

We consider a physical system that can be described by an weighted interaction network among nodes that can assume different dynamical states (in the case of a gene network the states $\sigma \in [0, 1]$ and we have models similar to spin models). In the simplest case, we introduce a stochastic dynamics using the probability $p_i(t)$ that the node i is in the state $\sigma_i = 1$ (then $1 - p_i(t)$ is the probability to get $\sigma_i = 0$) and we define a linear equation for the probability evolution

$$\dot{p}_i(t) = \sum_j \mathcal{P}_{ij} p_j(t) - \gamma_i p_i(t) \quad (1)$$

where \mathcal{P}_{ij} are transition probability rates and γ_i^{-1} defines the mean lifetime of the excited state. The meaning of the rates \mathcal{P}_{ij} is the rate at which the excited state of the node j increases (or decreases if $\mathcal{P}_{ij} < 0$) the probability of a transition to the excited state of the node i . Since $0 \leq p_i \leq 1$ for all i , this space should be invariant for the dynamics. This condition depends on the spectral properties of the matrix

$$\mathcal{P}_{ij} - \gamma_j \delta_{ij} \quad (2)$$

associated to the system. Let consider the case $\mathcal{L}_{ij} \geq 0$ (i.e. we have no inhibitory link), the first quadrant is clearly invariant and if we define

$$\sum_i \mathcal{P}_{ij} = \hat{\gamma}_j > 0$$

the matrix

$$\mathcal{L}_{ij} = \mathcal{P}_{ij} - \hat{\gamma}_j \delta_{ij}$$

is a Laplacian matrix and the system (1) can be written in the form

$$\dot{p}_i(t) = \sum_j \mathcal{L}_{ij} p_j(t) - \Delta \gamma_i p_i(t) \quad \Delta \gamma_i = \gamma_i - \hat{\gamma}_i$$

and by assumption we have $\gamma_i > \hat{\gamma}_i$. The eigenvalues of the matrix \mathcal{L}_{ij} have all negative real part except the null eigenvalue. It follows that all the eigenvalue of the matrix (2) has negative real part and the dynamics is a contraction towards the origin: a stable solution (i.e. without any external stimulus the system relaxes to the $\sigma_i = 0$ state). A non trivial stationary can be achieved only if an external stimulus is inserted

$$\dot{p}_i(t) = \sum_j \mathcal{P}_{ij} p_j(t) - \gamma_i p_i(t) + \epsilon f_i(t) \quad (3)$$

The stationary solution has to satisfy $p_i \in [0, 1]$ so that $f_i(t) \geq 0$ otherwise we can have negative probability when $p_i \simeq 0$. The case of a Laplacian matrix

$$\hat{\gamma}_i = \gamma_i$$

we get another possible stationary solution for $\mathcal{L}_{ij} p_j^* = 0$ in the first quadrant and the subspace $\sum p_i = 0$ is invariant and the dynamics is a contraction in this subspace (in general). Then the system a stable stationary solution even in absence of an external stimulus.

The presence of inhibitory links complicates the model and one has to prove that

- 1) there exists a physical space: an invariant cone in the first quadrant where the dynamics is a contraction towards the origin;
- 2) the external stimulus maintains the solution in the physical space.

Another solution could be to introduce boundary conditions so that $p_i \geq 0$ in any case (the system is non linear in such a case).

The eigenvalues of the matrix (2) define the different relaxation time scale the process and determine its reactivity to the change of the external stimulus: in a typical problem one consider a slowly varying external stimulus so that the system could be considered in a quasi stationary state

$$\sum_j \mathcal{L}_{ij} p_j - \Delta\gamma_i p_i = -\epsilon f_i(t) \quad \frac{df_i}{dt} \ll 1$$

the derivative is small with respect to the eigenvalues of the matrix (adiabatic approximation). On the other hand we have the effect of a correlated noise (we need to introduce a correlation in order to have a continuous function $f_i(t)$). The problem is to study the relation between the solution and the spectral properties of the matrix \mathcal{L}_{ij} : we simplify the equation by assuming $\Delta\gamma_i = \Delta\gamma$ so that if λ is an eigenvalue of \mathcal{L}_{ij} then $\lambda - \Delta\gamma$ is an eigenvalue of the matrix (2) and we assume that the dynamics is perturbed by

$$\dot{p}_i(t) = \sum_j (\mathcal{L}_{ij} + \Delta\mathcal{L}_{ij}) p_j(t) - \Delta\gamma p_i(t) + \epsilon f_i(t) \quad (4)$$

where the perturbation $\Delta\mathcal{L}_{ij}$ is a Laplacian matrix ($\sum_i \Delta\mathcal{L}_{ij} = 0$ and we assume $\langle \mathcal{L} \rangle = 0$) that can represent an error in the measure of the transition rates \mathcal{L}_{ij} or possible evolution of network due to in time. In the first case we have an ensemble of transition matrices and we have to study the eigenvalue distribution due to perturbation and the possible presence of bifurcation phenomena. In the second case we have a stochastic differential equation (since $\Delta\mathcal{L}_{ij}(t)$ can be represented as a realization of a stochastic process). The possible approach are Perturbation Theory, Random Matrix Theory and Statistical Physics Methods for random matrices. The external signal from the environment (the environmental node) can be considered in the adiabatic approximation (to be justified from a biological point of view).

The underlying stochastic process on the graph is defined by assigning the state $\xi_i(t) \in [0, 1]$ at each node i according to a probability distribution $\pi_i(t)$ that evolves as

$$\dot{\pi}_i(t) = \sum_j \mathcal{L}_{ij} \xi_j(t) - \Delta\gamma \xi_i(t) + \epsilon f_i(t)$$

By discretizing the dynamics for a time step Δt we have the evolution

$$\pi_i(t + \Delta t) = \pi_i(t) + \sum_j \mathcal{L}_{ij} \xi_j(t) \Delta t - \Delta\gamma \Delta t \xi_i(t) + \epsilon f_i(t) \Delta t$$

and $\xi(t + \Delta t)$ realized according to the distribution $\pi_i(t + \Delta t)$ (stochastic cellular automata). The average dynamics is computed by

$$\begin{aligned} \dot{\langle \pi_i(t) \rangle} &= \sum_j \mathcal{L}_{ij} \langle \xi_j(t) \rangle - \Delta\gamma \langle \xi_i(t) \rangle + \epsilon f_i(t) \\ &= \sum_j \mathcal{L}_{ij} p_j(t) - \Delta\gamma p_i(t) + \epsilon f_i(t) = \dot{p}_i(t) \end{aligned}$$

and we recover the average equation (1). But the stochastic dynamics gives information on the applicability of the average approximation and the variability at the critical states (at bifurcation of the spectrum of \mathcal{L}). The stochastic dynamics can be studied for stochastic connection matrices $\mathcal{L} + \Delta\mathcal{L}$.