AN ADELIC DESCRIPTION OF MODULAR CURVES

Riccardo Pengo, advised by Peter Bruin

July 24, 2017

Dipartimento di Matematica “Federigo Enriques”, Università degli Studi di Milano
Mathematical Institute , Universiteit Leiden

- Let $\mathfrak{h} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid \Im(z) > 0\}$. We have an action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) * z = {}^{az+b}/_{cz+d}$
- Affine modular curves are Riemann surfaces obtained as quotients of $\mathfrak{h}$ by the action of discrete subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$;
- These curves parametrize suitable isomorphism classes of elliptic curves with some extra structure. For example, $\mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ parametrizes the isomorphism classes of elliptic curves over $\mathbb{C}$.
- Usually the quotient $\Gamma\backslash\mathfrak{h}$ is not compact. For example if $\Gamma\backslash\mathfrak{h}$ is compact then for every $A \in \Gamma \setminus \{\pm I_2\}$ we have $|\mathbf{tr}(A)| \neq 2$.

- If K is a global field we define Σ_K (resp. Σ_K^∞) to be the set of all equivalence classes of (non Archimedean) norms $|\cdot| : K \to \mathbb{R}_{\geq 0}$.
- With respect to any place $v \in \Sigma_K$ we can define a completion K_v of K, with a ring of integers $\mathcal{O}_{K_v} \subseteq K_v$;
- Now we define the adèle rings $\mathbb{A}_K$ and $\mathbb{A}_K^\infty$ as

$$\mathbb{A}_K \stackrel{\text{def}}{=} \prod_{v \in \Sigma_K}{}' (K_v : \mathcal{O}_{K_v}) \qquad \text{and} \qquad \mathbb{A}_K^\infty \stackrel{\text{def}}{=} \prod_{v \in \Sigma_K^\infty}{}' (K_v : \mathcal{O}_{K_v})$$

- This ring is used in class field theory to prove that

$$\mathbf{Gal}\left({}^{K^{ab}}/_{K}\right) \cong \widehat{\left({}^{\mathbb{A}_K^\times}/_{K^\times}\right)}$$

for every global field K.

We can describe suitable disjoint unions of affine modular curves as double quotients of the topological group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

Theorem 1

Let $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ be a compact and open subgroup and let $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be such that $\{\det(A_j)\}_{j=1}^n$ is a set of representatives for the quotient $\widehat{\mathbb{Z}}^\times/_{\det(K^\infty)}$. Then we have a homeomorphism

$$\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h} \cong Y/K^\infty \qquad \text{with} \qquad Y \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty$$

where $\Gamma_j \stackrel{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\mathbb{Q})$ and $K_\infty \stackrel{\text{def}}{=} \mathbb{R}_{>0} \times \mathrm{SO}_2(\mathbb{R})$

- For every $\Gamma \leq \mathsf{SL}_2(\mathbb{R})$ we define $\mathcal{P}_\Gamma$ as the set of all the points of $\mathbb{P}^1(\mathbb{R})$ fixed by some matrix $A \in \Gamma$ with $|\mathrm{tr}(A)| = 2$.
- For every discrete $\Gamma \leq \mathsf{SL}_2(\mathbb{R})$ we define two toplogical spaces

$$\mathfrak{h}_\Gamma^* \stackrel{\text{def}}{=} \mathfrak{h} \cup \mathcal{P}_\Gamma \quad \text{and} \quad \mathfrak{h}_\Gamma^{**} \stackrel{\text{def}}{=} \mathfrak{h} \cup \bigsqcup_{x \in \mathcal{P}_\Gamma} \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$$

  endowed with an ad-hoc topology such that the inclusions $\mathfrak{h} \hookrightarrow \mathfrak{h}_\Gamma^*$ and $\mathfrak{h} \hookrightarrow \mathfrak{h}_\Gamma^{**}$ are open embeddings.

Theorem 2

For every Fuchsian group of the first kind $\Gamma \leq \mathsf{SL}_2(\mathbb{R})$ the quotient $\Gamma \backslash \mathfrak{h}_\Gamma^*$ is a compact Riemann surface and the quotient $\Gamma \backslash \mathfrak{h}_\Gamma^{**}$ is a compact real manifold with boundary such that the inclusion maps $\Gamma \backslash \mathfrak{h} \hookrightarrow \Gamma \backslash \mathfrak{h}_\Gamma^*$ and $\Gamma \backslash \mathfrak{h} \hookrightarrow \Gamma \backslash \mathfrak{h}_\Gamma^{**}$ are open embeddings.

THE GOAL OF THE THESIS

We want to generalise Theorem 1 to the Baily-Borel and the Borel-Serre compactifications of modular curves.

In particular we want to find four topological spaces C^{BB}, X^{BB}, C^{BS} and X^{BB} with a right action of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that for every compact and open subgroup $K^{\infty} \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ we have homeomorphisms

$$C^{BB}/K^{\infty} \cong \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \qquad X^{BB}/K^{\infty} \cong \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h}^*$$

$$C^{BS}/K^{\infty} \cong \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathcal{L} \qquad X^{BS}/K^{\infty} \cong \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h}^{**}$$

where $\mathfrak{h}^* \overset{\mathrm{def}}{=} \mathfrak{h}^*_{\mathrm{SL}_2(\mathbb{Z})}$, $\mathfrak{h}^{**} \overset{\mathrm{def}}{=} \mathfrak{h}^{**}_{\mathrm{SL}_2(\mathbb{Z})}$ and $\mathcal{L} \overset{\mathrm{def}}{=} \bigsqcup_{x \in \mathbb{P}^1(\mathbb{Q})} \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$.

Definition 3

For every $\mathbb{Q}$-algebra R we define the set

$$W(R) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_{2,2}(R) : \ Ra + Rb + Rc + Rd = R \text{ and } \\ (\alpha, \beta) \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (0,0) \text{ for some } (\alpha, \beta) \in \mathbb{Q}^2_{prim}\}.$$

where $A^2_{prim} \stackrel{\text{def}}{=} \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in A^2 \mid Ax + Ay = A\}$ for every ring A. We also define $\mathcal{W}(R) \stackrel{\text{def}}{=} W(R) \times R^\times$.

Observe that it is not immediate to define a topology on R^2_{prim}. Nevertheless if we have such a topology we endow W(R) with the disjoint union topology given by the bijections

$$W(R) = \bigsqcup_{(\alpha:\ \beta) \in \mathbb{P}^1(\mathbb{Q})} \left\{\left(\begin{smallmatrix} \alpha x & \alpha y \\ \beta x & \beta y \end{smallmatrix}\right) : \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in R^2_{prim}\right\} \cong \bigsqcup_{(\alpha:\ \beta) \in \mathbb{P}^1(\mathbb{Q})} R^2_{prim}$$

which hold for every $\mathbb{Q}$-algebra R.

A DESCRIPTION OF THE CLASSICAL CUSPS USING THE FINITE ADÈLES

For every global field K and every finite set $S \subseteq \Sigma_K$ we define a topology on $(\mathbb{A}_K^S)^2_{\mathrm{prim}}$ by observing that

$$(\mathbb{A}_K^S)^2_{\mathrm{prim}} = \prod_{v \in \Sigma_K^S}{}' ((K_v)^2_{\mathrm{prim}} : (\mathcal{O}_{K_v})^2_{\mathrm{prim}}).$$

Theorem 4

For every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ we have a homeomorphism

$$\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)/K^\infty.$$

and thus we can take $C^{BB} = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)$.

Theorem 5

Let $K^\infty \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a compact and open subgroup. Then we have a homeomorphism

$$\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \times \mathbb{R}_{>0} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q}) / K^\infty \times K_\infty .$$

We see from Theorem 5 that the quotient

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q}) / K^\infty \times K_\infty$$

is not compact. Thus we turn our attention to the Borel-Serre compactification, which seems better suited for these kind of problems.

To find the right description for the space C^{BS} we observe first of all that

$$C^{BS} \cong GL_2(\mathbb{Q}) \backslash \left(\mathcal{L}^{\pm} \times GL_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \right)$$

where we define $X^{\pm} \overset{\text{def}}{=} X \times \{\pm 1\}$ for every topological space X.

Definition 6

For every $(x_0 : x_1) \in \mathbb{P}^1(\mathbb{Q})$ we define

$$B_{(x_0 \,:\, x_1)} \overset{\text{def}}{=} \{M \in \mathcal{M}_{2,2}(\mathbb{R}) \mid \det(M) = 0,\ (-x_1, x_0) \cdot M \neq (0,0)\}$$

and we define $\mathcal{B}(\mathbb{R}) \overset{\text{def}}{=} \bigsqcup_{x \in \mathbb{P}^1(\mathbb{Q})} B_x$.

Theorem 7

We have a homeomorphism

$$\mathcal{B}(\mathbb{R})^{\pm}/K_{\infty} \xrightarrow{\sim} \mathcal{L}^{\pm} \qquad [(\varepsilon, x, M)] \mapsto [(x, \varepsilon, \varphi_x(M))] \tag{$\diamond$}$$

where $\varphi_{\infty} \colon B_{\infty} \to \mathbb{P}^1(\mathbb{R}) \setminus \{\infty\}$ is defined as

$$\varphi_{\infty}\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) \stackrel{\text{def}}{=} \frac{ac+bd}{c^2+d^2} = \Re\left(\frac{ai+b}{ci+d}\right).$$

and for every $x \in \mathbb{Q}$ we define $\varphi_x \colon B_x \to \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$ as

$$\varphi_x(M) \stackrel{\text{def}}{=} \left(\begin{smallmatrix} x & -1 \\ 1 & 0 \end{smallmatrix}\right) * \varphi_{\infty}(\left(\begin{smallmatrix} 0 & 1 \\ -1 & x \end{smallmatrix}\right) \cdot M)$$

.

We have found a space $\mathcal{C}(\mathbb{A}_{\mathbb{Q}}) \stackrel{\text{def}}{=} \mathcal{B}(\mathbb{R})^{\pm} \times GL_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that

$$C^{BS} \cong GL_2(\mathbb{Q})\backslash\mathcal{C}(\mathbb{A}_{\mathbb{Q}})/K_{\infty}.$$

We observe first of all that $X^{BS} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \left((\mathfrak{h}^{**})^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \right)$.

Definition 8

For every $(x_0 : x_1) \in \mathbb{P}^1(\mathbb{Q})$ we define

$$U_{(x_0 \,:\, x_1)} \overset{\text{def}}{=} \{ M \in \mathcal{M}_{2,2}(\mathbb{R}) \mid \mathbf{det}(M) \geq 0,\ (-x_1, x_0) \cdot M \neq (0,0) \}$$

and we set $\mathcal{G}(\mathbb{R}) \overset{\text{def}}{=} \left(\bigsqcup_{x \in \mathbb{P}^1(\mathbb{Q})} U_x \right) / \sim$ where $(x, A) \sim (y, B)$ if and only if $A, B \in \mathrm{GL}_2^+(\mathbb{R})$ and $A = B$.

For every $x \in \mathbb{P}^1(\mathbb{Q})$ we have $U_x = \mathrm{GL}_2^+(\mathbb{R}) \sqcup B_x$ as sets but not as topological spaces, and we have two continuous inclusions $\mathrm{GL}_2^+(\mathbb{R}) \hookrightarrow \mathcal{G}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R}) \hookrightarrow \mathcal{G}(\mathbb{R})$.

Theorem 9

The two maps

$$\mathrm{GL}_2^+(\mathbb{R}) \to \mathfrak{h} \qquad \mathcal{B}(\mathbb{R}) \to \mathcal{L}$$

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{ac+bd}{c^2+d^2} + \frac{c^2+d^2}{ad-bc} \cdot i \qquad (x, M) \mapsto (x, \varphi_x(M))$$

induce a homeomorphism $\mathcal{G}(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathfrak{h}^{**}$.

We have found a space $\mathcal{Z}(\mathbb{A}_\mathbb{Q}) \stackrel{\text{def}}{=} \mathcal{G}(\mathbb{R})^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ such that

$$X^{BS} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{Z}(\mathbb{A}_\mathbb{Q})/K_\infty.$$

We have found three spaces $\mathcal{W}(\mathbb{A}_{\mathbb{Q}}^{\infty}), \mathcal{C}(\mathbb{A}_{\mathbb{Q}})$ and $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ and we have described four homeomorphisms

$$Y = \varprojlim_{K^\infty} \left(\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h} \right) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty$$

$$C^{BB} = \varprojlim_{K^\infty} \left(\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \right) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_{\mathbb{Q}}^{\infty})$$

$$C^{BS} = \varprojlim_{K^\infty} \left(\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathcal{L} \right) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{C}(\mathbb{A}_{\mathbb{Q}})/K_\infty$$

$$X^{BS} = \varprojlim_{K^\infty} \left(\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h}^{**} \right) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{Z}(\mathbb{A}_{\mathbb{Q}})/K_\infty$$

such that $Y \hookrightarrow X^{BS}$ is an open embedding, $C^{BS} \twoheadrightarrow C^{BB}$ is a continuous surjection and $C^{BS} \hookrightarrow X^{BS}$ is a closed embedding.

- Make the definition of the spaces $\mathcal{C}(\mathbb{A}_{\mathbb{Q}})$ and $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ more symmetric in the archimedean and non-archimedean parts of $\mathbb{A}_{\mathbb{Q}}$;
- Is the fact that $\mathcal{B}(\mathbb{R})$ is a disjoint union over $\mathbb{P}^1(\mathbb{Q})$ an evidence of the global nature of the compactification of modular curves?
- Understand adelic automorphic forms on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ and relate them to the theory of adelic automorphic forms on $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

THANK YOU FOR YOUR ATTENTION!

Twenty years from now you will be more disappointed by the things you didn't do than by the ones you did do. So throw off the bowlines. Sail away from the safe harbor. Catch the trade winds in your sails. Explore. Dream. Discover.

H. Jackson Brown, P.S. I Love You