Musical Acoustics

HOMEWORK 2

Group Members:
Riccardo Sebastiani Croce 10806858
Andrea De Fabritiis 10823661

October 27, 2022

1 Introduction

The aim of the homework is to characterize the vibration of a circular membrane and a circular plate, according to their modes, and to investigate the coupling between a string and the plate in terms of frequency coupling.

2 Circular membrane characterization

2.1 Propagation speed

When a membrane, clamped at its edges, is excited by an external force, it starts vibrating according to the interaction between the excitement and the restoring force, the tension T. This transverse vibration is characterized by a propagation velocity, which is related to the restoring force and is defined as:

$$v = \sqrt{\frac{T}{\sigma}} \tag{1}$$

where T is the surface tension (per unit length) of the membrane and σ is the areal mass density of the membrane.
In the specific case of a circular membrane with a unit surface weight $\sigma = 11.95\,[kg/m^2]$ and a tension $T = 10\,[N/m]$, we obtain:

$$v \simeq 11.95\,m/s \tag{2}$$

2.2 Modal frequencies and mode shapes

The vibration of a 2D circular membrane is characterized by transverse waves, whose combination creates a pattern of standing waves. These standing waves are actually the normal modes of the membrane and its vibration can thus be decomposed in a combination of said modes. These modal frequencies are computed starting from the Bessel functions according to the equation:

$$f_{mn} = \frac{\boldsymbol{Z}_n(\boldsymbol{J}_m(kr))}{2\pi a}\sqrt{\frac{T}{\sigma}} \tag{3}$$

where $\boldsymbol{Z}_n(\boldsymbol{J}_m(kr))$ indicates the values of kr where the *n-th* zero of $J_m(kr)$ occurs.

According to the table of the zeros of the Bessel functions, the resulting frequencies of the circular membrane for the first 18 modes are:

$\boldsymbol{m}$	$\boldsymbol{n}$	$\boldsymbol{Z_n(J_m(kr))}$	$\boldsymbol{f_{mn}}[Hz]$
0	1	2.4048	30.5
1	1	3.8317	48.6
2	1	5.1356	65.1
0	2	5.5201	70.0
3	1	6.3802	80.9
1	2	7.0156	88.9
4	1	7.5883	96.2
2	2	8.4172	106.7
0	3	8.6357	109.5
5	1	8.7715	111.2
3	2	9.7610	123.8
6	1	9.9360	126.0
1	3	10.1735	129.0
4	2	11.0647	140.3
7	1	11.0860	140.6
2	3	11.6198	147.4
0	4	11.7915	149.5
8	1	12.2250	155.0

The corresponding natural modes are computed as:

$$\tilde{z}_{mn}(r,\phi) = \Phi(\phi)R(r) = Ae^{\pm jm\phi}J_m(k_n r) \tag{4}$$

where A is the maximum possible amplitude (imposed equal to 1) and k_n is such that $k_n a = \boldsymbol{Z}_n(\boldsymbol{J}_m(kr))$ in which a is the radius of the membrane.
Below we report the plot of the modeshapes for the first six modes, in which we can clearly see the effect of nodal diameters (value of m) and nodal circles (value of n).

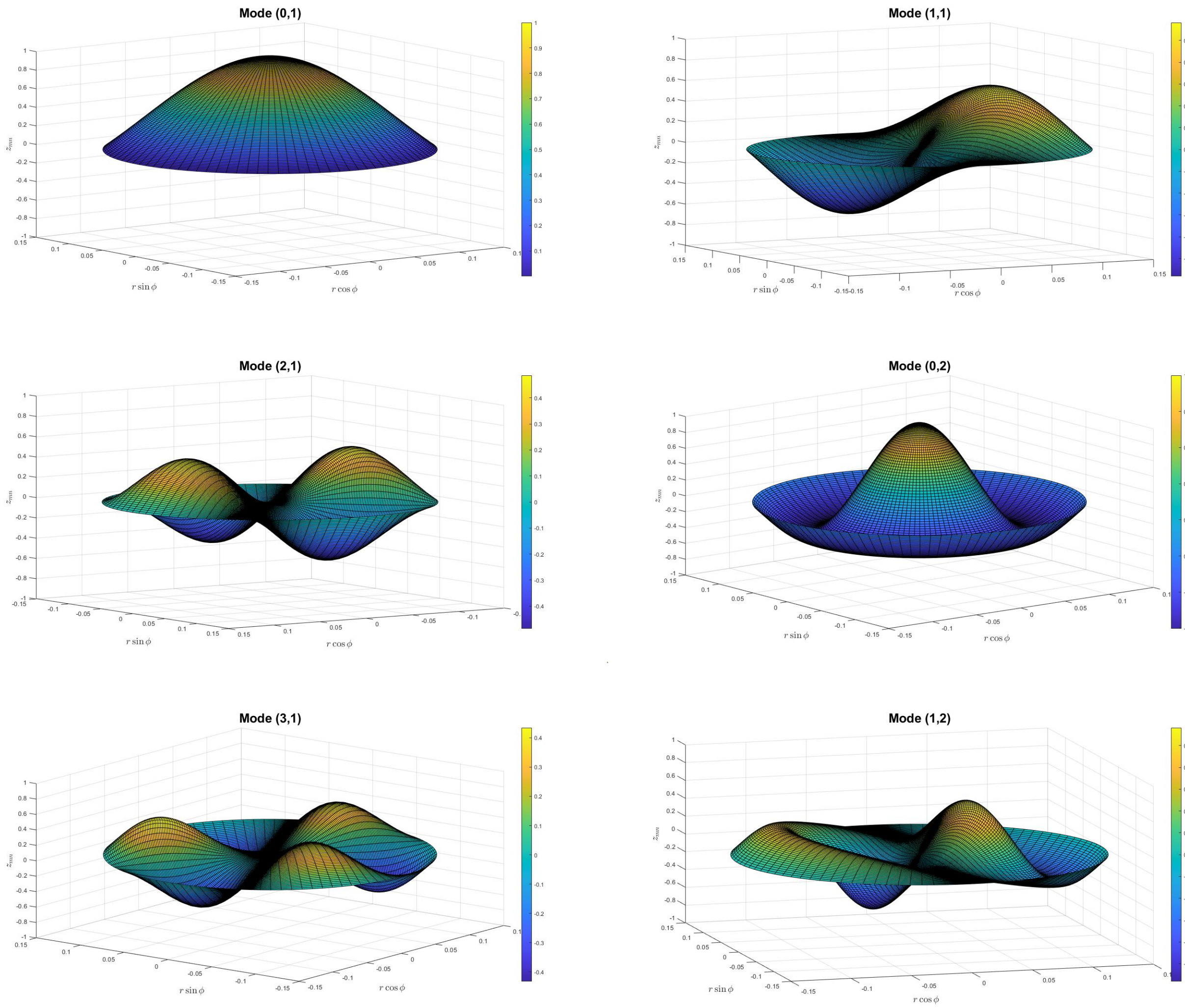

Figure 1: First six mode shapes of the circular membrane

2.3 Displacement when a force is applied

In order to analyze the behaviour of the membrane when it is subject to the application of an impulsive force

$$f(t) = 0.1 \cdot e^{-\frac{(t-0.03)^2}{0.01^2}} \tag{5}$$

let us consider the membrane as an N point degrees of freedom system for the purpose of investigating the vibrating characteristics of the membrane with respect to its first eighteen modes. The characteristics of a mode are the modal mass m_{mn}, the modal stiffness k_{mn} and the modal damping c_{mn}, which are related to one another by the equations

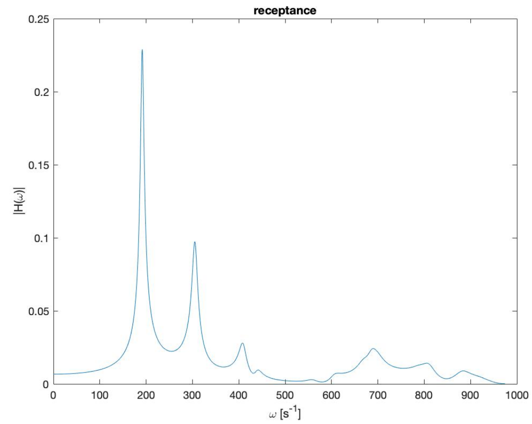

Figure 2: Receptance function evaluated over two points on the membrane

$$\begin{cases} m_{mn} = |A|^2\sigma \int_0^{2\pi} \int_0^a |e^{\pm jm\phi}|^2 |J_m(k_n r)|^2 r \, d\phi \, dr \\ \\ k_{mn} = \omega_{mn}^2 m_{mn} \\ \\ c_{mn} = \frac{\sqrt{m_{mn}k_{mn}}}{Q} \end{cases} . \tag{6}$$

where σ is the surface weight, A is the maximum amplitude of the modes, ω_{mn} is the frequency of said modes and Q their quality factor. The equation of the mass can be rewritten considering that the function is factorizable, thus the integrals are independent, and that the squared module of an exponential component of a complex number is equal to one:

$$m_{mn} = 2\pi|A|^2\sigma \int_0^a |J_m(k_n r)|^2 r \, dr \tag{7}$$

Starting from these variables and evaluating the modes values at the starting point ($r_0 = 0.075, \phi_0 = 15^o$) and at the point where the sensor is located ($r_1 = 0.075, \phi_1 = 195^o$), we can determine the receptance function in the frequency domain

$$H(\omega) = \sum_{m,n} \frac{\tilde{z}_{mn}(r_0, \phi_0)\tilde{z}_{mn}(r_1, \phi_1)}{-\omega^2 m_{mn} + j\omega c_{mn} + k_{mn}} \tag{8}$$

The receptance's curve is described in Fig.2. It is easy to address the composed nature of said function due to the superposition of the different modes receptance. The first modes appear to be more relevant in the shaping of the function, thus the final output according to the force.
Hence according to the convolution theorem

$$z(t) = f(t) * h(t) = \mathscr{F}^{-1}\{\mathscr{F}\{f(t)\} \cdot H(\omega)\} \tag{9}$$

where $\mathscr{F}$ is the Fourier transform and $\mathscr{F}^{-1}$ the inverse Fourier transform, we obtain the displacement time signal $z(t)$. According to Fig.3 it is clear how the impulsive nature of the force shapes

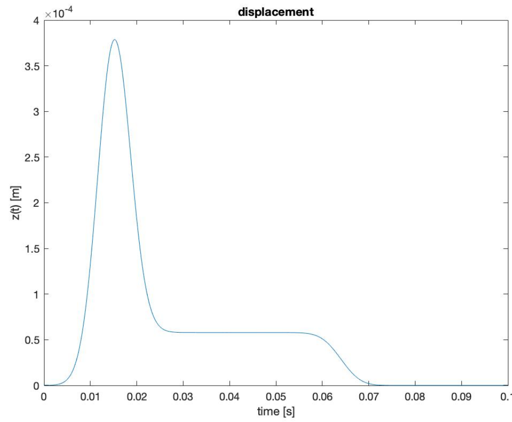

Figure 3: Displacement output function of the membrane vibration due to hammer impulsive force

the displacement at $2r_0$ distance and how, as said for the receptance function, its shape is a combination of the convolution of the interaction of each mode with the initial displacement displayed by the force.

3 Circular plate characterization

Thin plates differ from membrane for the influence the mechanical properties of the medium hold on the transmission of the waves. The waves transmitted by a plate can be longitudinal, torsional or bending waves. This type of medium is characterized by stiffness that can be expressed considering the elastic properties of the material.
In particular, these are described by the Young's modulus E, the Poisson's ratio ν and the shear or torsional modulus G.

3.1 Propagation speed of quasi-longitudinal and longitudinal waves

Due to its elastic properties, a thin plate can transmit two similar type of waves: longitudinal and quasi-longitudinal waves. In a pure longitudinal wave the vibration of the medium is parallel to the direction of the wave and the displacement of the medium follows the same (or opposite) direction of the wave propagation. These type of waves usually occur in solids whose dimensions in all directions are much greater than one wavelength. The propagation speed for a pure longitudinal wave can be computed according to the equation

$$c'_L = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} \tag{10}$$

In the specific case of a thin circular plate of density $\rho = 2700\ [kg/m^3]$, Young's modulus $E = 69\ [GPa]$ and Poisson's ration $\nu = 0.334$, the propagation speed is

$$c'_L \simeq 6.2 \cdot 10^3\ m/s \tag{11}$$

When a quasi-longitudinal wave occurs, the effect governed by Poisson's ratio determines that there must be displacement perpendicular to the line of force (wavefront) as well as an elongation in the direction of the force. The propagation speed for this type of wave, according to the specifics of the medium, is

$$c_L = \sqrt{\frac{E}{\rho(1-\nu^2)}} \simeq 5.4 \cdot 10^3 \; m/s \tag{12}$$

3.2 Speed of bending waves

As introduced in the previous chapter, a thin plate can transmits different type of waves, one of which is called bending wave. A bending wave is mainly responsible for the radiation of sound from vibrating structures since it has a displacement. The propagation velocity of a bending wave in a circular plate is dispersive and it is given by

$$c_b = \sqrt{1.8 f h c_L} \tag{13}$$

where f is the frequency, c_L is the quasi-longitudinal propagation velocity and h is the plate thickness, which in the specific case is $1\;mm$.

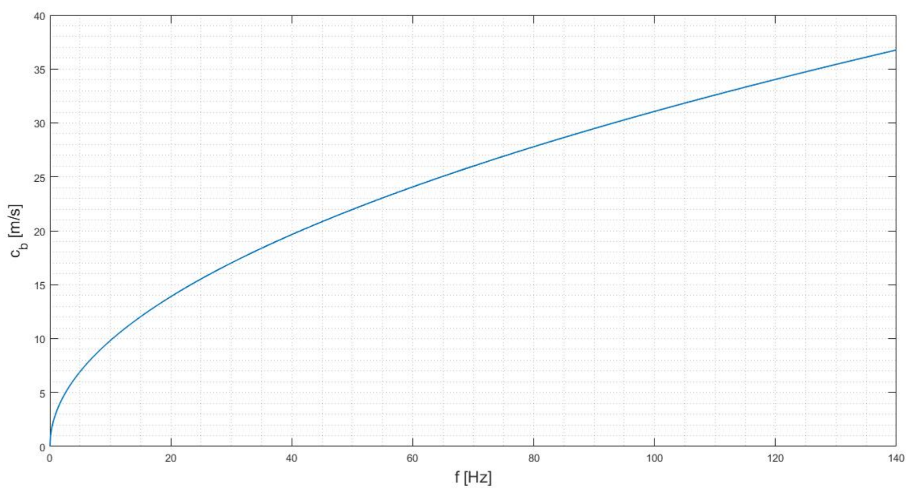

Figure 4: Curve of the propagation velocity of bending waves in a thin circular plate with respect to the frequency

The curve in Fig.4 shows how for high values of the frequency the propagation velocity curve behaves almost linearly. For smaller values of the frequency it shows the typical square root behaviour.

3.3 Modal frequencies

We are dealing with a circular plate in a clamped edge configuration and a radius equal to the previous case of a circular membrane ($a = 0.15\,m$).
By imposing the boundary conditions $Z|_{r=a} = 0$ and $\frac{\partial Z}{\partial r}|_{r=a} = 0$ to the solution $Z(r, \phi)$ of the motion equation, we can retrieve the wavenumbers that are needed to find the frequencies of the modes. The first five wavenumbers are the following:

k_{01}	$3.189/a$
k_{11}	$4.612/a$
k_{21}	$5.904/a$
k_{02}	$6.308/a$
k_{12}	$7.801/a$

Since the modal bending frequencies are proportional to k^2 through the following equation

$$f^b_{mn} = \frac{\omega}{2\pi} = 0.0459 h c_L k^2_{mn} \tag{14}$$

we obtain the values:

f^b_{01}	$0.4694 c_L h/a^2$	$111.9\,Hz$
f^b_{11}	$2.08 f^b_{01}$	$232.7\,Hz$
f^b_{21}	$3.41 f^b_{01}$	$381.5\,Hz$
f^b_{02}	$3.89 f^b_{01}$	$435.3\,Hz$
f^b_{12}	$5.00 f^b_{01}$	$559.5\,Hz$

As expected, we can see that they are not in harmonic relation.

4 Coupling between string and circular plate

In this case, we analyze the interaction between two vibrating systems that can exchange energy and so generate a modification of the vibrational behavior of the structures, with a coupling that depends on the type of systems involved and on frequency.

4.1 String tension

The string is attached to the previously considered soundboard and its fundamental mode is tuned with the first mode of the plate, that has a value of $f^b_{01} = 111.9\,Hz$.
We can compute the tension of the string starting from the equation of the frequency its fundamental mode when it is in fixed ends configuration:

$$f_1 = \frac{c}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho S}} = f^b_{01} \tag{15}$$

where $L = 0.4\,m$ is the length of the string, T is the unknown tension, $\rho = 5000\,kg/m^3$ is the density of the iron and $S = \pi r_s^2$ is the area of its circular cross section, with $r_s = 0.001\,m$. So the tension is obtained as:

$$T = 4L^2\rho S f_1^2 \simeq 125.9\,N \tag{16}$$

4.2 Modal frequencies of the string

Since the mass of the string is very small with respect to the soundboard one, we can neglect the mass behavior and consider the the string in ideal case. However, we have to consider its stiffness that also contributes to the restoring force.
So, having fixed ends, the natural frequencies are given by:

$$f_n = n f_1^0 \sqrt{1 + Bn^2}[1 + 2/\pi B^{0.5} + (4/\pi^2)B] \tag{17}$$

where f_1^0 is the natural frequency of a string without stiffness (tuned and $B = \pi ESK^2/TL^2 = 0.0245$ is a constant that depends on Young's modulus $E = 200 * 10^9\,Pa$ for the iron, $K = r_s/2$ is the gyration radius of the cross section, while the other values has been already introduced.

Finally we can report the values of the first five natural frequencies considering the stiffness of the string and compare them with their values without stiffness. As we can see from Table 1, they are slightly increased and the partials are not in harmonic relation anymore.

Table 1: Comparison between natural frequencies

f_n^0 [Hz]	f_n [Hz]
111.9	125.7
223.8	260.2
335.7	411.5
447.6	585.9
559.5	788.3

4.3 Modal frequencies of the coupled system

First of all, we have to know when we have weak or strong coupling. To do this, we solve for n the following inequation to have strong coupling:

$$m/(n^2 M) > \pi^2/(4Q_B^2) \implies n < \sqrt{\frac{m}{M}\frac{4Q_B^2}{\pi^2}} \tag{18}$$

where $m/(n^2M)$ is the *coupling coefficient*, in which m is the mass of the string, M is the mass of the soundboard and n refers to the considered mode, while Q_B is the merit factor of the resonance of the plate.
So, we obtain that for $n < 5.785$ that we approximate to $n < 6$ since it has to be an integer number, we have strong coupling, while for greater modes the system is weak coupled.

Then, we calculate the relative difference between the natural frequencies of the string ω_s and the plate ω_B as $\delta\omega_{rel} = \frac{\omega_s - \omega_B}{\omega_B}$ obtaining the following values for the first five modes (since we had only them from previous computation on string and plate):

ω_s [rad/s]	ω_B [rad/s]	$\delta\omega_{rel}$
703.021	703.021	0
1406.041	1462.283	-0.038
2109.062	2397.300	-0.120
2812.082	2734.750	0.028
3515.103	3515.103	0

Since the two frequencies of the first mode are the same and we are in a strong coupling situation, they are symmetrically splitted about $\omega_0 = 703.021\,rad/s$. In order to find their values, we use the curve in correspondence of $Q_B = 50$ in Fig.5 for $n = 1$ that give us a value of $m/(n^2M) \times 10^3 = 33$, so the normalized separation of frequency $\frac{\Omega_+ - \Omega_-}{\omega_0} \simeq 0.077$.
Finally, we obtain that $\Omega_+ - \Omega_- = 0.077\omega_0 \simeq 54.1\,rad/s$ and so $\Omega_+ = 730.1\,rad/s$ and $\Omega_- = 675.9\,rad/s$

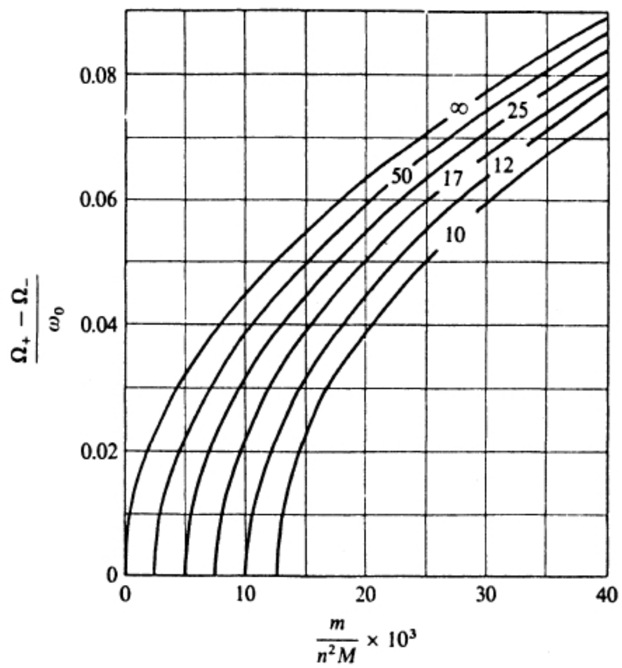

Figure 5: Normalized separation of the frequency of the normal modes when $\omega_n = \omega_B$ as a function of $m/(n^2M)$.

For what concern the remaining modes, we have to use the diagram in Fig.6. Having the values of relative differences $\delta\omega_{rel}$ between the frequencies, we read the corresponding values of $\frac{\Omega_\pm - \omega_B}{\omega_B}$ so that:

$$\Omega_+ = \omega_B(1 + \delta\omega_+)$$
$$\Omega_- = \omega_B(1 + \delta\omega_-)$$

where $\delta\omega_+$ and $\delta\omega_-$ refer respectively to the upper and lower solid lines.

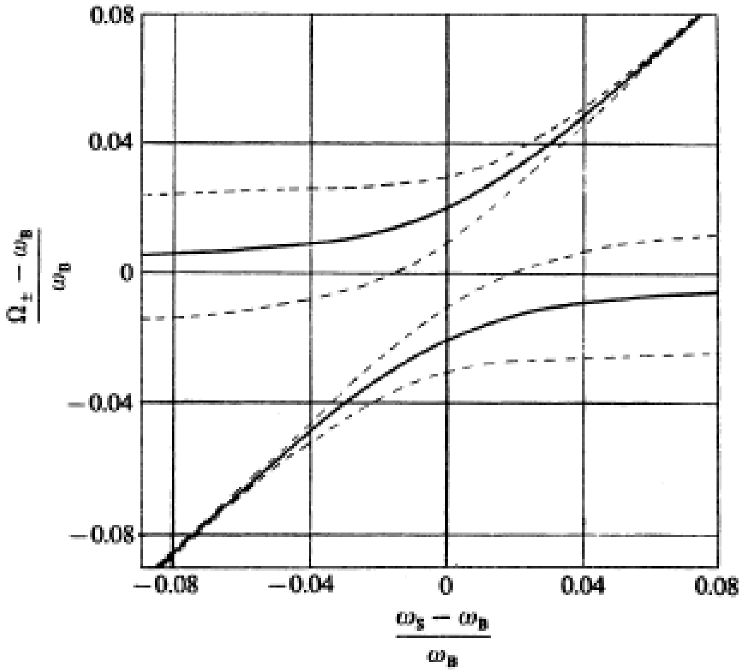

Figure 6: Normal mode frequencies of a string coupled to a soundboard as a function of their uncoupled frequencies ω_s and ω_B for strong coupling.

Finally, all the values of Ω_+ and Ω_- are reported in the table below:

Ω_+ [rad/s]	Ω_- [rad/s]
730.1	675.9
1479.8	1396.5
2399.7	2085.6
2844.1	2693.7
3585.4	3444.8