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The Perturbed Laplacian Matrix of a Graph

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For a graph G, we define its perturbed Laplacian matrix as $D - A(G)$ where $A(G)$ is the adjacency matrix of G and D is an arbitrary diagonal matrix. Both the Laplacian matrix and the negative of the adjacency matrix are special instances of the perturbed Laplacian. Several well-known results, contained in the classical work of Fiedler and in more recent contributions of other authors are shown to be true, with suitable modifications, for the perturbed Laplacian. An appropriate generalization of the monotonicity property of a Fiedler vector for a tree is obtained. Some of the results are applied to interval graphs.

Keywords: Laplacian matrix; Algebraic connectivity; Characteristic set; Interval graphs; Perturbed Laplacian matrix; Fiedler vector; Perron component

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1. INTRODUCTION

Let G be a connected weighted graph – *i.e.*, a graph with vertex set $V = \{1, 2, \ldots, n\}$ and edge set E such that each edge is associated with a positive number, the *weight* of the edge. (In the case that all of the weights are equal to 1, we refer to G as an *unweighted* graph.) For a weighted graph G, the corresponding adjacency matrix $A(G)$ is given by $A(G)=(a_{ij})$, where

$$a_{ij}=\begin{cases}\theta, & \text{if } (i,j)\in E \text{ and the weight of the edge is } \theta,\\ 0, & \text{otherwise.}\end{cases}$$

Given a real diagonal matrix D, we define the *perturbed Laplacian matrix of* G, denoted by $\mathfrak{L}(G)$, to be given by $\mathfrak{L}(G)=D-A(G)$. (In the sequel we will use $\mathfrak{L}$ instead of $\mathfrak{L}(G)$ when G is clear from the context.) Observe that if $d_{ii}=\sum_{j\neq i} a_{ij}$ for each $1\leq i\leq n$, then $\mathfrak{L}(G)$ coincides with the usual Laplacian matrix for G, while when D is 0, $\mathfrak{L}(G)$ is just $-A(G)$.

For a weighted graph G, the eigenvalues of both the Laplacian matrix and the adjacency matrix have been studied extensively (see [12] and [13], for example). Given that $\mathfrak{L}(G)$ encompasses both the Laplacian and adjacency matrices, it is natural to consider the spectral properties of that matrix as well.

Our results in the present paper deal primarily with the second smallest eigenvalue of $\mathfrak{L}(G)$ and the corresponding eigenspace, and thus serve to show how the corresponding results for both Laplacian and adjacency matrices fit into a more general framework. It follows readily from Perron–Frobenius theory that since G is connected, the smallest eigenvalue of $\mathfrak{L}(G)$ is simple and has a corresponding eigenvector with all entries positive (we will consistently use Z to denote that positive eigenvector). In keeping with the extensive literature on Laplacian matrices, we will refer to the second smallest eigenvalue of $\mathfrak{L}(G)$ as its *algebraic connectivity* and to the corresponding eigenvectors as *Fiedler vectors*. In this paper, we focus on the algebraic connectivity of $\mathfrak{L}(G)$ and on the structure of the corresponding Fiedler vectors, and provide a number of generalizations of known results for (ordinary) Laplacian matrices. We then apply these results to discuss a problem on interval graphs.

Throughout the paper, we will occasionally appeal to standard ideas and results from the theory of matrices and the theory of graphs. We refer the reader to [9] for the fundamentals of the former and to [3] for the basics of the latter.

2. PRELIMINARY RESULTS

We begin with some notation. Given a connected weighted graph G and a diagonal matrix D, we let μ denote the second smallest eigenvalue of $\mathfrak{L}(G)$ (G and D will be clear from the context). For a vector X, $X(v)$, will denote the coordinate of X corresponding to the vertex v. An edge between two vertices v and w in G is denoted by $[v, w]$. Let Y be a Fiedler vector of $\mathfrak{L}$. Then the *eigencondition* at a vertex v is the equation

$$\sum_{[i,v]\in E} \mathfrak{L}(v,i)Y(i) = [\mu - \mathfrak{L}(v,v)]Y(v),$$

where E is the edge set of G. Finally, given a symmetric matrix B, we denote its smallest eigenvalue by $\tau(B)$, and its largest eigenvalue by $\lambda(B)$; in the event that B is entrywise nonnegative, we denote its Perron value by $\rho(B)$.

Let Y be a Fiedler vector of $\mathfrak{L}$. A vertex v of G is called a *characteristic vertex* of G if $Y(v) = 0$ and if there is a vertex w, adjacent to v, such that $Y(w) \neq 0$. An edge $e = [u, w]$ is called a *characteristic edge* of G if $Y(u)Y(w) < 0$. By $C(G, D, Y)$ we denote the *characteristic set* of G which is defined as the collection of all characteristic vertices and characteristic edges of G with respect to the Fiedler vector Y of $\mathfrak{L}$. Observe that this notation emphasizes the fact that the characteristic set depends on:

(i) The graph structure. For example, let $D = 0$, G be the unweighted path on 4 vertices and H be the unweighted star on 4 vertices. Let Y_G be a Fiedler vector of $\mathfrak{L}(G)$ and Y_H be a Fiedler vector of $\mathfrak{L}(H)$. It is not hard to show that

$$C(G, D, Y_G) \neq C(H, D, Y_H).$$

(ii) The matrix D. To see this let G be the unweighted path on 4 vertices, First let D be the degree matrix $diag(1, 2, 2, 1)$ so that $\mathfrak{L}$ is the ordinary Laplacian matrix. Let Y be the Fiedler vector. It can be shown that $C(G, D, Y)$ contains the middle edge.

Now, let $D = diag(1, 2, 3, 4)$ and let Y be a Fiedler vector of $\mathfrak{L}$. It can be shown that $C(G, D, Y)$ does not contain the middle edge.

(iii) The Fiedler vector Y (if the multiplicity of μ is more than one). To see this take the unweighted complete graph on 4 vertices. Let D be the degree matrix. So $\mathfrak{L}$ is the ordinary Laplacian matrix.

One can check that $Y_1 = [1, 1, -1, -1]$ and $Y_2 = [0, 1, -1, 0]$ are two Fiedler vectors of $\mathfrak{L}$. Notice that $\mathcal{C}(G, D, Y_1)$ has 4 elements and $\mathcal{C}(G, D, Y_2)$ has 3 elements.

When D is the diagonal degree matrix, so that $\mathfrak{L}$ is the ordinary Laplacian matrix, we denote the characteristic set by $\mathcal{C}(G, Y)$; this notation follows that in [2].

Note 1 Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathfrak{L}$. If v is a characteristic vertex of G then the eigencondition for Y at v implies that there are at least two vertices u, w in G, adjacent to v such that $Y(u) > 0$ and $Y(w) < 0$.

The following is well known in the case of Laplacian and adjacency matrices (see, for example, [5, 12]).

LEMMA 1 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathfrak{L}$. Then the subgraph induced by the vertices v in G for which $Y(v) \geq 0$ is connected. Similarly the subgraph induced by the vertices v in G for which $Y(v) \leq 0$ is connected.*

Proof Let $L = \mathfrak{L} - \tau(\mathfrak{L})I$ and $\mu' = \mu - \tau(\mathfrak{L})$, where I is the identity matrix. Without loss of generality, suppose that the subgraph induced by the set of vertices v for which $Y(v) \geq 0$ is not connected. By performing a permutation similarity transformation if necessary, we get

$$LY = \mu' Y = \mu' \begin{bmatrix} Y_+ \\ Y_- \end{bmatrix}, \tag{1}$$

where Y_+ and Y_- are the subvectors of Y containing all the nonnegative and negative entries, respectively. The matrix L can be partitioned as

$$L = \begin{bmatrix} L_{11} & 0 & L_{13} \\ 0 & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix},$$

where $\begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix}$ corresponds to Y_+ and L_{33} corresponds to Y_-. Partition Y conformally as $Y = [Y_+^{1T} Y_+^{2T} Y_-{}^T]^T$.

From Eq. (1) we have $L_{11}Y_+^1 + L_{13}Y_- = \mu' Y_+^1$, so that $(L_{11} - \mu' I)Y_+^1 = -L_{13}Y_-$. Note that each entry of L_{13} is nonpositive and each entry of Y_- is negative. Further, since G is connected, at least one entry of L_{13} is negative, so that $-L_{13}Y_- \neq 0$. In particular, $Y_+^1 \neq 0$, nor is Y_+^1 a μ' eigenvector for L_{11}. We find that

$$(Y_+^1)^T(L_{11} - \mu' I)Y_+^1 = -(Y_+^1)^T L_{13} Y_- \leq 0, \tag{2}$$

so at least one eigenvalue of $L_{11} - \mu' I$ is nonpositive. Further, if every eigenvalue of $L_{11} - \mu' I$ is nonnegative, then in fact $\mu' = \tau(L_{11})$, with Y_+^1 as a μ' eigenvector for L_{11}, a contradiction. We conclude that $L_{11} - \mu' I$ has a negative eigenvalue, and similarly that $L_{22} - \mu' I$ also has a negative eigenvalue. By the Cauchy interlacing theorem we find that at least two eigenvalues of $L - \mu' I$ are negative. Since μ' is the second smallest eigenvalue of L, $\tau(L)$ must have multiplicity two, a contradiction. ■

Remark In the above lemma, keeping the graph unchanged, suppose we replace the diagonal matrix D by another diagonal matrix D_1 and let Y_1 be the new Fiedler vector. Then the subgraph of G induced by the vertices $\{v : Y_1(v) \geq 0\}$, may be a different connected subgraph than the subgraph induced by $\{v : Y(v) \geq 0\}$. For example, consider the unweighted path $P = [1, 2, 3, 4]$ on 4 vertices. When D is the degree matrix $diag(1, 2, 2, 1)$, the connected components of Lemma 1 are induced by $\{1, 2\}$ and $\{3, 4\}$. When $D_1 = diag(1, 2, 3, 4)$, those connected components are induced by $\{1\}$ and $\{2, 3, 4\}$.

The following is a generalization of Lemma 6 of [2].

LEMMA 2 *Let G be a connected graph and D be any diagonal matrix. Consider $\mathfrak{L}$ and μ. Let W be a set of vertices of G such that $G - W$ is disconnected. Let G_1, G_2 be two components of $G - W$ and L_1, L_2 be the principal submatrices of $\mathfrak{L}$ corresponding to G_1, G_2 respectively. Suppose $\tau(L_1) \leq \tau(L_2)$. Then either $\tau(L_2) > \mu$ or $\tau(L_1) = \tau(L_2) = \mu$. In particular, we always have $\tau(L_2) \geq \mu$.*

Proof It suffices to prove that if $\tau(L_2) \leq \mu$ then $\tau(L_1) = \tau(L_2) = \mu$, so we assume that $\tau(L_2) \leq \mu$. For $i = 1, 2$, let U_i be a positive eigenvector of L_i corresponding to $\tau(L_i)$. After a permutation similarity operation we have

$$\mathcal{L} = \left[\begin{array}{cccc|c} L_1 & 0 & \dots & 0 & \\ 0 & L_2 & \dots & 0 & \\ & & & & B \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & L_k & \\ \hline & & B^T & & C \end{array}\right],$$

where each L_i, $i=1,\dots,k$ corresponds to one of the connected components of $G-W$. Let V be a vector of the form $V=[U_1^T\ -xU_2^T\quad 0^T\cdots 0^T]^T$, where x is chosen so that $Z^TV=0$. Then $V^TLV=\tau(L_1)U_1^TU_1+\tau(L_2)x^2U_2^TU_2\le\mu V^TV$, with strict inequality if and only if either $\tau(L_1)<\mu$ or $\tau(L_2)<\mu$. We thus conclude that $\tau(L_1)=\tau(L_2)=\mu$, as desired.

The following is an interesting application of Lemma 2. ■

LEMMA 3 *Let G be a connected graph and D be any diagonal matrix. Let Y be a Fiedler vector of $\mathcal{L}$. Let W be a nonempty set of vertices of G such that $Y(u)=0$, for all $u\in W$ and suppose $G-W$ is disconnected with $t\ge 2$ components $G_1,G_2,\dots,G_t$ such that $Y(G_i)\ne 0$, $i=1,\dots,t$. Let L_i and Y_i be the principal submatrix of $\mathcal{L}$ and the subvector of Y corresponding to G_i, $i=1,2,\dots,t$. Then each Y_i, $i\in\{1,\dots,t\}$ is either all positive, or all negative, with $\tau(L_i)=\mu$ in either case. In particular, $C(G,D,Y)\subset W$.*

Proof Note that for $1\le i\le t$, L_i is irreducible. From the facts that $\mathcal{L}Y=\mu Y$ and $Y(u)=0$, for all $u\in W$, it follows that $L_iY_i=\mu Y_i$ for each $1\le i\le t$. Note that if $\tau(L_i)>\mu$, then necessarily $Y_i=0$. By hypothesis, Y_1 is nonzero, as is Y_i for each $2\le i\le t$. Hence $\tau(L_1)$, $\tau(L_i)\le\mu$, for each such i, and we deduce from Lemma 2 that $\tau(L_1)=\tau(L_i)=\mu$. In particular, for each $1\le i\le t$, Y_i is either all positive or all negative. ■

It is proved in Lemma 1 that when Y is a Fiedler vector of a connected graph G then the vertices v such that $Y(v)\ge 0$ induce a connected subgraph. In particular when Y contains no zero entry we find that the vertices v such that $Y(v)>0$ induce a connected subgraph. It is natural to wonder whether there are other conditions which imply the existence of such a subgraph. The following lemma answers that question in the affirmative.

LEMMA 4 *Let G be connected. Let Y be a Fiedler vector of $\mathfrak{L}$. Suppose that $C(G,D,Y)$ contains an edge $[u,w]$. Then the vertices v such that $Y(v)>0$ induce a connected subgraph.*

Proof The result is immediate from Lemma 1 if Y has no zero entries, so we assume henceforth that the set $W=\{v|Y(v)=0\}$ is nonempty. If we have two components C_1, C_2 of $G-W$ such that $Y(C_1)$, $Y(C_2)\neq 0$, then by Lemma 3, $C(G,D,Y)\subset W$ and thus $C(G,D,Y)$ cannot have an edge, contrary to the hypothesis. Thus $G-W$ has exactly one component C such that $Y(C)\neq 0$. Let L_1 be the principal submatrix of $\mathfrak{L}$ corresponding to C. Clearly $L_1Y(C)=\mu Y(C)$. Since we know that the eigenvector corresponding to $\tau(L_1)$ is positive, it follows that the second smallest eigenvalue of L_1 is at most μ. On the other hand since L_1 is a principal submatrix of $\mathfrak{L}$, μ is at most the second smallest eigenvalue of L_1. Thus we see that $Y(C)$ is a Fiedler vector for L_1 and $Y(C)$ does not contain any zero entry. Applying Lemma 1, yields the result. ■

The following result discusses the characteristic set of a graph in relation to the structure of its blocks and cutpoints.

LEMMA 5 *Let G be a connected graph and D be any diagonal matrix. Consider a Fiedler vector Y of $\mathfrak{L}$ and let $S=C(G,D,Y)$. Then*

(i) *Any two characteristic elements lie on a simple cycle which contains no other characteristic elements and*
(ii) *Either S is a single vertex or S is contained in a block of G.*

Proof To prove (i), first suppose that S contains only vertices and let $v_1, v_2\in S$. Delete all characteristic vertices from G except v_1. By Lemma 3, in the resulting graph there is only one component, say H, such that $Y(H)\neq 0$. Let u and w be vertices adjacent to v_2 such that $Y(u)>0$ and $Y(w)<0$. Since both u, w are in H, there is a path, say P, joining them in H. Since G has no characteristic edge, at least one vertex on P has to be a zero vertex. Thus P contains a characteristic vertex. Since all characteristic vertices except v_1 have been deleted, v_1 is the only characteristic vertex on P. Note that the edges $[w,v_2]$ and $[v_2,u]$ along with the path P form a simple cycle with just two characteristic elements, v_1 and v_2.

In the case that S contains at least one edge, by Lemma 4, the vertices v such that $Y(v) < 0$ induce a connected subgraph G_- while the vertices u such that $Y(u) > 0$ induce a connected subgraph G_+. Now since any characteristic vertex is adjacent to a vertex in G_- and a vertex in G_+, and any characteristic edge is incident with a vertex in G_- and a vertex in G_+, the desired cycle is readily constructed.

Now we prove item (ii). Suppose that S is not a single vertex. If $S = \{e\}$ for some edge e, then trivially S is contained in a block of G. If S has at least two elements, then from item (i), we see that for any two elements in S, there exists a simple cycle in G containing both of them. Thus a block which contains one element of S must contain all of S. ∎

From Lemma 5 we see that if Y_1 and Y_2 are Fiedler vectors of $\mathfrak{L}$, then each of $\mathcal{C}(G,D,Y_1)$ and $\mathcal{C}(G,D,Y_2)$ is either a single vertex or is contained in a single block. In the next section we will show that in fact the vertex or block identified by Y_1 coincides with that identified by Y_2.

3. FIEDLER VECTORS

Let G be a connected graph, D be a diagonal matrix, and μ be the algebraic connectivity of $\mathfrak{L}$. Suppose that v is a cutpoint of G, with components $G_1, \ldots, G_k$ as the connected components of $G - v$. For $i = 1, \ldots, k$, let L_i be the principal submatrix of $\mathfrak{L}$ corresponding to G_i. A component G_j is called a *Perron component at* v if $\tau(L_j)$ is the minimum among all $\tau(L_i)$. We remark that for the ordinary Laplacian matrix, this notion coincides with that of [10]; in that paper, it is observed that each L_i^{-1} is an entrywise positive matrix, and G_j is said to be a Perron component at v if $\rho(L_j^{-1})$ is maximum among all $\rho(L_i^{-1})$.

The following result gives a connection between Perron components and algebraic connectivity.

LEMMA 6 *Let G be a connected graph and let μ be the algebraic connectivity of $\mathfrak{L}$. Suppose that v is a cutpoint, with components $G_1, \ldots, G_k$ at v. Then G_i is a Perron component at v if and only if $\tau(L_i) \leq \mu$, where L_i is the principal submatrix of $\mathfrak{L}$ corresponding to G_i.*

Proof If G_i is a Perron component then by the Cauchy interlacing theorem $\tau(L_i) \le \mu$. Conversely, if G_i is a component with $\tau(L_i) \le \mu$, and if G_j is another component at v, it follows from Lemma 2 that $\tau(L_j) \ge \mu$. Thus $\tau(L_i)$ is the smallest among all $\tau(L_j)$, which means that G_i is a Perron component. ■

The following result is one of the important ones and has many uses.

THEOREM 7 *Let v be a cut point of the connected graph G. The following are equivalent*:

(i) *For some Fiedler vector Y of $\mathfrak{L}$, $C(G, D, Y) = \{v\}$.*
(ii) *There is a component C_1 at v with corresponding principal submatrix L_1 of $\mathfrak{L}$, such that $\tau(L_1) = \mu$.*
(iii) *For every Fiedler vector X, $C(G, D, X) = \{v\}$.*
(iv) *There are two or more Perron components at v.*

Proof Suppose that (i) holds and let $L_1, L_2, \ldots, L_k$ be the principal submatrices and $Y_1, Y_2, \ldots, Y_k$ be the subvectors of Y corresponding to the connected components $G_1, G_2, \ldots, G_k$ at v. Observe that since no G_i contains a characteristic element, each Y_i is either all positive, all negative, or 0. Since $Z^T Y = 0$, at least two Y_i's must be nonzero, say Y_1 and Y_2. Now $Y(v) = 0$, so we find that $L_i Y_i = \mu Y_i$, $\forall i$. Thus $\tau(L_1)$, $\tau(L_2) \le \mu$, and so by Lemma 3 we have $\tau(L_1) = \tau(L_2) = \mu$, so (ii) holds.

Now suppose (ii) holds and that X is a Fiedler vector. Let L_1 be the principal submatrix of $\mathfrak{L}$ and X_1 be the subvector of X corresponding to C_1. Then $\mu = \tau(L_1)$; let W_1 be a positive vector such that $L_1 W_1 = \mu W_1$. We also have $L_1 X_1 - L(C_1, v) X(v) = \mu X_1$, where $L(C_1, v)$ is the part of the vth column of $\mathfrak{L}$ corresponding to C_1 and $X(v)$ is the vth entry of X. Notice that all the entries of $L(C_1, v)$ are nonpositive. It follows by multiplying the vector W_1^T from left that $X(v) = 0$, and hence that X_1 is a scalar multiple of W_1. Since $Z^T X$ must be 0, there is another component C' such that $X(C') \ne 0$. Thus, applying Lemma 3 we see that for each component G_i at v, $X(G_i)$ is either all positive all negative, or all zero, and it now follows that (iii) holds.

Suppose that (iii) holds and let X be a Fiedler vector. As in the proof that (i) implies (ii), we find that there are at least two Perron components at v, so that (iv) holds.

Finally, if (iv) holds, then there are at least two Perron components at v, with corresponding principal submatrices of $\hat{L}$ given by L_1 and L_2, say. Applying Lemmas 6 and 2 we find that $\tau(L_1)=\tau(L_2)=\mu$. Letting W_1 and W_2 be corresponding positive eigenvectors, respectively, we readily construct a Fiedler vector Y which can be reordered to have the form $[W_1^T - xW_2^T \quad 0^T \cdots 0^T]^T$, where x is chosen so that $Z^T Y=0$. Observe now that the characteristic set for Y is v, so that (i) holds. ■

Corollary 8 *Let G be connected and suppose that there are $t \geq 2$ Perron components at a vertex v. Then the multiplicity of the algebraic connectivity is exactly $t-1$.*

Proof Let $G_1, \ldots, G_t$ be the Perron components at v, let $L_1, \ldots, L_t$ be the corresponding principal submatrices of $\hat{L}$, and let $W_1, \ldots, W_t$ be the corresponding positive eigenvectors. Consider the vectors Y_i, $i=2, \ldots, t$, where

$$Y_i(x) = \begin{cases} \gamma_i W_1(x) & \text{if } x \in G_1 \\ -W_i(x) & \text{if } x \in G_i \\ 0 & \text{else} \end{cases},$$

where γ_i is chosen so that $Z^T Y_i=0$, $i=2, \ldots, t$. It is readily verified that $Y_1, \ldots, Y_t$ is a linearly independent set of Fiedler vectors for $\hat{L}$. It remains only to show that every Fiedler vector is a linear combination of Y_i's. Let X be a Fiedler vector; by Theorem 7, $C(G,D,X)=\{v\}$. Thus for any component C at v with corresponding principal submatrix L_C of $\hat{L}$, we have $L_C X(C)=\mu X(C)$. If C is not a Perron component at v, then $\tau(L_C) > \mu$, so that necessarily $X(C)=0$, while if C is a Perron component at v, say G_i, then $\tau(L_C)=\mu$ and $X(C)$ is a scalar multiple of W_i. Applying the fact that $Z^T X=0$, it now follows that X is a linear combination of Y_i's. ■

We now turn to the case that for some Fiedler vector Y, $C(G,D,Y)$ is not a single vertex.

Theorem 9 *Let G be connected and Y, X be Fiedler vectors of $\hat{L}$. Suppose that $C(G,D,Y)$ is contained in a block B. Then $C(G,D,X)$ is also contained in the same block B.*

Proof If G itself is only one block then we have nothing to prove. Let B_y, B_x be the blocks containing the characteristic sets $C(G,D,Y)$, $C(G,D,X)$ respectively. Suppose that v is a cutpoint such that $G-v$ has two different components G_y and G_x containing at least one vertex of B_y and B_x, respectively.

Note that there is only one Perron component at v, for if there were two or more Perron components at v then by Theorem 7 $C(G,D,X) = \{v\}$, for every Fiedler vector X, contrary to the assumption. Similarly, for any component C at v with corresponding principal submatrix L_1 of $\mathfrak{L}$, $\tau(L_1) \neq \mu$.

Select a and b so that the vector $U=aX-bY$ has a zero in the position corresponding to v. If there are at least two components C_1, C_2 such that $U(C_1)$ and $U(C_2)$ are nonzero, then by Lemmas 3 and 6, C_1 and C_2 are Perron components at v, a contradiction. Thus there is just one component C at v such that $U(C)$ is nonzero. As a result, either $aX(G_x)=bY(G_x)$ or $aX(G_y)=bY(G_y)$; in the former case we see that a must be zero and that $Y(G_x)=0$, while in the latter case, we have $b=0$ and $X(G_y)=0$. So suppose without loss of generality that $X(G_y)=0$. Then for a sufficiently small $\varepsilon>0$ we see that the Fiedler vector $X+\varepsilon Y$ has characteristic elements in both G_x and G_y, contradicting Lemma 5. ■

The following corollary is a generalization of Corollary 2.5 of [6].

Corollary 10 *Let G be connected and Y be a Fiedler vector of $\mathfrak{L}$. Suppose that $C(G,D,Y)=\{[u,v]\}$. Then μ is a simple eigenvalue of $\mathfrak{L}$, so that in particular for any Fiedler vector X, $C(G,D,X)=\{[u,v]\}$.*

Proof We claim that $[u,v]$ is not on a cycle. To see the claim, note that if $[u,v]$ is on a cycle, then $G-[u,v]$ is connected and thus there is a path joining u and v. This path necessarily contains another characteristic element, contradicting the fact that $C(G,D,Y)$ consists of a single edge. Thus the claim holds. By Theorem 9, it follows that for any Fiedler vector X, $C(G,D,X)$ is contained in the block $[u,v]$, and by Theorem 7 $C(G,D,X)$ cannot be a vertex. Thus $C(G,D,X) = \{[u,v]\}$. In particular, if μ is not a simple eigenvalue, then there is a Fiedler vector X such that $X(u)=0$, a contradiction. Hence μ must be simple. ■

Remark In view of Theorems 7 and 9, a connected graph G either has a particular vertex as a characteristic vertex for every Fiedler vector or it has a particular block which contains the characteristic elements of G for every Fiedler vector.

Let G be connected, Y be a Fiedler vector and suppose that $Y(u) > 0$, where u is a cut point of G. Let $v(C(G, D, Y))$ denote the set consisting of the characteristic vertices and end points of characteristic edges of $C(G, D, Y)$. We claim that there is exactly one component at u containing a vertex in $v(C(G, D, Y))$. To see the claim, note that if $C(G, D, Y)$ is a singleton vertex w, it cannot be u since $Y(u) > 0$, so that w belongs to exactly one component at u. If G has a single characteristic block, say C, suppose C_1 and C_2 are two components at u, each of which has a negatively valuated vertex. Thus both the graph induced by $C_1 \bigcup \{u\}$ and the graph induced by $C_2 \bigcup \{u\}$ must contain characteristic elements, say s_1, s_2. It follows from Lemma 5 that s_1, s_2 must lie on a cycle, a contradiction. Thus at u there is exactly one component, say, C which has a negatively valuated vertex. Further, the other components at u are necessarily positive (otherwise, there would be a characteristic element outside of C). Thus in each component at u which is distinct from C, the vertices are neither characteristic vertices nor adjacent to any negative vertex, so no such component can contain a vertex from $v(C(G, D, Y))$.

A similar argument shows that if G is connected, Y is a Fiedler vector and $Y(u) = 0$ but $u \notin C(G, D, Y)$, then there is exactly one component at u which contains vertices from $v(C(G, D, Y))$.

We summarize these observations in the following.

LEMMA 11 *Let G be connected, Y be a Fiedler vector and suppose that u is a cut point of G and $u \notin C(G, D, Y)$. Then there is exactly one component C at u which contains vertices from $v(C(G, D, Y))$.*

We note that when u is a cut point which is not a characteristic vertex for some Fiedler vector, it is not a characteristic vertex for any Fiedler vector. And since $C(G, D, Y)$ is either always a unique singleton vertex or lies in a unique block independent of the choice of the Fiedler vector, it follows that the component at u in the above lemma, which contains some vertices from $v(C(G, D, Y))$, remains unchanged for any choice of the Fiedler vector Y. Also, we know from Theorem 7 that there is exactly one Perron component at u. Theorem 12 asserts that these two components coincide.

Henceforth we assume that the smallest eigenvalue of $\mathfrak{L}$ is 0. Observe that this amounts to considering $\mathfrak{L}-\tau(\mathfrak{L})I$ instead of $\mathfrak{L}$; in particular, both matrices have the same Fiedler vectors, and the same Perron components at each vertex of G. For a connected component C at a vertex v of G, the *bottleneck matrix of C* is the inverse of the principal submatrix of $\mathfrak{L}$ corresponding to C; observe that since $\mathfrak{L}$ is an irreducible singular M-matrix, the bottleneck matrix of C is an entry-wise positive matrix.

Theorem 12 *Let Y be a Fiedler vector and let C_0 be the unique component at v containing some entries of $v(C(G,D,Y))$. Let $C_1=G-C_0$. Partition $\mathfrak{L}$ accordingly as $\left[\begin{array}{c|c} L_1 & -e_v\theta^T \\ \hline -\theta e_v & L_0 \end{array}\right]$ and Y as $\left[\begin{array}{c} Y_1 \\ \hline Y_0 \end{array}\right]$. Let $A_1,\ldots,A_k$ be the bottleneck matrices for the components at v without C_0. Then there exists a unique $\gamma>0$ such that*

$$\rho\left(\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & 0 & 0 \\ 0 & \cdots & & A_k & 0 \\ \hline 0 & \cdots & & 0 & 0 \end{array}\right]+\gamma Z_1Z_1^T\right)=\lambda(L_0^{-1}-\gamma Z_0Z_0^T)=\frac{1}{\mu},$$

where $Z=\left[\begin{array}{c} Z_1 \\ \hline Z_0 \end{array}\right]$ is partitioned conformally with Y and is the positive eigenvector of $\mathfrak{L}$ corresponding to the eigenvalue 0. Further Y_0 is a $(1/\mu)$ eigenvector for $L_0^{-1}-\gamma Z_0Z_0^T$, and Y_1 has the form $((Z_0^TY_0)/(Z_1^TU))(-U)$,

where U is a Perron vector for $\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & 0 & 0 \\ 0 & \cdots & & A_k & 0 \\ \hline 0 & \cdots & & 0 & 0 \end{array}\right]+\gamma Z_1Z_1^T.$

Proof Note that Y_1 is positive, negative or zero according as $Y(v)$ is positive, negative or zero. Suppose first that $Y(v)>0$, so that $Y_1>0$ as well. We have $Z_1^TY_1+Z_0^TY_0=0$, $L_1Z_1=(\theta^TZ_0)e_v$ and $L_0Z_0=Z(v)\theta$. $L_1Y_1-e_v(\theta^TY_0)=\mu Y_1$ and $L_0Y_0-Y(v)\theta=\mu Y_0$. Premultiplying the first by Z_1^T yields $Z_1^TL_1Y_1-Z(v)(\theta^TY_0)=\mu Z_1^TY_1$ and hence $(\theta^TZ_0)Y(v)-Z(v)(\theta^TY_0)=\mu Z_1^TY_1$. Since $Z_1^TY_1>0$, we have $(\theta^TZ_0)Y(v)-Z(v)$

$(\theta^T Y_0) > 0$ as well. Now

$$\begin{aligned}\frac{1}{\mu}Y_1 &= L_1^{-1}Y_1 + \frac{1}{\mu}L_1^{-1}e_v(\theta^T Y_0) = L_1^{-1}Y_1 + \frac{\theta^T Y_0}{\mu(\theta^T Z_0)}Z_1 \\ &= L_1^{-1}Y_1 + \frac{\theta^T Y_0}{(\theta^T Z_0)}\left(\frac{1}{(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)}\right)Z_1 Z_1^T Y_1 \\ &= L_1^{-1}Y_1 + \left(\frac{(\theta^T Y_0)Z(v)}{(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)}\right)\frac{1}{(\theta^T Z_0)Z(v)}Z_1 Z_1^T Y_1.\end{aligned}$$

Let $A_1, \ldots, A_k$ be the bottleneck matrices for the components at v not containing C_0. It is straightforward to show that

$$L_1^{-1} = \left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \frac{1}{(\theta^T Z_0)Z(v)}Z_1 Z_1^T,$$

so that

$$\begin{aligned}&L_1^{-1} + \left(\frac{(\theta^T Y_0)Z(v)}{(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)}\right)\frac{1}{(\theta^T Z_0)Z(v)}Z_1 Z_1^T \\ &= \left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \frac{Y(v)}{(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)}\frac{1}{Z(v)}Z_1 Z_1^T,\end{aligned}$$

which is a positive matrix. In particular the Perron value of that matrix is $(1/\mu)$ and Y_1 is a corresponding Perron vector.

Let $\gamma = \frac{Y(v)}{(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)}\frac{1}{Z(v)}$. Then we also have

$$\begin{aligned}\frac{1}{\mu}Y_0 &= L_0^{-1}Y_0 + \frac{1}{\mu}Y(v)L_0^{-1}\theta = L_0^{-1} + \frac{1}{\mu}\frac{Y(v)}{Z(v)}Z_0 \\ &= L_0^{-1}Y_0 - \frac{Y(v)}{Z(v)[(\theta^T Z_0)Y(v) - (\theta^T Y_0)Z(v)]}Z_0 Z_0^T Y_0 \\ &= L_0^{-1}Y_0 - \gamma Z_0 Z_0^T Y_0.\end{aligned}$$

In particular, $\rho\left(\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \gamma Z_1 Z_1^T\right)$ is an eigenvalue of $L_0^{-1} - \gamma Z_0 Z_0^T$, with corresponding eigenvector Y_0. We thus find that C_0 is the unique Perron component at v. As in Lemmas 2.1, 2.2 of [4], it now follows that in fact

$$\rho\left(\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \gamma Z_1 Z_1^T\right) = \lambda(L_0^{-1} - \gamma Z_0 Z_0^T) = \frac{1}{\mu}.$$

Finally, suppose that $Y(v) = 0$, so that $Y_1 = 0$ as well. Then $L_0^{-1} Y_0 = (1/\mu) Y_0$, $\theta^T Y_0 = 0$ and $Z_0^T Y_0 = 0$, so for any $\gamma > 0$, $(1/\mu)$ is an eigenvalue of $L_0^{-1} - \gamma Z_0 Z_0^T$ with corresponding eigenvector Y_0. Hence $\lambda(L_0^{-1} - \gamma Z_0 Z_0^T) \geq (1/\mu)$, $\forall \gamma > 0$. Since $\tau(L_0) \leq \mu$, we find that C_0 is the unique Perron component at v. Consequently, there exists a unique $\gamma > 0$ such that

$$\rho\left(\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \gamma Z_1 Z_1^T\right) = \lambda(L_0^{-1} - \gamma Z_0 Z_0^T) = \frac{1}{\mu}.$$

■

Corollary 13 *Let γ be as in Theorem 12 and let $W_1, \ldots, W_k$ be a basis for the eigenspace of $\lambda(L_0^{-1} - \gamma Z_0 Z_0^T)$, and let Y_1 be a Perron vector for* $\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \gamma Z_1 Z_1^T$. *Then the vectors* $\frac{\frac{(Z_1^T W_i)}{Z_1^T Y_1}(-Y_1)}{W_i}$ *are a basis for the Fiedler eigenspace.*

Proof By our arguments given in Theorem 12, these vectors are all Fiedler vectors and span the μ-eigenspace. Their independence follows from that of $W_1, \ldots, W_k$. ■

COROLLARY 14 *The multiplicity of μ is the same as the multiplicity of $(1/\mu)$ as an eigenvalue of $L_0^{-1} - \gamma Z_0 Z_0^T$.*

We now consider a couple of interesting applications of Theorem 12.

COROLLARY 15 *Let G be connected. Let u be a cutpoint which is not a characteristic vertex for some (and hence any) Fiedler vector. Then there is a unique component at u, namely the Perron component, which contains vertices from $v(\mathcal{C}(G,D,Y))$ for every Fiedler vector Y. Thus for any Fiedler vector Y, all non-Perron components at u do not contain any vertices from $v(\mathcal{C}(G,D,Y))$.*

Proof There exists a unique component (namely C_0 in Theorem 12) which contains vertices from $v(\mathcal{C}(G,D,Y))$ for every Y and by Theorem 12 $\tau(L_0) < \mu$. By Lemma 2, if C is any other component at u with corresponding principal submatrix $\bar{L}$ of $\hat{\mathcal{L}}$, then $\tau(\bar{L}) > \mu$. Hence C_0 is the unique Perron component at u. ■

COROLLARY 16 *Let v be a cut point of G and C be a component at v. Assume that $Y(C) > 0$ for some Fiedler vector Y. Let Z be the unique positive eigenvector corresponding to the smallest eigenvalue of $\hat{\mathcal{L}}$. Let u be any vertex in C. Then*

$$\frac{Y(v)}{Z(v)} < \frac{Y(u)}{Z(u)}.$$

Proof If $Y(v) \leq 0$ we have nothing to prove. So let $Y(v) > 0$ and this ensures that C is not the component which contains some vertices of $v(\mathcal{C}(G,D,Y))$.

Let C_0 be the component at v which contains some vertices from $v(\mathcal{C}(G,D,Y))$. Then by Theorem 12, we have a $\gamma > 0$ such that $Y_1 = Y(G - C_0)$ is the Perron vector of

$$\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & 0 & & 0 \\ 0 & \cdots & A_k & & 0 \\ \hline 0 & \cdots & 0 & & 0 \end{array}\right] + \gamma Z_1 Z_1^T,$$

where the A_i's are the bottleneck matrices of components at v distinct from C_0, Z_1 is the part of Z corresponding to $G - C_0$, and where the last row and column correspond to vertex v. Observe that $Y(v)$ is the last entry of Y_1, while $Y(u)$ is some other entry of that vector.

Since

$$\left(\left[\begin{array}{cccc|c} A_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & 0 & 0 \\ 0 & \cdots & & A_k & 0 \\ \hline 0 & \cdots & & 0 & 0 \end{array}\right] + \gamma Z_1 Z_1^T\right) Y_1 = \frac{1}{\mu} Y_1, \tag{3}$$

and since each A_i is a positive matrix, we have $(1/\mu)Y_1(v) = \gamma Z_1(v) Z_1^T Y_1$ and $(1/\mu)Y_1(u) > \gamma Z_1(u) Z_1^T Y_1$. The result now follows. ∎

We now show how Corollary 16 can be used to obtain a more general version of the well known monotonicity result for Fiedler vectors when G is a tree.

THEOREM 17 *Let T be a tree with vertices $1, 2, \ldots, n$ and let D be any diagonal matrix. Consider a Fiedler vector Y of $\tilde{L}(T)$ and let Z be the eigenvector of $\tilde{L}(T)$ corresponding to $\tau(\tilde{L}(T))$. Let*

$$\frac{Y}{Z} = \begin{bmatrix} \frac{Y(1)}{Z(1)} & \frac{Y(2)}{Z(2)} & \cdots & \frac{Y(n)}{Z(n)} \end{bmatrix}^T.$$

Then one of the following cases occur.

(*a*) *No entry of Y is zero. In this case, there is a unique pair of vertices i and j such that i and j are adjacent in T with $Y(i) > 0$ and $Y(j) < 0$. Further, the entries of $\frac{Y}{Z}$ increase along any path in T which starts at i and does not contain j, while the entries of $\frac{Y}{Z}$ decrease along any path in T which starts at j and does not contain i.*

(*b*) *Some entry of Y is zero. In this case the subgraph of T induced by the set of vertices corresponding to the 0's in Y is connected. Moreover, there is a unique vertex k such that $Y(k) = 0$ and k is adjacent to a vertex m such that $Y(m) \neq 0$. The entries of $\frac{Y}{Z}$ either increase, decrease or are identically zero along any path in T which starts at k.*

Proof In view of Lemma 5 we know that $C(T, D, Y)$ is either a singleton vertex or a block (which is an edge here).

First we prove case (a). Here there is only one characteristic edge, $e = [i, j]$ with, say, $Y(i) > 0$. Consider any edge $e' = [v, u]$ on a path P which starts from i and does not contain j. At the vertex v the component which contains u is positively valuated. Thus $(Y(v)/Z(v)) < (Y(u)/Z(u))$. The rest of the proof of the case (a) is routine.

Next we prove the case (b). Here we have only one characteristic vertex, say, k. The graph $T-k$, obtained by deleting k from T has at least two Perron components, and for each component C at k, $Y(C)$ is either all positive, all negative or all zero. Thus a path starting from k is either a zero path or a positive path (except the starting vertex) or a negative path (except the starting vertex). The rest of the proof is similar to that of the case (a). ∎

Example 18 Here we give two weighted trees and consider the negative adjacency matrix to illustrate Theorem 17.

Case 1 Negative adjacency matrix and characteristic edge.

(See Fig. 1). The weights are given according to the following description:

$\theta_{10,7}=4$, $\theta_{8,7}=1$, $\theta_{7,1}=7$, $\theta_{1,2}=6$, $\theta_{2,3}=8$, $\theta_{2,4}=7$, $\theta_{2,9}=1$, $\theta_{2,5}=9$ and $\theta_{5,6}=5$. Here $\mathcal{L}=-A$, $\tau(-A)=-15.7994$, $\mu=-7.6442$. The vectors Z, Y and $\frac{Y}{Z}$ are given below; observe that $[1,2]$ is the characteristic edge.

$$Z=[.3255 \quad .6765 \quad .3426 \quad .2997 \quad .4283 \quad .1355 \quad .1547 \quad .0098 \quad .0428 \quad .0392]^T$$

$$Y=[.5241 \quad -.1219 \quad -.1276 \quad -.1117 \quad -.2509 \quad -.1641 \quad .6769 \quad .0886 \quad -.0160 \quad .3542]^T$$

$$\frac{Y}{Z}=[1.6104 \quad -.1803 \quad -.3726 \quad -.3726 \quad -.5859 \quad -1.2110 \quad 4.3744 \quad 9.0412 \quad -.3726 \quad 9.0412]^T$$

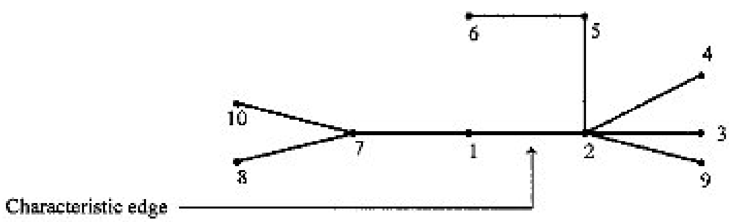

FIGURE 1 Case 1.

Case 2 Negative adjacency matrix and characteristic vertex.

(See Fig. 2). This tree is unweighted. Here $\mathcal{L} = -A$, $\tau(-A) = -2.0743$ and $\mu = -1.618$. The vectors Z, Y and $\frac{Y}{Z}$ are given below. We see that vertex 5 is the characteristic vertex.

$$Z = [.0837 \quad .1735 \quad .2763 \quad .3996 \quad .5526 \quad .3996$$
$$.2763 \quad .1735 \quad .0837 \quad .3470 \quad .1673]^T,$$
$$Y = [-.2629 \quad -.4253 \quad -.4253 \quad -.2629 \quad 0 \quad .2629$$
$$.4253 \quad .4253 \quad .2629 \quad 0 \quad 0]^T,$$
$$\frac{Y}{Z} = [-3.1423 \quad -2.4511 \quad -1.5394 \quad -.6578 \quad 0 \quad .6578$$
$$1.5394 \quad 2.4511 \quad 3.1423 \quad 0 \quad 0]^T.$$

■

Consider a tree T, and a diagonal matrix D, and let Y be a Fiedler vector for $\mathcal{L}(T)$. From the results above we see that $|C(T, D, Y)| = 1$ and indeed that $C(T, D, Y)$ is independent of the choice of Y. The following result presents an upper bound on the cardinality of the characteristic set for a general graph G. For completeness we include the proof, though it is the same as the proof of the corresponding result in [2] for Laplacian matrices.

Consider a connected graph G. By $\mathcal{N}_G$ denote the number of chords in G (with respect to some spanning tree). Thus $\mathcal{N}_G = m - n + 1$, where m and n are the number of edges and vertices in the graph.

THEOREM 19 *Let G be a connected graph and D be any diagonal matrix. Consider $\mathcal{L}$, Y. Let $S = C(G, D, Y)$. Suppose S lies in the block B. Then $1 \leq |S| \leq \mathcal{N}_B + 1$, where $|S|$ is the number of elements in S.*

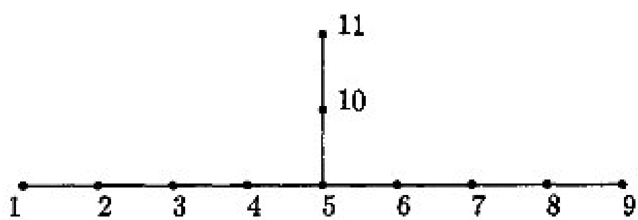

FIGURE 2 Case 2.

Proof If $|S|=1$ there is nothing to prove. Let $S=\{s_1,s_2,\ldots,s_r\}$, $r>1$. By Lemma 5 we know that for any two elements of S there is a simple cycle in G which contains these two elements and contains no more elements of S. Denote by $\Gamma_{i,r}$ a cycle of the above type which contains s_i, s_r; $i=1,\ldots,r-1$. From the definition of a block it is clear that these cycles are contained in B. For $i=1,2,\ldots,r-1$, define

$$e_i = \begin{cases} s_i, & \text{if } s_i \text{ is an edge,} \\ \text{the edge on } \Gamma_{i,r}, \text{ joining} & \\ s_i \text{ and a positive vertex,} & \text{if } s_i \text{ is a vertex.} \end{cases}$$

Let us delete the edge e_1 from B to obtain B_1. Note that none of the cycles $\Gamma_{i,r}$, $i=2,\ldots,r-1$ contain e_1, because otherwise they have to contain s_1, which is not possible (by Lemma 5). Let us delete the edge e_2 from B_1 to obtain B_2. None of the cycles $\Gamma_{i,r}$, $i=3,\ldots,r-1$ contain e_2, because otherwise they have to contain s_2, which is not possible (by Lemma 5). Thus repeating this process some more times, we conclude that the deletion of $e_1,\ldots,e_{r-1}$ will result in the graph, say B_{r-1}, which is connected (because each time we are deleting an edge from a cycle only). Let T_{r-1} be a spanning tree of B_{r-1}, thus of B. The edges $e_1,\ldots,e_{r-1}$ are chords of B with respect to T_{r-1}. Hence $r-1\le\mathcal{N}_B$ and the proof is complete. ∎

We note here that the reader can find a class of examples in [2], where the inequality given by the above theorem is an equality. The characterization of the graphs for which the equality holds remains open.

The following result presents a situation parallel to that for trees.

PROPOSITION 20 *Let G be a connected graph such that each block is either an edge or a cycle. Let X, Y be two Fiedler vectors of $\mathfrak{L}$. Then* $|\mathcal{C}(G,D,Y)|=|\mathcal{C}(G,D,X)|$.

Proof If $|\mathcal{C}(G,D,Y)|=1$, then by Theorem 7, and Corollary 10, $\mathcal{C}(G,D,X)=\mathcal{C}(G,D,Y)$ and thus both the characteristic sets have the same cardinality. If $|\mathcal{C}(G,D,Y)|=2$, then necessarily $\mathcal{C}(G,D,Y)$ lies on a cycle. By Theorem 9, $\mathcal{C}(G,D,X)$ also lies on the same cycle, and cannot have cardinality 1, by our discussion in the preceding paragraph. Thus $|\mathcal{C}(G,D,X)|$ is also 2. ∎

4. INTERVAL GRAPHS

In this section we apply some of our results above to a certain class of graphs. An *interval graph* is a collection of complete graphs $\{K_i : i=1,\ldots,n\}$ such that $K_i \bigcap K_{i+1} \neq \emptyset$ for $1 \leq i \leq n-1$. We note that some properties of the Laplacian matrices of interval graphs have been addressed in [1]. In this section we discuss the algebraic connectivity and Fiedler vectors of certain perturbed Laplacian matrices for an interval graph. Throughout the sequel, we will suppose that $n \geq 3$.

We begin with the following useful definition.

Definition Consider an interval graph $\mathcal{I} = \{K_i : i = 1,\ldots,n\}$. For each $1 \leq i \leq n-1$, the *overlapping* $O_{i,i+1}$ is the subgraph $K_i \bigcap K_{i+1}$ of $\mathcal{I}$. For each $1 \leq i \leq n$, the *mid-part* M_i is the subgraph $K_i - \{u : u \in O_{i-1,i} \bigcup O_{i,i+1}\}$.

We label the vertices of the interval graph $\mathcal{I} = \{K_i : i = 1,\ldots,n\}$ so that the vertices in M_i have labels less than the vertices in $O_{i,i+1}$ and M_{i+1} and the vertices in $O_{i,i+1}$ have labels less than the vertices in M_{i+1} and $O_{i+1,i+2}$. Let m_i denote the number of vertices in M_i, and let $w_{i,i+1}$ be the number of vertices in $O_{i,i-1}$. Then the adjacency matrix for $\mathcal{I}$ can be written as:

$$A = \begin{array}{c} m_1 \\ w_{1,2} \\ m_2 \\ w_{2,3} \\ m_3 \\ w_{3,4} \\ \vdots \\ w_{n-1,n} \\ m_n \end{array} \left[\begin{array}{c|c|c|c|c|c|c|c|c} J-I & J & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline J & J-I & J & J & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & J & J-I & J & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & J & J & J-I & J & J & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & J & J-I & J & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & J & J & J-I & \cdots & 0 & 0 \\ \hline \vdots & & & & & & \ddots & & \\ \hline 0 & 0 & 0 & 0 & 0 & & \cdots & J-I & J \\ \hline 0 & 0 & 0 & 0 & 0 & & \cdots & J & J-I \end{array}\right].$$

Observe that the degree matrix can be written as

$$D = \begin{bmatrix} d_1 I_{m_1} & & & \\ & d_{12} I_{w_{1,2}} & & \\ & & \ddots & \\ & & & d_n I_{m_n} \end{bmatrix},$$

where

$d_1 = m_1 + w_{1,2} - 1;$
$d_i = w_{i-1,i} + m_i + w_{i,i+1} - 1$, for $2 \leq i \leq n-1$;
$d_n = w_{n-1,n} + m_n - 1;$
$d_{1,2} = m_1 + w_{1,2} + m_2 + w_{2,3} - 1;$
$d_{i,i+1} = w_{i-1,i} + m_i + w_{i,i+1} + m_{i+1} + w_{i+1,i+2} - 1$, for $2 \leq i \leq n-3$;
and $d_{n-1,n} = w_{n-2,n-1} + m_{n-1} + w_{n-1,n} + m_n - 1.$

Form $L_\alpha = \alpha D - A$, where $0 \leq \alpha \leq 1$. It follows that L_α has the eigenvalues $\alpha d_i + 1$ with multiplicities $m_i - 1$, $1 \leq i \leq n$ and $\alpha d_{i,i+1} + 1$ with multiplicities $w_{i,i+1} - 1$, $1 \leq i \leq n-1$; further the corresponding eigenspaces are spanned by vectors of the form $[0 \ \cdots \ 0|1 \ \ 0 \ \cdots \ 0 \ \ -1 \ \ 0 \ \cdots \ 0|0 \ \cdots \ 0]^T$, where the nonzero portion of the vector is contained within a single block of the partition. It follows that the remaining eigenvalues of L_α coincide with those of

$$M(\alpha) = \begin{bmatrix} (\alpha d_1 - m_1 - 1) & -\sqrt{w_{1,2}m_1} & 0 & \cdots & & \\ -\sqrt{w_{1,2}m_1} & (\alpha d_{12} - w_{1,2} + 1) & -\sqrt{w_{1,2}m_2} & \sqrt{w_{1,2}w_{2,3}} & 0 & \cdots \\ 0 & -\sqrt{w_{1,2}m_2} & (\alpha d_2 - m_2 + 1) & -\sqrt{w_{2,3}m_2} & 0 & \cdots \\ 0 & -\sqrt{w_{2,3}w_{1,2}} & -\sqrt{w_{2,3}m_2} & (\alpha d_{23} - w_{2,3} + 1) & 0 & \cdots \\ \vdots & & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & -\sqrt{w_{n-1,n}m_n} & (\alpha d_n - m_n + 1) \end{bmatrix}.$$

Further $X = [x_1 \ \ x_{1,2} \ \ x_2 \ \ x_{2,3} \ \ \cdots \ \ x_{n-1,n} \ \ x_n]^T$, is an eigenvector of $M(\alpha)$ corresponding to ν if and only if

$$\begin{bmatrix} (x_1/\sqrt{m_1})1_{m_1} \\ (x_{1,2}/\sqrt{w_{1,2}})1_{w_{1,2}} \\ (x_2/\sqrt{m_2})1_{m_2} \\ \vdots \\ (x_n/\sqrt{m_n})1_{m_n} \end{bmatrix}$$ is an eigenvector of L_α corresponding to ν, where

1_k denotes the vector of size k with each entry 1.

Certainly the unique positive eigenvector of $M(\alpha)$ corresponds to $\tau(M(\alpha))$, and hence to $\tau(L_\alpha)$. Evidently the algebraic connectivity of L_α is the minimum of the algebraic connectivity of $M(\alpha)$ and $\min_{1 \leq i \leq n}\{\alpha d_i + 1\}$.

Note that the graph of $M(\alpha)$ has the form shown in Figure 3.

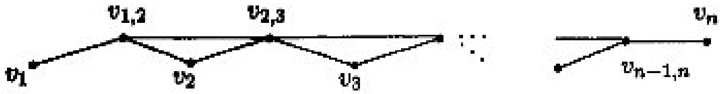

FIGURE 3 Graph of the compressed matrix.

Fix $1 \leq i \leq n-1$ and delete vertex $v_{i,i+1}$ and let M_1 and M_2 be the submatrices of $M(\alpha)$ on vertices $\{v_1, v_{1,2}, \ldots, v_i\}$ and $\{v_{i+1}, v_{i+1,i+2}, \ldots, v_n\}$, respectively. Letting $a_1 = \begin{bmatrix} \sqrt{m_1} \\ \sqrt{w_{1,2}} \\ \vdots \\ \sqrt{m_i} \end{bmatrix}$, we find from a computation that $a_1{}^T M_1 a_1 \leq \alpha w_{i,i+1} a_1{}^T a_1$, with strict inequality provided that $i \neq 1$. Thus it follows that $\tau(M_1) \leq \alpha d_i + 1$, with strict inequality if M_1 is not 1×1. A similar argument applies to M_2, and we find that $\tau(M_1), \tau(M_2) \leq \alpha d_i + 1$, with strict inequality for one of them since $n \geq 3$. We now find from Lemma 2 that the algebraic connectivity of $M(\alpha)$ is strictly less than $\alpha d_i + 1$. Since i was arbitrary, it follows that the algebraic connectivity of L_α coincides with that of $M(\alpha)$.

The following result is motivated by Theorem 4.6 in [1].

THEOREM 21 *Suppose that $n \geq 3$, and consider the interval graph $\mathcal{I} = \{K_i : i = 1, \ldots, n\}$. If $0 \leq \alpha \leq 1$, then the algebraic connectivity of L_α is simple.*

Proof In view of the discussion above, it is sufficient to show that the algebraic connectivity of $M(\alpha)$ is simple. Assume that the multiplicity is at least two. So let Y, Y' be two linearly independent Fiedler vectors of $M(\alpha)$ and ν be the algebraic connectivity. Let X be a linear combination of Y and Y' such that $x_1 = 0$. From the eigenequation at v_1, we thus have $x_{1,2} = 0$. From the eigenequation at v_2, we have $(\alpha d_2 - m_2 + 1)x_2 - \sqrt{m_2 w_{2,3}} x_{2,3} = \nu x_2$, and from the eigenequation at v_{12}, we have $\sqrt{w_{1,2} m_2} x_2 - \sqrt{w_{1,2} w_{2,3}} x_{2,3} = 0$. Putting these together yields $(\alpha d_2 - m_2 + 1)x_2 + m_2 x_2 = \nu x_2$, *i.e.* $(\alpha d_2 + 1)x_2 = \nu x_2$. Since $\nu < \alpha d_2 + 1$, we find that $x_2 = 0$ and hence $x_{2,3} = 0$. Repeating the argument at v_3 and $v_{2,3}$ now yields $x_3 = 0 = x_{3,4}$, since $\nu < \alpha d_3 + 1$. In this way we find that $X = 0$, a contradiction. ∎

Our final result applies some of the results in Section 3.

THEOREM 22 *Suppose that $n \geq 3$, and consider the interval graph $\mathcal{I} = \{K_i : i = 1, \ldots, n\}$. Fix $0 \leq \alpha \leq 1$, let Y be a Fiedler vector of L_α, and let Z be the eigenvector corresponding to $\tau(L_\alpha)$. Let G_1 be a component of $G - O_{i,i+1}$ which contains $K_{i+1} - O_{i,i+1}$. Suppose that for each vertex w in G_1, $Y(w) > 0$. Let $u \in O_{i,i+1}$ and $v \in G_1$. Then $(Y(u)/Z(u)) \leq (Y(v)/Z(v))$.*

Proof In view of the discussion done earlier in this section, it is sufficient to prove that if Y is Fiedler vector of $M(\alpha)$ such that $y_j > 0$, $y_{j,j-1} > 0$, $\forall j > i$, and Z is the eigenvector corresponding to $\tau(M(\alpha))$, then $(y_{i,i+1}/z_{i,i+1}) \leq (y_j/z_j)$, $\forall j > i$. This statement follows readily from Corollary 16 by considering $M(\alpha) - \tau(M(\alpha))I$. ■

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