)$ for every $\theta \in \mathbb{R}$. Thus we have $\varphi = \rho(A_{\varphi(i)}) \circ \widetilde{\varphi} = \rho(A_{\varphi(i)} \cdot r_\theta)$ and this proves that $\rho\colon \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{h})$ is surjective, as we wanted to prove.

To conclude it is sufficient to observe that the map

$$\mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) \qquad A \mapsto \frac{1}{\sqrt{\det(A)}} \cdot A$$

induces an isomorphism

$$\frac{\mathrm{GL}_2^+(\mathbb{R})}{\ker(\rho)} = \frac{\mathrm{GL}_2^+(\mathbb{R})}{\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\}} \cong \frac{\mathrm{SL}_2(\mathbb{R})}{\{\pm I_2\}}$$

which, combined with the map $\rho\colon \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{h})$ induces the isomorphism $\mathrm{Aut}(\mathfrak{h}) \cong \mathrm{PSL}_2(\mathbb{R})$ that we wanted to prove. □

The previous theorem suggests to consider the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$ instead of the action of $\mathrm{GL}_2(\mathbb{R})$. Thus for every subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ we will consider the topological space $\Gamma \backslash \mathfrak{h}$ whose elements are the orbits of the action $\Gamma \circlearrowright \mathfrak{h}$ and whose topology is the finest topology such that the quotient map $\mathfrak{h} \twoheadrightarrow \Gamma \backslash \mathfrak{h}$ is continuous. In order to study the topological properties of this quotient we shall review the topological properties of $\mathrm{SL}_2(\mathbb{R})$.

We can define a topology on the space of 2×2 matrices $M_{2,2}(\mathbb{R})$ using the bijection $M_{2,2}(\mathbb{R}) \longleftrightarrow \mathbb{R}^4$. With this topology we have that $\mathrm{SL}_2(\mathbb{R}) \subseteq M_{2,2}(\mathbb{R})$ is a closed subset since $\mathrm{SL}_2(\mathbb{R}) = \det^{-1}(1)$ and the map $\det\colon M_{2,2}(\mathbb{R}) \to \mathbb{R}$ is continuous. Observe moreover that $\mathrm{SL}_2(\mathbb{R})$ endowed with the subspace topology is itself a topological group.

Recall now that the action $G \circlearrowleft X$ of a group G on a topological space X is said to be *properly discontinuous* if for every two compact subsets $K_1, K_2 \subseteq X$ the set

$$\{g \in G : g * K_1 \cap K_2 \neq \emptyset\}$$

is finite. We can characterize the subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ such that the action $\Gamma \circlearrowleft \mathfrak{h}$ is properly discontinuous in the following topological way.

Theorem 2. *Let* $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ *be a subgroup. Then* $\Gamma \circlearrowleft \mathfrak{h}$ *is properly discontinuous if and only if* Γ *is a discrete subspace of* $\mathrm{SL}_2(\mathbb{R})$.

Proof. Suppose first of all that the action $\Gamma \circlearrowleft \mathfrak{h}$ is properly discontinuous, and take an open subset $U \subseteq \mathrm{SL}_2(\mathbb{R})$ such that $I_2 \in U$ and $\overline{U} \subseteq \mathrm{SL}_2(\mathbb{R})$ is compact. We can do so because $M_{2,2}(\mathbb{R}) \cong \mathbb{R}^4$ is a locally compact topological space and thus $\mathrm{SL}_2(\mathbb{R})$ is a locally compact topological group. Observe now that for every point $z \in \mathfrak{h}$ the set

$$\overline{U}\, z = \{A * z \mid A \in \overline{U}\}$$

is compact, since it is the image of $\overline{U}$ under the continuous map $\mathrm{SL}_2(\mathbb{R}) \to \mathfrak{h}$ which sends A to $A * z$. Thus for every $z \in \mathfrak{h}$ the set $\{A \in \Gamma \mid A * z \in \overline{U}\, z\}$ is finite since both $\{z\} \subseteq \mathfrak{h}$ and $\overline{U}\, z \subseteq \mathfrak{h}$ are compact and the action $\Gamma \circlearrowleft \mathfrak{h}$ is properly discontinuous by hypothesis. Thus the set $\Gamma \cap U \subseteq \{A \in \Gamma \mid A * z \in \overline{U}\, z\}$ is finite, say $\Gamma \cap U = \{I_2, A_1, \ldots, A_n\}$. This implies that $U \setminus \{A_1, \ldots, A_n\}$ is open and that $\Gamma \cap (U \setminus \{A_1, \ldots, A_n\}) = \{I_2\} \in \Gamma$ is an open point. This implies that Γ is a discrete group since for every $A \in \Gamma$ the map $\iota_A : \Gamma \to \Gamma$ such that $\iota_A(B) = A \cdot B$ is a homeomorphism and thus every point $\{A\} = \iota_A(\{I_2\})$ is open.

Suppose now that $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a discrete subgroup and let $K_1, K_2 \subseteq \mathfrak{h}$ be two compact subsets. Observe that $\{A \in \Gamma \mid A * K_1 \cap K_2 \neq \emptyset\} = \Gamma \cap H_2 H_1^{-1}$ where

$$H_j \stackrel{\text{def}}{=} \{A \in \mathrm{SL}_2(\mathbb{R}) \mid A * i \in K_j\} \subseteq \mathrm{SL}_2(\mathbb{R})$$

and $H_2 H_1^{-1} = \{A \cdot B \in \mathrm{SL}_2(\mathbb{R}) \mid A \in H_2 \text{ and } B^{-1} \in H_1\}$. Indeed, if $M \in \Gamma \cap H_2 H_1^{-1}$ then $M = A \cdot B$ with $A \in H_2$ and $B^{-1} \in H_1$ which means that $B^{-1} * i \in K_1$ and $A * i \in K_2$. Thus

$$M * (B^{-1} * i) = (A \cdot B) * (B^{-1} * i) = A * i \in K_2$$

which implies that $M * K_1 \cap K_2 \neq \emptyset$ and so that $\Gamma \cap H_2 H_1^{-1} \subseteq \{A \in \Gamma \mid A * K_1 \cap K_2 \neq \emptyset\}$. To prove the opposite inclusion let $M \in \{A \in \Gamma \mid A * K_1 \cap K_2 \neq \emptyset\}$ and let $z \in K_1$ such that $M * z \in K_2$. Observe now that $A_z \stackrel{\text{def}}{=} \frac{1}{\sqrt{y}} \cdot \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in H_1$ since $A_z * i = z \in K_1$. Observe moreover that $M \cdot A_z \in H_2$ since $(M \cdot A_z) * i = M * (A_z * i) = M * z \in K_2$. Thus $M \in H_2 H_1^{-1}$ since $M = (M \cdot A_z) \cdot A_z^{-1}$ and $M \cdot A_z \in H_2$ whereas $(A_z^{-1})^{-1} = A_z \in H_1$. This implies that $\{A \in \Gamma \mid A * K_1 \cap K_2 \neq \emptyset\} \subseteq \Gamma \cap H_2 H_1^{-1}$ and thus that $\{A \in \Gamma \mid A * K_1 \cap K_2 \neq \emptyset\} = \Gamma \cap H_2 H_1^{-1}$.

To conclude we have to prove that $\Gamma \cap H_2 H_1^{-1}$ is finite, and to do so we recall that Γ is a discrete topological space, and thus it is sufficient to show that the two subsets $H_j \subseteq \mathrm{SL}_2(\mathbb{R})$ are compact to prove that $\Gamma \cap H_2 H_1^{-1}$ is both discrete and compact, and hence finite. Observe first of all that $H_j \subseteq \mathrm{SL}_2(\mathbb{R})$ is closed for every $j = 1, 2$ since $H_j = \varphi^{-1}(K_j)$ where $\varphi : \mathrm{SL}_2(\mathbb{R}) \to \mathfrak{h}$ is the continuous map which sends $A \in \mathrm{SL}_2(\mathbb{R})$ to $A * i \in \mathfrak{h}$.

Recall now that $\mathrm{SL}_2(\mathbb{R})$ is a locally compact Hausdorff topological space, as we have shown above, and thus for every $l = 1, 2$ we can find some open subsets $\{U_j^l\}_{j \in \mathcal{J}}$ such that

$$H_l \subseteq \bigcup_{j \in \mathcal{J}} U_j^l$$

and $\overline{U_j^l} \subseteq \mathrm{SL}_2(\mathbb{R})$ is compact. This implies that

$$K_l = H_l * i \subseteq \bigcup_{j \in \mathscr{J}} U_j^l * i$$

and thus we can find some $j_1, \ldots, j_n \in \mathscr{J}$ such that $K_l \subseteq \bigcup_{k=1}^n U_{j_k}^l * i$ because K_l is compact and $U_j^l * i$ is open for every $j \in \mathscr{J}$ and $l = 1,2$. This is true because in general $U * x \subseteq \mathfrak{h}$ is open for every $x \in \mathfrak{h}$ and every open subset $U \subseteq \mathrm{SL}_2(\mathbb{R})$, as it is shown in **Theorem 1.2.1** of [1].

Using the fact that $K_l \subseteq \bigcup_{k=1}^n U_{j_k}^l * i$ we obtain that $H_l \subseteq \bigcup_{k=1}^n U_{j_k}^l \cdot \mathrm{SO}_2(\mathbb{R})$, where

$$\mathrm{SO}_2(\mathbb{R}) = \{A \in \mathrm{SL}_2(\mathbb{R}) \mid {}^t A \cdot A = I_2\}$$

is the real special orthogonal group. Indeed for every $l = 1,2$ and every $A \in H_l$ we have that $A * i \in K_l$ which implies that $A * i \in U_{j_k}^l * i$ for some $k \in \{1, \ldots, n\}$. Thus there exists a matrix $B \in U_{j_k}^l$ such that $A * i = B * i$, and this implies that $(A \cdot B^{-1}) * i = i$. Observe now that for every $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have

$$M * i = \frac{a i + b}{c i + d} = i \quad \text{if and only if} \quad a = d,\ b = -c \quad \text{and} \quad \det(M) = a d - b c = a^2 + b^2 = 1$$

and thus if and only if

$${}^t M \cdot M = \begin{pmatrix} a^2 + c^2 & a b + c d \\ a b + c d & b^2 + d^2 \end{pmatrix} = I_2$$

that is if and only if $M \in \mathrm{SO}_2(\mathbb{R})$.

To conclude we have only to observe that the set $\overline{U_{j_k}^l} \cdot \mathrm{SO}_2(\mathbb{R})$ is compact for every $l \in \{1,2\}$ and $k \in \{1, \ldots, n\}$, and so

$$H_l \subseteq \bigcup_{k=1}^n U_{j_k}^l \cdot \mathrm{SO}_2(\mathbb{R}) \subseteq \bigcup_{k=1}^n \overline{U_{j_k}^l} \cdot \mathrm{SO}_2(\mathbb{R})$$

which implies that H_l is compact since it is a closed subset of a compact set. To prove that $\overline{U_{j_k}^l} \cdot \mathrm{SO}_2(\mathbb{R})$ is compact we observe first of all that $\overline{U_{j_k}^l}$ is compact (by hypothesis) and that $\mathrm{SO}_2(\mathbb{R})$ is compact because under the homeomorphism $M_{2,2}(\mathbb{R}) \cong \mathbb{R}^4$ the subgroup $\mathrm{SO}_2(\mathbb{R}) \subseteq M_{2,2}(\mathbb{R})$ corresponds to the subset

$$\left\{ \left(\begin{smallmatrix} x \\ y \\ z \\ w \end{smallmatrix}\right) \in \mathbb{R}^4 \mid x^2 + z^2 = y^2 + w^2 = 1,\ x y + z w = 0 \right\}$$

which is a closed and bounded subset of $\mathbb{R}^4$, and hence is compact.

To conclude that the set $\overline{U_{j_k}^l} \cdot \mathrm{SO}_2(\mathbb{R})$ is compact we have only to observe that for every two compact subsets $X, Y \subseteq \mathrm{SL}_2(\mathbb{R})$ the set $X \times Y \subseteq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is compact, and thus the subset $X \cdot Y \subseteq \mathrm{SL}_2(\mathbb{R})$ is also compact because the multiplication on SL_2 is continuous. Hence we have proved that H_l is compact for $l = 1,2$, and thus we can finally conclude that $H_2 H_1^{-1}$ is also compact since $H_2 \times H_1 \subseteq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ is compact and the map $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ which sends a couple $(A, B) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ to $B \cdot A^{-1}$ is continuous. $\square$

We have seen in the previous theorem that discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ play an important role when looking at the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$, and thus they have the special name of *Fuchsian groups*. Some important examples of Fuchsian groups are given by subgroups of the *full modular group*

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}(\mathbb{Z}) \mid a d - b c = 1 \right\}$$

which is clearly a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. In particular for every $N \in \mathbb{N}$ we can define three families of groups $\Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z})$, $\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(N) \leq \mathrm{SL}_2(\mathbb{Z})$ by setting

$$\begin{aligned}
\Gamma_0(N) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\} \\
\Gamma_1(N) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \text{ and } a \equiv d \equiv 1 \bmod N \right\} \\
\Gamma(N) &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv c \equiv 0 \text{ and } a \equiv d \equiv 1 \bmod N \right\}.
\end{aligned}$$

The group $\Gamma(N)$ is called the *principal congruence subgroup* of level N and the groups $\Gamma_0(N)$ and $\Gamma_1(N)$ are called *modular groups of Hecke type* of level N. The importance of these subgroups will become clear in the following section, where we will describe the structure of the modular curves as *moduli spaces* of elliptic curves.

Let's return for now to the general setting in which $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. As we said, the fact that the action $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous is important to prove that the quotient topological space $\Gamma \backslash \mathfrak{h}$ maintains some of the topological properties of $\mathfrak{h}$, as shown in the following theorem.

Theorem 3. *Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group. Then the quotient $\Gamma \backslash \mathfrak{h}$ is a locally compact, connected and Hausdorff topological space. Moreover, it admits a Riemann surface structure such that the quotient map $\mathfrak{h} \twoheadrightarrow \Gamma \backslash \mathfrak{h}$ is holomorphic.*

Proof. Observe first of all that $\Gamma \backslash \mathfrak{h}$ is a connected topological space since $\mathfrak{h}$ is connected and the projection map $\mathfrak{h} \twoheadrightarrow \Gamma \backslash \mathfrak{h}$ is a continuous surjective map. In the same way it is very easy to prove that $\Gamma \backslash \mathfrak{h}$ is a locally compact topological space, since for every equivalence class $[z] \in \Gamma \backslash \mathfrak{h}$ we can take any representative element $z \in \mathfrak{h}$ and we can take a compact neighbourhood $z \in K \subseteq \mathfrak{h}$ and project it to a compact neighbourhood of $[z] \in \Gamma \backslash \mathfrak{h}$ using the projection map $\mathfrak{h} \twoheadrightarrow \Gamma \backslash \mathfrak{h}$ which is continuous and hence sends compact sets to compact sets.

Suppose now that $z, w \in \mathfrak{h}$ are two points such that $A * z \neq w$ for every $A \in \Gamma$. This implies in particular that $z \neq w$ and thus we can take two open neighbourhoods $z \in U \subseteq \mathfrak{h}$ and $w \in V \subseteq \mathfrak{h}$ such that $U \cap V = \emptyset$ and $\overline{U} \subseteq \mathfrak{h}$ and $\overline{V} \subseteq \mathfrak{h}$ are compact sets. Thus the set $\{A \in \Gamma \mid A * \overline{U} \cap \overline{V} \neq \emptyset\}$ is finite, namely $\{A \in \Gamma \mid A * \overline{U} \cap \overline{V} \neq \emptyset\} = \{A_1, \ldots, A_n\}$. By hypothesis we know that $A_j * z \neq w$ for every $j \in \{1, \ldots, n\}$ and thus there exist some open neighbourhoods $A_j * z \in U_j \subseteq \mathfrak{h}$ and $w \in V_j \subseteq \mathfrak{h}$ such that $U_j \cap V_j = \emptyset$ for every $j \in \{1, \ldots, n\}$. Hence the sets $A_j^{-1} * U_j \subseteq \mathfrak{h}$ are open neighbourhoods of $z \in \mathfrak{h}$. If we define now the two open subsets

$$U' \stackrel{\text{def}}{=} U \cap A_1^{-1} * U_1 \cap \ldots \cap A_n^{-1} * U_n \subseteq \mathfrak{h} \qquad \text{and} \qquad V' \stackrel{\text{def}}{=} V \cap V_1 \cap \ldots \cap V_n \subseteq \mathfrak{h}$$

we have $z \in U'$, $w \in V'$ and $A * U' \cap V' = \emptyset$ for every $A \in \Gamma$, which implies that $\Gamma \backslash \mathfrak{h}$ is an Hausdorff topological space.

Recall now that an *holomorphic atlas* $\mathbb{A}$ on a topological space X is given by a collection

$$\mathbb{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathscr{A}} \qquad \text{with} \qquad U_\alpha \subseteq X \qquad \text{and} \qquad \varphi_\alpha : U_\alpha \to \mathbb{C}^n$$

where U_α is open, φ_α is an homeomorphism for every $\alpha \in \mathscr{A}$, $n \in \mathbb{N}$ doesn't change when $\alpha \in \mathscr{A}$ changes and for every $\alpha, \beta \in \mathscr{A}$ the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic. Such an atlas is called *maximal* if for every open subset $U \subseteq X$ and every homeomorphism $\varphi : U \to \mathbb{C}^n$ such that $\varphi_\alpha \circ \varphi^{-1} : \mathbb{C}^n \to \mathbb{C}^n$ is holomorphic for every $\alpha \in \mathscr{A}$ we have that $(U, \varphi) \in \mathbb{A}$. We say then that a topological space X with a maximal holomorphic atlas $\mathbb{A}$ is a complex manifold of complex dimension n. Observe finally that every atlas $\mathbb{A}$ for a topological space X can be completed to a unique maximal atlas containing it, by adding to $\mathbb{A}$ all the possible couples (V, ψ) with $V \subseteq X$ open, $\psi : V \to \mathbb{C}^n$

homeomorphism and $\psi \circ \varphi^{-1}$ holomorphic for every $(U, \varphi) \in \mathbb{A}$. Thus giving a complex manifold structure on a topological space X is the same as giving an holomorphic atlas on X.

In our case we can easily give a complex atlas on $\Gamma\backslash\mathfrak{h}$. Let first of all $z \in \mathfrak{h}$ be a point such that $A * z \neq z$ for every $A \in \Gamma \setminus \{\pm I_2\}$ and observe that we can find an open neighbourhood $z \in U \subseteq \mathfrak{h}$ such that $A * U \cap U = \emptyset$ for every $A \in \Gamma \setminus \{\pm I_2\}$. Indeed we know from the previous theorem that the action $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous, which implies that for every open neighbourhood $z \in V \subseteq \mathfrak{h}$ such that $\overline{V} \subseteq \mathfrak{h}$ is compact the set

$$\{A \in \Gamma \mid A * V \cap V \neq \emptyset\}$$

is finite, say $\{A \in \Gamma \mid A * V \cap V \neq \emptyset\} = \{A_1, \ldots, A_n\}$. Thus for every $j \in \{1, \ldots, n\}$ we can find two open neighbourhoods $z \in W_j \subseteq \mathfrak{h}$ and $A_j * z \in W_j' \subseteq \mathfrak{h}$ such that $W_j \cap W_j' = \emptyset$. This implies that the open set

$$U = \left(\bigcap_{j=1}^{n} W_j \cap A_j^{-1} * W_j' \right) \cap V$$

is an open neighbourhood of z such that $A * U \cap U = \emptyset$ for every $A \in \Gamma \setminus \{\pm I_2\}$. Let now $z \in B_z \subseteq U$ any open ball (which exists because U is open and not empty) and observe that the restriction $\pi|_{B_z} : B_z \to \Gamma\backslash B_z$ of the quotient map $\pi : \mathfrak{h} \to \Gamma\backslash\mathfrak{h}$ to this ball is an homeomorphism, because the quotient map $\pi : \mathfrak{h} \to \Gamma\backslash\mathfrak{h}$ is open and continuous and moreover $\pi|_{B_z}$ is a bijection since $A * B_z \cap B_z = \emptyset$ for every $A \in \Gamma$. Finally, since $B_z \subseteq \mathfrak{h}$ is an open ball there exists an homeomorphism $\varphi_z : B_z \xrightarrow{\sim} \mathbb{C}$ and so the map $\varphi_z \circ \pi|_{B_z}{}^{-1} : \Gamma\backslash B_z \to \mathbb{C}$ is an homeomorphism.

Let now $z \in \mathfrak{h}$ be a point such that the stabilizing subgroup $\Gamma_z \stackrel{\text{def}}{=} \{A \in \Gamma \mid A * z = z\} \leq \Gamma$ is not trivial. Using again the fact that the action $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous we can find an open neighbourhood $z \in U_z \subseteq \mathfrak{h}$ such that $\{A \in \Gamma \mid A * U_z \cap U_z \neq \emptyset\} = \Gamma_z$. Thus we have an injection $\Gamma_z\backslash U_z \hookrightarrow \Gamma\backslash U_z$ which identifies $\Gamma_z\backslash U_z$ with an open neighbourhood of $[z] \in \Gamma\backslash\mathfrak{h}$. Observe now that for every $A \in \mathrm{SL}_2(\mathbb{R})$ we have $A * z = z$ if and only if $(A_z^{-1} * A \cdot A_z) * i = i$, where $A_z = \Im(z)^{-1/2} \cdot \left(\begin{smallmatrix} \Im(z) & \Re(z) \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ is the transformation defined above, such that $A_z * i = z$. This is equivalent to say that

$$\mathrm{SL}_2(\mathbb{R})_z = A_z^{-1} \cdot \mathrm{SL}_2(\mathbb{R})_i \cdot A_z = A_z^{-1} \cdot \mathrm{SO}_2(\mathbb{R}) \cdot A_z$$

which in turn implies that $\mathrm{SL}_2(\mathbb{R})_z$ is compact, since the map $\mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ which sends $X \in \mathrm{SL}_2(\mathbb{R})$ to $A_z^{-1} \cdot X \cdot A_z$ is an homeomorphism, and that $\Gamma_z = \Gamma \cap \mathrm{SL}_2(\mathbb{R})_z$ is finite since it is a discrete subgroup of a compact group.

Recall now that the map

$$\eta_z : \mathfrak{h} \to \mathfrak{b} \qquad \text{such that} \qquad \eta_z(w) = \frac{w - z}{w + z}$$

is bi-holomorphic, where $\mathfrak{b} = \{z \in \mathbb{C} \mid |z| < 1\}$. We claim that the map

$$U_z \to \mathfrak{b} \qquad \text{such that} \qquad w \mapsto \eta_z(w)^{\#\Gamma_z}$$

induces an homeomorphism $\Gamma_z\backslash U_z \xrightarrow{\sim} V$, where $V \subseteq \mathfrak{b}$ is an open subset. This implies that (after restricting U_z and V to suitable opens) we obtain an homeomorphism $\varphi_z : U_z \xrightarrow{\sim} \mathbb{C}$ such that the set $\left\{(U_z, \varphi_z)\right\}_{[z] \in \Gamma\backslash\mathfrak{h}}$ is an holomorphic atlas for $\Gamma\backslash\mathfrak{h}$.

To prove our claim it is sufficient to observe that for every $\phi \in \Gamma_z$ seen as an element of $\mathrm{Aut}(\mathfrak{h})$ the map $\psi = \eta_z \circ \phi \circ \eta_z^{-1} : \mathfrak{b} \to \mathfrak{b}$ is an automorphism of $\mathfrak{b}$ such that $\psi(0) = 0$. Thus, using again *Schwarz's lemma* we obtain that there exists $t \in [0, 1)$ such that $\psi(z) = e^{2\pi i t} \cdot z$ for every $z \in \mathfrak{b}$. Observe now that

$$\overbrace{\phi \circ \phi \circ \ldots \circ \phi}^{\#\Gamma_z \text{ times}} = 1_{\mathfrak{h}} \qquad \text{since} \qquad \phi \in \Gamma_z$$

which implies that

$$\overbrace{\psi \circ \psi \circ \ldots \circ \psi}^{\#\Gamma_z \text{ times}} = 1_{\mathfrak{b}} \qquad \text{and so that} \qquad (e^{2\pi i t})^{\#\Gamma_z} = 1$$

and this is equivalent to say that $t = {}^k/_{\#\Gamma_z}$ with $k \in \{0, \ldots, \#\Gamma_z - 1\}$. Thus for every $w, w' \in U_z$ we have that $w' \in \Gamma_z * w$ if and only if there exists $k \in \{0, \ldots, \#\Gamma_z - 1\}$ such that

$$w' = \eta_z^{-1}(\zeta^k \cdot \eta_z(w)) \qquad \text{with} \qquad \zeta = e^{2\pi i/_{\#\Gamma_z}}$$

and this happens if and only if $\eta_z(w)^{\#\Gamma_z} = \eta_z(w')^{\#\Gamma_z}$. This implies that the continuous and open map $U_z \to \mathfrak{b}$ such that $w \mapsto \eta_z(w)^{\#\Gamma_z}$ defines an injective map $\Gamma_z \backslash U_z \to \mathfrak{b}$ which is also continuous and open, and thus defines an homeomorphism $\Gamma_z \backslash U_z \to V \subseteq \mathfrak{b}$ as we wanted to prove. □

Using Theorem 3 we can define a Riemann surface $Y_\Gamma \stackrel{\text{def}}{=} \Gamma \backslash \mathfrak{h}$ for every Fuchsian group $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$. In particular we define

$$Y(N) \stackrel{\text{def}}{=} Y_{\Gamma(N)} \qquad \text{and} \qquad Y_0(N) \stackrel{\text{def}}{=} Y_{\Gamma_0(N)} \qquad \text{and} \qquad Y_1(N) \stackrel{\text{def}}{=} Y_{\Gamma_1(N)}$$

for every $N \in \mathbb{N}$. We recall now that many important results from the theory of Riemann surfaces hold only if a Riemann surface is compact. For example, every compact Riemann surface is isomorphic (as a complex manifold) to an algebraic curve inside the three-dimensional projective space $\mathbb{P}^3_{\mathbb{C}}$, and this allows us to use many tools from algebraic geometry to study compact Riemann surfaces. Unfortunately in our case not many of the surfaces Y_Γ are compact, as it is shown in the following proposition.

Proposition 4. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group such that the Riemann surface Y_Γ is compact. Then for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ we have $(a+d)^2 \neq 4$.*

Proof. Suppose by contradiction that there exists a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ such that $(a+d)^2 = 4$ and suppose initially that $c = 0$. Then $\det(A) = ad = 1$ together with the fact that $(a+d)^2 = 4$ imply that $a = d = 1$ or $a = d = -1$. Observe now that the set $G = \{\pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma\}$ is a subgroup of Γ such that $G/\{\pm I_2\}$ is not trivial because $A \in G$. Observe moreover that $G/\{\pm I_2\}$ is also a non trivial discrete subgroup of $\{\pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}_2 \mathbb{R}\}/\{\pm I_2\} \cong (\mathbb{R}, +)$ and thus $G/\{\pm I_2\} \cong \mathbb{Z}$. This implies that there exists $h \in \mathbb{R}^\times$ such that $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$ and $G = \{\pm\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^m \mid m \in \mathbb{Z}\}$.

Let now $\left(\begin{smallmatrix} a_0 & b_0 \\ c_0 & d_0 \end{smallmatrix}\right) \in \Gamma$ and suppose that $|c_0 h| < 1$. We define a sequence $\{A_n\} \subseteq \Gamma$ by setting

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \qquad \text{and} \qquad A_{n+1} = A_n \cdot \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \cdot A_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

and we obtain the recursive relations

$$\begin{aligned} a_{n+1} &= 1 - a_n c_n h & \qquad b_{n+1} &= a_n^2 h \\ c_{n+1} &= -c_n^2 h & \qquad d_{n+1} &= 1 + a_n c_n h \end{aligned}$$

which imply that

$$\begin{aligned} &\lim_{n\to+\infty} c_n = 0 && \text{since} \quad c_n = -c(ch)^{2^n - 1} \quad \text{for every} \quad n \geq 1 \\ &\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} d_n = 1 && \text{since} \quad |a_{n+1} - 1| = |d_{n+1} - 1| = |a_n||c_n h| \leq (|a| + n)|ch|^{2^n} \\ &\lim_{n\to+\infty} b_n = h && \text{since} \quad b_{n+1} = a_n^2 h \quad \text{and} \quad \lim_{n\to+\infty} a_n = 1 \end{aligned}$$

and this finally implies that $A_n \to \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ when $n \to +\infty$. Since Γ is a discrete group this shows that there exists $n_0 \in \mathbb{N}$ such that $A_n = \left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ for every $n \geq n_0$ and thus we have shown that for every matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ either $c = 0$ or $|c| \geq {}^1/_{|h|}$.

We will use this fact to obtain a contradiction. To do so let $U = \{z \in \mathfrak{h} \mid \Im(z) > |h|\}$ and observe that for every matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ with $\gamma \neq 0$ and every $z \in U$ we have that

$$\Im(A * z) = \left|\gamma z + \delta\right|^{-2} \Im(z) \leq (\gamma \Im(z))^{-2} \Im(z) \leq h^2 \Im(z)^{-1} \leq |h|$$

which implies that $A * z \notin U$. Let now $z_n = i \cdot n$ and observe that $z_n \in U$ for every $n \geq \lceil |h| \rceil$ and that for every matrix $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ we have $A * z_n \notin U$ if $\gamma \neq 0$ and $\Im(A * z_n) = n$ if $\gamma = 0$ which implies that $A * z_n \neq z_m$ for every $A \in \Gamma$ and $m \neq n$. If we denote by $\pi \colon \mathfrak{h} \to \Gamma \backslash \mathfrak{h}$ the quotient map what we have shown implies that the image $\{\pi(z_n)\}$ of the sequence $\{z_n\}$ in the quotient $Y_\Gamma = \Gamma \backslash \mathfrak{h}$ is still an infinite sequence. This sequence admits a convergent subsequence $\{\pi(z_{n_k})\}$ because Y_Γ is compact by our assumption. Thus there exists a point $w \in \mathfrak{h}$ such that $\pi(z_{n_k}) \to \pi(w)$ as $k \to +\infty$. Let now $w \in W \subseteq \mathfrak{h}$ be an open neighbourhood such that the closure $\overline{W} \subseteq \mathfrak{h}$ is compact and observe that there exist $r < s \in \mathbb{R}_{>0}$ such that $r < \Im(z) < s$ for every $z \in \overline{W}$. So if we define

$$V = \left\{z \in \mathfrak{h} \mid \Im(z) > \max(s, {}^{h^2}/_r)\right\}$$

we have that $V \cap \Gamma * \overline{W} = \emptyset$ because for every $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma$ and every $z \in \overline{W}$ we have that $\Im(A * z) \leq h^2 \Im(z)^{-1} = {}^{h^2}/_r$ if $\gamma \neq 0$ whereas $\Im(A * z) = \Im(z) < s$ if $\gamma = 0$ and thus in both cases $A * z \notin V$ which implies that $V \cap \Gamma * \overline{W} = \emptyset$ as we wanted to show. Hence we obtain that $\pi(\overline{W}) \cap \pi(V) = \emptyset$ which is a contradiction because $\pi(z_{n_k}) \to \pi(w)$ as $k \to +\infty$ and thus there exists $k_0 \in \mathbb{N}$ such that $\pi(z_{n_k}) \in \pi(\overline{W}) \cap \pi(V)$ for every $k \geq k_0$.

Suppose now that there exists a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ such that $(a+d)^2 = 4$ with $c \neq 0$. Then there exists exactly one $\alpha \in \mathbb{R}$ such that

$$\frac{a\alpha + b}{c\alpha + d} = \alpha \qquad \text{because the equation} \qquad c x^2 + (d-a)x - b = 0$$

has discriminant $(a-d)^2 + 4bc = (a+d)^2 - 4(ad - bc) = 0$ since $\det(A) = ad - bc = 1$. Let now $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & \alpha \end{smallmatrix}\right)$ and let $A' = B \cdot A \cdot B^{-1} = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. An easy calculation shows that $c' = c\alpha^2 + (d-a)\alpha - b = 0$ which implies that the subgroup $\Gamma' = B \cdot \Gamma \cdot B^{-1}$ contains a matrix $A' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ such that $c' = 0$ and $(a' + b')^2 = 4$. Observe now that $\Gamma' \leq \mathrm{SL}_2(\mathbb{R})$ is also a Fuchsian subgroup and that $Y_{\Gamma'} \cong Y_\Gamma$ since the bi-holomorphic map $\mathfrak{h} \to \mathfrak{h}$ defined as $z \mapsto B * z$ induces an isomorphism $Y_\Gamma \cong Y_{\Gamma'}$. Thus $Y_{\Gamma'}$ is itself compact and this gives us a contradiction using what we have proved in the previous paragraph. $\square$

Corollary 5. *For every $N \in \mathbb{N}$ the Riemann surfaces $Y(N)$, $Y_0(N)$ and $Y_1(N)$ are not compact.*

Proof. We have $A_N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$ and $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$ for every $N \in \mathbb{N}$ which implies that $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ contain a matrix A_N such that $\mathrm{tr}(A_N)^2 = 4$. $\square$

What we have just proved shows the necessity to find a good candidate for the *compactification* of the Riemann surface $Y_\Gamma = \Gamma \backslash \mathfrak{h}$ where $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. In particular we want to find a compact Riemann surface X_Γ together with an embedding $\iota \colon Y_\Gamma \hookrightarrow X_\Gamma$ such that $\iota(Y_\Gamma)$ is an open subset of X_Γ. To do so, we'll extend the space

$\mathfrak{h}$ on which Γ acts by adding some points. Observe first of all that $\mathrm{SL}_2(\mathbb{R})$ acts also on the set $\mathbb{R}\cup\{\infty\}$ by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * x = \begin{cases} \infty, \text{ if } c \neq 0 \text{ and } x = -{}^{d}/_{c} \text{ or } x = \infty \text{ and } c = 0 \\ {}^{a}/_{c}, \text{ if } x = \infty \text{ and } c \neq 0 \\ {}^{ax+b}/_{cx+d}, \text{ otherwise} \end{cases}$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and every $x \in \mathbb{R}\cup\{\infty\}$. Observe in particular that for every $x \in \mathbb{R}$ we have $B_x * x = \infty$ where $B_x \overset{\text{def}}{=} \left(\begin{smallmatrix} 0 & 1 \\ -1 & x \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

We can easily classify the fixed points of the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}\cup\mathbb{R}\cup\{\infty\}$ using the following proposition.

Proposition 6. *For every matrix* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}) \setminus \{\pm I_2\}$ *we have:*

- $(a+d)^2 < 4$ *if and only if there exists* $z \in \mathfrak{h}$ *such that* $A * z = z$ *and* $A * w \neq w$ *for every* $w \in \mathfrak{h}\cup\mathbb{R}\cup\{\infty\}\setminus\{z\}$*;*
- $(a+d)^2 = 4$ *if and only if there exists* $x \in \mathbb{R}\cup\{\infty\}$ *such that* $A*x = x$ *and* $A*w \neq w$ *for every* $w \in \mathfrak{h}\cup\mathbb{R}\cup\{\infty\}\setminus\{x\}$*;*
- $(a+d)^2 > 4$ *if and only if there exist* $x, y \in \mathbb{R}\cup\{\infty\}$ *such that* $A * x = x$, $A * y = y$ *and* $A * w \neq w$ *for every* $w \in \mathfrak{h}\cup\mathbb{R}\cup\{\infty\}\setminus\{x,y\}$.

Proof. Suppose first of all that $c \neq 0$ and observe that $A * \infty \neq \infty$ and that for every $z \in \mathfrak{h}\cup\mathbb{R}$ we have $A * z = z$ if and only if $cz^2 + (d-a)z - b = 0$. Since this quadratic equation has discriminant given by

$$(a-d)^2 + 4bc = (a+d)^2 - 4(ad-bc) = (a+d)^2 - 4 \qquad \text{because} \qquad ad - bc = \det(A) = 1$$

we get that $A*z = z$ for two real distinct numbers if and only if $(a+d)^2 > 4$ and $c \neq 0$, that $A*z = z$ for exactly one $z \in \mathbb{R}$ if and only if $(a+d)^2 = 4$ and $c \neq 0$ and finally that $A*z = z$ for exactly one $z \in \mathfrak{h}$ if and only if $(a+d)^2 < 4$ and $c \neq 0$. If $c = 0$ then $A * \infty = \infty$ by definition and for every $z \in \mathfrak{h}\cup\mathbb{R}$ we have $A * z = z$ if and only if $(d-a)z = b$. Thus if $A*z = z$ then $z \in \mathbb{R}\cup\{\infty\}$ and we have $A*z = z$ for some $z \in \mathbb{R}$ if and only if $d \neq a$. Since $c = 0$ we have that $\det(A) = ad = 1$ and thus $d = a$ if and only if $(a+d)^2 = 4$, whereas $d \neq a$ if and only if $(a+d)^2 > 4$, which concludes our proof. □

Let now $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a subgroup. We call a point $x \in \mathbb{R}\cup\{\infty\}$ a *cusp* of Γ if there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ such that $(a+d)^2 = 4$ and $A * x = x$ and we call a point $z \in \mathfrak{h}$ an *elliptic point* of Γ if there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $(a+d)^2 < 4$ and $A * z = z$. Let now $\mathfrak{E}_\Gamma$ be the set of all elliptic points of Γ, $\mathfrak{P}_\Gamma$ be the set of all cusps of Γ and let $\mathfrak{h}^*_\Gamma = \mathfrak{h}\cup\mathfrak{P}_\Gamma$. We can define a topology on the set $\mathfrak{h}^*_\Gamma$ by taking as a fundamental system of neighbourhoods of a cusp $s \in \mathfrak{P}_\Gamma$ the system $\{B_s^{-1} * U_l^*\}_{l>0}$ where

$$U_l^* = U_l \cup \{\infty\} \qquad \text{and} \qquad U_l = \{z \in \mathfrak{h} \mid \Im(z) > l\}$$

and as a fundamental system of neighbourhoods of a point $z \in \mathfrak{h}$ the usual system of open balls $\{B(z;\varepsilon)\}_{\varepsilon>0}$. Observe now that $\Im(B_x * z) = \Im(z)\cdot|x-z|^{-2}$ for every $x \in \mathbb{R}$ and every $z \in \mathfrak{h}$, which implies that for every cusp $s \in \mathfrak{P}_\Gamma \setminus \{\infty\}$ we have

$$B_s^{-1} * U_l = \left\{ z \in \mathfrak{h} \mid |s-z| < \sqrt{\frac{\Im(z)}{l}} \right\} = B\left(s + \frac{1}{2l}i; \frac{1}{2l}\right)$$

and thus in particular that $B_s^{-1} * U_l$ is an open ball of radius ${}^1/_{2l}$ whose boundary ∂U_l is a circle tangent to the real line at the point s. This shows immediately that $\mathfrak{h}_\Gamma^*$ is an Hausdorff topological space, since the neighbourhoods $B(z;\varepsilon)$ with $\varepsilon < \Im(z)$, U_l^* with $l \geq \Im(z) + \varepsilon$ and $B_s^{-1} * U_{l'}$ with $l\,l' \geq 1$ separate ∞ and any $z \in \mathfrak{h}$ and $s \in \mathfrak{P}_\Gamma \setminus \{\infty\}$ if $\infty \in \mathfrak{P}_\Gamma$, whereas the neighbourhoods $B_s^{-1} * U_l^*$ and $B(z;\varepsilon)$ clearly separate any real cusp $s \in \mathfrak{P}_\Gamma \setminus \{\infty\}$ from any point $z \in \mathfrak{h}$ for sufficiently small $l > 0$ and $\varepsilon > 0$ and the neighbourhoods $B_s^{-1} * U_l^*$ and $B_t^{-1} * U_l^*$ with $l > |s-t|^{-1}$ separate two real cusps $s, t \in \mathfrak{P}_\Gamma \setminus \{\infty\}$. Observe moreover that $\mathfrak{h}_\Gamma^*$ is not locally compact since for every open neighbourhood $\infty \in U \subseteq \mathfrak{h}_\Gamma^*$ the closure $\overline{U} \subseteq \mathfrak{h}_\Gamma^*$ is not compact. it is also easy to prove that the action $\Gamma \circlearrowright \mathfrak{h}_\Gamma^*$ is still continuous. We can thus consider the quotient space $\Gamma \backslash \mathfrak{h}_\Gamma^*$, whose main topological properties are summarized in the following theorem.

Theorem 7. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group. Then the quotient space $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is a locally compact and Hausdorff topological space with a natural structure of a Riemann surface. Moreover, if $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is compact then the sets $\Gamma\backslash\mathfrak{P}_\Gamma$ and $\Gamma\backslash\mathfrak{E}_\Gamma$ are finite.*

Proof. Let first of all $\pi\colon \mathfrak{h}_\Gamma^* \to \Gamma\backslash\mathfrak{h}_\Gamma^*$ be the quotient map and suppose that $\infty \in \mathfrak{P}_\Gamma$. Then let $U = \{z \in \mathfrak{h} \mid \Im(z) > |h|\} \cup \{\infty\}$ where $h \in \mathbb{R}^\times$ is such that

$$\Gamma_\infty = \{A \in \Gamma \mid A * \infty = \infty\} = \left\{\pm\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^m \mid m \in \mathbb{Z}\right\}.$$

We know from the proof of Proposition 4 that if $A \notin \Gamma_\infty$ we have that $A * z \notin U$ for every $z \in U$. This implies that for every $c > |h|$ if we define $V = \{z \in \mathfrak{h} \mid \Im(z) > c\} \cup \{\infty\}$ we can identify $\overline{\pi(V)}$ with $\Gamma_\infty\backslash\overline{V}$. Using the fact that $\Gamma_\infty = \left\{\pm\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^m \mid m \in \mathbb{Z}\right\}$ we obtain that the map π becomes an homeomorphism when restricted to the closure of

$$W = \{z \in \mathfrak{h} \mid 0 < \Re(z) < |h|,\ \Im(z) > c\} \cup \{\infty\}$$

which is given by $\overline{W} = \{z \in \mathfrak{h} \mid 0 \leq \Re(z) \leq |h|,\ \Im(z) \geq c\} \cup \{\infty\}$. Observe that $\overline{W}$ is compact since the open neighbourhoods of ∞ are given by $U_l = \{z \in \mathfrak{h} \mid \Im(z) > l\}$ and

$$\overline{W} \setminus U_l = \{z \in \mathfrak{h} \mid 0 \leq \Re(z) \leq |h|,\ c \leq \Im(z) \leq l\}$$

is clearly compact for every $l \in \mathbb{R}_{>0}$. This implies that $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is a locally compact topological space.

We want now to prove that $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is an Hausdorff topological space. We have already proved in Theorem 3 that $\Gamma\backslash\mathfrak{h}$ is an Hausdorff space, so we have only to show that for every cusp $s \in \mathfrak{P}_\Gamma$ and every $z \in \mathfrak{h}_\Gamma^*$ we can find two disjoint open neighbourhoods of $\pi(s)$ and $\pi(z)$. We already know from the proof of Proposition 4 that for every $z \in \mathfrak{h}$ and every Fuchsian group $\Gamma' \leq \mathrm{SL}_2(\mathbb{R})$ such that $\infty \in \mathfrak{P}_{\Gamma'}$ there exist an open neighbourhood $z \in U \subseteq \mathfrak{h}$ and an open subset $V \subseteq \mathfrak{h}$ such that $V \cup \{\infty\}$ is an open neighbourhood of ∞ and $W \cap \Gamma' * (V \cup \{\infty\}) = \emptyset$. Observe now that for every cusp $s \in \mathfrak{P}_\Gamma$ we have that $\infty \in \mathfrak{P}_{\Gamma'}$ where $\Gamma' = B_s \cdot \Gamma \cdot B_s^{-1}$ and thus we can find two open subsets $z \in U \subseteq \mathfrak{h}$ and $V \subseteq \mathfrak{h}$ such that $W \cap \Gamma' * (V \cup \{\infty\}) = \emptyset$, which implies that $B_z^{-1} * W \cap \Gamma * (B_z^{-1} * (\mathfrak{h} \cup \{\infty\})) = \emptyset$. This implies that for every cusp $s \in \mathfrak{P}_\Gamma$ and every $z \in \mathfrak{h}$ we can find two disjoint open neighbourhoods of $\pi(s)$ and $\pi(z)$.

Let now $s, t \in \mathfrak{P}_\Gamma$ such that $\pi(s) \neq \pi(t)$ and observe that if $\Gamma' = B_t \cdot \Gamma \cdot B_t^{-1}$ then $\infty \in \mathfrak{P}_{\Gamma'}$ and $B_t * s \in \mathfrak{P}_{\Gamma'}$. Moreover if we can find two open neighbourhoods $B_t * s \in U \subseteq \mathfrak{h}_{\Gamma'}^*$ and $\infty \in V \subseteq \mathfrak{h}_{\Gamma'}^*$ such that $U \cap \Gamma' * V = \emptyset$ then $s \in B_t^{-1} * U \subseteq \mathfrak{h}_\Gamma^*$ and $t \in B_t^{-1} * V \subseteq \mathfrak{h}_\Gamma^*$ are two open neighbourhoods of s and t such that $B_t^{-1} * U \cap \Gamma * (B_t^{-1} * V) = \emptyset$. This implies that we can suppose without loss of generality that $t=\infty$ and thus that $s\in\mathfrak{P}_\Gamma\cap\mathbb{R}$. Let now $l\in\mathbb{R}_{>0}$ be any positive real number and let

$$K=\{z\in\mathfrak{h}\mid 0\le\Re(z)\le|h|,\ \Im(z)=l\}$$

where $h\in\mathbb{R}^\times$ is such that $\Gamma_\infty=\{\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)^m\mid m\in\mathbb{Z}\}$. Since K is compact we can find an open neighbourhood $s\in U\subseteq\mathfrak{h}_\Gamma^*$ such that $K\cap\Gamma*U=\emptyset$ as we have done in the proof of Proposition 4 and we can assume that ∂U is a circle tangent to the real line. We want now to prove that $V\cap\Gamma*U=\emptyset$ where $V=\{z\in\mathfrak{h}\mid\Im(z)>l\}\cup\{\infty\}$. To do so we argue by contradiction that there exists $A\in\Gamma$ such that $A*U\cap V\neq\emptyset$. Observe that $A*s\neq\infty$ because $\pi(s)\neq\pi(\infty)$ by hypothesis. This implies that $A*s\in\mathfrak{P}_\Gamma\cap\mathbb{R}$ and that $\partial(A*U)$ is again a circumference tangent to the real line. This implies that there exists $z\in U$ such that $\Im(A*z)=l$, and so there exists an element $B\in\Gamma_\infty$ such that $B*A*z\in K$ which is a contradiction since $K\cap\Gamma*U=\emptyset$, and this eventually proves that $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is an Hausdorff topological space.

We now have to define the structure of a Riemann surface on $\Gamma\backslash\mathfrak{h}_\Gamma^*$. Let $s\in\mathfrak{P}_\Gamma$ be a cusp and observe from the proofs given above that there exists an open neighbourhood $s\in U_s\subseteq\mathfrak{h}_\Gamma^*$ such that

$$\{A\in\Gamma\mid A*U_s\cap U_s\neq\emptyset\}=\Gamma_s=\{A\in\Gamma\mid A*s=s\}$$

and this gives us an open injection $\Gamma_s\backslash U_s\hookrightarrow\Gamma\backslash\mathfrak{h}_\Gamma^*$. Thus we can see $\Gamma_s\backslash U_s$ as an open neighbourhood of $s\in\mathfrak{h}_\Gamma^*$. Let now $h\in\mathbb{R}_{>0}$ be such that

$$(B_s\cdot\Gamma_s\cdot B_s^{-1})/\{\pm I_2\}=\left\{\pm\left(\begin{smallmatrix}1&h\\0&1\end{smallmatrix}\right)^m\mid m\in\mathbb{Z}\right\}/\{\pm I_2\}$$

and observe that the map $\phi\colon U_s\to\mathbb{C}$ defined as $\phi(z)=\exp\left({}^{2\pi i\cdot(B_s*z)}/_h\right)$ factors through the quotient $U_s\twoheadrightarrow\Gamma_s\backslash U_s$ defining a continuous, open and injective map $\Gamma_s\backslash U_s\to\mathbb{C}$. By restricting U_s we can suppose that the image of this map is homeomorphic to $\mathbb{C}$ and thus we obtain an homeomorphism $\varphi_s\colon\Gamma_s\backslash U_s\to\mathbb{C}$. Let now $\mathbb{A}$ be the atlas of the Riemann surface $\Gamma\backslash\mathfrak{h}$ defined in Theorem 3. it is now straightforward to check that the set $A\cup\{(U_s,\varphi_s)_{s\in\mathfrak{P}_\Gamma}\}$ is an holomorphic atlas for the topological space $\Gamma\backslash\mathfrak{h}_\Gamma^*$ which makes is into a Riemann surface.

Suppose now that $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is compact. Recall from what we have proven before that for every $z\in\mathfrak{h}$ and $s\in\mathfrak{P}_\Gamma$ we can take open neighbourhoods $z\in U_z\subseteq\mathfrak{h}$ and $s\in U_s\subseteq\mathfrak{h}_\Gamma^*$ such that

$$\{A\in\Gamma\mid A*U_z\cap U_z\neq\emptyset\}=\{A\in\Gamma\mid A*z=z\}\qquad\text{and}\qquad\{A\in\Gamma\mid A*U_s\cap U_s\neq\emptyset\}=\Gamma_s$$

and this implies in particular that $U_s\cap\mathfrak{E}_\Gamma=\emptyset$ for every $s\in\mathfrak{P}_\Gamma$ and that $U_z\cap\mathfrak{E}_\Gamma=\emptyset$ if $z\notin\mathfrak{E}_\Gamma$ whereas $U_z\cap\mathfrak{E}_\Gamma=\{z\}$ if $z\in\mathfrak{E}_\Gamma$. Recall now that $\Gamma*\mathfrak{E}_\Gamma=\mathfrak{E}_\Gamma$ and $\Gamma*\mathfrak{P}_\Gamma=\mathfrak{P}_\Gamma$, which implies that for every point $x\in\mathfrak{E}_\Gamma\cup\mathfrak{P}_\Gamma$ and every $y\in\mathfrak{h}_\Gamma^*$ with $x\neq y$ we have that $\pi(x)\notin\pi(U_y)$. Observe finally that, since $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is compact there exist $\{z_1,\ldots,z_n\}\in\mathfrak{h}_\Gamma^*$ such that $\Gamma\backslash\mathfrak{h}_\Gamma^*=\pi(U_{z_1})\cup\ldots\cup\pi(U_{z_n})$ and this implies that $\pi(\mathfrak{E}_\Gamma\cup\mathfrak{P}_\Gamma)$ is finite, which is equivalent to say that $\Gamma\backslash\mathfrak{E}_\Gamma$ and $\Gamma\backslash\mathfrak{P}_\Gamma$ are both finite, as we wanted to prove. $\square$

Using Theorem 7 we can see immediately that for every Fuchsian group $\Gamma\subseteq\mathrm{SL}_2(\mathbb{R})$ the Riemann surface $X_\Gamma\overset{\text{def}}{=}\Gamma\backslash\mathfrak{h}_\Gamma^*$ is not compact. For instance we can take insert the (2,3,7) group?. We'll call *Fuchsian groups of the first kind* the Fuchsian groups $\Gamma\le\mathrm{SL}_2(\mathbb{R})$ such that X_Γ is compact, and we'll call *Fuchsian groups of the second kind* the rest of the Fuchsian groups. Do we know any example of a Fuchsian group of the first kind? The following lemma and theorem answer to this question in a positive way.

Lemma 8. *Let* $\Gamma, \Gamma' \leq \mathrm{SL}_2(\mathbb{R})$ *be two* commensurable subgroups, *i.e. two subgroups of* $\mathrm{SL}_2(\mathbb{R})$ *such that* $\Gamma \cap \Gamma'$ *is a subgroup of finite index in* Γ *and* Γ'*. Then* Γ *is Fuchsian if and only if* Γ' *is Fuchsian, and* Γ *is of the first kind if and only if* Γ' *is of the first kind.*

Proof. Suppose first of all that $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is Fuchsian, which is to say that Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$. Then $\Gamma \cap \Gamma'$ has also the discrete topology and if $n = [\Gamma' : \Gamma \cap \Gamma']$ we have that

$$\Gamma' = (\Gamma \cap \Gamma') \cdot A_1 \cup \ldots \cup (\Gamma \cap \Gamma') \cdot A_n \qquad \text{for some} \qquad A_1, \ldots, A_n \in \Gamma'$$

which implies that Γ' is also discrete. Indeed if $B \in \Gamma'$ then

$$\Gamma' \setminus \{B\} = ((\Gamma \cap \Gamma') \cdot A_1 \setminus \{B\}) \cup \ldots \cup ((\Gamma \cap \Gamma') \cdot A_n \setminus \{B\})$$

and all the sets $(\Gamma \cap \Gamma') \cdot A_j \setminus \{B\}$ are closed inside $(\Gamma \cap \Gamma') \cdot A_j$. Thus to conclude that $\{B\}$ is open and that Γ' is discrete it is sufficient to prove that all the subsets $(\Gamma \cap \Gamma') \cdot A_j \subseteq \Gamma'$ are closed.

To show this observe first of all that the map $\Gamma' \xrightarrow{\cdot A_j} \Gamma'$ is an automorphism of Γ' as a topological group, and thus it is sufficient to prove that the discrete subgroup $\Gamma \cap \Gamma' \leq \Gamma'$ is closed in Γ'. This is generally true for every Hausdorff topological group G and every discrete subgroup $H \leq G$. Indeed we can suppose by contradiction that $\overline{H} \setminus H \neq \emptyset$, where $\overline{H}$ is the closure of H in G, so that there exists $g \in \overline{H} \setminus H$. Observe that since H is discrete we can find an open neighbourhood $1_G \in U \subseteq G$ such that $U \cap H = \{1_G\}$. Let $V = U \cap \iota(U)$ where $\iota: G \to G$ is the inversion map and observe that if $\mathfrak{l}_g: G \to G$ is defined as $\mathfrak{l}_g(x) = g \cdot x$ we have that $\mathfrak{l}_g(V)$ is an open neighbourhood of g in G. Since $\overline{H}$ is the closure of H in G we obtain that $\mathfrak{l}_g(V) \cap H \neq \emptyset$ and thus we can find $h \in \mathfrak{l}_g(V) \cap H$. In particular $g \neq h$ since $g \notin H$ and thus we can find an open subset $g \in W \subseteq G$ such that $h \notin W$ because G is an Hausdorff topological space. We obtain that $\mathfrak{l}_g(V) \cap W$ is an open neighbourhood of g in G and thus there exists $k \in \mathfrak{l}_g(V) \cap W \cap H$ such that

$$\begin{aligned} k^{-1} h \in \iota(\mathfrak{l}_g(V) \cap H) \cdot (\mathfrak{l}_g(V) \cap H) &= (\mathfrak{l}_{g^{-1}}(V) \cdot \mathfrak{l}_g(V)) \cap H = \\ &= (\iota(V) \cdot V) \cap H = V \cap H \subseteq U \cap H = \{1_G\} \end{aligned}$$

which implies that $k = h$, which is finally a contradiction because $h \notin W$ whereas $k \in W$ by definition. This finally shows that if Γ is Fuchsian then Γ' is Fuchsian, and it is clearly perfectly analogous to prove the converse.

Suppose now that $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group and observe first of all that $\Gamma \cap \Gamma'$ is a Fuchsian group which is a subgroup of Γ and Γ'. Hence we can assume without loss of generality that $\Gamma' \leq \Gamma$ is a subgroup of finite index. Observe now that $\mathfrak{P}_\Gamma = \mathfrak{P}_{\Gamma'}$ because clearly $\mathfrak{P}_{\Gamma'} \subseteq \mathfrak{P}_\Gamma$ and if $s \in \mathfrak{P}_\Gamma$ we can find $A \in \Gamma$ such that $A * s = s$. Since $\Gamma' \leq \Gamma$ has finite index $n = [\Gamma : \Gamma']$ we have that $A^n \in \Gamma'$ and clearly $A^n * s = s$ which implies that $s \in \mathfrak{P}_{\Gamma'}$. This shows that $\mathfrak{P}_\Gamma = \mathfrak{P}_{\Gamma'}$ and thus in particular that $\mathfrak{h}_\Gamma^* = \mathfrak{h}_{\Gamma'}^*$. Moreover since $\Gamma' \leq \Gamma$ we have a continuous surjective map $X_{\Gamma'} \to X_\Gamma$ and thus if Γ' is a Fuchsian group of the first kind so it is Γ. If conversely we suppose that Γ is a Fuchsian group of the first kind we can find a finite number of open subsets $U_1, \ldots, U_m \subseteq \mathfrak{h}_\Gamma^*$ such that $\overline{\pi(U_j)}$ is compact for every $j \in \{1, \ldots, n\}$ and $X_\Gamma = \pi(U_1) \cup \ldots \cup \pi(U_m)$ where $\pi: \mathfrak{h}_\Gamma^* \to X_\Gamma$ is the quotient map. Then if we let $A_1, \ldots, A_n \in \Gamma$ be such that $\Gamma = \Gamma' \cdot A_1 \cup \ldots \cup \Gamma' \cdot A_n$ we have that

$$X_{\Gamma'} = \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} \overline{\pi'(A_j \cdot U_k)} \qquad \text{where} \qquad \pi': \mathfrak{h}_{\Gamma'}^* \to X_{\Gamma'} \qquad \text{is the quotient map}$$

and thus $X_{\Gamma'}$ is compact. $\square$

Theorem 9. *The groups* $\Gamma(N)$, $\Gamma_0(N)$ *and* $\Gamma_1(N)$ *are Fuchsian groups of the first kind for all* $N \in \mathbb{N}_{\geq 1}$.

Proof. Observe first of all that $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$ and that $\Gamma(N) \leq \mathrm{SL}_2(\mathbb{Z})$ for every $N \in \mathbb{N}_{\geq 1}$. We want now to prove that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] < \infty$ for every $N \in \mathbb{N}_{\geq 1}$. Clearly $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$ and thus we can suppose that $N \geq 2$. We define a map $f_N : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2\left(\mathbb{Z}/_{N\mathbb{Z}}\right)$ such that for every $A \in \mathrm{SL}_2(\mathbb{Z})$ the matrix $f_N(A)$ is obtained by applying the quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/_{N\mathbb{Z}}$ to every element of A. Observe now that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \ker(f_N)$ if and only if

$$a \equiv d \equiv 1 \bmod N \qquad \text{and} \qquad b \equiv c \equiv 0 \bmod N$$

and thus $\ker(f_N) = \Gamma(N)$. This implies immediately that $\Gamma(N) \leq \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of finite index because

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = \#\mathrm{Im}(f_N) \leq \mathrm{SL}_2\left(\mathbb{Z}/_{N\mathbb{Z}}\right)$$

and $\mathrm{SL}_2\left(\mathbb{Z}/_{N\mathbb{Z}}\right)$ is clearly a finite group. Actually, it could be proved that f_N is surjective, but this is not needed for our purposes. Observe finally that also $\Gamma_0(N) \leq \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_1(N) \leq \mathrm{SL}_2(\mathbb{Z})$ are subgroups of finite index.

Using what we have proved above and applying Lemma 8 we see that the groups $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$ are Fuchsian groups of the first kind if and only if $\mathrm{SL}_2(\mathbb{Z})$ is a Fuchsian group of the first kind. To show this we show that $X_{\mathrm{SL}_2(\mathbb{Z})} \cong \mathbb{P}^1_{\mathbb{C}}$ which implies in particular that $X_{\mathrm{SL}_2(\mathbb{Z})}$ is compact.

Observe fist of all that $\mathfrak{P}_{\mathrm{SL}_2(\mathbb{Z})} = \mathbb{Q} \cup \{\infty\}$ because $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) * \infty = \infty$ and for every couple of coprime integers $p, q \in \mathbb{Z}$ we have that $p/_q \in \mathfrak{P}_{\mathrm{SL}_2(\mathbb{Z})}$ since

$$\left(\begin{pmatrix} p & r \\ q & s \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p & r \\ q & s \end{pmatrix}^{-1}\right) * \frac{p}{q} = \frac{p}{q}$$

where $r, s \in \mathbb{Z}$ are such that $ps - qr = 1$. Observe in particular that $\left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right) * \infty = p/_q$, which implies that

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{P}_{\mathrm{SL}_2(\mathbb{Z})} = \pi(\{\infty\})$$

where $\pi : \mathfrak{h}^*_{\mathrm{SL}_2(\mathbb{Z})} \to X_{\mathrm{SL}_2(\mathbb{Z})}$ is the canonical quotient map. This shows in particular that $X_{\mathrm{SL}_2(\mathbb{Z})} = \mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{h} \cup \{\pi(\infty)\}$. We'll thus define a bi-holomorphic map $X_{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{P}^1_{\mathbb{C}}$ by defining a bi-holomorphic map $Y_{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{C}$ and then extending it by setting $\pi(\infty) \mapsto (1:0)$.

Consider now the power series

$$G_n(z) = \sum_{\substack{(a,b) \in \mathbb{Z}^2 \\ a^2 + b^2 \neq 0}} \frac{1}{(a + bz)^n}$$

which converge absolutely on all $\mathfrak{h}$ and thus define a family of holomorphic functions $G_n : \mathfrak{h} \to \mathbb{C}$ for every $n \geq 3$ such that $G_{2k+1}(z) = 0$ for every $k \geq 1$ and every $z \in \mathfrak{h}$ because

$$\frac{1}{(a + bz)^{2k+1}} + \frac{1}{(-a - bz)^{2k+1}} = 0$$

for every $(a, b) \in \mathbb{Z}^2 \setminus (0, 0)$, every $z \in \mathfrak{h}$ and every $k \geq 1$. Define now a map $j : \mathfrak{h} \to \mathbb{C}$ by setting

$$j(z) = 1728 \cdot \frac{g_2(z)^3}{g_2(z)^3 - 27 g_3(z)^2} \qquad \text{where} \qquad g_2(z) = 60 G_4(z) \quad \text{and} \quad g_3(z) = 140 G_6(z)$$

and observe that $j(z) = j(w)$ if and only if $z \in \mathrm{SL}_2(\mathbb{Z}) * \{w\}$ and that j is surjective. Insert proof of this fact.

This shows that $j\colon \mathfrak{h} \to \mathbb{C}$ factors through the quotient map $\mathfrak{h} \twoheadrightarrow Y_{\mathrm{SL}_2(\mathbb{Z})}$ and gives us a bijective and holomorphic map $Y_{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{C}$. We can also show that the inverse of this map is holomorphic, since should we insert a proof of this fact?. This implies that $Y_{\mathrm{SL}_2(\mathbb{Z})} \cong \mathbb{C}$ as Riemann surfaces. Using this fact we can show that the map

$$\mathfrak{h}^*_{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{P}^1(\mathbb{C}) \qquad \text{such that} \qquad z \mapsto \begin{cases} (j(z):1), & \text{if } z \notin \mathfrak{P}_{\mathrm{SL}_2(\mathbb{Z})} \\ (1:0), & \text{if } z \in \mathfrak{P}_{\mathrm{SL}_2(\mathbb{Z})} \end{cases}$$

defines an isomorphism of Riemann surfaces $X_{\mathrm{SL}_2(\mathbb{Z})} \cong \mathbb{P}^1_{\mathbb{C}}$, as we wanted to prove. □

What we have proved shows that $X(N)$, $X_0(N)$ and $X_1(N)$ are compact and connected Riemann surfaces. Thus these Riemann surfaces can be embedded into $\mathbb{P}^3_{\mathbb{C}}$ and the image of this embedding would be an algebraic curve, as shown in insert reference for the proof of this theorem. Maybe Forster?. The importance of these surfaces will be clear in the following section, in which we will analyze their role as moduli spaces of elliptic curves over the complex numbers.

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