# A very brief outline of the first steps of the PhD project

**Abstract**

A brief summary of what we discussed with Ian on the 13th and 14th of July, 2017.

First of all, the background studies needed. This means learning more about *modular forms* first of all, by studying for example on Diamond and Shurman's book [3]. Then maybe something more computational about modular forms will be needed, and thus it would be useful to look at Stein's book [4]. It might also be necessary for me to learn some K-theory in the future.

Then, the initial program will be to try to generalise Brunault's article [2] to (hopefully) all the modular curves $X_0(n)$ and $X_1(n)$. The article is still pretty general in its setting, but we will probably encounter two major obstacles:

- How to choose the appropriate equation for the modular curves $X_0(n)$ and $X_1(n)$ and how does the Mahler measure change if we change the equation without changing the curve?
- How to calculate explicitly the modular units in $\mathcal{O}^\times(Y_0(n)(\mathbb{C})) \otimes \mathbb{C}$ and $\mathcal{O}^\times(Y_1(n)(\mathbb{C})) \otimes \mathbb{C}$ as it is done in §5 of [2]?

Another interesting problem will be to understand whether we will get formulas as simple as the ones outlined in Boyd's article [1]. From that article it seems that from every polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$ which satisfies some suitable conditions we should be able to find another "object" $X$ such that $m(P) = r \cdot L'(X, s_0)$ where $r \in \mathbb{Q}^\times$, $m(P)$ is the Mahler measure of the polynomial $P$ defined as

$$m(P) \overset{\text{def}}{=} \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}\right) \right| dt_1 \cdots dt_n$$

and $L'(X, s_0)$ is a value of the derivative of the L-function associated to $X$. The first examples of these identities involve polynomials $P(x, y) \in \mathbb{Z}[x, y]$ such that the projective closure of the variety defined by $P(x, y) = 0$ has genus 0, and in this case it seems that the right objects to which we should attach an L-function are Dirichlet characters. Then going to genus 1 the polynomial $P(x, y)$ defines an elliptic curve $E$ which could be the right object $X_P$ to associate to $P$. Finally for the modular curve $X_1(13)$ considered in [2] it seems that the right objects to consider are modular forms.

In particular in the last part of Brunault's article he shows in equation (17) that numerical computations suggest that the Mahler measure of the polynomial

$$P_{25}(x, y) = y^2x^4 + (y^3 + y^2)x^3 + (3y^3 - y^2 - 2y)x^2 + (y^4 - 4y^2 + y - 1)x - y^3$$

which is one of the polynomials that define the modular curve $X_1(25)$ should be related to a linear combination of special values of L-functions attached to a modular form and to two Dirichlet characters. This indeed could be one of the reasons why we shouldn't expect in general formulas as neat as the ones that Boyd outlines.

Finally it seems that at the beginning of the project we will need to do some numerical experiments to verify our conjectures and to try to formulate new ones. In particular we could start by trying to see if there is any relation between the Mahler measures of different Weierstrass equations of the same elliptic curve. If the conjectures of Boyd would be true we would expect to get Mahler measures which are rational multiples of each other, but this is not clear at the first sight. Then we could try to go to curves of higher genus by studying for instance $X_1(17)$ which has genus three. We would like for instance to make more numerical experiments to replicate the ones done for $X_1(13)$ in [2] for this new modular curve.

## References


[1] David W. Boyd. Mahler's measure and special values of $L$-functions. *Experiment. Math.*, 7(1):37–82, 1998. ISSN 1058-6458. URL `http://projecteuclid.org/euclid.em/1047674271`.

[2] F. Brunault. On the Mahler measure associated to $X\_1(13)$. *ArXiv e-prints*, March 2015.

[3] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. ISBN 0-387-23229-X.

[4] William Stein. *Modular forms, a computational approach*, volume 79 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2007. ISBN 978-0-8218-3960-7; 0-8218-3960-8. doi: 10.1090/gsm/079. URL `http://dx.doi.org/10.1090/gsm/079`. With an appendix by Paul E. Gunnells.