To the memory of Lina and Michele Quarteroni,
and to Enrica and Franco Valli,
our parents

PREFACE

One must never exhaust a subject to the point
that the reader has nothing left to do.
It is not a matter of inducing to read, but of inducing to think.

Charles-Louis de Montesquieu

The numerical solution of differential problems of practical interest may often lead to large-scale algebraic systems. Modern supercomputers make it possible nowadays to afford a wide range of problems that were unaffordable until recently. However, the size of most of such problems is so large that substantial attention needs to be paid to the improvement of existing numerical algorithms, as well as to the development of new ones that may better fit the architecture of available supercomputers.

Domain decomposition for the numerical solution of partial differential equations is a relatively new field (the first important ideas emerged in the early eighties). In particular, it is one of the most significant ways for devising parallel algorithms that can benefit strongly from multiprocessor computers. Parallel approaches are mandatory for very large-scale numerical problems like those that arise very often in many branches of physics and engineering.

Any domain decomposition method is based on the assumption that the given computational domain, say Ω, is partitioned into subdomains Ω_i, $i = 1, \ldots, M$, which may or may not overlap. Next, the original problem can be reformulated upon each subdomain Ω_i, yielding a family of subproblems of reduced size that are coupled one to another through the values of the unknown solution at subdomain interfaces.

Very often the interface coupling is removed at the expense of introducing an iterative process among subdomains, yielding at each step independent subproblems (of lower complexity) upon subdomains, which can be efficiently faced by multiprocessor systems.

When properly devised, these iterative procedures intrinsically embody a preconditioner for the system induced on the interface unknowns. A distinguishing feature of a domain decomposition method is the property of optimality of such a preconditioner; that is, its capability of generating a sequence that converges at a rate that does not depend on the size of the original system.

A zonal multi-domain approach can better account for multiple-scale solutions such as those occurring in highly structured flows in fluid dynamics, or in

fractured materials in structural mechanics.

Furthermore, domain decomposition can easily encompass the use of different numerical schemes within different subdomains, accounting for diverse behaviours of the physical solution. Even further, they allow the use of different kinds of equations in different subdomains whenever the physics behind the problem has a variable nature therein. This is the case, for example, for viscous–inviscid flow interactions in boundary layers, or the coexistence of heterogeneous materials (insulator and conductor) in electromagnetism.

In this book we illustrate the basic mathematical concepts behind domain decomposition. For any given partial differential equation we derive its multidomain formulation from the analysis of transmission conditions at subdomain interfaces. The relationship with the Steklov–Poincaré problem at subdomain interfaces is illustrated.

Concerning the finite dimensional approximation of the problems that are treated in this book, we confine ourselves to finite elements, purely for the sake of exposition. As a matter of fact, most of our analysis can be applied to any family of Galerkin approximation, such as, for example, spectral element methods or the h–p version of finite elements.

A large variety of boundary value problems is addressed, including symmetric elliptic equations, advection–diffusion equations, the elasticity problem, the Stokes problem for incompressible and compressible fluids, the time-harmonic Maxwell equations, parabolic and hyperbolic equations, and suitable couplings of heterogeneous equations.

We consider both overlapping and non-overlapping subdomain decompositions (although we pay more attention to the latter), and analyse the convergence of several iterative procedures among subdomains.

The reader may feel that this book is mainly confined to a simple decomposition in two subdomains. Actually, this has been a deliberate choice, stemming from the consideration that difficult concepts can be more easily addressed in this context. Moreover, this simple partition is suitable for carrying out the sensitivity of domain decomposition algorithms with respect to the grid refinement. The scalable property of the algorithms in terms of the number of subdomains requires instead an ad hoc analysis, which we sketch for elliptic problems only.

We develop the algebraic part of the algorithms, but mostly focus on the differential interpretation of the numerical methods that we propose. In other words, our attention focuses more on the solvers than on the preconditioners. Indeed, this viewpoint is more suitable for envisaging how to extend these methods to new types of equations and even to heterogeneous situations such as those described earlier.

An outline of this book is sketched below. In Chapter 1 we present the mathematical foundation of domain decomposition methods for both overlapping and non-overlapping domain partitions, in the case of symmetric elliptic equations. Also, we illustrate the relationship with the Steklov–Poincaré problem at subdomains interfaces.

In Chapter 2 we consider the finite element approximation of the Steklov–Poincaré interface problem, as well as its relation with the Schur complement matrix. A brief presentation of non-conforming domain decomposition methods is also included.

In Chapter 3 the finite element approximation of the domain decomposition algorithms presented in Chapter 1 is addressed. In particular, we introduce many preconditioners for the Schur complement matrix based on a substructuring strategy.

The main theoretical convergence results are contained in Chapter 4. First of all, we focus on three extension theorems, which prepare the basis for the analysis of the convergence of substructuring iterative methods. Then we give some abstract convergence theorems for Krylov-type iterations, which are used for showing the convergence of Dirichlet–Neumann and Neumann–Neumann algorithms. The analysis of Robin and Schwarz domain decomposition methods is also presented.

Chapter 5 is devoted to the formulation and analysis of domain decomposition methods for other boundary value problems; precisely, non-symmetric elliptic problems, the linear elasticity problem, the Stokes problem (for both incompressible and compressible flows), the first-order advection problem, and the time-harmonic Maxwell problem.

Some domain decomposition methods specifically suited for advection–diffusion equations are presented and analysed in Chapter 6.

Time-dependent problems are addressed in Chapter 7, for both parabolic and hyperbolic operators. In particular, we evaluate the impact of both implicit and explicit time-advancing finite difference schemes on domain partitioning in space. Non-linear evolution problems, especially the Navier–Stokes and Euler equations in fluid dynamics, are also considered.

Finally, in Chapter 8 we describe several types of heterogeneous domain decomposition methods, which are of particular interest in fluid dynamics and electromagnetism.

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Milan and Trento — Alfio Quarteroni
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CONTENTS

1

THE MATHEMATICAL FOUNDATION OF DOMAIN DECOMPOSITION METHODS

In this chapter the reader is encouraged to discover the mathematical foundation of domain decomposition methods, which are based on partitions of the computational domain into subdomains with or without overlap.

We introduce the concept of transmission conditions at subdomain interfaces and the Steklov–Poincaré problem for the interface variables. Both differential and variational formulations are addressed.

Then we present substructuring iterative methods for disjoint subdomains, and the Schwarz alternating method for overlapping subdomains. The convergence analysis of these methods will be carried out in Chapter 4. We also comment on two other approaches: the fictitious domain method and the so-called three-field method.

We deal mainly with *symmetric* linear elliptic boundary value problems, and, in particular, with the Poisson problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, and in the rest of this book, Ω is a d-dimensional domain ($d = 2, 3$), with a Lipschitz boundary $\partial\Omega$, whose outer unit normal direction is denoted by $\mathbf{n}^*$, f is a given function of $L^2(\Omega)$, $\Delta := \sum_{j=1}^{d} D_j D_j$ is the Laplace operator and D_j denotes the partial derivative with respect to x_j, $j = 1, \ldots, d$. To start

FIG. 1.1. Non-overlapping partition of the domain Ω into two subdomains.

with, we assume that Ω is partitioned into two non-overlapping subdomains Ω_1 and Ω_2, and denote by $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ (see Fig. 1.1). We also assume that Γ is a Lipschitz $(d-1)$-dimensional manifold.

The generalisation to other boundary value problems will be done in later chapters, particularly Chapters 5, 6 and 8.

The finite dimensional approximation is addressed in the next two chapters.

1.1 Multi-domain formulation and the Steklov–Poincaré interface equation

We indicate by u_i the restriction to Ω_i, $i = 1, 2$, of the solution u to (1.1), and by $\mathbf{n}^i$ the normal direction on $\partial\Omega_i \cap \Gamma$, oriented outward. For simplicity of notation we also set $\mathbf{n} = \mathbf{n}^1$.

It is easily seen that the Poisson problem (1.1) can be reformulated in the equivalent multi-domain form:

$$
\text{(1.1.1)} \qquad \begin{cases} -\Delta u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1 = u_2 & \text{on } \Gamma \\ \dfrac{\partial u_2}{\partial n} = \dfrac{\partial u_1}{\partial n} & \text{on } \Gamma \\ u_2 = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ -\Delta u_2 = f & \text{in } \Omega_2. \end{cases}
$$

Equations $(1.1.1)_3$ and $(1.1.1)_4$ are the *transmission conditions* for u_1 and u_2 on Γ.

The physical meaning of this split formulation is clear as soon as the original solution of problem (1.1) is smooth enough (say, $u \in C^1(\overline{\Omega})$). In a more general framework the equivalence between (1.1) and (1.1.1) is shown in the next section by resorting to the weak formulation of both problems.

We will see in Section 1.3 that domain decomposition methods are generally amenable to iterative procedures for an *interface* equation that is associated with the given differential problem. This interface problem can be defined in terms of the Steklov–Poincaré operator that we are going to introduce.

Let us refer to the model problem (1.1) and its multi-domain formulation (1.1.1), which corresponds to the domain partition of Fig. 1.1. The same arguments apply to the other boundary value problems that will be described in Section 1.4 (for further details see also Quarteroni and Valli 1991*a*, *b*).

Let λ denote the unknown value of u on Γ. We consider the two Dirichlet problems:

$$\begin{cases} -\Delta w_i = f & \text{in } \Omega_i \\ w_i = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ w_i = \lambda & \text{on } \Gamma, \end{cases} \tag{1.1.2}$$

for $i = 1, 2$. We can obviously state that

$$w_i = u_i^0 + u_i^*, \tag{1.1.3}$$

where we have defined u_i^0 and u_i^* to be the solutions of the following Dirichlet problems:

$$\begin{cases} -\Delta u_i^0 = 0 & \text{in } \Omega_i \\ u_i^0 = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ u_i^0 = \lambda & \text{on } \Gamma, \end{cases} \tag{1.1.4}$$

and

$$\begin{cases} -\Delta u_i^* = f & \text{in } \Omega_i \\ u_i^* = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ u_i^* = 0 & \text{on } \Gamma. \end{cases} \tag{1.1.5}$$

For each $i = 1, 2$, u_i^0 is the *harmonic extension* of λ into Ω_i, and will be denoted $H_i\lambda$. We will also write $\mathcal{G}_i f$ instead of u_i^*.

If we proceed formally, comparing (1.1.1) with (1.1.2), it follows that

$$w_i = u_i \ \text{ for } i = 1, 2 \ \text{ if and only if } \ \frac{\partial w_1}{\partial n} = \frac{\partial w_2}{\partial n} \text{ on } \Gamma. \tag{1.1.6}$$

The latter condition amounts to the requirement that λ satisfies the *Steklov–Poincaré interface equation*

$$S\lambda = \chi \qquad \text{on } \Gamma, \tag{1.1.7}$$

where

$$\begin{aligned} \chi :&= \frac{\partial}{\partial n}\mathcal{G}_2 f - \frac{\partial}{\partial n}\mathcal{G}_1 f \\ &= -\sum_{i=1}^{2} \frac{\partial}{\partial n^i}\mathcal{G}_i f \end{aligned} \tag{1.1.8}$$

and S is the *Steklov–Poincaré operator*, which is formally defined as

$$
\begin{aligned}
S\eta :&= \frac{\partial}{\partial n} H_1 \eta - \frac{\partial}{\partial n} H_2 \eta \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial n^i} H_i \eta.
\end{aligned} \tag{1.1.9}
$$

In particular, S can be split as $S = S_1 + S_2$, with

$$
S_i \eta := \frac{\partial}{\partial n^i} H_i \eta, \quad i = 1, 2. \tag{1.1.10}
$$

This operator, which was introduced a century ago (1896–1900), has been more recently analysed by Agoshkov and Lebedev (1985) in the framework of iterative methods. It is, however, worthwhile to point out that Agoshkov and Lebedev in fact considered the inverse operators S_1^{-1} and S_2^{-1}, and called them Poincaré–Steklov operators.

Remark 1.1.1 (Generalisation) The analysis that we are going to carry out on the Poisson equation will be applied to a far more general differential problem of the form

$$
\mathcal{L} u = f \quad \text{in } \Omega, \tag{1.1.11}
$$

where $\mathcal{L}$ is a partial differential operator, f is a given datum, and u is the unknown solution. A broad range of problems can be considered (see Chapter 5), including non-symmetric elliptic problems, the linear elasticity problem, the Stokes problem for incompressible flows, the viscous and inviscid Stokes problem for compressible flows, and the time-harmonic Maxwell system.

Should Ω be partitioned into two disjoint subdomains Ω_1 and Ω_2 as indicated in Fig. 1.1, we can go along the same lines presented above to generate a split version of problem (1.1.11). Denoting again for $i = 1, 2$ by u_i the restriction of u to Ω_i, it follows from (1.1.11) that

$$
\begin{aligned}
\mathcal{L} u_1 &= f \quad \text{in } \Omega_1 \\
\mathcal{L} u_2 &= f \quad \text{in } \Omega_2.
\end{aligned} \tag{1.1.12}
$$

To guarantee the equivalence with (1.1.11) we need to enforce transmission conditions between u_1 and u_2 across Γ. In an abstract form, such conditions can be expressed by the two relationships

$$
\begin{aligned}
&\Phi(u_1) = \Phi(u_2) \text{ on } \Gamma \\
&\Psi(u_1) = \Psi(u_2) \text{ on } \Gamma,
\end{aligned} \tag{1.1.13}
$$

where the functions Φ and Ψ will depend upon the nature of the problem.

Typically, for second-order elliptic operators, (1.1.13) expresses the continuity across Γ of u and of the normal 'flux' (namely, the normal stress) involving first-order derivatives of u_1 and u_2. More generally, these interface conditions are most often determined noting that:

- *The solution u belongs to a space of functions defined over the whole Ω. This requires that $u_{|\Omega_1}$ in Ω_1 and $u_{|\Omega_2}$ in Ω_2 enjoy a certain regularity therein, and in addition that they satisfy a suitable matching on Γ.*
- *The restrictions $u_{|\Omega_1}$ and $u_{|\Omega_2}$ are distributional solutions to the given equation in Ω_1 and Ω_2, respectively. Another interface condition between them comes from the fact that u in fact satisfies the equation in the sense of distributions in the whole Ω; namely, through the interface Γ and not only separately in Ω_1 and Ω_2.*

For the Poisson problem (1.1.1) the obvious identification

$$\Phi(v) = v, \quad \Psi(v) = \frac{\partial v}{\partial n}$$

holds true. Keeping in mind this correspondence, all iterative substructuring methods that we are going to introduce for the Poisson problem in Sections 1.3, 1.4 can actually be extended to the more general problems (1.1.12), (1.1.13) in a straightforward manner. This is the case, in particular, for the classical methods like Dirichlet–Neumann, Neumann–Neumann or Robin, originally introduced for the Laplace operator, and here applied to a very general family of boundary value problems (see Chapter 5). □

Remark 1.1.2 In transforming the Dirichlet boundary value problem (1.1) into an equation on Γ, we have chosen as interface unknown the *physical* variable that has to match on Γ as described in the first condition in Remark 1.1.1, while the interface equation (1.1.7) for the Steklov–Poincaré operator S is based on ensuring that the second condition is satisfied. The same procedure will be constantly followed in the remainder of this book, but clearly other choices of interface equation could be devised. These are amenable to different forms of the Steklov–Poincaré operator, and, correspondingly, different iterative substructuring methods. □

1.2 Variational formulation of the multi-domain problem

In this section we formulate (1.1) in a variational way. This requires us to introduce Sobolev spaces and to take into account some of their properties. We will not dwell here on this argument, and refer the interested reader to the comprehensive presentation of this theory that can be found, for example, in J.-L. Lions and Magenes (1972) (see also Chapter 9).

By integrating by parts in Ω, it is easily seen that the weak formulation of (1.1) reads

$$\text{find } u \in V \ : \ a(u,v) = (f,v) \qquad \forall\, v \in V, \tag{1.2.1}$$

where

$$
\begin{aligned}
(w,v) &:= \int_\Omega w\, v \\
a(w,v) &:= (\nabla w, \nabla v) \\
H^1(\Omega) &:= \{v \in L^2(\Omega) \,|\, D_j v \in L^2(\Omega),\ j = 1,\ldots,d\} \\
H^1_0(\Omega) &:= \{v \in H^1(\Omega) \,|\, v_{|\partial\Omega} = 0\} \\
V &:= H^1_0(\Omega)
\end{aligned}
$$

and $v_{|\partial\Omega}$ denotes the trace of v (that is, its restriction) on $\partial\Omega$. The norm of $H^1(\Omega)$ will be denoted by $||\cdot||_{1,\Omega}$, while $||\cdot||_{0,\Omega}$ will indicate the norm of $L^2(\Omega)$. We recall that

$$||v||_{0,\Omega} = (v,v)^{1/2},$$

while

$$||v||_{1,\Omega} = \left(||v||^2_{0,\Omega} + \sum_{j=1}^{d} ||D_j v||^2_{0,\Omega} \right)^{1/2}$$

for each $v \in H^1(\Omega)$.

The Poincaré inequality states that there exists a constant $C_\Omega > 0$ such that

$$\int_\Omega v^2 \le C_\Omega \int_\Omega \sum_{j=1}^{d} (D_j v)^2 \qquad \forall\ v \in H^1_0(\Omega). \tag{1.2.2}$$

Therefore, the norm $||v||_{1,\Omega}$ is equivalent to the norm $||\nabla v||_{0,\Omega}$ for each $v \in H^1_0(\Omega)$. It is worthwhile to note that the same result is true for functions that vanish only on an open and non-empty subset Σ of $\partial\Omega$.

We also recall that the trace space of $H^1(\Omega)$ on the boundary $\partial\Omega$ is denoted $H^{1/2}(\partial\Omega)$. In an analogous way, the trace space on an open and non-empty subset $\Sigma \subset \partial\Omega$ is indicated by $H^{1/2}(\Sigma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is surjective and continuous; that is, the following trace inequality holds

$$||v_{|\partial\Omega}||_{1/2,\partial\Omega} \le C^*_\Omega ||v||_{1,\Omega} \qquad \forall\ v \in H^1(\Omega), \tag{1.2.3}$$

where $||\cdot||_{1/2,\partial\Omega}$ denotes the norm in $H^{1/2}(\partial\Omega)$. Moreover, it can be shown that there exist injective, linear, and continuous extension operators from $H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$.

Let us also consider the weak multi-domain formulation equivalent to (1.2.1). First of all, let us set

$$
\begin{aligned}
(w_i, v_i)_{\Omega_i} &:= \int_{\Omega_i} w_i\, v_i \\
a_i(w_i, v_i) &:= (\nabla w_i, \nabla v_i)_{\Omega_i} \\
V_i &:= \{v_i \in H^1(\Omega_i) \,|\, v_{i|\partial\Omega\cap\partial\Omega_i} = 0\} \\
V_i^0 &:= H^1_0(\Omega_i) \\
\Lambda &:= \{\eta \in H^{1/2}(\Gamma) \,|\, \eta = v_{|\Gamma} \text{ for a suitable } v \in V\}.
\end{aligned} \tag{1.2.4}
$$

When $\Gamma \cap \partial\Omega = \emptyset$ we have $\Lambda = H^{1/2}(\Gamma)$; instead, when $\Gamma \cap \partial\Omega \neq \emptyset$ the space Λ, which in this case is usually denoted by $H_{00}^{1/2}(\Gamma)$, is strictly included in $H^{1/2}(\Gamma)$, and is endowed with a norm $||\cdot||_\Lambda$, which is larger than the norm of $H^{1/2}(\Gamma)$. We refer to J.-L. Lions and Magenes (1972) for an intrinsic definition of the trace space Λ and for its principal properties. Note, in particular, that the trace inequality holds as well when considered on the interface Γ; that is, there exist $C_i^* > 0$, $i = 1, 2$, such that

$$||v_{i|\Gamma}||_\Lambda \leq C_i^* ||v_i||_{1,\Omega_i} \qquad \forall\, v_i \in V_i. \tag{1.2.5}$$

Finally, for $i = 1, 2$ denote by $\mathcal{R}_i$ any possible (continuous) operator from Λ to V_i that satisfies $(\mathcal{R}_i\eta)_{|\Gamma} = \eta$. Any such operator (which can be proved to exist) will be called an extension operator from Λ to V_i.

Lemma 1.2.1 *The Poisson problem* (1.2.1) *can be equivalently reformulated as: find $u_1 \in V_1$, $u_2 \in V_2$ such that*

$$\begin{cases} a_1(u_1, v_1) = (f, v_1)_{\Omega_1} \qquad \forall\, v_1 \in V_1^0 \\ \\ u_1 = u_2 \qquad \text{on } \Gamma \\ \\ a_2(u_2, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\, v_2 \in V_2^0 \\ \\ a_2(u_2, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} - a_1(u_1, \mathcal{R}_1\mu) \qquad \forall\, \mu \in \Lambda, \end{cases} \tag{1.2.6}$$

where $\mathcal{R}_i$ denotes any possible extension operator from Λ to V_i.

Proof Let us start by considering the solution u to (1.2.1). Setting $u_i := u_{|\Omega_i}$, $i = 1, 2$, we have that $u_i \in V_i$ and that $(1.2.6)_1$, $(1.2.6)_2$ and $(1.2.6)_3$ are trivially satisfied. Moreover, for each $\mu \in \Lambda$ the function $\mathcal{R}\mu$ defined as

$$\mathcal{R}\mu := \begin{cases} \mathcal{R}_1\mu & \text{in } \Omega_1 \\ \mathcal{R}_2\mu & \text{in } \Omega_2 \end{cases}$$

belongs to V, therefore we have

$$a(u, \mathcal{R}\mu) = (f, \mathcal{R}\mu),$$

which is equivalent to $(1.2.6)_4$.

On the other hand, let u_i, $i = 1, 2$ be the solutions to (1.2.6). Setting

$$u := \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases},$$

from $(1.2.6)_2$ it follows that $u \in V$. Then, taking $v \in V$, we have that $\mu := v_{|\Gamma} \in \Lambda$. Define $\mathcal{R}\mu$ as before; clearly $(v_{|\Omega_i} - \mathcal{R}_i\mu) \in V_i^0$, and from $(1.2.6)_1$, $(1.2.6)_3$, and $(1.2.6)_4$ it follows that

$$\begin{aligned} a(u,v) &= \sum_{i=1}^{2}[a_i(u_i, v_{|\Omega_i} - \mathcal{R}_i\mu) + a_i(u_i, \mathcal{R}_i\mu)] \\ &= \sum_{i=1}^{2}[(f, v_{|\Omega_i} - \mathcal{R}_i\mu)_{\Omega_i} + (f, \mathcal{R}_i\mu)_{\Omega_i}] \\ &= (f, v). \end{aligned}$$

Let us point out, moreover, that $(1.2.6)_1$ and $(1.2.6)_3$ are the weak counterparts of $(1.1.1)_1$ and $(1.1.1)_6$, respectively, whereas $(1.2.6)_4$ is the weak form of the Neumann condition $(1.1.1)_4$. □

We stress that the extension operator $\mathcal{R}_i : \Lambda \to V_i$ used in $(1.2.6)_4$ can actually be chosen arbitrarily.

Note that the weak form $(1.2.6)_4$ of the continuity of the Neumann condition makes sense also for corner points, where the pointwise relation $(1.1.1)_4$ would be ambiguous or even meaningless.

The Steklov–Poincaré operator S introduced in (1.1.9) can be characterised as follows. It acts between the space of trace functions Λ (introduced in (1.2.4)) and its dual Λ'. More precisely, applying Green's formula and recalling that $H_i\eta$ is harmonic in Ω_i for all $\eta \in \Lambda$, and $i = 1, 2$ we have

$$\begin{aligned} \langle S\eta, \mu\rangle &= \sum_{i=1}^{2}\left\langle \frac{\partial}{\partial n^i}H_i\eta, \mu \right\rangle = \sum_{i=1}^{2}\int_{\Omega_i} \nabla H_i\eta \cdot \nabla\mathcal{R}_i\mu \\ &= a_i(H_i\eta, \mathcal{R}_i\mu) \qquad \forall\ \eta, \mu \in \Lambda, \end{aligned}$$

where the extension operators $\mathcal{R}_i$ are as in Lemma 1.2.1. Hereafter $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Λ' and Λ. In particular, taking $\mathcal{R}_i\mu = H_i\mu$, we obtain the following variational representation of S:

$$\langle S\eta, \mu\rangle = \sum_{i=1}^{2} a_i(H_i\eta, H_i\mu) \qquad \forall\ \eta, \mu \in \Lambda, \tag{1.2.7}$$

hence the operator S is *symmetric*. Moreover, from the Poincaré inequality (1.2.2) it holds that

$$\langle S\eta, \eta\rangle = \sum_{i=1}^{2} ||\nabla H_i\eta||^2_{0,\Omega_i} \geq \sum_{i=1}^{2} \frac{1}{1 + C_{\Omega_i}} ||H_i\eta||^2_{1,\Omega_i}.$$

Taking into consideration the trace inequality (1.2.5) we finally have

$$\langle S\eta, \eta\rangle \geq \alpha||\eta||^2_\Lambda \qquad \forall\ \eta \in \Lambda,$$

for a suitable constant $\alpha > 0$, therefore S is a *coercive* operator.

Proceeding in an analogous way, from (1.1.10) we have

$$\langle S_i\eta,\mu\rangle = \int_{\Omega_i} \nabla H_i\eta\cdot\nabla H_i\mu = a_i(H_i\eta,H_i\mu) \qquad \forall\ \eta,\mu\in\Lambda. \tag{1.2.8}$$

Clearly, each S_i is symmetric and coercive; that is, it satisfies

$$\langle S_i\eta,\eta\rangle \geq \alpha_i||\eta||^2_\Lambda \qquad \forall\ \eta\in\Lambda. \tag{1.2.9}$$

Another relevant property of the Steklov–Poincaré operators S_i is that they are *continuous*; that is, there exist constants $\beta_i > 0$ such that

$$\langle S_i\eta,\mu\rangle \leq \beta_i||\eta||_\Lambda\,||\mu||_\Lambda. \tag{1.2.10}$$

In fact

$$\langle S_i\eta,\mu\rangle \leq ||H_i\eta||_{1,\Omega_i}\,||H_i\mu||_{1,\Omega_i},$$

and from well known estimates for the solution of elliptic boundary value problems (see, for example, J.-L. Lions and Magenes 1972) it follows that

$$||H_i\eta||_{1,\Omega_i} \leq C||\eta||_\Lambda \qquad \forall\ \eta\in\Lambda, \tag{1.2.11}$$

hence (1.2.10) holds.

These properties are of great interest, because they can be exploited to obtain a numerical solution to the Steklov–Poincaré problem (1.1.7). Clearly, as soon as an approximation of λ is available (1.1.2) can be reduced to the solution of two *independent* Dirichlet problems.

Finally, we can also give a variational interpretation of the right-hand side χ in (1.1.7). From (1.1.8) it can be expressed through the functions f and $\mathcal{G}_i f$ as follows:

$$\begin{aligned}\langle\chi,\mu\rangle &= -\sum_{i=1}^{2}\left\langle\frac{\partial}{\partial n^i}\mathcal{G}_i f,\mu\right\rangle = \sum_{i=1}^{2}\int_{\Omega_i}(f\,\mathcal{R}_i\mu - \nabla\mathcal{G}_i f\cdot\nabla\mathcal{R}_i\mu)\\ &= \sum_{i=1}^{2}[(f,\mathcal{R}_i\mu)_{\Omega_i} - a_i(\mathcal{G}_i f,\mathcal{R}_i\mu)] \qquad \forall\ \mu\in\Lambda.\end{aligned} \tag{1.2.12}$$

Therefore, the Steklov–Poincaré equation (1.1.7) can be written in a variational form as

$$\text{find}\,\lambda\in\Lambda\ :\ \langle S\lambda,\mu\rangle = \langle\chi,\mu\rangle \qquad \forall\ \mu\in\Lambda. \tag{1.2.13}$$

Let us also note that, from a variational point of view, the functions $u_i^0 = H_i\lambda$ and $u_i^* = \mathcal{G}_i f$ introduced in (1.1.4) and (1.1.5) are the solutions to the following problems:

$$\begin{cases} H_i\lambda\in V_i\ : \\ \\ a_i(H_i\lambda,v_i) = 0 \qquad \forall\ v_i\in V_i^0 \\ \\ H_i\lambda = \lambda \qquad \text{on}\ \Gamma \end{cases} \tag{1.2.14}$$

and

$$\mathcal{G}_i f \in V_i^0 \ : \ a_i(\mathcal{G}_i f, v_i) = (f, v_i) \qquad \forall\, v_i \in V_i^0. \tag{1.2.15}$$

Remark 1.2.2 The variational form (1.2.13) of the Steklov–Poincaré equation could also be obtained directly from the interface relationship $(1.2.6)_4$. Indeed, from this relation using the splitting $u_i = u_i^0 + u_i^*$ we obtain

$$\sum_{i=1}^{2} a_i(u_i^0, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} - \sum_{i=1}^{2} a_i(u_i^*, \mathcal{R}_i\mu) \qquad \forall\, \mu \in \Lambda.$$

Integrating by parts on each side, and using (1.1.4) and (1.1.5), we deduce that

$$\begin{aligned} a_i(u_i^0, \mathcal{R}_i\mu) &= \int_\Gamma \frac{\partial u_i^0}{\partial n}\mu \\ a_i(u_i^*, \mathcal{R}_i\mu) &= (f, \mathcal{R}_i\mu)_{\Omega_i} + \int_\Gamma \frac{\partial u_i^*}{\partial n}\mu \end{aligned}$$

(the integrals over Γ are formal expressions: indeed, they should be replaced by the duality pairing between Λ' and Λ). We therefore conclude with the following integral equations at the interface Γ:

$$\int_\Gamma \left(\frac{\partial u_1^0}{\partial n}\mu - \frac{\partial u_2^0}{\partial n}\mu \right) \mu = -\int_\Gamma \left(\frac{\partial u_1^*}{\partial n} - \frac{\partial u_2^*}{\partial n} \right) \mu \qquad \forall\, \mu \in \Lambda,$$

or, equivalently,

$$\int_\Gamma \left(\frac{\partial}{\partial n} H_1\lambda - \frac{\partial}{\partial n} H_2\lambda \right) \mu = -\int_\Gamma \left(\frac{\partial}{\partial n}\mathcal{G}_1 f - \frac{\partial}{\partial n}\mathcal{G}_2 f \right) \mu \qquad \forall\, \mu \in \Lambda,$$

which coincides with (1.2.13). □

In Table 1.2.1 we summarise the relation between the differential and the variational forms of the single domain problem, the multi-domain problem and the Steklov–Poincaré equation.

1.3 Iterative substructuring methods based on transmission conditions at the interface

We face now the task of solving the multi-domain problem (1.1.1) by iterative procedures that will then be replicated at the finite dimensional level.

These methods are traditionally referred to as *iterative substructuring methods*. Typically, they introduce a sequence of subproblems in Ω_1 and Ω_2 for which conditions $(1.1.1)_3$ and $(1.1.1)_4$ provide, respectively, Dirichlet or Neumann data at the internal boundary Γ.

This can be accomplished in several ways, some of which are presented below.

Table 1.2.1 *The differential and the variational forms of the single domain, the multi-domain and the Steklov–Poincaré problems*

	Differential vs variational form	
Single domain	$-\Delta u = f$ in Ω $u = 0$ on $\partial\Omega$	$u \in H_0^1(\Omega)\ :\ \forall\ v \in H_0^1(\Omega)$ $a(u,v) = (f,v)$
Multi-domain	$-\Delta u_1 = f$ in Ω_1 $u_1 = 0$ on $\partial\Omega_1 \cap \partial\Omega$ $-\Delta u_2 = f$ in Ω_2 $u_2 = 0$ on $\partial\Omega_2 \cap \partial\Omega$ $u_1 = u_2$ on Γ $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$ on Γ	$u_1 \in V_1\ :\ \forall\ v_1 \in V_1^0$ $a_1(u_1,v_1) = (f,v_1)_{\Omega_1}$ $u_2 \in V_2\ :\ \forall\ v_2 \in V_2^0$ $a_2(u_2,v_2) = (f,v_2)_{\Omega_2}$ $u_1 = u_2$ on Γ $\sum_i a_i(u_i, \mathcal{R}_i\mu)$ $= \sum_i (f, \mathcal{R}_i\mu)_{\Omega_i} \quad \forall\ \mu \in \Lambda$
Steklov–Poincaré	$S\lambda = \chi$ on Γ $\left[S\eta := \frac{\partial}{\partial n}(H_1\eta - H_2\eta) \right.$ $\left. \chi := -\frac{\partial}{\partial n}(\mathcal{G}_1 f - \mathcal{G}_2 f) \right]$	$\lambda \in \Lambda\ :\ \forall\ \mu \in \Lambda$ $\langle S\lambda, \mu\rangle = \langle \chi, \mu\rangle$ $\left[\langle S\eta, \mu\rangle := \sum_i a_i(H_i\eta, \mathcal{R}_i\mu) \right.$ $\langle \chi, \mu\rangle := \sum_i (f, \mathcal{R}_i\mu)_{\Omega_i}$ $\left. - \sum_i a_i(\mathcal{G}_i f, \mathcal{R}_i\mu) \right]$

In general, two sequences of functions $\{u_1^k\}$, $\{u_2^k\}$ are generated starting from an initial guess u_1^0, u_2^0, and will converge to u_1 and u_2, respectively.

1. *The Dirichlet–Neumann method*

Given λ^0, solve for each $k \geq 0$:

$$\begin{cases} -\Delta u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma, \end{cases} \tag{1.3.1}$$

then

$$\begin{cases} -\Delta u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial u_2^{k+1}}{\partial n} = \dfrac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma, \end{cases} \tag{1.3.2}$$

with

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k, \tag{1.3.3}$$

θ being a positive acceleration parameter.

This method was considered, for example, by Bjørstad and Widlund (1986); Bramble *et al.* (1986*a*); Funaro *et al.* (1988); and Marini and Quarteroni (1988, 1989). The same method without relaxation (that is, with $\theta = 1$) does not necessarily converge, unless special assumptions are made about Ω_1 and Ω_2. This can be easily seen already for one-dimensional problems: if the length of Ω_1 (the Dirichlet subdomain) is larger than that of Ω_2 we have convergence, otherwise the unrelaxed method diverges. An example is furnished in Fig. 1.3.1, where the method is applied to approximate the (null) solution of the problem

$$\begin{cases} -u''(x) = 0 & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

Note that for this particular problem, whatever partition is chosen, a suitable parameter θ yields exact convergence in two iterations.

FIG. 1.3.1. Divergence (left) and convergence (right) for the unrelaxed Dirichlet–Neumann iterations.

Let us point out that a similar iterative procedure can be obtained by applying a relaxation procedure on the Neumann boundary condition, defining $\hat{u}_1^{k+1}$

and $\hat{u}_2^{k+1}$ as the solutions to

$$\begin{cases} -\Delta \hat{u}_2^{k+1} = f & \text{in } \Omega_2 \\ \hat{u}_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial \hat{u}_2^{k+1}}{\partial n} = \mu^k & \text{on } \Gamma \end{cases}$$

$$\begin{cases} -\Delta \hat{u}_1^{k+1} = f & \text{in } \Omega_1 \\ \hat{u}_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \hat{u}_1^{k+1} = \hat{u}_2^{k+1} & \text{on } \Gamma, \end{cases}$$

setting then

$$\mu^{k+1} := \theta \frac{\partial \hat{u}_1^{k+1}}{\partial n} + (1-\theta)\mu^k.$$

When we present the finite dimensional version of the Dirichlet–Neumann scheme and its algebraic formulation (see Sections 3.2 and 3.3), it will be useful to resort to the weak formulation of the iterative procedure (1.3.1), (1.3.2). It reads as follows:

$$\begin{cases} \text{find } u_1^{k+1} \in V_1 \ : \\ a_1(u_1^{k+1}, v_1) = (f, v_1)_{\Omega_1} \qquad \forall\ v_1 \in V_1^0 \\ u_1^{k+1} = \lambda^k \qquad \text{on } \Gamma \end{cases}$$

$$\begin{cases} \text{find } u_2^{k+1} \in V_2 \ : \\ a_2(u_2^{k+1}, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\ v_2 \in V_2^0 \\ a_2(u_2^{k+1}, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} - a_1(u_1^{k+1}, \mathcal{R}_1\mu) \qquad \forall\ \mu \in \Lambda, \end{cases}$$

where $a_i(\cdot,\cdot)$ have been introduced in (1.2.4) and $\mathcal{R}_i$ denotes any possible extension operator from Λ to V_i, and the notations are the same as in Section 1.2.

Let us show how this iterative scheme can be interpreted as a preconditioned Richardson procedure. For the sake of exposition, we refer to the differential formulation (1.3.1)–(1.3.3); however, the same result can be proved for the weak formulation introduced above. Since we have chosen $\mathbf{n} = \mathbf{n}^1 = -\mathbf{n}^2$, we have

$$\begin{cases} -\Delta(u_2^{k+1} - \mathcal{G}_2 f) = 0 & \text{in } \Omega_2 \\ u_2^{k+1} - \mathcal{G}_2 f = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial(u_2^{k+1} - \mathcal{G}_2 f)}{\partial n^2} = -\dfrac{\partial u_1^{k+1}}{\partial n} + \dfrac{\partial \mathcal{G}_2 f}{\partial n} & \text{on } \Gamma. \end{cases}$$

Noting that $u_1^{k+1} = H_1\lambda^k + \mathcal{G}_1 f$, it follows that

$$\begin{aligned} u_{2|\Gamma}^{k+1} &= (u_2^{k+1} - \mathcal{G}_2 f)_{|\Gamma} \\ &= S_2^{-1}\left(-\frac{\partial H_1\lambda^k}{\partial n} - \frac{\partial \mathcal{G}_1 f}{\partial n} + \frac{\partial \mathcal{G}_2 f}{\partial n}\right) \\ &= S_2^{-1}(-S_1\lambda^k + \chi), \end{aligned}$$

and therefore

$$\begin{aligned} \lambda^{k+1} &= \lambda^k + \theta[S_2^{-1}(-S_1\lambda^k + \chi) - \lambda^k] \\ &= \lambda^k + \theta S_2^{-1}(-S\lambda^k + \chi). \end{aligned}$$

We are thus left with a Richardson procedure for the Steklov–Poincaré equation (1.1.7) with the operator S_2 as a preconditioner. This interpretation plays a central role in the convergence analysis of the Dirichlet–Neumann method.

The proof that the Dirichlet–Neumann iterations converge for very general problems will be given in Section 4.3, as a consequence of Theorem 4.2.2, which is concerned with Steklov–Poincaré iterations. Besides, we prove the convergence of this iterative scheme in the framework of finite element discretisation, and in particular it will be shown that its rate of convergence is independent of the discretisation parameter h.

2. *The Neumann–Neumann method*

This method was considered by Bourgat *et al.* (1989); a former version had been investigated already in Agoshkov and Lebedev (1985).

In this case, for each $k \geq 0$ we have to solve

$$\begin{cases} -\Delta u_i^{k+1} = f & \text{in } \Omega_i \\ u_i^{k+1} = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ u_i^{k+1} = \lambda^k & \text{on } \Gamma, \end{cases} \tag{1.3.4}$$

and then

$$\begin{cases} -\Delta\psi_i^{k+1} = 0 & \text{in } \Omega_i \\ \psi_i^{k+1} = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ \dfrac{\partial\psi_i^{k+1}}{\partial n} = \dfrac{\partial u_1^{k+1}}{\partial n} - \dfrac{\partial u_2^{k+1}}{\partial n} & \text{on } \Gamma, \end{cases} \tag{1.3.5}$$

for $i = 1, 2$, with

$$\lambda^{k+1} := \lambda^k - \theta(\sigma_1 \psi_{1|\Gamma}^{k+1} - \sigma_2 \psi_{2|\Gamma}^{k+1}). \tag{1.3.6}$$

As before, $\theta > 0$ is an acceleration parameter, σ_1 and σ_2 are two positive averaging coefficients, while λ^0 is a given datum.

Using the same notations as for the Dirichlet–Neumann scheme, the weak formulation of the Neumann–Neumann method reads:

$$\begin{cases} \text{find } u_i^{k+1} \in V_i \ : \\ a_i(u_i^{k+1}, v_i) = (f, v_i)_{\Omega_i} \qquad \forall\ v_i \in V_i^0 \\ u_i^{k+1} = \lambda^k \qquad \text{on } \Gamma \end{cases}$$

for $i = 1, 2$, and

$$\begin{cases} \text{find } \psi_1^{k+1} \in V_1 \ : \\ a_1(\psi_1^{k+1}, v_1) = 0 \qquad \forall\ v_1 \in V_1^0 \\ a_1(\psi_1^{k+1}, \mathcal{R}_1\mu) = -(f, \mathcal{R}_1\mu)_{\Omega_1} - (f, \mathcal{R}_2\mu)_{\Omega_2} \\ \qquad\qquad\qquad + a_1(u_1^{k+1}, \mathcal{R}_1\mu) + a_2(u_2^{k+1}, \mathcal{R}_2\mu) \qquad \forall\ \mu \in \Lambda, \end{cases}$$

$$\begin{cases} \text{find } \psi_2^{k+1} \in V_2 \ : \\ a_2(\psi_2^{k+1}, v_2) = 0 \qquad \forall\ v_2 \in V_2^0 \\ a_2(\psi_2^{k+1}, \mathcal{R}_2\mu) = (f, \mathcal{R}_1\mu)_{\Omega_1} + (f, \mathcal{R}_2\mu)_{\Omega_2} \\ \qquad\qquad\qquad - a_1(u_1^{k+1}, \mathcal{R}_1\mu) - a_2(u_2^{k+1}, \mathcal{R}_2\mu) \qquad \forall\ \mu \in \Lambda. \end{cases}$$

Also, the Neumann–Neumann scheme can be interpreted as a preconditioned Richardson scheme. In fact, we have already noted that $u_i^{k+1} = H_i\lambda^k + \mathcal{G}_i f$, therefore

$$\begin{aligned} \psi_{1|\Gamma}^{k+1} &= S_1^{-1}\left(\frac{\partial H_1\lambda^k}{\partial n} + \frac{\partial \mathcal{G}_1 f}{\partial n} - \frac{\partial H_2\lambda^k}{\partial n} - \frac{\partial \mathcal{G}_2 f}{\partial n}\right) \\ &= -S_1^{-1}(-S\lambda^k + \chi), \end{aligned}$$

and similarly, recalling that $\mathbf{n} = -\mathbf{n}^2$,

$$\psi_{2|\Gamma}^{k+1} = S_2^{-1}(-S\lambda^k + \chi).$$

Therefore

$$\lambda^{k+1} = \lambda^k + \theta(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})(\chi - S\lambda^k),$$

which is a Richardson procedure for the Steklov–Poincaré equation (1.1.7) with the operator $(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ as a preconditioner.

The proof of the convergence of this algorithm will be presented in Section 4.4, also for the finite element discretisation. In that case, the rate of convergence is shown to be independent of the grid-size h.

3. *The Robin method*

This time, for each $k \geq 0$, we solve (see P.-L. Lions 1990 for the presentation and the analysis of the algorithm; see also Agoshkov 1988 for a similar formulation at the algebraic level)

$$\text{(1.3.7)} \qquad \begin{cases} -\Delta u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \dfrac{\partial u_1^{k+1}}{\partial n} + \gamma_1 u_1^{k+1} = \dfrac{\partial u_2^k}{\partial n} + \gamma_1 u_2^k & \text{on } \Gamma, \end{cases}$$

and then

$$\text{(1.3.8)} \qquad \begin{cases} -\Delta u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial u_2^{k+1}}{\partial n} - \gamma_2 u_2^{k+1} = \dfrac{\partial u_1^{k+1}}{\partial n} - \gamma_2 u_1^{k+1} & \text{on } \Gamma, \end{cases}$$

where u_2^0 is given, and γ_1 and γ_2 are non-negative acceleration parameters satisfying $\gamma_1 + \gamma_2 > 0$. For the sake of parallelisation, in (1.3.8) we could also consider u_1^k instead of u_1^{k+1}, assigning in that case also u_1^0.

The proof of the convergence of this scheme will be given in Section 4.5.

4. *The method by Agoshkov and Lebedev*

Along the same lines, we mention the following iteration-by-subdomain algorithm that has been proposed by Agoshkov and Lebedev (1985).

Given u_1^0 and u_2^0, for each $k \geq 0$ we have to solve

$$\text{(1.3.9)} \qquad \begin{cases} -\Delta u_1^{k+1/2} = f & \text{in } \Omega_1 \\ u_1^{k+1/2} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \dfrac{\partial u_1^{k+1/2}}{\partial n} + p_k u_1^{k+1/2} = \dfrac{\partial u_2^k}{\partial n} + p_k u_2^k & \text{on } \Gamma, \end{cases}$$

$$\text{(1.3.10)} \qquad u_1^{k+1} = u_1^k + \alpha_{k+1}(u_1^{k+1/2} - u_1^k) \quad \text{in } \Omega_1,$$

$$
\begin{cases}
-\Delta u_2^{k+1/2} = f & \text{in } \Omega_2 \\
u_2^{k+1/2} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\
-q_k \dfrac{\partial u_2^{k+1/2}}{\partial n} + u_2^{k+1/2} = -q_k \dfrac{\partial u_1^{k+1}}{\partial n} + u_1^{k+1} & \text{on } \Gamma,
\end{cases}
\tag{1.3.11}
$$

$$
u_2^{k+1} = u_2^k + \beta_{k+1}(u_2^{k+1/2} - u_2^k) \quad \text{in } \Omega_2. \tag{1.3.12}
$$

In the above procedure, $p_k \geq 0$, $q_k \geq 0$, α_{k+1}, and β_{k+1} are free parameters.

In fact, this algorithm encompasses and generalises many other methods: the Dirichlet–Neumann method (1.3.1)–(1.3.3) is a particular case of the Agoshkov–Lebedev one when $p_k = q_k = 0$ and $\alpha_{k+1} = 1$ (note, however, that in the Agoshkov–Lebedev method the roles of Ω_1 and Ω_2 are reversed, and that the relaxation is carried out on the Neumann boundary datum). Similarly, the Robin method (1.3.7), (1.3.8) can be obtained from (1.3.9)–(1.3.12) by taking $\alpha_k = \beta_k = 1$ and $\gamma_1 = p_k$, $\gamma_2 = 1/q_k$ ($q_k > 0$).

Remark 1.3.1 (Subdomain iterations and parallelism) With the exception of the Neumann–Neumann method, the different iterative procedures introduced thus far share the feature of generating at each step two boundary value problems, the former set in Ω_1, the latter in Ω_2, to be solved *sequentially*.

A simple modification of this procedure making it more interesting in view of parallel implementation is in order. As a matter of fact, when solving the boundary value problem in Ω_2 at the new step $k+1$ it is enough to use as data on Γ those generated by u_1^k (rather than u_1^{k+1}).

For instance, following this approach, the Dirichlet–Neumann method (1.3.1)–(1.3.3) should be modified by simply replacing in (1.3.2) the Neumann condition on Γ by the new one

$$
\frac{\partial u_2^{k+1}}{\partial n} = \frac{\partial u_1^k}{\partial n} \qquad \text{on } \Gamma.
$$

Similarly, for the Robin method the interface condition in (1.3.8) should become

$$
\frac{\partial u_2^{k+1}}{\partial n} - \gamma_2 u_2^{k+1} = \frac{\partial u_1^k}{\partial n} - \gamma_2 u_1^k \qquad \text{on } \Gamma.
$$

A similar modification arises in problem (1.3.11) of the Agoshkov–Lebedev algorithm.

However, this 'parallel' procedure in the case of two subdomains generates a sequence that has as a subsequence the 'sequential' procedure. For instance, if for $i = 1, 2$ and $k \geq 1$ we denote by $\{u_{i,s}^k\}$ the sequence generated by the 'sequential' Dirichlet–Neumann method and by $\{u_{i,p}^k\}$ the sequence generated by the 'parallel' Dirichlet–Neumann method, we have that $u_{1,s}^k = u_{1,p}^{2k-1}$ and $u_{2,s}^k = u_{2,p}^{2k}$, $k \geq 1$.

On the other hand, the issue of parallelism is relevant in the case of partitions of Ω using many (more than two) subdomains, and will be addressed in a more general framework in Section 1.4.2. □

1.4 Generalisations

The whole set of considerations that we have developed so far for the Poisson equation can be straightforwardly extended to the homogeneous Dirichlet boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4.1}$$

associated with the more general *symmetric elliptic* operator

$$Lw := -\sum_{l,j=1}^{d} D_l(a_{lj} D_j w) + a_0 w, \tag{1.4.2}$$

where the coefficients a_0 and a_{lj} are assumed to belong to $L^\infty(\Omega)$.

The operator L is elliptic if the coefficients a_{lj} satisfy

$$\sum_{l,j=1}^{d} a_{lj}(\mathbf{x})\xi_j\xi_l \geq \alpha_0|\boldsymbol{\xi}|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbf{R}^d, \text{ for almost all } \mathbf{x} \in \Omega$$

for a suitable constant $\alpha_0 > 0$. The operator L is symmetric if its coefficients satisfy

$$a_{lj}(\mathbf{x}) = a_{jl}(\mathbf{x}) \qquad \forall\, l, j = 1, \ldots, d,$$

for almost all $\mathbf{x} \in \Omega$.

The associated bilinear form is

$$a^*(w, v) := \int_\Omega \left(\sum_{l,j=1}^{d} a_{lj} D_j w D_l v + a_0\, w\, v \right). \tag{1.4.3}$$

We assume that $a_0(\mathbf{x}) \geq 0$ for almost all $\mathbf{x} \in \Omega$, hence $a^*(\cdot,\cdot)$ turns out to be symmetric, continuous, and coercive in $H_0^1(\Omega)$, because from the Poincaré inequality (1.2.2) it follows that

$$a^*(v, v) \geq \alpha_0 ||\nabla v||_{0,\Omega}^2 \geq \alpha_0 (1 + C_\Omega)^{-1} ||v||_{1,\Omega}^2 \qquad \forall\, v \in H_0^1(\Omega).$$

In particular, the form $a^*(\cdot,\cdot)$ induces a scalar product in $H_0^1(\Omega)$.

The week formulation of (1.4.1) reads:

$$\text{find } u \in H_0^1(\Omega) \;:\; a^*(u, v) = (f, v) \qquad \forall\, v \in H_0^1(\Omega), \tag{1.4.4}$$

and, as a consequence of the Lax–Milgram lemma, there exists a unique solution.

Denoting by u_i the restriction of u to Ω_i, $i = 1, 2$, u being the solution to problem (1.4.1), the interface conditions satisfied on $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ are

$$\begin{aligned} & u_1 = u_2 && \text{on } \Gamma \\ & \frac{\partial u_1}{\partial n_L} = \frac{\partial u_2}{\partial n_L} && \text{on } \Gamma, \end{aligned} \tag{1.4.5}$$

where the conormal derivative $\frac{\partial w}{\partial n_L}$ is defined as

$$\frac{\partial w}{\partial n_L} := \sum_{l,j=1}^{d} a_{lj} D_j w \, n_l \tag{1.4.6}$$

and, as usual, we have set $\mathbf{n} = \mathbf{n}^1$ on Γ. Note that the conormal derivative coincides with the normal derivative $\frac{\partial}{\partial n}$ when $L = -\Delta$.

Having defined for $i = 1, 2$ the bilinear forms

$$a_i^*(w_i, v_i) := \int_{\Omega_i} \left(\sum_{l,j=1}^{d} a_{lj} D_j w_i \, D_l v_i + a_0 w_i \, v_i \right), \tag{1.4.7}$$

the multi-domain weak formulation of (1.4.1) can be written in the following way: find $u_1 \in V_1$, $u_2 \in V_2$ such that

$$\begin{cases} a_1^*(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\ v_1 \in V_1^0 \\ \\ u_1 = u_2 \qquad \text{on } \Gamma \\ \\ a_2^*(u_2, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\ v_2 \in V_2^0 \\ \\ a_2^*(u_2, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} - a_1^*(u_1, \mathcal{R}_1\mu) \qquad \forall\ \mu \in \Lambda, \end{cases} \tag{1.4.8}$$

where $\mathcal{R}_i$ denotes any extension operator from Λ to V_i, and the notations are the same as in Section 1.2. As we have already observed, the last equation is the weak form of the interface condition $(1.4.5)_2$.

The more general mixed non-homogeneous boundary value problem

$$\begin{cases} Lu = f \qquad & \text{in } \Omega \\ \\ u = \varphi_D \qquad & \text{on } \Gamma_D \\ \\ \dfrac{\partial u}{\partial n_L^*} = \varphi_N \quad & \text{on } \Gamma_N \end{cases} \tag{1.4.9}$$

can be considered as well. Here, f, φ_D and φ_N are given functions, $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$, $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $\mathbf{n}^*$ is the unit outward normal vector on $\partial\Omega$.

In this case, the variational formulation reads:

$$\text{find } W \in H^1_{\Gamma_D}(\Omega) \ : \ a^*(W,v) = (f,v) + (\varphi_N, v_{|\Gamma_N})_{\Gamma_N} - a^*(\widehat{\varphi}_D, v) \qquad \forall\, v \in H^1_{\Gamma_D}(\Omega), \tag{1.4.10}$$

where

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v_{|\Gamma_D} = 0\} \tag{1.4.11}$$

$$(\phi, \psi)_{\Gamma_N} := \int_{\Gamma_N} \phi\, \psi, \tag{1.4.12}$$

and $\widehat{\varphi}_D \in H^1(\Omega)$ denotes any extension in Ω of the non-homogeneous Dirichlet datum φ_D. The solution u is obtained by adding W and $\widehat{\varphi}_D$.

Again, we can go through the formalism needed in the two-domain formulation, with the simple warning that, since the right-hand side (f, v) of the one-domain weak formulation (1.4.4) has been substituted in (1.4.10) by

$$(f,v) + (\varphi_N, v_{|\Gamma_N})_{\Gamma_N} - a^*(\widehat{\varphi}_D, v), \tag{1.4.13}$$

similarly the right-hand side $(f, v_i)_{\Omega_i}$ in (1.4.7) must be substituted by the restriction of (1.4.13) to the domain Ω_i, $i = 1, 2$. However, it is worthwhile to note that, in the case $a_0(\mathbf{x}) = 0$, to avoid the compatibility conditions on the data that have to be satisfied when solving a pure Neumann problem, the subdomain decomposition has to be chosen in such a way that $\Gamma_D \cap \partial\Omega_2 \neq \emptyset$.

Finally, the pure Neumann case (that is, when $\Gamma_D = \emptyset$) deserves a more careful analysis. No change occurs if the coefficient a_0 satisfies

$$a_0(\mathbf{x}) \geq \mu_0 > 0 \qquad \text{for almost all } \mathbf{x} \in \Omega,$$

because the bilinear form $a^*(\cdot, \cdot)$ would be coercive in $H^1(\Omega)$ in this case. In the case where $a_0 = 0$, it is well known that the boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial n^*_L} = \varphi_N & \text{on } \partial\Omega \end{cases} \tag{1.4.14}$$

is solvable if and only if the *compatibility condition*

$$\int_\Omega f + \int_{\partial\Omega} \varphi_N = 0 \tag{1.4.15}$$

is satisfied. In this case, a possible variational formulation reads:

(1.4.16) find $u \in H^1_m(\Omega)$: $a^*(u,v) = (f,v) + (\varphi_N, v_{|\partial\Omega})_{\partial\Omega} \qquad \forall\, v \in H^1_m(\Omega)$,

where $H^1_m(\Omega) := H^1(\Omega) \cap L^2_0(\Omega)$ and

$$L^2_0(\Omega) := \left\{ v \in L^2(\Omega) \mid \int_\Omega v = 0 \right\}. \tag{1.4.17}$$

The compatibility condition has to be satisfied also in every subdomain where a pure Neumann problem has to be solved. For instance, in the present situation the Dirichlet–Neumann scheme reads

$$\begin{cases} Lu_1^{k+1} = f & \text{in } \Omega_1 \\ \dfrac{\partial u_1^{k+1}}{\partial n_L^*} = \varphi_N & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma, \end{cases} \tag{1.4.18}$$

then

$$\begin{cases} Lu_2^{k+1} = f & \text{in } \Omega_2 \\ \dfrac{\partial u_2^{k+1}}{\partial n_L^*} = \varphi_N & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial u_2^{k+1}}{\partial n_L} = \dfrac{\partial u_1^{k+1}}{\partial n_L} & \text{on } \Gamma, \end{cases} \tag{1.4.19}$$

with

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k. \tag{1.4.20}$$

Therefore, the compatibility condition has to be assumed for the pure Neumann problem (1.4.19), and is given by

$$\int_{\Omega_2} f + \int_{\partial\Omega_2 \cap \partial\Omega} \varphi_N - \int_\Gamma \frac{\partial u_1^{k+1}}{\partial n_L} = 0. \tag{1.4.21}$$

(Recall that we have chosen $\mathbf{n} = \mathbf{n}^1$ on Γ.)

On the other hand, we can write

$$\begin{aligned} \int_\Gamma \frac{\partial u_1^{k+1}}{\partial n_L} &= \int_\Gamma \frac{\partial u_1^{k+1}}{\partial n_L} + \int_{\partial\Omega_1 \cap \partial\Omega} \frac{\partial u_1^{k+1}}{\partial n_L^*} - \int_{\partial\Omega_1 \cap \partial\Omega} \varphi_N \\ &= -\int_{\Omega_1} f - \int_{\partial\Omega_1 \cap \partial\Omega} \varphi_N, \end{aligned}$$

therefore the local compatibility condition (1.4.21) is a consequence of the global condition (1.4.15).

We can verify that the same result holds for the Neumann–Neumann iterative scheme as well.

1.4.1 The Steklov–Poincaré equation for the Neumann boundary value problem

The construction of the Steklov–Poincaré operator associated with problems (1.4.9) and (1.4.14) mimics that for problem (1.1), which has been presented in Sections 1.1 and 1.2. In the case of the mixed boundary value problem (1.4.9), the Steklov–Poincaré operator is symmetric, continuous, and coercive in the trace space Λ. The same happens for the Neumann problem (1.4.14), provided that $a_0(\mathbf{x}) \geq \mu_0 > 0$.

Instead, if $a_0 = 0$ the Steklov–Poincaré operator turns out to be singular. In fact, let us introduce for $i = 1, 2$ and $\eta \in H^{1/2}(\Gamma)$ the function $E_i^*\eta$, solution to

$$\begin{cases} E_i^*\eta \in H^1(\Omega_i) \ : \\ \\ a_i^*(E_i^*\eta, v_i) = 0 \qquad \forall\ v_i \in H^1_\Gamma(\Omega_i) \\ \\ E_i^*\eta = \eta \qquad \text{on } \Gamma. \end{cases}$$

The Steklov–Poincaré operator associated with the Neumann boundary value problem (1.4.14) is defined as $S = S_1 + S_2$, with

$$\langle S_i\eta, \mu\rangle := a_i^*(E_i^*\eta, E_i^*\mu) \qquad \forall\ \eta, \mu \in H^{1/2}(\Gamma), \quad i = 1, 2.$$

Clearly, $E_i^*1 = 1$, consequently $S_i 1 = 0$ and $\langle S_i\eta, 1\rangle = 0$ for each $\eta \in H^{1/2}(\Gamma)$. The operator S_i is therefore symmetric and continuous, but not coercive in $H^{1/2}(\Gamma)$.

To derive the associated Steklov–Poincaré equation, let us start by noting that the solutions to (1.4.14) are defined up to an additive constant. However, since $S1 = 0$, the Steklov–Poincaré equation for the trace $\lambda = u_{|\Gamma}$ of any such solution reads as usual

$$\text{find } \lambda \in H^{1/2}(\Gamma) \ : \ \langle S\lambda, \mu\rangle = \langle \chi, \mu\rangle \qquad \forall\ \mu \in H^{1/2}(\Gamma), \tag{1.4.22}$$

where

$$\langle \chi, \mu\rangle := \sum_{i=1}^{2} [(f, E_i^*\mu)_{\Omega_i} + (\varphi_N, E_i^*\mu)_{\partial\Omega_i \cap \partial\Omega} - a_i^*(u_i^*, E_i^*\mu)] \qquad \forall\ \mu \in H^{1/2}(\Gamma),$$

and u_i^* is the solution of

$$u_i^* \in H^1_\Gamma(\Omega_i) \ : \ a_i^*(u_i^*, v_i) = (f, v_i)_{\Omega_i} + (\varphi_N, v_i)_{\partial\Omega_i \cap \partial\Omega} \qquad \forall\ v_i \in H^1_\Gamma(\Omega_i).$$

Since the operator S is singular in $H^{1/2}(\Gamma)$, it is convenient to consider a different space of functions on Γ, defined as

$$\hat{H}^{1/2}(\Gamma) := \left\{ \eta \in H^{1/2}(\Gamma) \,|\, \int_\Gamma \eta = 0 \right\}, \tag{1.4.23}$$

endowed with the norm of $H^{1/2}(\Gamma)$. In this space, the Steklov–Poincaré operators S_i are coercive. In fact, taking $\eta \in \hat{H}^{1/2}(\Gamma)$ we have

$$\begin{aligned} \langle S_i\eta, \eta\rangle &= a_i^*(E_i^*\eta, E_i^*\eta) \geq \alpha_0 ||\nabla E_i^*\eta||^2_{0,\Omega_i} \\ &\geq \alpha_0 (1 + C_{\Omega_i})^{-1} ||E_i^*\eta||^2_{1,\Omega_i}, \end{aligned}$$

because the Poincaré inequality (1.2.2) still holds in

$$\hat{H}^1(\Omega_i) := \left\{ \hat{v}_i \in H^1(\Omega_i) \,|\, \int_\Gamma \hat{v}_{i|\Gamma} = 0 \right\}.$$

Finally, from the trace inequality (1.2.3) we find that

$$\langle S_i\eta, \eta\rangle \geq \alpha_0 (1 + C_{\Omega_i})^{-1} (C^*_{\Omega_i})^{-2} ||\eta||^2_{1/2,\Gamma}.$$

Hence, we can find the unique solution $\hat{\lambda}$ of the reduced Steklov–Poincaré equation

$$\text{find } \hat{\lambda} \in \hat{H}^{1/2}(\Gamma) \;:\; \langle S\hat{\lambda}, \hat{\mu}\rangle = \langle \chi, \hat{\mu}\rangle \qquad \forall\, \hat{\mu} \in \hat{H}^{1/2}(\Gamma). \tag{1.4.24}$$

The solution $\hat{\lambda}$ indeed satisfies

$$\langle S\hat{\lambda}, \mu\rangle = \langle \chi, \mu\rangle \qquad \forall\, \mu \in H^{1/2}(\Gamma). \tag{1.4.25}$$

In fact, denoting by $\mu_\Gamma := \frac{1}{\text{meas }\Gamma} \int_\Gamma \mu$, it holds that

$$\begin{aligned} \langle S\hat{\lambda}, \mu\rangle &= \langle S\hat{\lambda}, \mu - \mu_\Gamma\rangle + \mu_\Gamma \langle S\hat{\lambda}, 1\rangle \\ &= \langle \chi, \mu - \mu_\Gamma\rangle = \langle \chi, \mu\rangle - \mu_\Gamma \langle \chi, 1\rangle \\ &= \langle \chi, \mu\rangle, \end{aligned}$$

because

$$\begin{aligned} \langle \chi, 1\rangle &= \sum_{i=1}^{2} [(f,1)_{\Omega_i} + (\varphi_N, 1)_{\partial\Omega_i \cap \partial\Omega}] \\ &= (f,1)_\Omega + (\varphi_N, 1)_{\partial\Omega} = 0, \end{aligned}$$

due to the compatibility condition (1.4.15).

The solution $\hat{\lambda}$ does not coincide with the trace $u_{|\Gamma}$ of the solution to (1.4.16), and the difference is a constant, because $\hat{\lambda}$ is in fact equal to the trace on Γ of the solution to

$$\text{find } \hat{u} \in \hat{H}^1(\Omega) \;:\; a^*(\hat{u}, \hat{v}) = (f, \hat{v}) + (\varphi_N, \hat{v}_{|\partial\Omega})_{\partial\Omega} \qquad \forall\, \hat{v} \in \hat{H}^1(\Omega), \tag{1.4.26}$$

where

$$\hat{H}^1(\Omega) := \left\{ \hat{v} \in H^1(\Omega) \mid \int_\Gamma \hat{v}_{|\Gamma} = 0 \right\}.$$

1.4.2 Iterations on many subdomains

In view of a *parallel implementation*, a desirable situation is that in which Ω is partitioned into *many* subdomains Ω_i, $i = 1, \ldots, M$ (see, for example, Fig. 1.4.1). In this case, denoting by $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$ the common interface between two adjoining subdomains Ω_i and Ω_j, the split form of

$$Lu = f \qquad \text{in } \Omega,$$

for L introduced in (1.4.2), is given by

$$\begin{cases} Lu_i = f & \text{in } \Omega_i, \quad \forall\, i = 1, \ldots, M \\ \Phi(u_i) = \Phi(u_j) & \text{on } \Gamma_{ij}, \quad \forall\, j \text{ such that } \Gamma_{ij} \neq \emptyset \\ \Psi(u_i) = \Psi(u_j) & \text{on } \Gamma_{ij}, \quad \forall\, j \text{ such that } \Gamma_{ij} \neq \emptyset, \end{cases} \tag{1.4.27}$$

where

$$\Phi(v) = v, \quad \Psi(v) = \frac{\partial v}{\partial n_L}. \tag{1.4.28}$$

As already suggested in Remark 1.1.1, we prefer to keep this general notation, since it is convenient to treat also a larger family of boundary value problems that will be considered in Chapter 5.

The extension of substructuring iterative techniques to the case of many subdomains will be addressed in Section 3.3.2. However, the reader may find it interesting to have a preliminary idea of how the previous Dirichlet–Neumann method can be adapted to the multi-domain case. This iteration-by-subdomain procedure generalises accordingly as follows: after using a black and white colouring, set $I_B := \{1 \leq i \leq M \mid \Omega_i \text{ is black}\}$ and $I_W := I \setminus I_B$, and solve

$$\begin{cases} Lu_i^{k+1} = f & \text{in } \Omega_i, \quad \forall\, i \in I_B \\ \Phi(u_i^{k+1}) = \theta\Phi(u_j^k) + (1-\theta)\Phi(u_i^k) & \text{on } \Gamma_{ij}, \quad \forall\, j \in I_W : \Gamma_{ij} \neq \emptyset. \end{cases} \tag{1.4.29}$$

This yields a family of Φ-type subproblems that, when Φ is given as in (1.4.28), are *independent* of one another and can be solved simultaneously, allowing an effective treatment within a multi-processor environment. Then solve

$$\begin{cases} Lu_j^{k+1} = f & \text{in } \Omega_j, \quad \forall\, j \in I_W \\ \Psi(u_j^{k+1}) = \Psi(u_i^{k+1}) & \text{on } \Gamma_{ij}, \quad \forall\, i \in I_B : \Gamma_{ij} \neq \emptyset. \end{cases} \tag{1.4.30}$$

FIG. 1.4.1. Black and white subdomain decomposition of the domain Ω.

On each white subdomain, we have, therefore, a Ψ-type subproblem; when Ψ is given as in (1.4.28), all these subproblems are coupled (although mildly) at the cross-points.

We point out that the algorithm (1.4.29), (1.4.30) is *block-sequential*: each problem of the second block (1.4.30) (on the white subdomains) can be faced only after having solved those from the first block (1.4.29) (on the black subdomains). A *block-parallel* variant is easily achieved by replacing (1.4.30) by

$$\begin{cases} Lu_j^{k+1} = f & \text{in } \Omega_j, \quad \forall\, j \in I_W \\ \\ \Psi(u_j^{k+1}) = \Psi(u_i^k) & \text{on } \Gamma_{ij}, \quad \forall\, i \in I_B : \Gamma_{ij} \neq \emptyset. \end{cases} \tag{1.4.31}$$

However, as already pointed out in Remark 1.3.1 for the two-domain case, the block-parallel sequence generated by (1.4.29) and (1.4.31) contains as a subsequence the block-sequential sequence generated by (1.4.29) and (1.4.30).

A third, *fully parallel* algorithm would consist of solving both a Φ-type problem and a Ψ-type problem in *all* subdomains (investing at each step twice as much computational work as in the previous cases).

In this case, however, since at the end of each step the values of Φ and/or Ψ are not necessarily single-valued at subdomain interfaces, we need to adopt an averaging procedure at interfaces among the values of Φ from both sides (and similarly for Ψ).

The new algorithm can be defined as follows. Given the values $\{u_i^k \mid i = 1, \ldots, M\}$, define for each $i = 1, \ldots, M$

$$\Phi_i^{\text{av}} := \alpha\Phi(u_i^k) + (1-\alpha)\Phi(u_j^k) \qquad \text{on } \Gamma_{ij}, \tag{1.4.32}$$

where α is an averaging parameter, and Γ_{ij} is the part of $\partial\Omega_i$ separating Ω_i from Ω_j. Obviously, $\Phi_i^{\text{av}} = \Phi(u_i^k)$ on $\partial\Omega_i \cap \partial\Omega$.

Then solve the M subproblems of the Φ-type:

$$\begin{cases} Lu_i^{k+1/2} = f & \text{in } \Omega_i \\ \\ \Phi(u_i^{k+1/2}) = \Phi_i^{\text{av}} & \text{on } \partial\Omega_i \end{cases} \tag{1.4.33}$$

FIG. 1.5.1. Two examples of overlapping partitions.

for $i = 1, \ldots, M$.

Now compute for each i the value $\Psi(u_i^{k+1/2})$, and average them as follows

$$\Psi_i^{\rm av} := \beta\Psi(u_i^{k+1/2}) + (1-\beta)\Psi(u_j^{k+1/2}) \qquad \text{on } \Gamma_{ij}, \tag{1.4.34}$$

where β is another averaging parameter. Again, we set $\Psi_i^{\rm av} = \Psi(u_i^{k+1/2})$ on $\partial\Omega_i \cap \partial\Omega$.

Finally, solve the M subproblems of the Ψ-type:

$$\begin{cases} Lu_i^{k+1} = f & \text{in } \Omega_i \\ \\ \Psi(u_i^{k+1}) = \Psi_i^{\rm av} & \text{on } \partial\Omega_i \end{cases} \tag{1.4.35}$$

for $i = 1, \ldots, M$.

1.5 The Schwarz method for overlapping subdomains

The Schwarz method is undoubtedly the earliest example of a domain decomposition approach for partial differential equations. It was introduced by Schwarz (1869); however, among others, let us also mention the early contributions of Sobolev (1936); Mikhlin (1951); and Matsokin and Nepomnyaschikh (1985).

This time we decompose Ω into two overlapping subdomains Ω_1 and Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$, and denote by $\Gamma_1 := \partial\Omega_1 \cap \Omega_2$, $\Gamma_2 := \partial\Omega_2 \cap \Omega_1$, $\Omega_{1,2} := \Omega_1 \cap \Omega_2$ (see Fig. 1.5.1).

1.5.1 The multiplicative and additive forms of the Schwarz method

The original form of the Schwarz iterative procedure, known as the *alternating* Schwarz method, consists of solving successively the following problems.

Let u^0 be an initialisation function defined in Ω and vanishing on $\partial\Omega$, and set $\widehat{u}_2^0 := u^0_{|\Omega_2}$. For $k \geq 0$ we define two sequences $\widehat{u}_1^{k+1}$ and $\widehat{u}_2^{k+1}$ by solving respectively:

$$\begin{cases} -\Delta \widehat{u}_1^{k+1} = f & \text{in } \Omega_1 \\ \widehat{u}_1^{k+1} = \widehat{u}_2^{k} & \text{on } \Gamma_1 \\ \widehat{u}_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \end{cases} \tag{1.5.1}$$

and

$$\begin{cases} -\Delta \widehat{u}_2^{k+1} = f & \text{in } \Omega_2 \\ \widehat{u}_2^{k+1} = \widehat{u}_1^{k+1} & \text{on } \Gamma_2 \\ \widehat{u}_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega. \end{cases} \tag{1.5.2}$$

On the other hand, having set $\widehat{U}_1^0 := u^0_{|\Omega_1}$ and $\widehat{U}_2^0 := u^0_{|\Omega_2}$, we could make the two steps independent of each other by solving

$$\begin{cases} -\Delta \widehat{U}_1^{k+1} = f & \text{in } \Omega_1 \\ \widehat{U}_1^{k+1} = \widehat{U}_2^{k} & \text{on } \Gamma_1 \\ \widehat{U}_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \end{cases} \tag{1.5.3}$$

and

$$\begin{cases} -\Delta \widehat{U}_2^{k+1} = f & \text{in } \Omega_2 \\ \widehat{U}_2^{k+1} = \widehat{U}_1^{k} & \text{on } \Gamma_2 \\ \widehat{U}_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega. \end{cases} \tag{1.5.4}$$

The two approaches (1.5.1), (1.5.2) and (1.5.3), (1.5.4) form the basis of modern formulations of the Schwarz method, named respectively *multiplicative* and *additive*, which will be addressed in the finite dimensional case in Section 3.4.

The alternating Schwarz method (1.5.1), (1.5.2) converges to the solution u of (1.1), provided some mild assumptions on the subdomains Ω_1 and Ω_2 are satisfied. Precisely, there exist $C_1, C_2 \in (0,1)$ such that for all $k \geq 0$

$$||u_{|\Omega_1} - \widehat{u}_1^{k+1}||_{L^\infty(\Omega_1)} \leq C_1^k C_2^k ||u - \widehat{u}^0||_{L^\infty(\Gamma_1)} \tag{1.5.5}$$

$$||u_{|\Omega_2} - \widehat{u}_2^{k+1}||_{L^\infty(\Omega_2)} \leq C_1^{k+1} C_2^k ||u - \widehat{u}^0||_{L^\infty(\Gamma_2)}. \tag{1.5.6}$$

The error reduction constants C_1 and C_2 can be quite close to one if the overlapping region $\Omega_{1,2}$ is thin. In Fig. 1.5.2 we present an example showing how the

rate of convergence depends on the dimension of the overlapping region (denoted by a shaded interval). The Schwarz method is applied to approximate the (null) solution of the problem

$$\begin{cases} -u''(x) = 0 & \text{in } (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

FIG. 1.5.2. Error behaviour for the Schwarz method.

The proof of estimates (1.5.5), (1.5.6) can be obtained via the maximum principle (see, for example, Kantorovich and Krylov 1964, and P.-L. Lions 1989, where some convergence results for rather general geometrical configurations are also obtained). We will not adopt this point of view in our presentation, resorting instead to a variational interpretation due to P.-L. Lions (1988).

For an introduction and analysis of the additive Schwarz method we refer to Matsokin and Nepomnyaschikh (1985) and Dryja and Widlund (1990).

1.5.2 Variational interpretation of the Schwarz method

First of all, note that the alternating Schwarz method (1.5.1), (1.5.2) can obviously be generalised to the case of the symmetric elliptic operator (1.4.2). The variational formulation of the Schwarz method for the homogeneous Dirichlet boundary value problem associated with it can be stated as follows: set, as usual, $V := H_0^1(\Omega)$, $V_i^0 := H_0^1(\Omega_i)$, $i = 1, 2$, and moreover

$$V_i^* := \{v \in V \mid v = 0 \text{ in } \Omega \setminus \overline{\Omega_i}\}. \tag{1.5.7}$$

Method (1.5.1), (1.5.2) reads: given $u^0 \in V$, solve for each $k \geq 0$

$$
\begin{aligned}
&w_1^k \in V_1^0 \ : \ a_1^*(w_1^k, v_1) = (f, v_1)_{\Omega_1} - a_1^*(u^k, v_1) \qquad \forall\ v_1 \in V_1^0 \\
&u^{k+1/2} = u^k + \widetilde{w_1^k} \\
&w_2^k \in V_2^0 \ : \ a_2^*(w_2^k, v_2) = (f, v_2)_{\Omega_2} - a_2^*(u^{k+1/2}, v_2) \qquad \forall\ v_2 \in V_2^0 \\
&u^{k+1} = u^{k+1/2} + \widetilde{w_2^k},
\end{aligned}
\tag{1.5.8}
$$

where $\widetilde{w_i^k}$ denotes the extension of w_i^k by 0 in $\Omega \setminus \Omega_i$.

Similarly, method (1.5.3), (1.5.4) is obtained by solving

$$
\begin{aligned}
&W_1^k \in V_1^0 \ : \ a_1^*(W_1^k, v_1) = (f, v_1)_{\Omega_1} - a_1^*(U^k, v_1) \qquad \forall\ v_1 \in V_1^0 \\
&W_2^k \in V_2^0 \ : \ a_2^*(W_2^k, v_2) = (f, v_2)_{\Omega_2} - a_2^*(U^k, v_2) \qquad \forall\ v_2 \in V_2^0 \\
&U^{k+1} = U^k + \widetilde{W_1^k} + \widetilde{W_2^k},
\end{aligned}
\tag{1.5.9}
$$

having set $U^0 := u^0$.

The proof that these variational formulations are equivalent to the original ones (1.5.1), (1.5.2) and (1.5.3), (1.5.4) is obtained via the verification of the following relations:

$$
\begin{aligned}
u^{k+1/2} &= \begin{cases} \widehat{u}_1^{k+1} & \text{in } \Omega_1 \\ \widehat{u}^k_{2|\Omega_2 \setminus \Omega_1} & \text{in } \Omega \setminus \Omega_1 \end{cases} \\
u^{k+1} \quad &= \begin{cases} \widehat{u}_2^{k+1} & \text{in } \Omega_2 \\ \widehat{u}^{k+1}_{1|\Omega_1 \setminus \Omega_2} & \text{in } \Omega \setminus \Omega_2, \end{cases}
\end{aligned}
\tag{1.5.10}
$$

and

$$
U^{k+1} = \begin{cases} \widehat{U}^{k+1}_{1|\Omega_1 \setminus \Omega_2} & \text{in } \Omega \setminus \Omega_2 \\ \widehat{U}^{k+1}_{1|\Omega_{1,2}} + \widehat{U}^{k+1}_{2|\Omega_{1,2}} - U^k{}_{|\Omega_{1,2}} & \text{in } \Omega_{1,2} \\ \widehat{U}^{k+1}_{2|\Omega_2 \setminus \Omega_1} & \text{in } \Omega \setminus \Omega_2. \end{cases}
\tag{1.5.11}
$$

1.5.3 The Schwarz method as a projection method

Concerning the alternating Schwarz method (1.5.1), (1.5.2), using $(1.5.8)_1$ and $(1.5.8)_2$ we see that for each $v \in V_1^*$

$$
\begin{aligned}
a^*(u^{k+1/2} - u^k, v) = a^*(\widetilde{w_1^k}, v) &= a_1^*(w_1^k, v_{|\Omega_1}) \\
&= (f, v_{|\Omega_1})_{\Omega_1} - a_1^*(u^k, v_{|\Omega_1}) \\
&= (f, v) - a^*(u^k, v) \\
&= a^*(u - u^k, v).
\end{aligned}
$$

Similarly, from $(1.5.8)_3$, $(1.5.8)_4$ it follows that

$$
a^*(u^{k+1} - u^{k+1/2}, v) = a^*(u - u^{k+1/2}, v) \qquad \forall\ v \in V_2^*.
$$

Therefore, the sequences $u^{k+1/2}$ and u^{k+1} satisfy

$$\begin{aligned} u^{k+1/2} - u^k &= \mathcal{P}_1^*(u - u^k) \\ u^{k+1} - u^{k+1/2} &= \mathcal{P}_2^*(u - u^{k+1/2}), \end{aligned} \tag{1.5.12}$$

where $\mathcal{P}_i^*$, $i = 1, 2$ is the orthogonal projection of V onto V_i^* with respect to the scalar product induced by the bilinear form $a^*(\cdot, \cdot)$; that is, for any $w \in V$ it holds that

$$\mathcal{P}_i^* w \in V_i^* \ : \ a^*(\mathcal{P}_i^* w - w, v) = 0 \qquad \forall\ v \in V_i^*.$$

Let us denote by I the identity operator, by $\mathcal{J}_i$, $i = 1, 2$, the (non-dense) immersion of V_i^* into V (that is, $\mathcal{J}_i v = v$ for each $v \in V_i^*$), and by $\mathcal{J}_i^T$ its transpose operator; that is, the (non-injective) map from V' into $(V_i^*)'$ defined by

$$\langle \mathcal{J}_i^T F, v \rangle = \langle F, \mathcal{J}_i v \rangle \qquad \forall\ F \in V', v \in V_i^*. \tag{1.5.13}$$

If we set

$$\mathcal{P}_i := \mathcal{J}_i \mathcal{P}_i^* : V \to V, \tag{1.5.14}$$

from (1.5.12) it follows at once that

$$\begin{aligned} u^{k+1/2} &= (I - \mathcal{P}_1)u^k + \mathcal{P}_1 \mathcal{G} f \\ &= u^k + \mathcal{P}_1 \mathcal{G}(f - Lu^k) \end{aligned} \tag{1.5.15}$$

and

$$\begin{aligned} u^{k+1} &= (I - \mathcal{P}_2)u^{k+1/2} + \mathcal{P}_2 \mathcal{G} f \\ &= u^{k+1/2} + \mathcal{P}_2 \mathcal{G}(f - Lu^{k+1/2}), \end{aligned} \tag{1.5.16}$$

where $\mathcal{G}$ is the resolvent operator associated with problem (1.4.1); that is, $\mathcal{G} = L^{-1}$ and, in particular, $\mathcal{G} f = u$.

Therefore, the alternating Schwarz method (1.5.8) reads

$$\begin{aligned} u^{k+1} &= (I - \mathcal{P}_2)[(I - \mathcal{P}_1)u^k + \mathcal{P}_1 \mathcal{G} f] + \mathcal{P}_2 \mathcal{G} f \\ &= (I - \mathcal{P}_2)(I - \mathcal{P}_1)u^k + (I - \mathcal{P}_2)\mathcal{P}_1 \mathcal{G} f + \mathcal{P}_2 \mathcal{G} f \\ &= u^k + \mathcal{Q}_{\mathrm{m}}(\mathcal{G} f - u^k) = u^k + \mathcal{Q}_{\mathrm{m}} \mathcal{G}(f - Lu^k), \end{aligned} \tag{1.5.17}$$

where

$$\mathcal{Q}_{\mathrm{m}} := \mathcal{P}_1 + \mathcal{P}_2 - \mathcal{P}_2 \mathcal{P}_1. \tag{1.5.18}$$

In a similar way, the alternating Schwarz method (1.5.9) can be written as

$$\widetilde{W_1^k} = \mathcal{P}_1^*(u - U^k), \quad \widetilde{W_2^k} = \mathcal{P}_2^*(u - U^k),$$

therefore

$$\begin{aligned} U^{k+1} &= (I - \mathcal{P}_1 - \mathcal{P}_2)U^k + (\mathcal{P}_1 + \mathcal{P}_2)\mathcal{G}f \\ &= U^k + \mathcal{Q}_{\rm a}(\mathcal{G}f - U^k) = U^k + \mathcal{Q}_{\rm a}\mathcal{G}(f - LU^k), \end{aligned} \tag{1.5.19}$$

where

$$\mathcal{Q}_{\rm a} := \mathcal{P}_1 + \mathcal{P}_2. \tag{1.5.20}$$

Concerning the error equations for the Schwarz method (1.5.1), (1.5.2), from (1.5.15), (1.5.16) we have

$$\begin{aligned} u - u^{k+1/2} &= (I - \mathcal{P}_1)(u - u^k) \\ u - u^{k+1} \quad &= (I - \mathcal{P}_2)(u - u^{k+1/2}). \end{aligned} \tag{1.5.21}$$

Introducing the error $e^k := u - u^k$, the previous relations yield the error recursion formula:

$$e^{k+1} = (I - \mathcal{P}_2)(I - \mathcal{P}_1)e^k \qquad \forall\ k \geq 0. \tag{1.5.22}$$

These equations are the basis of the proof of the convergence of u^k and $u^{k+1/2}$ to u in $H^1(\Omega)$ (see Section 4.6).

Similarly, setting $E^k := u - U^k$ and using (1.5.19), for the Schwarz method (1.5.3), (1.5.4) it holds that

$$E^{k+1} = (I - \mathcal{P}_1 - \mathcal{P}_2)E^k \qquad \forall\ k \geq 0. \tag{1.5.23}$$

1.5.4 The Schwarz method as a Richardson method

On the basis of (1.5.17), the alternating Schwarz method (1.5.1), (1.5.2) can also be regarded as a Richardson iterative procedure for the solution of the new problem

$$\mathcal{Q}_{\rm m}u = g := \mathcal{Q}_{\rm m}\mathcal{G}f. \tag{1.5.24}$$

The multiplicative term $\mathcal{P}_2\mathcal{P}_1$ is responsible for $\mathcal{Q}_{\rm m}$ being a second-degree polynomial operator, and prevents the parallelisation of method (1.5.1), (1.5.2). As a matter of fact (1.5.1), (1.5.2) is a sequential algorithm. The presence of the term $\mathcal{P}_2\mathcal{P}_1$ justifies the adjective *multiplicative*, which is attributed to the alternating Schwarz method.

On the other hand, the algorithm (1.5.3), (1.5.4) is called the *additive* Schwarz method since it can be regarded as a Richardson iterative procedure for solving another problem that reads

$$\mathcal{Q}_{\rm a} u = g^* := \mathcal{Q}_{\rm a} \mathcal{G} f. \tag{1.5.25}$$

It is easy to see that $g^* = g_1^* + g_2^*$, where $g_1^* \in V_1^*$, $g_2^* \in V_2^*$ and

$$a^*(g_i^*, v_i) = (f, v_i) \qquad \forall\ v_i \in V_i^*, \quad i = 1, 2,$$

hence g^* can be computed by two local solves.

The projection operators $\mathcal{P}_i = \mathcal{J}_i \mathcal{P}_i^*$ are symmetric with respect to the scalar product induced by the form $a^*(\cdot, \cdot)$, because

$$\begin{aligned} a^*(\mathcal{P}_i v, w) &= a^*(\mathcal{P}_i^* v, w) \\ &= a^*(\mathcal{P}_i^* v, \mathcal{P}_i^* w) = a^*(v, \mathcal{P}_i^* w) \\ &= a^*(v, \mathcal{P}_i w) \qquad \forall\ v, w \in V. \end{aligned} \tag{1.5.26}$$

Consequently, the operator $\mathcal{Q}_{\rm a}$ is symmetric and positive definite with respect to the bilinear form $a^*(\cdot, \cdot)$. This property suggests basing on $\mathcal{Q}_{\rm a}^{-1}$ the construction, at the finite dimensional level, of a *preconditioner* of the original problem, see Section 3.5.

1.5.5 A characterisation of the projection operators

In view of the discretisation of the alternating Schwarz method that will be discussed in Section 3.4, it is useful to rewrite the projection operators and, consequently, $\mathcal{Q}_{\rm m}$ in a different form. This involves the original differential operator L, and its restrictions to the subspaces V_i^0. Let us introduce the operators $L_i : V_i^0 \to (V_i^0)'$ associated with the bilinear forms $a_i^*(\cdot, \cdot)$:

$$\langle L_i w_i, v_i \rangle := a_i^*(w_i, v_i) \qquad \forall\ w_i, v_i \in V_i^0, \tag{1.5.27}$$

as well as the extension operators $\rho_i^T : V_i^0 \to V_i^*$

$$\rho_i^T v_i := \widetilde{v_i} \qquad \forall\ v_i \in V_i^0 \tag{1.5.28}$$

and the transpose restriction operators $\rho_i : (V_i^*)' \to (V_i^0)'$

$$\langle \rho_i G, v_i \rangle := \langle G, \rho_i^T v_i \rangle \qquad \forall\ G \in (V_i^*)', v_i \in V_i^0. \tag{1.5.29}$$

It is easily seen that

$$L_i = \rho_i \mathcal{J}_i^T L \mathcal{J}_i \rho_i^T. \tag{1.5.30}$$

We claim that the following identities hold:

$$\mathcal{P}_i^* = \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L, \quad i = 1, 2. \tag{1.5.31}$$

In fact, define j_i^0 and j_i^*, the identification operators between V_i^0 and its dual $(V_i^0)'$ and V_i^* and its dual $(V_i^*)'$, respectively. In particular, we have that

$$j_i^0 = \rho_i j_i^* \rho_i^T, \quad (j_i^*)^{-1} = \rho_i^T (j_i^0)^{-1} \rho_i, \tag{1.5.32}$$

and that, from (1.5.30),

$$\mathcal{J}_i^T L \mathcal{J}_i = j_i^* \rho_i^T (j_i^0)^{-1} L_i (j_i^0)^{-1} \rho_i j_i^*. \tag{1.5.33}$$

Then, using (1.5.32) and (1.5.33) for each $v \in V_i^*, w \in V$ it holds that

$$\begin{aligned} a^*(\rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L w, v) &= a^*(\mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L w, \mathcal{J}_i v) \\ &= \langle L \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L w, \mathcal{J}_i v \rangle \\ &= \langle \mathcal{J}_i^T L \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L w, v \rangle \\ &= \langle \mathcal{J}_i^T L w, v \rangle = \langle L w, \mathcal{J}_i v \rangle \\ &= a^*(w, \mathcal{J}_i v) = a^*(w, v), \end{aligned}$$

and the proof of (1.5.31) is complete.

As a consequence, we also obtain

$$\mathcal{P}_i = \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T L, \quad \mathcal{P}_i \mathcal{G} = \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T. \tag{1.5.34}$$

1.5.6 The Schwarz method for many subdomains

The generalisation of the Schwarz method to the case of when Ω is partitioned into $M > 2$ subdomains (see Fig. 1.5.3 for an example) is straightforward.

FIG. 1.5.3. Overlapping decomposition into 16 subdomains.

Precisely, the multiplicative Schwarz method is given by

$$u^{k+\frac{i}{M}} = (I - \mathcal{P}_i) u^{k+\frac{i-1}{M}} + \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T f, \quad i = 1, \ldots, M \tag{1.5.35}$$

(here we have used (1.5.34)), and the additive Schwarz method reads

FIG. 1.6.1. Domain embedding.

$$U^{k+1} = \left(I - \sum_{i=1}^{M} \mathcal{P}_i \right) U^k + \sum_{i=1}^{M} \mathcal{J}_i \rho_i^T L_i^{-1} \rho_i \mathcal{J}_i^T f. \tag{1.5.36}$$

The corresponding error equations read

$$u - u^{k+1} = (I - \mathcal{P}_{\mathrm{m}}) \cdots (I - \mathcal{P}_1)(u - u^k) \tag{1.5.37}$$

for the multiplicative case, and

$$u - U^{k+1} = \left(I - \sum_{i=1}^{M} \mathcal{P}_i \right) (u - U^k) \tag{1.5.38}$$

for the additive case.

1.6 The fictitious domain method

This is one of the earliest ideas closely related to domain decomposition. The main motivation is that whenever a problem needs to be solved on a domain $\hat{\Omega}$ having a complex boundary, it may be useful to embed it into a larger domain Ω of a simpler shape; say, for instance, a rectangle as in Fig. 1.6.1, and then solve a problem of similar type in Ω (see, for example, Buzbee *et al.* 1971; Matsokin 1972; Proskurowski and Widlund 1976; Kuznetsov 1989; and Glowinski *et al.* 1994).

Suppose, for instance, that the given problem in $\hat{\Omega}$ is the following elliptic problem with a non-homogeneous Dirichlet boundary condition

$$\begin{cases} \hat{L}\hat{u} = \hat{f} & \text{in } \hat{\Omega} \\ \\ \hat{u} = \hat{\varphi}_D & \text{on } \partial\hat{\Omega}. \end{cases} \tag{1.6.1}$$

Here, we assume that $\hat{f} \in L^2(\hat{\Omega})$, $\hat{\varphi}_D \in H^{1/2}(\partial\hat{\Omega})$. The operator $\hat{L}$ is given by

$$\hat{L}\hat{w} := -\sum_{l,j=1}^{d} D_l(\hat{a}_{lj} D_j \hat{w}) + \hat{a}_0 \hat{w}, \tag{1.6.2}$$

where the coefficients $\hat{a}_{ij}$, $\hat{a}_0$ belong to $L^\infty(\hat{\Omega})$ and satisfy the ellipticity condition

$$\sum_{l,j=1}^{d} \hat{a}_{lj}(\mathbf{x})\xi_j\xi_l \geq \alpha_0|\boldsymbol{\xi}|^2 \qquad \forall\, \boldsymbol{\xi} \in \mathbf{R}^d, \text{ for almost all } \mathbf{x} \in \hat{\Omega}$$

for a suitable constant $\alpha_0 > 0$. Moreover, we assume that

$$\hat{a}_{lj}(\mathbf{x}) = \hat{a}_{jl}(\mathbf{x}) \qquad \forall\, l, j = 1, \ldots, d$$

for almost all $\mathbf{x} \in \hat{\Omega}$, and that $\hat{a}_0(\mathbf{x}) \geq 0$ for almost all $\mathbf{x} \in \hat{\Omega}$.

As a consequence, the associated bilinear form

$$\hat{a}^*(\hat{w}, \hat{v}) := \int_{\hat{\Omega}} \left(\sum_{l,j=1}^{d} \hat{a}_{lj} D_j \hat{w} D_l \hat{v} + \hat{a}_0\, \hat{w}\, \hat{v} \right)$$

turns out to be symmetric, continuous, and coercive in $H_0^1(\hat{\Omega})$.

Therefore, there exists a unique solution $\hat{u}$ of the corresponding weak formulation

$$\begin{cases} \text{find } \hat{u} \in H^1(\hat{\Omega}) \ : \\ \hat{a}^*(\hat{u}, \hat{v}) = \displaystyle\int_{\hat{\Omega}} \hat{f}\hat{v} \qquad \forall\, \hat{v} \in H_0^1(\hat{\Omega}) \\ \hat{u}_{|\partial\hat{\Omega}} = \hat{\varphi}_D \qquad \text{on } \partial\hat{\Omega}. \end{cases} \tag{1.6.3}$$

Let f, a_{lj}, a_0 be suitable extensions in Ω of $\hat{f}$, $\hat{a}_{lj}$ and $\hat{a}_0$, respectively, and consider the extended bilinear form

$$a^*(w, v) := \int_{\Omega} \left(\sum_{l,j=1}^{d} a_{lj} D_j w D_l v + a_0\, w\, v \right),$$

which is now defined in $H^1(\Omega)$. Then consider the extended problem in Ω:

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \ : \\ a^*(u, v) = \displaystyle\int_{\Omega} fv \qquad \forall\, v \in \hat{H}_0^1(\Omega) \\ u_{|\partial\hat{\Omega}} = \hat{\varphi}_D \qquad \text{on } \partial\hat{\Omega}, \end{cases} \tag{1.6.4}$$

where we have introduced the Hilbert space

$$\hat{H}_0^1(\Omega) := \{v \in H_0^1(\Omega) \,|\, v_{|\partial\hat{\Omega}} = 0\}. \tag{1.6.5}$$

If the extensions f, a_{ij}, a_0 satisfy in Ω the same assumptions that $\hat{f}$, $\hat{a}_{ij}$, $\hat{a}_0$ satisfy in $\hat{\Omega}$, then the bilinear form $a^*(\cdot,\cdot)$ is symmetric, continuous, and coercive in $\hat{H}_0^1(\Omega)$. Therefore, problem (1.6.4) has a unique solution u, whose restriction $u_{|\hat{\Omega}}$ coincides in $\hat{\Omega}$ with the solution $\hat{u}$ of the original problem (1.6.3).

Another point of view when addressing the fictitious domain method is based on the use of *Lagrange multipliers*. For instance, again referring to the Dirichlet problem (1.6.3), we can instead consider the following problem in the larger domain Ω:

$$\begin{cases} \text{find } u \in H_0^1(\Omega),\ \hat{\tau} \in H^{-1/2}(\partial\hat{\Omega}) : \\ a^*(u,v) + \displaystyle\int_{\partial\hat{\Omega}} \hat{\tau} v = \int_{\hat{\Omega}} f v \qquad \forall\ v \in H_0^1(\Omega) \\ \displaystyle\int_{\partial\hat{\Omega}} u\hat{\mu} = \int_{\partial\hat{\Omega}} \hat{\varphi}_D \hat{\mu} \qquad \forall\ \hat{\mu} \in H^{-1/2}(\partial\hat{\Omega}), \end{cases} \tag{1.6.6}$$

where $H^{-1/2}(\partial\hat{\Omega})$ is the dual space of the trace space $H^{1/2}(\partial\hat{\Omega})$. Clearly, the solution u to (1.6.6) satisfies $u_{|\hat{\Omega}} = \hat{u}$, and $\hat{\tau}$ is a Lagrange multiplier associated with the constraint $\hat{u}_{|\partial\hat{\Omega}} = \hat{\varphi}_D$. It represents the jump of the conormal derivative of u across $\partial\hat{\Omega}$; namely

$$\hat{\tau} = \frac{\partial u_{|\Omega\setminus\hat{\Omega}}}{\partial n_L^*} - \frac{\partial u_{|\hat{\Omega}}}{\partial n_L^*},$$

where $\mathbf{n}^*$ is the unit normal vector on $\partial\hat{\Omega}$ directed towards $\Omega \setminus \hat{\Omega}$. For this approach we refer, for example, to Glowinski *et al.* (1994).

The fictitious domain method can also be applied to the Neumann problem:

$$\begin{cases} \hat{L}\hat{u} = \hat{f} & \text{in } \hat{\Omega} \\ \dfrac{\partial \hat{u}}{\partial n_{\hat{L}}^*} = \hat{\varphi}_N & \text{on } \partial\hat{\Omega}, \end{cases} \tag{1.6.7}$$

where

$$\frac{\partial \hat{u}}{\partial n_{\hat{L}}^*} := \sum_{l,j=1}^{d} \hat{a}_{lj} D_j \hat{u}\, n_l^*$$

is the conormal derivative of $\hat{u}$. The data $\hat{a}_{lj}$, $\hat{a}_0$ and $\hat{f}$ satisfy the same assumptions as before, with the extra requirement that $\hat{a}_0(\mathbf{x}) \geq \mu_0 > 0$ for almost all $\mathbf{x} \in \hat{\Omega}$, to guarantee the existence and uniqueness of the solution. Furthermore, we suppose that $\hat{\varphi}_N \in L^2(\partial\hat{\Omega})$.

It is convenient to formulate problem (1.6.7) in a different form. Setting $\hat{p}_l := \sum_j \hat{a}_{lj} D_j \hat{u}$, we have

$$-\operatorname{div}\hat{\mathbf{p}} + \hat{a}_0\hat{u} = \hat{f} \qquad \text{in } \hat{\Omega}$$

and

$$\sum_{j=1}^{d} \hat{a}^{lj}\hat{p}_j - D_l\hat{u} = 0 \qquad \text{in } \hat{\Omega}, \quad \forall\, l = 1, \ldots, d,$$

where $\hat{a}^{lj}$ are the entries of the inverse matrix of $\{\hat{a}_{lj}\}$. Eliminating the unknown $\hat{u}$ we finally find

$$\sum_{j=1}^{d} \hat{a}^{lj}\hat{p}_j - D_l(\hat{a}_0^{-1}\operatorname{div}\hat{\mathbf{p}}) = D_l(\hat{a}_0^{-1}\hat{f}) \qquad \text{in } \hat{\Omega}, \quad \forall\, l = 1, \ldots, d. \tag{1.6.8}$$

Let us introduce the spaces

$$\begin{aligned} H(\operatorname{div};\hat{\Omega}) &:= \{\hat{\mathbf{q}} \in (L^2(\hat{\Omega}))^d \mid \operatorname{div}\hat{\mathbf{q}} \in L^2(\hat{\Omega})\} \\ H_0(\operatorname{div};\hat{\Omega}) &:= \{\hat{\mathbf{q}} \in H(\operatorname{div};\hat{\Omega}) \mid (\hat{\mathbf{q}}\cdot\mathbf{n}^*)_{|\partial\hat{\Omega}} = 0\}, \end{aligned}$$

and the bilinear form

$$\hat{\mathcal{B}}(\hat{\mathbf{w}}, \hat{\mathbf{q}}) := \int_{\hat{\Omega}} \sum_{l,j=1}^{d} (\hat{a}^{lj}\hat{w}_j\hat{q}_l + \hat{a}_0^{-1}\operatorname{div}\hat{\mathbf{w}}\ \operatorname{div}\hat{\mathbf{q}}).$$

The weak formulation of (1.6.8) reads

$$\begin{cases} \text{find } \hat{\mathbf{p}} \in H(\operatorname{div};\hat{\Omega}) \ : \\ \hat{\mathcal{B}}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = -\displaystyle\int_{\hat{\Omega}} \hat{a}_0^{-1}\hat{f}\ \operatorname{div}\hat{\mathbf{q}} \qquad \forall\ \hat{\mathbf{q}} \in H_0(\operatorname{div};\hat{\Omega}) \\ \hat{\mathbf{p}}\cdot\mathbf{n}^* = \hat{\varphi}_N \qquad \text{on } \partial\hat{\Omega}. \end{cases} \tag{1.6.9}$$

Introducing the space

$$\hat{H}(\operatorname{div};\Omega) := \{\mathbf{q} \in H(\operatorname{div};\Omega) \mid (\mathbf{q}\cdot\mathbf{n}^*)_{|\partial\hat{\Omega}} = 0\},$$

the fictitious domain formulation of (1.6.9) reads

$$\begin{cases} \text{find } \mathbf{p} \in H(\operatorname{div};\Omega) \ : \\ \mathcal{B}(\mathbf{p}, \mathbf{q}) = -\displaystyle\int_{\Omega} a_0^{-1} f\ \operatorname{div}\mathbf{q} \qquad \forall\ \mathbf{q} \in \hat{H}(\operatorname{div};\Omega) \\ \mathbf{p}\cdot\mathbf{n}^* = \hat{\varphi}_N \qquad \text{on } \partial\hat{\Omega}, \end{cases} \tag{1.6.10}$$

where $\mathcal{B}$ is defined through the extensions of the coefficients a_{lj} and a_0 to Ω.

All problems (1.6.4), (1.6.6) and (1.6.10) can be solved iteratively. For instance, the Lagrange multiplier $\hat{\tau}$ in (1.6.6) can be shown to solve a variational problem for (the inverse of) a suitable Steklov–Poincaré operator, symmetric and coercive in $H^{-1/2}(\partial\hat{\Omega})$. Therefore, it can be determined by means of preconditioned conjugate gradient iterations, which lead to the solution of a sequence of boundary value problems in the simple-shaped domain Ω (see Glowinski *et al.* 1994).

We will not discuss the fictitious domain approach further in what follows. Nonetheless, it is worthwhile to note that this method has received considerable attention in the past few years. For the early developments we refer to Marchuk *et al.* (1986). More recently, the analysis is carried out in several papers; in particular, in Börgers and Widlund (1990); Glowinski and Pan (1992); Glowinski *et al.* (1994); and Girault and Glowinski (1995). See also Girault *et al.* (1997) for a coupling of fictitious domain and domain decomposition techniques. Furthermore, see Kuznetsov (1997) for a discussion on the efficient preconditioned solution of systems arising from the Lagrange multiplier approach (1.6.6), and Rieder (1997) for a multi-scale Galerkin approximation to the variational approach (1.6.4) (or (1.6.10)).

1.7 The three-field method

The formulation that we present here is inspired by the so-called *hybrid* finite element formulation for elasticity problems (see, for example, Tong 1970), and has been introduced in the domain decomposition context by Brezzi and Marini (1994).

The original problem is reformulated by relaxing the continuity requirements on both u and $\frac{\partial u}{\partial n_L}$ (see (1.4.5)), at the expense of introducing two Lagrange multipliers for each subdomain.

The new weak formulation allows independent approximations within the subdomains, including the possibility of using different methods and different meshes from one subdomain to another. Still using the notations of Sections 1.2 and 1.4 we introduce the following formulation that generalises (1.4.8): for $i = 1, 2$ find $u_i \in V_i$, $\sigma_i \in \Lambda'$, $\lambda \in \Lambda$ such that

$$
\text{(1.7.1)} \qquad \begin{cases} a_1^*(u_1, v_1) - \langle \sigma_1, v_{1|\Gamma} \rangle_\Gamma = (f, v_1)_{\Omega_1} & \forall\ v_1 \in V_1 \\ \langle \rho_1, \lambda - u_{1|\Gamma} \rangle_\Gamma = 0 & \forall\ \rho_1 \in \Lambda' \\ \langle \sigma_1 + \sigma_2, \mu \rangle_\Gamma = 0 & \forall\ \mu \in \Lambda \\ \langle \rho_2, \lambda - u_{2|\Gamma} \rangle_\Gamma = 0 & \forall\ \rho_2 \in \Lambda' \\ a_2^*(u_2, v_2) - \langle \sigma_2, v_{2|\Gamma} \rangle_\Gamma = (f, v_2)_{\Omega_2} & \forall\ v_2 \in V_2. \end{cases}
$$

Here, we have denoted by $\langle \rho, \mu \rangle_\Gamma$ the duality pairing between Λ' and Λ; when $\rho \in L^2(\Gamma)$, then $\langle \rho, \mu \rangle_\Gamma = \int_\Gamma \rho\mu$.

In the formulation above, ρ_i is the Lagrange multiplier that is used to 'glue' the value of u_i and λ on Γ, $i = 1, 2$. As a matter of fact, the following equivalence result holds between the weak form of the unsplit problem (1.4.1) and the problem with Lagrange multipliers.

Proposition 1.7.1 *For every $f \in L^2(\Omega)$, problem* (1.7.1) *admits a unique solution (u_i, σ_i, λ), which is related to the solution of the problem*

$$u \in V \ : \ a^*(u, v) = (f, v) \qquad \forall\ v \in V \tag{1.7.2}$$

through the following relations:

$$\begin{aligned} & u_i = u_{|\Omega_i}, \quad i = 1, 2 \\ & \sigma_i = \left(\frac{\partial u}{\partial n_L^i} \right)_{|\Gamma}, \quad i = 1, 2 \\ & \lambda = u_{|\Gamma}, \end{aligned} \tag{1.7.3}$$

where $\left(\frac{\partial u}{\partial n_L^i}\right)_{|\Gamma}$ denotes the conormal derivative of u with respect to the unit normal vector $\mathbf{n}^i$ (issuing from Ω_i) on Γ.

Proof It is straightforward to prove that, if u is the solution to (1.7.2), then (u_i, σ_i, λ) is a solution to (1.7.1),

Uniqueness follows by inspecting the homogeneous problem. Therefore, let us take $f = 0$. From $(1.7.1)_2$ and $(1.7.1)_4$ we have that $\lambda = u_{1|\Gamma} = u_{2|\Gamma}$, thus the function

$$w := \begin{cases} u_1 & \text{in } \Omega_1 \\ u_2 & \text{in } \Omega_2 \end{cases}$$

belongs to $H_0^1(\Omega)$ and satisfies $w_{|\Gamma} = \lambda$. Taking $\mu = \lambda$ in $(1.7.1)_3$ and summing up $(1.7.1)_1$ (with $v_1 = u_1$) and $(1.7.1)_5$ (with $v_2 = u_2$) we obtain that

$$a_1^*(u_1, u_1) + a_2^*(u_2, u_2) = a^*(w, w) = 0,$$

whence $w = 0$ (by the coerciveness of the bilinear form $a^*(\cdot, \cdot)$), and therefore $u_i = 0$ and $\lambda = 0$. From $(1.7.1)_1$ we have now $\langle \sigma_1, v_{1|\Gamma} \rangle_\Gamma = 0$ for each $v_1 \in V_1$, which implies $\sigma_1 = 0$ on Γ. The last equality, $\sigma_2 = 0$ on Γ, follows similarly. □

There is a clear equivalence between the three-field method and the classical way of relaxing Dirichlet boundary conditions through Lagrange multipliers. Indeed, let us consider, for instance, equations $(1.7.1)_1$, $(1.7.1)_2$ for known values of f and λ. The problem reads: find $u_1 \in V_1$, $\sigma_1 \in \Lambda'$ such that

$$\begin{cases} a_1^*(u_1, v_1) - \langle \sigma_1, v_{1|\Gamma} \rangle_\Gamma = (f, v_1)_{\Omega_1} \qquad \forall\ v_1 \in V_1 \\ \\ \langle \rho_1, u_{1|\Gamma} \rangle_\Gamma = \langle \rho_1, \lambda \rangle_\Gamma \qquad \forall\ \rho_1 \in \Lambda', \end{cases} \tag{1.7.4}$$

and we see at once that this is the variational formulation of the boundary value problem

$$\begin{cases} Lu_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \\ u_1 = \lambda & \text{on } \Gamma, \end{cases}$$

where the last equation has been imposed via the Lagrange multiplier λ_1, which turns out to coincide with the conormal derivative of u_1 on Γ (see Babuška 1973).

2

DISCRETISED EQUATIONS AND DOMAIN DECOMPOSITION METHODS

In this chapter the general principles previously introduced will be accommodated to treat the finite dimensional approximation of differential problems.

We confine our analysis to the Poisson problem. However, its extension to the case of the more general symmetric elliptic boundary value problem (1.4.1) is straightforward and will be left to the interested reader. On the other hand, other kind of equations, either stationary or time-dependent, will be addressed from Chapter 5 onwards.

After presenting the Galerkin finite element approximation, we introduce the discrete Steklov–Poincaré operator and show that its algebraic counterpart is the Schur complement of the finite element stiffness matrix.

Our analysis will be restricted to the so-called conforming finite element spaces (see, for example, Ciarlet 1978). In the rest of the book, an exception will be made for the boundary value problems addressed in Section 5.4, where the approximation method can also be regarded as a mixed finite element method for the Laplace operator.

We also address the case of domain decompositions that are geometrically non-conforming, that is, where the subdomain grids do not match at the interfaces, and consider both the mortar method and the three-field method.

2.1 Finite element approximation of elliptic equations

In this section we give a brief presentation of the finite element approximation theory. For more details, we refer the interested reader, for example, to Ciarlet (1978) and Quarteroni and Valli (1994).

To start with, assume that the set $\Omega \subset \mathbf{R}^d$, $d = 2, 3$, is a *polygonal* domain, that is, Ω is a bounded open connected subset such that $\overline{\Omega}$ is the union of a finite number of polygons (for $d = 2$) or polyhedra (for $d = 3$).

The finite element approximation is based on a finite decomposition

$$\overline{\Omega} = \bigcup_{K \in \mathcal{T}_h} K,$$

where

- each K is a polygon or a polyhedron with a non-empty internal part $\overset{\circ}{K}$
- $\overset{\circ}{K}_1 \cap \overset{\circ}{K}_2 = \emptyset$ for each distinct $K_1, K_2 \in \mathcal{T}_h$

FIG. 2.1.1. Triangulation of Ω: admissible (left), non-admissible (right).

- if $F = K_1 \cap K_2 \neq \emptyset$ (K_1 and K_2 being distinct elements of $\mathcal{T}_h$) then F is a common face, side, or vertex of K_1 and K_2
- $\mathrm{diam}(K) \leq h$ for each $K \in \mathcal{T}_h$.

$\mathcal{T}_h$ is called a *triangulation* of $\overline{\Omega}$ (see Fig. 2.1.1).

In what follows we assume further that each element K of $\mathcal{T}_h$ can be obtained as $K = T_K(\hat{K})$, where $\hat{K}$ is a reference polygon or polyhedron and T_K is a suitable invertible affine map, i.e. $T_K(\hat{\mathbf{x}}) = B_K\hat{\mathbf{x}} + \mathbf{b}_K$, B_K being a non-singular matrix.

Moreover, we will confine ourselves to considering two different cases:

- The reference element $\hat{K}$ is the unit d-simplex; that is, the triangle with vertices (0,0), (1,0), (0,1) (when $d = 2$), or the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1) (when $d = 3$). As a consequence, each $K = T_K(\hat{K})$ is a triangle or a tetrahedron.
- The reference element $\hat{K}$ is the unit d-cube $[0,1]^d$. As a consequence, each $K = T_K(\hat{K})$ is a parallelogram (when $d = 2$) or a parallelepiped (when $d = 3$).

Let V_h denote a finite dimensional subspace of $H_0^1(\Omega)$. A Galerkin finite element approximation to (1.2.1) is defined as follows:

$$\text{find } u_h \in V_h \ : \ a(u_h, v_h) = (f, v_h) \qquad \forall\ v_h \in V_h. \tag{2.1.1}$$

The most frequent example is when V_h is given by piecewise polynomials. They can be introduced as follows. When the reference element $\hat{K}$ is the unit d-simplex, let us define

$$X_h^r := \{v_h \in C^0(\overline{\Omega}) \mid v_{h|K} \in \mathbf{P}_r(K)\ \forall\ K \in \mathcal{T}_h\}, \quad r \geq 1, \tag{2.1.2}$$

where $\mathbf{P}_r(K)$ denotes the set of polynomials defined in K and of degree less than or equal to r globally with respect to all space coordinates. Then we set

$$\begin{aligned} V_h :&= \{v_h \in X_h^r \mid v_{h|\partial\Omega} = 0\} \\ &= X_h^r \cap H_0^1(\Omega). \end{aligned} \tag{2.1.3}$$

When the reference element $\hat{K}$ is the unit d-cube, the space V_h is defined in the same way, but in this case the space X_h^r is given by

$$X_h^r := \{v_h \in C^0(\overline{\Omega}) \mid v_{h|K} \circ T_K \in \mathbf{Q}_r(K) \ \forall\ K \in \mathcal{T}_h\}, \tag{2.1.4}$$

where $\mathbf{Q}_r(K)$ denotes the set of polynomials defined in K and of degree less than or equal to r with respect to each variable $x_1, \ldots, x_d$.

The family of triangulations $\mathcal{T}_h$ is said to be regular if there exists a constant $\sigma \geq 1$ such that

$$\frac{h_K}{\rho_K} \leq \sigma \qquad \forall\ K \in \mathcal{T}_h, \quad \forall\ h > 0,$$

where h_K denotes the diameter of K and ρ_K the maximum diameter of a ball contained in K. Under this assumption, denoting by $\pi_h v \in V_h$ the interpolant of a continuous function v at the nodes of $\mathcal{T}_h$, the following interpolation error estimate holds:

$$||u - \pi_h u||_{0,\Omega} + h||u - \pi_h u||_{1,\Omega} \leq Ch^{r+1}|u|_{r+1,\Omega}. \tag{2.1.5}$$

It is well known from the Lax–Milgram lemma that problem (2.1.1) has a unique solution under the assumption that the bilinear form $a(\cdot,\cdot)$ is continuous and coercive in $H_0^1(\Omega)$. Under the further assumption that $a(\cdot,\cdot)$ is symmetric, (2.1.1) is a minimisation problem, because its solution u_h minimises the discrete energy

$$\mathcal{J}(v_h) := \frac{1}{2}a(v_h, v_h) - (f, v_h)$$

over the finite dimensional space V_h.

Besides, from the Céa lemma it follows that

$$||u - u_h||_{1,\Omega} \leq \frac{\gamma}{\alpha} \inf_{v_h \in V_h} ||u - v_h||_{1,\Omega},$$

where γ and α are the continuity and coerciveness constants of $a(\cdot,\cdot)$, respectively. By the interpolation error estimate (2.1.5), we finally find that

$$||u - u_h||_{1,\Omega} \leq Ch^r|u|_{r+1,\Omega}, \tag{2.1.6}$$

provided that $u \in H^{r+1}(\Omega)$.

2.1.1 The multi-domain formulation for finite elements

As we have done in Chapter 1, we split Ω into two subdomains Ω_1 and Ω_2, such that $\overline{\Omega_1} \cup \overline{\Omega_2} = \overline{\Omega}$, $\Omega_1 \cap \Omega_2 = \emptyset$, and we set $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ (see, for example, Fig. 1.1). We also suppose that the interface Γ does not cut any finite element T. This implies that the global triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ induces two triangulations $\mathcal{T}_{h,1}$ of $\overline{\Omega_1}$ and $\mathcal{T}_{h,2}$ of $\overline{\Omega_2}$ that are compatible on Γ; that is, they share the same edges on Γ (see Fig. 2.1.2, where another domain, different from those in Fig. 1.1, is considered).

FIG. 2.1.2. Splitting of Ω and finite element triangulation.

Let us start with the variational formulation of our problem (an alternative characterisation based on a purely algebraic argument is given in Section 2.3). To this purpose let us define

$$\Lambda_h := \{v_{h|\Gamma} \mid v_h \in V_h\}, \quad V_{i,h} := \{v_{h|\Omega_i} \mid v_h \in V_h\}, \tag{2.1.7}$$

and set

$$V_{i,h}^0 := \{v_h \in V_{i,h} \mid v_{h|\Gamma} = 0\}. \tag{2.1.8}$$

By repeating the lines of the proof of Lemma 1.2.1, we see that the finite element problem (2.1.1), after identifying $u_{1,h}$ with $u_{h|\Omega_1}$ and $u_{2,h}$ with $u_{h|\Omega_2}$, is equivalent to the multi-domain problem

$$\begin{cases} a_1(u_{1,h}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} & \forall\ v_{1,h} \in V_{1,h}^0 \\ u_{1,h} = u_{2,h} \quad \text{on } \Gamma & \\ a_2(u_{2,h}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} & \forall\ v_{2,h} \in V_{2,h}^0 \\ \displaystyle\sum_{i=1}^{2} a_i(u_{i,h}, \mathcal{R}_{i,h}\mu_h) = \sum_{i=1}^{2} (f, \mathcal{R}_{i,h}\mu_h)_{\Omega_i} & \forall\ \mu_h \in \Lambda_h, \end{cases} \tag{2.1.9}$$

where $\mathcal{R}_{i,h}$, $i = 1, 2$, is any extension operator from Λ_h into $V_{i,h}$. In practical implementation, these extension operators will be taken equal to the finite element interpolant $\pi_{i,h}\mu_h$, which belongs to $V_{i,h}$, equals μ_h at the nodes on the interface Γ, and vanishes at the internal nodes in Ω_i.

The formulation (2.1.9) may be generalised to many subdomains, possibly including cross-points. Note, in particular, that if $\mathcal{R}_{i,h}\mu_h$ is the restriction to Ω_i of the finite element shape function associated with a cross-point P, then $(2.1.9)_4$ enforces the continuity of the 'normal' derivative in P in a natural form.

Remark 2.1.1 A multi-domain approach based on a dual variational formulation and the use of Lagrange multipliers, called FETI (Finite Element Tearing and Interconnecting), has been proposed by Farhat and Roux (1991, 1994), and is extensively used nowadays. □

2.1.2 Algebraic formulation of the discrete problem

The unknowns of the finite dimensional problem (2.1.1) are given by the point-values of u_h at the finite element nodes $\mathbf{a}_j$. In fact, denoting by N_h the total number of the nodes and by φ_j the basis functions of V_h; that is, the unique functions in V_h satisfying $\varphi_j(\mathbf{a}_i) = \delta_{ij}$ for each $i, j = 1, \ldots, N_h$, each element $u_h \in V_h$ can be represented through

$$u_h(\mathbf{x}) = \sum_{j=1}^{N_h} u_h(\mathbf{a}_j)\varphi_j(\mathbf{x}). \tag{2.1.10}$$

Introducing the notation

$$\mathbf{u} := \{u_h(\mathbf{a}_j)\}_{j=1,\ldots,N_h} \tag{2.1.11}$$

and

$$\mathbf{f} := \{(f, \varphi_j)\}_{j=1,\ldots,N_h}, \tag{2.1.12}$$

problem (2.1.1) can be rewritten as

$$A\mathbf{u} = \mathbf{f}. \tag{2.1.13}$$

The matrix A is called the finite element *stiffness* matrix and is given by

$$A_{lj} := a(\varphi_j, \varphi_l), \quad l, j = 1, \ldots, N_h. \tag{2.1.14}$$

The stiffness matrix A is positive definite; that is, for any $\mathbf{v} \in \mathbf{R}^{N_h}$, $\mathbf{v} \neq \mathbf{0}$, $(A\mathbf{v}, \mathbf{v}) > 0$, where $(\cdot, \cdot)$ denotes the Euclidean scalar product. Indeed, let $v_h \in V_h$ be the function defined as

$$v_h(\mathbf{x}) = \sum_{j=1}^{N_h} v_j \, \varphi_j(\mathbf{x}).$$

Then

$$\begin{aligned}(A\mathbf{v}, \mathbf{v}) &= \sum_{l,j=1}^{N_h} v_i \, a(\varphi_j, \varphi_l) \, v_j \\ &= a(v_h, v_h) \geq 0,\end{aligned}$$

and $(A\mathbf{v}, \mathbf{v}) = 0$ if and only if $v_h = 0$, or, equivalently, $\mathbf{v} = \mathbf{0}$.

In particular, any eigenvalue of A has a positive real part.

Besides, when the bilinear form $a(\cdot,\cdot)$ is symmetric (and this is the case in (1.2.1)), it follows immediately that A is also symmetric.

Another important remark is concerned with the condition number

$$\kappa_2(A) := ||A||_2 \, ||A^{-1}||_2 = \frac{\sqrt{\lambda_{\max}(A^T A)}}{\sqrt{\lambda_{\min}(A^T A)}}. \tag{2.1.15}$$

In the symmetric case, we have the simplified relation

$$\kappa_2(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},$$

and it can be proved that

$$\kappa_2(A) = O(h^{-2}). \tag{2.1.16}$$

2.2 Finite element approximation of the Steklov–Poincaré operator

In the framework of the finite element problem (2.1.1), the finite dimensional counterpart of the Steklov–Poincaré operator can be easily characterised. For each $\eta_h \in \Lambda_h$ let us introduce its *finite element harmonic extension* in Ω_i ($i = 1, 2$):

$$\begin{cases} H_{i,h}\eta_h \in V_{i,h} \ : \\ a_i(H_{i,h}\eta_h, v_{i,h}) = 0 \qquad \forall \ v_{i,h} \in V_{i,h}^0 \\ H_{i,h}\eta_h{}_{|\Gamma} = \eta_h \qquad \text{on } \Gamma. \end{cases} \tag{2.2.1}$$

Similarly, we define

$$\mathcal{G}_{i,h} f \in V_{i,h}^0 \ : a_i(\mathcal{G}_{i,h} f, v_{i,h}) = (f, v_{i,h})_{\Omega_i} \quad \forall \ v_{i,h} \in V_{i,h}^0. \tag{2.2.2}$$

In other words, $\mathcal{G}_{i,h} f$ is the finite element solution of the homogeneous Dirichlet boundary value problem ($i = 1, 2$)

$$\begin{cases} -\Delta \mathcal{G}_{i,h} f = f \quad \text{in } \Omega_i \\ \mathcal{G}_{i,h} f = 0 \qquad \text{on } \partial\Omega_i. \end{cases}$$

Now we can define (formally) the finite element approximation of (1.1.7) as follows:

$$S_h \lambda_h = \chi_h \quad \text{on } \Gamma, \tag{2.2.3}$$

where

$$\begin{aligned}\chi_h &:= -\sum_{i=1}^{2} \frac{\partial}{\partial n^i}\mathcal{G}_{i,h} f \\ S_h\eta_h &:= \sum_{i=1}^{2} S_{i,h}\eta_h \\ S_{i,h}\eta_h &:= \sum_{i=1}^{2} \frac{\partial}{\partial n^i} H_{i,h}\eta_h.\end{aligned}$$

As in (1.2.7) and (1.2.12) we can easily get for all $\eta_h, \mu_h \in \Lambda_h$,

$$\begin{aligned}\langle S_h\eta_h, \mu_h\rangle &= \sum_{i=1}^{2} a_i(H_{i,h}\eta_h, \mathcal{R}_{i,h}\mu_h) \\ &= \sum_{i=1}^{2} a_i(H_{i,h}\eta_h, H_{i,h}\mu_h) = \sum_{i=1}^{2}\langle S_{i,h}\eta_h, \mu_h\rangle,\end{aligned} \tag{2.2.4}$$

and

$$\langle \chi_h, \mu_h\rangle = \sum_{i=1}^{2}[(f, \mathcal{R}_{i,h}\mu_h)_{\Omega_i} - a_i(\mathcal{G}_{i,h} f, \mathcal{R}_{i,h}\mu_h)],$$

where the discrete extension operators $\mathcal{R}_{i,h}$ were introduced in (2.1.9).

Therefore, the operators S_h and $S_{i,h}$, $i = 1, 2$, are *symmetric*. Moreover, due to the Poincaré inequality (1.2.2) and the trace inequality (1.2.5) it follows that

$$\langle S_{i,h}\eta_h, \eta_h\rangle = ||\nabla H_{i,h}\eta_h||^2_{0,\Omega_i} \geq \frac{1}{(1 + C_{\Omega_i})(C_i^*)^2}||\eta_h||^2_\Lambda,$$

namely, the Steklov–Poincaré operators are *coercive*, uniformly with respect to h.

In Theorem 4.1.3 it will be shown that there exist two positive constants $\hat{C}_1$, $\hat{C}_2$, independent of h, such that

$$\hat{C}_1||\eta_h||_\Lambda \leq ||H_{i,h}\eta_h||_{1,\Omega_i} \leq \hat{C}_2||\eta_h||_\Lambda \qquad \forall\ \eta_h \in \Lambda_h, \quad i = 1, 2. \tag{2.2.5}$$

This property is known as the *finite element uniform extension theorem* (see, for example, Bramble *et al.* 1986*a*; Bjørstad and Widlund (1986); or Marini and Quarteroni 1988, 1989). In view of (2.2.4) we also immediately obtain that there exists a constant $\hat{C} > 0$, independent of h, such that, for all $\eta_h \in \Lambda_h$

$$\langle S_{i,h}\eta_h, \eta_h\rangle \leq \langle S_h\eta_h, \eta_h\rangle \leq \hat{C}\langle S_{i,h}\eta_h, \eta_h\rangle, \quad i = 1, 2. \tag{2.2.6}$$

In other words, the operators $S_{i,h}$ and S_h are spectrally equivalent, and therefore either $S_{1,h}$ or $S_{2,h}$ can serve as an *optimal preconditioner* of S_h (see also the related result (2.3.14) below, which deals with the corresponding matrix problem).

2.2.1 Eigenvalue analysis for the finite element Steklov–Poincaré operator

Let us introduce the eigenvalue problem for the discrete Steklov–Poincaré operator S_h: find $\delta_h \in \mathbf{R}$ and $\omega_h \in \Lambda_h$, $\omega_h \neq 0$, such that

$$\langle S_h \omega_h, \mu_h \rangle = \delta_h \int_\Gamma \omega_h \, \mu_h \qquad \forall \ \mu_h \in \Lambda_h. \tag{2.2.7}$$

Let us indicate by N_Γ the dimension of the interface space Λ_h. For example, in the case of piecewise-linear finite elements ($r = 1$), N_Γ is given by the number of finite element nodes lying on Γ, excluding the nodes on the boundary $\partial\Omega$ (see Section 2.3).

We have the following result:

Proposition 2.2.1 *Since the bilinear form $a(\cdot,\cdot)$ is symmetric, continuous, and coercive in V, the associated discrete Steklov–Poincaré operator S_h has real and positive eigenvalues $\delta_{j,h}$, $j = 1, \ldots, N_\Gamma$, which obey the following bounds*

$$\hat{K}_1 \leq \delta_{j,h} \leq \hat{K}_2 h^{-1}, \tag{2.2.8}$$

where $\hat{K}_1$ and $\hat{K}_2$ are two positive constants, both independent of h.

Proof Since S_h is symmetric and coercive in Λ, its eigenvalues are real and positive, and are given by the Rayleigh quotient

$$\delta_{j,h} = \frac{\langle S_h \omega_{j,h}, \omega_{j,h} \rangle}{\int_\Gamma \omega_{j,h}^2}, \tag{2.2.9}$$

where $\omega_{j,h} \in \Lambda_h$ denotes the eigenvector associated with $\delta_{j,h}$. Using (2.2.4), the first inequality in (2.2.5) and Poincaré inequality (1.2.2) it follows that

$$\delta_{j,h} \geq \hat{K}_1 \frac{||\omega_{j,h}||_\Lambda^2}{\int_\Gamma \omega_{j,h}^2} \geq \hat{K}_1,$$

having set $\hat{K}_1 := \hat{C}_1^2 \sum_i (1 + C_{\Omega_i})^{-1}$. On the other hand, using now the second inequality in (2.2.5) we obtain

$$\langle S_h \omega_{j,h}, \omega_{j,h} \rangle = \sum_{i=1}^{2} ||H_{i,h} \omega_{j,h}||_{1,\Omega_i}^2 \leq 2\hat{C}_2^2 ||\omega_{j,h}||_\Lambda^2.$$

By following the lines of the proof presented in Ciarlet (1978) and recalling that the space Λ is a subspace of $H^{1/2}(\Gamma)$, we can obtain the inverse inequality

$$||\eta_h||_\Lambda \leq C h^{-1/2} \left(\int_\Gamma \eta_h^2 \right)^{1/2},$$

which yields at once (2.2.8). □

It can be proved that the bounds of $\delta_{j,h}$ are sharp, and indeed cannot be improved (see, for example, Bjørstad and Widlund 1986; and Smith *et al.* 1996, pp. 119–20, for piecewise-linear finite element or five-point finite difference approximation of the Poisson equation on an uniform grid; see also Section 3.3.1 and Brenner 1998*a*).

2.3 Algebraic formulation of the discrete Steklov–Poincaré operator: the Schur complement matrix

In order to give the algebraic interpretation of the finite dimensional operator S_h let us distinguish between the finite element nodes belonging to Γ and those of Ω_1 and Ω_2. We denote the corresponding vector of the finite element unknowns by $\mathbf{u}_\Gamma$, $\mathbf{u}_1$ and $\mathbf{u}_2$, respectively, and their lengths by N_Γ, N_1 and N_2 (see Fig. 2.1.2 for an illustrative example, where bold nodes indicate finite element nodes belonging to Γ). Analogously, the vector corresponding to the datum f will be denoted by $\mathbf{f} \in \mathbf{R}^{N_h}$.

As we have already seen, the finite element problem (2.1.1) can be written in the algebraic form (2.1.13), where A is the $N_h \times N_h$ stiffness matrix, with $N_h = N_\Gamma + N_1 + N_2$.

Our system can be written in block form as

$$\begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_\Gamma \end{pmatrix}, \tag{2.3.1}$$

where we have used the following notations: for $i = 1, 2$

$$(A_{ii})_{lj} := a_i(\varphi_j^{(i)}, \varphi_l^{(i)}), \quad l, j = 1, \ldots, N_i, \tag{2.3.2}$$

where $a_i(\cdot, \cdot)$ is the restriction of the bilinear form $a(\cdot, \cdot)$ to Ω_i defined in (1.2.4), and $\varphi_j^{(i)}$ are the basis functions associated with the nodes lying in Ω_i. Moreover, we have

$$(A_{\Gamma\Gamma})_{sr} := a_1(\varphi_r^{(\Gamma)}, \varphi_s^{(\Gamma)}) + a_2(\varphi_r^{(\Gamma)}, \varphi_s^{(\Gamma)}), \quad s, r = 1, \ldots, N_\Gamma, \tag{2.3.3}$$

where $\varphi_r^{(\Gamma)}$ are the basis functions associated with the nodes lying on Γ. For $i = 1, 2$ we have set, moreover,

$$(A_{i\Gamma})_{lr} := a_i(\varphi_r^{(\Gamma)}, \varphi_l^{(i)}), \quad l = 1, \ldots, N_i, r = 1, \ldots, N_\Gamma, \tag{2.3.4}$$

while $A_{\Gamma i}$ denotes the transpose of $A_{i\Gamma}$, $i = 1, 2$.

The algebraic system (2.3.1) can be regarded as directly associated with the set of equations (2.1.9), where the extension $\mathcal{R}_{i,h}\mu_h$ has been taken equal to the finite element interpolant $\pi_{i,h}\mu_h$, which takes values 0 at all the nodes internal at Ω_i. In fact, note that we can write

$$u_{i,h} = \hat{u}_{i,h} + \pi_{i,h}\lambda_h, \quad i = 1, 2,$$

where $\hat{u}_{i,h} \in V^0_{i,h}$ takes the same values of $u_{i,h}$ at the nodes internal at Ω_i and value 0 at the nodes on Γ, and $\lambda_h \in \Lambda_h$ is the common value of $u_{1,h}$ and $u_{2,h}$ on Γ. Then, denoting by $\mathbf{u}_i \in \mathbf{R}^{N_i}$ the vector of grid values of $u_{i,h}$ in Ω_i, and by $\mathbf{u}_\Gamma \in \mathbf{R}^{N_\Gamma}$ that of $u_{i,h}$ on Γ, we obtain (2.3.1) from (2.1.9).

After eliminating $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain the reduced system:

$$\Sigma_h \mathbf{u}_\Gamma = \boldsymbol{\chi}_\Gamma, \tag{2.3.5}$$

with

$$\boldsymbol{\chi}_\Gamma := \mathbf{f}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\mathbf{f}_1 - A_{\Gamma 2}A_{22}^{-1}\mathbf{f}_2 \tag{2.3.6}$$

and

$$\Sigma_h := A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}. \tag{2.3.7}$$

Once the solution $\mathbf{u}_\Gamma$ of (2.3.5) is available, the subdomain components $\mathbf{u}_1$ and $\mathbf{u}_2$ can be immediately recovered from (2.3.1) at the expense of two independent solves A_{11}^{-1} and A_{22}^{-1}.

We can split the matrix of the interface contributions as follows

$$A_{\Gamma\Gamma} = A^{(1)}_{\Gamma\Gamma} + A^{(2)}_{\Gamma\Gamma}, \tag{2.3.8}$$

where $A^{(i)}_{\Gamma\Gamma}$ denotes the contribution from the subdomain Ω_i, $i = 1, 2$. Then we can write

$$\Sigma_h = \Sigma_{1,h} + \Sigma_{2,h}, \tag{2.3.9}$$

with

$$\Sigma_{i,h} := A^{(i)}_{\Gamma\Gamma} - A_{\Gamma i}A_{ii}^{-1}A_{i\Gamma}, \quad i = 1, 2. \tag{2.3.10}$$

The matrix Σ_h is the *Schur complement matrix*: it is precisely the algebraic counterpart of the discrete Steklov–Poincaré operator S_h. In turn, (2.3.5) is called the Schur complement system. Σ_h is symmetric and positive definite, because it results from (2.3.7), and also from the relationship

$$[\Sigma_h \boldsymbol{\eta}, \boldsymbol{\mu}] = \langle S_h \eta_h, \mu_h \rangle \quad \forall\ \eta_h, \mu_h \in \Lambda_h, \tag{2.3.11}$$

where $[\cdot, \cdot]$ is the Euclidean scalar product in $\mathbf{R}^{N_\Gamma}$, and for each $\mu_h \in \Lambda_h$, $\boldsymbol{\mu}$ denotes the set of its values at the nodes on Γ. Similarly, we have for $i = 1, 2$

$$[\Sigma_{i,h} \boldsymbol{\eta}, \boldsymbol{\mu}] = \langle S_{i,h} \eta_h, \mu_h \rangle \quad \forall\ \eta_h, \mu_h \in \Lambda_h. \tag{2.3.12}$$

For each $i = 1, 2$, the matrix $\Sigma_{i,h}$ is dense: to compute its entries is expensive because we need as many solves as the number of nodes on Γ. Indeed, from

(2.3.12) we see that for each node on Γ we should compute the action of $S_{i,h}$ on the Lagrangian basis function associated with that node.

When the Schur complements are explicitly computed, we have the direct method called *substructuring*. It is common practice in the engineering community, and corresponds to having all interiors nodes eliminated by *static condensation* (see, for example, Przemieniecki 1985).

Most often, however, especially when the number of nodes lying on the interface Γ is high, conjugate gradient methods are used for an iterative solution of the Schur complement system (2.3.5). Each matrix–vector multiplication with Σ_h involves two subdomain solves, A_{11}^{-1} and A_{22}^{-1}, which can be performed in parallel.

However, the matrix Σ_h is ill-conditioned. Actually, it inherits a number of properties from S_h; in particular, from (2.2.8), its *spectral condition number* $\kappa(\Sigma_h)$ (that is, the ratio between the maximum and minimum eigenvalue) is bounded by

$$\kappa(\Sigma_h) \leq \widehat{C}_0 h^{-1} \tag{2.3.13}$$

for a suitable constant $\widehat{C}_0$ independent of h. The use of preconditioners is therefore mandatory. Their construction is addressed in Section 3.3.

At this stage, we can already propose the preconditioners $\Sigma_{i,h}$, $i = 1, 2$, which, from (2.3.11), (2.3.12) and (2.2.6), satisfy

$$\kappa\left(\Sigma_{i,h}^{-1}\Sigma_h\right) \leq \hat{C}, \quad i = 1, 2. \tag{2.3.14}$$

Hence, either $\Sigma_{1,h}$ or $\Sigma_{2,h}$ are *optimal preconditioners* of Σ_h.

Remark 2.3.1 In the special case of a scalar second-order elliptic boundary value problem with constant coefficients, using a uniform mesh and two equal-sized subdomains that are mirror images of each other (about Γ), it follows that $\Sigma_{1,h} = \Sigma_{2,h} = \frac{1}{2}\Sigma_h$. Thus, either $\Sigma_{1,h}$ or $\Sigma_{2,h}$ are ideal preconditioners for Σ_h. □

In Table 2.3.1 we summarise the relation between the Galerkin and the algebraic forms of the single domain problem, the multi-domain problem and the interface equation.

In the forthcoming Section 2.3.1 we show how to obtain a preconditioner for the stiffness matrix A once a preconditioner for the Schur complement Σ_h is available.

2.3.1 Preconditioners of the stiffness matrix derived from preconditioners of the Schur complement matrix

After (2.3.1) and (2.3.8), the stiffness matrix in the two-substructure case reads

$$A = \begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)} \end{pmatrix}. \tag{2.3.15}$$

Table 2.3.1 *The Galerkin and the algebraic forms of the single domain, the multi-domain and the interface problems*

	Galerkin vs algebraic	
Single domain (stiffness)	$u_h \in V_h \ : \ \forall\ v_h \in V_h$ $a(u_h, v_h) = (f, v_h)$	$A\mathbf{u} = \mathbf{f}$
Multi-domain (block stiffness)	$u_{1,h} \in V_{1,h} \ : \ \forall\ v_{1,h} \in V^0_{1,h}$ $a_1(u_{1,h}, v_{1,h}) = (f, v_{1,h})_{\Omega_1}$ $u_{2,h} \in V_{2,h} \ : \ \forall\ v_{2,h} \in V^0_{2,h}$ $a_2(u_{2,h}, v_{2,h}) = (f, v_{2,h})_{\Omega_2}$ $u_{1,h} = u_{2,h} \quad \text{on } \Gamma$ $\sum_i a_i(u_{i,h}, \pi_{i,h}\mu_h)$ $= \sum_i (f, \pi_{i,h}\mu_h)_{\Omega_i} \ \forall\ \mu_h \in \Lambda_h$	$\begin{bmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_\Gamma \end{bmatrix}$ $= \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_\Gamma \end{bmatrix}$
Interface problem (Schur complement)	$\lambda_h \in \Lambda_h \ : \ \forall\ \mu_h \in \Lambda_h$ $\langle S_h \lambda_h, \mu_h \rangle = \langle \chi_h, \mu_h \rangle$	$\Sigma_h \mathbf{u}_\Gamma = \boldsymbol{\chi}_\Gamma$

It can be expressed in factored form as

$$A = LDL^T,$$

where, denoting by I_1, I_2 and I_Γ the identity matrix of dimension N_1, N_2 and N_Γ, respectively,

$$L := \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ A_{\Gamma 1}A_{11}^{-1} & A_{\Gamma 2}A_{22}^{-1} & I_\Gamma \end{pmatrix}$$

(2.3.16)

$$D := \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & \Sigma_{1,h} + \Sigma_{2,h} \end{pmatrix}.$$

Equivalently, we have the following block LU decomposition

$$A = LU,$$

where

$$U := DL^T = \begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ 0 & 0 & \Sigma_{1,h} + \Sigma_{2,h} \end{pmatrix}. \tag{2.3.17}$$

Assume that a convenient preconditioner P_h is available for the Schur complement matrix Σ_h. Then we can devise the following preconditioner Q_h of A:

$$Q_h := L\widetilde{U}, \tag{2.3.18}$$

where L is given in (2.3.16) and $\widetilde{U}$ is obtained from U in (2.3.17) by approximating Σ_h with P_h; that is,

$$\widetilde{U} := \begin{pmatrix} A_{11} & 0 & A_{1\Gamma} \\ 0 & A_{22} & A_{2\Gamma} \\ 0 & 0 & P_h \end{pmatrix}. \tag{2.3.19}$$

Note that the blocks of Q_h coincide with those of A, except for the block $(3,3)$, which equals

$$(Q_h)_{33} = A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} + A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma} + P_h.$$

(In the special case in which the preconditioner is the Dirichlet–Neumann preconditioner $\Sigma_{2,h}$, see Section 3.2, we obtain $(Q_h)_{33} = A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} + A_{\Gamma\Gamma}^{(2)}$.)

Let λ be an eigenvalue of $Q_h^{-1}A$, and $\mathbf{w} \in \mathbf{R}^{N_h}$ a corresponding eigenvector. Writing $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_\Gamma)$, with obvious meaning of notation, from

$$A\mathbf{w} = \lambda Q_h \mathbf{w}$$

we have

$$\begin{cases} (1-\lambda)(A_{11}\mathbf{w}_1 + A_{1\Gamma}\mathbf{w}_\Gamma) = \mathbf{0} \\ \\ (1-\lambda)(A_{22}\mathbf{w}_2 + A_{2\Gamma}\mathbf{w}_\Gamma) = \mathbf{0} \\ \\ (1-\lambda)(A_{\Gamma 1}\mathbf{w}_1 + A_{\Gamma 2}\mathbf{w}_2) + A_{\Gamma\Gamma}\mathbf{w}_\Gamma \\ \qquad\qquad = \lambda(P_h\mathbf{w}_\Gamma + A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\mathbf{w}_\Gamma + A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\mathbf{w}_\Gamma). \end{cases}$$

From (2.3.7) the last equation can be rewritten as

$$\begin{aligned} (1-\lambda)(A_{\Gamma 1}\mathbf{w}_1 &+ A_{\Gamma 2}\mathbf{w}_2 + A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\mathbf{w}_\Gamma + A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\mathbf{w}_\Gamma) + \Sigma_h\mathbf{w}_\Gamma \\ &= \lambda P_h\mathbf{w}_\Gamma. \end{aligned}$$

Therefore, if $\lambda \neq 1$ we have

$$\begin{cases} A_{11}\mathbf{w}_1 + A_{1\Gamma}\mathbf{w}_\Gamma = \mathbf{0} \\ \\ A_{22}\mathbf{w}_2 + A_{2\Gamma}\mathbf{w}_\Gamma = \mathbf{0}, \end{cases}$$

hence, $\mathbf{w}_\Gamma \neq \mathbf{0}$ and

$$\Sigma_h \mathbf{w}_\Gamma = \lambda P_h \mathbf{w}_\Gamma.$$

We conclude that the matrix $Q_h^{-1}A$ has the same eigenvalues as $P_h^{-1}\Sigma_h$, plus the eigenvalue 1 (which is also an eigenvalue of $P_h^{-1}\Sigma_h$ provided that the corresponding eigenvector $\mathbf{w}$ satisfies $\mathbf{w}_\Gamma \neq \mathbf{0}$).

If we assume that P_h is spectrally equivalent to Σ_h; that is, there exist two positive constants K_1 and K_2, independent of h, such that

$$K_1[P_h\boldsymbol{\eta},\boldsymbol{\eta}] \leq [\Sigma_h\boldsymbol{\eta},\boldsymbol{\eta}] \leq K_2[P_h\boldsymbol{\eta},\boldsymbol{\eta}] \qquad \forall\, \boldsymbol{\eta} \in \mathbf{R}^{N_\Gamma}, \tag{2.3.20}$$

then we derive from the characterisation of the eigenvalues of $Q_h^{-1}A$ that

$$\kappa(Q_h^{-1}A) \leq \frac{\widetilde{K}_2}{\widetilde{K}_1}, \tag{2.3.21}$$

where

$$\widetilde{K}_1 := \min(1, K_1), \quad \widetilde{K}_2 := \max(1, K_2), \tag{2.3.22}$$

and therefore we conclude that Q_h is spectrally equivalent to A.

From the computational viewpoint, the solution of the preconditioned system

$$Q_h\widetilde{\mathbf{x}} = \widetilde{\mathbf{b}}$$

entails the solution of the following system

$$L\widetilde{\mathbf{y}} = \widetilde{\mathbf{b}}, \quad \widetilde{U}\widetilde{\mathbf{x}} = \widetilde{\mathbf{y}},$$

that is,

$$\widetilde{\mathbf{y}}_1 = \widetilde{\mathbf{b}}_1, \quad \widetilde{\mathbf{y}}_2 = \widetilde{\mathbf{b}}_2, \quad \widetilde{\mathbf{y}}_\Gamma = \widetilde{\mathbf{b}}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\widetilde{\mathbf{b}}_1 - A_{\Gamma 2}A_{22}^{-1}\widetilde{\mathbf{b}}_2 \tag{2.3.23}$$

and

$$P_h\widetilde{\mathbf{x}}_\Gamma = \widetilde{\mathbf{y}}_\Gamma, \quad A_{11}\widetilde{\mathbf{x}}_1 = \widetilde{\mathbf{b}}_1 - A_{1\Gamma}\widetilde{\mathbf{x}}_\Gamma, \quad A_{22}\widetilde{\mathbf{x}}_2 = \widetilde{\mathbf{b}}_2 - A_{2\Gamma}\widetilde{\mathbf{x}}_\Gamma. \tag{2.3.24}$$

Both steps (2.3.23) and (2.3.24) require the solution of two independent Dirichlet problems, one for each subdomain. Furthermore, step (2.3.24) requires the solution of a problem associated with P_h. The whole process is therefore amenable to independent solves in Ω_1 and Ω_2, provided P_h splits into independent solves as well.

In conclusion, the preconditioner Q_h inherits all the good properties enjoyed by P_h in terms of both parallelism and spectral equivalence.

Remark 2.3.2 In the case where we take $P_h = \Sigma_{2,h}$ (this corresponds to the Dirichlet–Neumann preconditioner, see Section 3.2), step (2.3.24) requires us to solve

$$\Sigma_{2,h}\widetilde{\mathbf{x}}_\Gamma = \widetilde{\mathbf{y}}_\Gamma,$$

which corresponds to a Neumann problem in Ω_2. □

2.4 The case of many subdomains

The case of partitions with $M > 2$ subdomains Ω_i can be dealt with similarly. Let Ω be partioned into M non-overlapping subdomains Ω_i of diameter H_i with interface Γ separating them, $\Gamma = \cup_{i=1}^M \Gamma_i$, $\Gamma_i := \partial\Omega_i \setminus \partial\Omega$. Let $I = \cup_{i=1}^M N_i$ denote the indices corresponding to the internal nodes (see Fig. 2.4.1).

FIG. 2.4.1. Multi-domain partition and finite element triangulation (bold lines denote subdomain interfaces).

Then, with obvious meaning of notation, we can write the algebraic problem $A\mathbf{u} = \mathbf{f}$ blockwise as follows:

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Gamma \end{pmatrix}, \tag{2.4.1}$$

where $A_{\Gamma I} = A_{I\Gamma}^T$. Since the interior nodes in each subdomain will be decoupled from the interior nodes in other subdomains, we will have

$$A_{II} = \text{blockdiag}\,(A_{ii}) = \begin{pmatrix} A_{11} & \cdots & 0 \\ . & \cdots & . \\ 0 & \cdots & A_{MM} \end{pmatrix}. \tag{2.4.2}$$

The ith block A_{ii} is the principal submatrix of the local stiffness matrices that are associated with either Dirichlet or Neumann problems in the subdomains Ω_i. Indeed, the matrix

$$A_i := \begin{pmatrix} A_{ii} & A_{i\Gamma} \\ A_{\Gamma i} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix} \tag{2.4.3}$$

is the finite element matrix associated with the Poisson problem in Ω_i with the Neumann datum on Γ_i (and the homogeneous Dirichlet datum on $\partial\Omega \cap \partial\Omega_i$). The notations have the following meaning:

$$\begin{aligned} &(A_{ii})_{lj} := a_i(\varphi_j, \varphi_l), \quad 1 \le j, l \le N_i, \\ &(A_{i\Gamma})_{lr} := a_i(\psi_r, \varphi_l), \quad 1 \le r \le J_i, 1 \le l \le N_i, \\ &(A_{\Gamma\Gamma}^{(i)})_{sr} := a_i(\psi_r, \psi_s), \quad 1 \le r, s \le J_i, \end{aligned} \tag{2.4.4}$$

where φ_j, $j = 1, \ldots, N_i$, are the finite element basis functions associated with the internal nodes $\mathbf{x}_j$ in Ω_i, while ψ_r, $r = 1, \ldots, J_i$, are those associated with the nodes $\mathbf{y}_r$ belonging to Γ_i.

Similarly, the matrix

$$\mathcal{D}_i := \begin{pmatrix} A_{ii} & A_{i\Gamma} \\ 0 & I_\Gamma^{(i)} \end{pmatrix}, \tag{2.4.5}$$

where $I_\Gamma^{(i)}$ is the identity matrix of order J_i, is the matrix associated with the finite element discretisation of the Poisson problem in Ω_i with the Dirichlet datum prescribed on Γ_i (and the homogeneous Dirichlet datum on $\partial\Omega \cap \partial\Omega_i$).

The Schur complement matrix Σ_h associated with the interface variables $\mathbf{u}_\Gamma$ of (2.4.1) is

$$\Sigma_h = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I\Gamma}, \tag{2.4.6}$$

and the associated interface problem reads:

$$\Sigma_h \mathbf{u}_\Gamma = \boldsymbol{\chi}_\Gamma, \tag{2.4.7}$$

where

$$\boldsymbol{\chi}_\Gamma := \mathbf{f}_\Gamma - A_{\Gamma I} A_{II}^{-1} \mathbf{f}_I. \tag{2.4.8}$$

With the aim of representing Σ_h as a sum of local contributions, we start by splitting the stiffness matrix A into the contributions given by each subdomain Ω_i, $i = 1, \ldots, M$. Since

$$a(\varphi_j, \varphi_l) = \sum_{i=1}^{M} a_i(\varphi_j, \varphi_l),$$

it is easily checked that A can be split as

$$A = \sum_{i=1}^{M} R_i^T A_i R_i, \tag{2.4.9}$$

where A_i is the local stiffness matrix introduced in (2.4.3), R_i is the restriction matrix from the full vector in Ω to local vectors in $\Omega_i \cup \Gamma_i$ and consequently the matrix R_i^T denotes prolongation by 0 on the nodes external to $\Omega_i \cup \Gamma_i$.

In a similar way, we can split the Schur complement matrix Σ_h. In fact, the Steklov–Poincaré operator satisfies

$$\langle S_h \eta_h, \mu_h \rangle = \sum_{i=1}^{M} \langle S_{i,h} \eta_h, \mu_h \rangle = \sum_{i=1}^{M} a_i(H_{i,h}\eta_h, H_{i,h}\mu_h) \tag{2.4.10}$$

(see (2.2.4)). Denote now by $\Sigma_{i,h}$ the *local* Schur complement; that is, the Schur complement of the local stiffness matrix A_i with respect to the nodes of Γ_i. From (2.4.3), it can be written as

$$\Sigma_{i,h} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma i} A_{ii}^{-1} A_{i\Gamma}, \quad i = 1, \ldots, M. \tag{2.4.11}$$

It is easily seen that

$$\Sigma_h = \sum_{i=1}^{M} R_{\Gamma_i}^T \Sigma_{i,h} R_{\Gamma_i}. \tag{2.4.12}$$

Here, R_{Γ_i} is the restriction matrix from the vector of coefficient unknowns related to the nodes on Γ to only those associated with Γ_i, and $R_{\Gamma_i}^T$ is the matrix that extends by 0 a nodal vector from Γ_i to Γ.

Note that in the case of two subdomains ($M = 2$) we have $\Gamma = \Gamma_1 = \Gamma_2$, therefore both the restriction and the extension operators R_{Γ_i} and $R_{\Gamma_i}^T$ coincide with the identity matrix, and consequently (2.4.12) yields $\Sigma_h = \Sigma_{1,h} + \Sigma_{2,h}$, as already stated in (2.3.9).

As reported in Le Tallec (1994) (see also Brenner 1998*a*), the condition number of Σ_h satisfies

$$\kappa(\Sigma_h) \leq C \frac{H}{h H_{\min}^2}, \tag{2.4.13}$$

where $H_{\min}$ and H denote, respectively, the minimum and maximum diameters of the subdomains.

If the family of partitions of Ω is quasi-uniform; that is, there exists a positive constant τ, independent of H, such that $H_{\min} \geq \tau H$, then $\kappa(\Sigma_h) = O(h^{-1}H^{-1})$. The challenging point is therefore the construction of a suitable preconditioner for the Schur complement Σ_h such that the convergence rate of the preconditioned iterative method is independent of both h and H (see Section 3.3 for the development of this issue).

When an iterative method is applied to (2.3.5) (for example, the Richardson or the conjugate gradient method), the action of Σ_h on a given vector has to be computed. In view of (2.4.11) and (2.4.12), this amounts to inverting M

independent matrices A_{ii}, or, equivalently, to solve M independent Dirichlet problems.

Following a direct approach, generalising what was carried out in Section 2.3.1 for a two-domain subdivision, we see that the stiffness matrix A can be factored as follows:

$$A = LDL^T, \tag{2.4.14}$$

with

$$L := \begin{pmatrix} I_{\Omega\setminus\Gamma} & 0 \\ A_{\Gamma I}A_{II}^{-1} & I_\Gamma \end{pmatrix}, \quad D := \begin{pmatrix} A_{II} & 0 \\ 0 & \Sigma_h \end{pmatrix}, \tag{2.4.15}$$

or $A = LU$, with

$$U := \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & \Sigma_h \end{pmatrix}.$$

As noted in Section 2.3.1, should P_h be a preconditioner of the Schur complement Σ_h, then the matrix

$$Q_h := L\widetilde{U}, \tag{2.4.16}$$

with

$$\widetilde{U} := \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & P_h \end{pmatrix} \tag{2.4.17}$$

is a preconditioner of the stiffness matrix A. The eigenvalues of $Q_h^{-1}A$ are the same of those of $P_h^{-1}\Sigma_h$, plus the eigenvalue 1.

When an iterative method is used to solve the linear system associated with $Q_h^{-1}A$, at each iteration step a system with the matrix Q_h has to be solved. Using the factorisation (2.4.16), this requires us to solve first the L factor by forward substitution, which involves the inversion of A_{ii} (see (2.4.2)) for each $i = 1, \ldots, M$, requiring therefore the solution of M independent Dirichlet problems.

As for the $\widetilde{U}$ factor, this is solved by backward substitution, starting from the block P_h, followed by the block A_{II}, which again yields the solution of M independent Dirichlet problems in the subdomains.

As for P_h, in the special case where $P_h = \Sigma_{2,h}$, with

$$\Sigma_{2,h} := \sum_{i\in I_W} \Sigma_{i,h}$$

(see Section 1.4.2 for notation), the inversion of P_h corresponds to a Neumann problem in $\Omega_W := \cup_{i\in I_W}\Omega_i$. This global problem cannot be split into independent problems in the subdomains, because they are coupled through corner points (and edges in three dimensions).

2.5 Non-conforming domain decomposition methods

In some circumstances it is useful to approximate a given partial differential equation, in a domain Ω split into non-overlapping domains, by means of a numerical method that makes use of different finite dimensional spaces and/or different grids in different subdomains. The distinguishing feature of this approximation is that, in principle, the approximate solution will not be continuous across subdomain interfaces.

For the sake of exposition we refer to the case of a domain Ω partitioned into two subdomains Ω_1 and Ω_2, whose interface is Γ (see Fig. 2.5.1).

Another case happens when, although the grids match each other on Γ, the structures of the approximate solutions are different in Ω_1 and Ω_2. Referring to the situation depicted in Fig. 2.5.2, one could use, for example, linear finite elements in Ω_1 and quadratic in Ω_2.

Any other combination of the above situations is admissible as well and yields the same kind of difficulty.

We now refer to the Poisson problem (1.1) and its weak formulation (1.2.1). A

FIG. 2.5.1. Non-matching grids: unstructured (left), structured (right).

FIG. 2.5.2. Substructuring with matching grids but different finite element approximation: piecewise-linear in Ω_1 and piecewise-quadratic in Ω_2.

Galerkin approximation like that introduced in (2.1.1) cannot be used this time because the finite dimensional space, being made up of discontinuous functions, cannot be a subspace of $H^1(\Omega)$.

The alternative approach consists of giving a new weak formulation of (1.1), which, being compatible with the use of spaces defined independently in Ω_1 and Ω_2, is more adapted to the finite dimensional level. This can be done in different ways: we will consider two of them, the *mortar method* and the *three-field method* (the latter has been introduced already, at the continuous level, in Section 1.7).

2.5.1 The mortar method

This method has been introduced by Bernardi *et al.* (1994) with the aim of combining spectral methods having different polynomial degrees, or the spectral method with the finite element method. Its generality, however, goes beyond these two specific examples. The idea is to approximate (1.2.1) by the following discrete problem:

$$\text{find } u_\delta \in V_\delta \ : \ \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_\delta \cdot \nabla v_\delta = \sum_{i=1}^{2} \int_{\Omega_i} f v_\delta \quad \forall\, v_\delta \in V_\delta. \tag{2.5.1}$$

Here, $\delta > 0$ is a parameter describing the quality of the discretisation, and V_δ is a finite dimensional space that approximates $H_0^1(\Omega)$ without being contained into $C^0(\overline{\Omega})$. More precisely, V_δ is a subspace of the following space:

$$Y_\delta := \{v_\delta \in L^2(\Omega) \,|\, v_{\delta|\Omega_i} \in Y_{i,\delta},\ i = 1, 2\}, \tag{2.5.2}$$

where, for each $i = 1, 2$, $Y_{i,\delta}$ is a finite dimensional subspace of the space V_i introduced in (1.2.4). The subspace $Y_{i,\delta}$ can be either a finite element space, or a polynomial spectral space. In any case, no requirement of compatibility is made for the restriction on Γ of the functions of $Y_{1,\delta}$ and $Y_{2,\delta}$.

Heuristically, the space V_δ will be made up of functions belonging to Y_δ that satisfy some kind of matching across Γ. Precisely, if $v_\delta \in V_\delta$ and $v_\delta^{(1)} \in Y_{1,\delta}$, $v_\delta^{(2)} \in Y_{2,\delta}$ denotes its restriction to Ω_1 and Ω_2, respectively, for a certain fixed index i the following integral matching conditions should be satisfied

$$\int_\Gamma (v_\delta^{(1)} - v_\delta^{(2)}) \mu_\delta^{(i)} = 0 \qquad \forall\, \mu_\delta^{(i)} \in \Lambda_\delta^{(i)}, \tag{2.5.3}$$

where $\Lambda_\delta^{(i)}$ denotes the restriction to Γ of the functions of $Y_{i,\delta}$.

If we take $i = 2$ in (2.5.3), this amounts to letting Ω_1 play the role of *master* and Ω_2 that of *slave*, and (2.5.3) has to be intended as the way of generating the value of $v_{\delta|\Gamma}^{(2)}$ once $v_{\delta|\Gamma}^{(1)}$ is available. The other way round, i.e. taking $i = 1$ in (2.5.3), is also admissible. Depending upon the choice of index i made in (2.5.3), the method will produce different solutions.

The mathematical rationale behind the choice of the matching condition (2.5.3) (rather than a more 'natural' condition of pointwise continuity at one

set of grid nodes on Γ) becomes clear from the convergence analysis on problem (2.5.1).

With this aim we introduce

$$||v||_* := (||v||^2_{0,\Omega} + ||\nabla v_{|\Omega_1}||^2_{0,\Omega_1} + ||\nabla v_{|\Omega_2}||^2_{0,\Omega_2})^{1/2}, \tag{2.5.4}$$

which is a norm (the 'graph' norm) on the Hilbert space

$$H_* := \{v \in L^2(\Omega) \,|\, v_{|\Omega_1} \in H^1(\Omega_1),\ v_{|\Omega_2} \in H^1(\Omega_2)\}. \tag{2.5.5}$$

Owing to the Poincaré inequality (1.2.2), we have that

$$\sum_{i=1}^{2} \int_{\Omega_i} |\nabla v_\delta|^2 \geq \alpha_* ||v_\delta||^2_* \qquad \forall\ v_\delta \in V_\delta, \tag{2.5.6}$$

whence the discrete problem (2.5.1) admits a unique solution by a straightforward application of the Lax–Milgram lemma.

For any $v_\delta \in V_\delta$ we now have

$$\begin{aligned} \alpha_* ||u_\delta - v_\delta||^2_* &\leq \sum_{i=1}^{2} \int_{\Omega_i} |\nabla(u_\delta - v_\delta)|^2 \\ &\leq \sum_{i=1}^{2} \int_{\Omega_i} \nabla u_\delta \cdot \nabla(u_\delta - v_\delta) - \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_\delta \cdot \nabla(u_\delta - v_\delta) \\ &= \sum_{i=1}^{2} \int_{\Omega_i} f(u_\delta - v_\delta) - \sum_{i=1}^{2} \int_{\Omega_i} \nabla v_\delta \cdot \nabla(u_\delta - v_\delta). \end{aligned} \tag{2.5.7}$$

Replacing f by $-\Delta u$ and integrating by parts on each Ω_i we obtain:

$$\begin{aligned} \sum_{i=1}^{2} \int_{\Omega_i} f(u_\delta - v_\delta) = \sum_{i=1}^{2} \int_{\Omega_i} \nabla u \cdot \nabla(u_\delta - v_\delta) \\ - \int_\Gamma \frac{\partial u}{\partial n} \left[(u_\delta - v_\delta)^{(1)} - (u_\delta - v_\delta)^{(2)} \right] \end{aligned} \tag{2.5.8}$$

(we recall that $\frac{\partial}{\partial n}$ is the normal derivative on Γ and $\mathbf{n}$ is the unit normal vector on Γ pointing into Ω_2).

Denoting by

$$[v_\delta]_\Gamma := v^{(1)}_{\delta|\Gamma} - v^{(2)}_{\delta|\Gamma}$$

the jump across Γ of a function $v_\delta \in V_\delta$, from (2.5.7) and (2.5.8) we have that

$$\alpha_* ||u_\delta - v_\delta||^2_* \leq ||u - v_\delta||_* ||u_\delta - v_\delta||_* + \left| \int_\Gamma \frac{\partial u}{\partial n} [u_\delta - v_\delta]_\Gamma \right|,$$

and also

$$||u_\delta - v_\delta||_* \leq \frac{1}{\alpha_*}\left(||u - v_\delta||_* + \sup_{w_\delta \in V_\delta} \frac{\left|\int_\Gamma \frac{\partial u}{\partial n}[w_\delta]_\Gamma\right|}{||w_\delta||_*}\right).$$

By the triangle inequality

$$||u - u_\delta||_* \leq ||u - v_\delta||_* + ||u_\delta - v_\delta||_*$$

we then obtain the following inequality for the error $u - u_\delta$:

$$||u - u_\delta||_* \leq \left(1 + \frac{1}{\alpha_*}\right) \inf_{v_\delta \in V_\delta} ||u - v_\delta||_* + \frac{1}{\alpha_*} \sup_{w_\delta \in V_\delta} \frac{\left|\int_\Gamma \frac{\partial u}{\partial n}[w_\delta]_\Gamma\right|}{||w_\delta||_*}. \tag{2.5.9}$$

The approximation error of (2.5.1) is therefore bounded (up to a multiplicative constant) by the best approximation error (that is, the distance between the exact solution u and the finite dimensional space V_δ) plus an extra error involving interface jumps. The latter would not appear in the framework of classical Galerkin approximation (like (2.1.1)), and is the price to pay for the violation of the conforming property; that is, for the fact that $V_\delta \not\subset H_0^1(\Omega)$.

The error estimate (2.5.9) is *optimal* if each one of the two terms on the right can be bounded by the norm of *local* errors arising from the approximations in Ω_1 and Ω_2, without the presence of terms that combine them in a multiplicative fashion. In this way, we can take advantage of the local regularity of the exact solution as well as the approximation properties enjoyed by the local subspaces $Y_{i,\delta}$ of $H^1(\Omega_i)$.

Let us focus on the last term of (2.5.9) (the interface error). Owing to (2.5.3) we have

$$\int_\Gamma \frac{\partial u}{\partial n}[w_\delta]_\Gamma = \int_\Gamma \left(\frac{\partial u}{\partial n} - \mu_\delta^{(2)}\right)[w_\delta]_\Gamma$$

for all $\mu_\delta^{(2)} \in \Lambda_\delta^{(2)}$ (the space of the restrictions to Γ of the functions of $Y_{2,\delta}$). We can therefore take as $\mu_\delta^{(2)}$ any convenient approximation of $\frac{\partial u}{\partial n}_{|\Gamma}$. For instance, if $Y_{2,\delta}$ is made up of finite elements of degree $s_2 \geq 1$, then $\mu_\delta^{(2)}$ can be chosen as the finite element interpolant of $\frac{\partial u}{\partial n}_{|\Gamma}$ at the finite element nodes induced on Γ from the mesh used in Ω_2. This would give the estimate

$$\begin{aligned}\left|\left|\frac{\partial u}{\partial n} - \mu_\delta^{(2)}\right|\right|_{0,\Gamma} &\leq C h_2^{s_2 - 1/2} \left|\left|\frac{\partial u}{\partial n}\right|\right|_{s_2 - 1/2, \Gamma} \\ &\leq C h_2^{s_2 - 1/2} ||u_{|\Omega_2}||_{s_2+1, \Omega_2},\end{aligned}$$

provided $u_{|\Omega_2} \in H^{s_2+1}(\Omega_2)$ and h_2 is the maximum grid size in $\overline{\Omega_2}$.

On the other hand, the identity (2.5.3) states that $w_{\delta|\Gamma}^{(2)}$ can be regarded as the orthogonal projection of $w_{\delta|\Gamma}^{(1)}$ upon $\Lambda_\delta^{(2)}$ with respect to the scalar product of $L^2(\Gamma)$. Therefore,

$$||[w_\delta]_\Gamma||_{0,\Gamma} = ||w^{(2)}_{\delta|\Gamma} - w^{(1)}_{\delta|\Gamma}||_{0,\Gamma} \le Ch_2^{1/2}||w^{(1)}_{\delta|\Gamma}||_{1/2,\Gamma}$$
$$\le Ch_2^{1/2}||w^{(1)}_\delta||_{1,\Omega_1},$$

and we conclude that

$$\sup_{w_\delta \in V_\delta} \frac{\left|\int_\Gamma \frac{\partial u}{\partial n}[w_\delta]_\Gamma\right|}{||w_\delta||_*} \le Ch_2^{s_2}||u_{|\Omega_2}||_{s_2+1,\Omega_2}. \tag{2.5.10}$$

Concerning the best approximation error, a first step is to find a convenient approximation $\Pi^{(1)}_\delta u$ of u in $Y_{1,\delta}$, the space of the master domain, that satisfies an optimal local estimate. For instance, we can take $\Pi^{(1)}_\delta u$ as the finite element interpolant of degree s_1 of u in Ω_1, in order to find (see (2.1.5))

$$||u - \Pi^{(1)}_\delta u||_{1,\Omega_1} \le Ch_1^{s_1}||u_{|\Omega_1}||_{s_1+1,\Omega_1}, \tag{2.5.11}$$

where h_1 is the maximum grid size in $\overline{\Omega_1}$.

Then we define a suitable function $\Pi^{(2)}_\delta u \in Y_{2,\delta}$, a piecewise polynomial of degree s_2 that satisfies the matching condition

$$\int_\Gamma (\Pi^{(2)}_\delta u - \Pi^{(1)}_\delta u)\mu^{(2)}_\delta = 0 \qquad \forall\ \mu^{(2)}_\delta \in \Lambda^{(2)}_\delta, \tag{2.5.12}$$

and is such that the function

$$v^*_\delta = \begin{cases} \Pi^{(1)}_\delta u & \text{in } \Omega_1 \\ \Pi^{(2)}_\delta u & \text{in } \Omega_2 \end{cases}$$

satisfies

$$||u - v^*_\delta||_* \le C(h_1^{s_1}||u_{|\Omega_1}||_{s_1+1,\Omega_1} + h_2^{s_2}||u_{|\Omega_2}||_{s_2+1,\Omega_2}). \tag{2.5.13}$$

The construction of $\Pi^{(2)}_\delta u$ can be done by a slight modification of Proposition 4.6 in Bernardi *et al.* (1990).

Finally, we can bound the norm $||u-u_\delta||_*$ as in (2.5.13), using (2.5.9), (2.5.10) and (2.5.13).

To generate a basis for the finite dimensional space V_δ, we can proceed as follows. For $i = 1, 2$, let us denote by $\mathcal{N}_i$ the set of nodes in the interior of Ω_i, and by $\mathcal{N}^{(i)}_\Gamma$ the set of nodes on Γ, whose cardinality will be indicated by N_i and $N^{(i)}_\Gamma$, respectively. Note that, in general, $\mathcal{N}^{(1)}_\Gamma$ and $\mathcal{N}^{(2)}_\Gamma$ can be totally unrelated.

Now, denote by $\{\varphi^{(1)}_{k'}\}$, $k' = 1, \ldots, N_1$, the Lagrange functions associated with the nodes of $\mathcal{N}_1$; since they vanish on Γ, they can be extended by 0 in $\overline{\Omega_2}$. These extended functions are denoted by $\{\widetilde{\varphi}^{(1)}_{k'}\}$, and can be taken as a first set of basis functions for V_δ.

Symmetrically, we can generate as many basis functions for V_δ as the number of nodes of $\mathcal{N}_2$ by extending by 0 in $\overline{\Omega_1}$ the Lagrange functions associated with these nodes. These new functions are denoted by $\{\widetilde{\varphi}^{(2)}_{k''}\}$, $k'' = 1, \ldots, N_2$.

Finally, always supposing that Ω_1 is the master domain and Ω_2 its slave, for every Lagrange function $\{\varphi^{(1)}_{m,\Gamma}\}$ in $\overline{\Omega_1}$, $m = 1, \ldots, N^{(1)}_\Gamma$, we obtain a basis function $\{\widetilde{\varphi}_{m,\Gamma}\}$ as follows

$$\widetilde{\varphi}_{m,\Gamma} := \begin{cases} \varphi^{(1)}_{m,\Gamma} & \text{in } \overline{\Omega_1} \\ \widetilde{\varphi}^{(2)}_{m,\Gamma} & \text{in } \overline{\Omega_2} \end{cases},$$

where

$$\widetilde{\varphi}^{(2)}_{m,\Gamma} := \sum_{j=1}^{N^{(2)}_\Gamma} \xi_j \varphi^{(2)}_{j,\Gamma},$$

$\varphi^{(2)}_{j,\Gamma}$ are the Lagrange functions in $\overline{\Omega_2}$ associated with the nodes of $\mathcal{N}^{(2)}_\Gamma$, and ξ_j are unknown coefficients that should be determined through the fulfilment of the matching equations (2.5.3). Precisely, they must satisfy

$$\int_\Gamma \left(\sum_{j=1}^{N^{(2)}_\Gamma} \xi_j \varphi^{(2)}_{j,\Gamma} - \varphi^{(1)}_{m,\Gamma} \right) \varphi^{(2)}_{l,\Gamma} = 0 \qquad \forall\, l = 1, \ldots, N^{(2)}_\Gamma. \tag{2.5.14}$$

A basis for V_δ is therefore provided by the set of all functions $\{\widetilde{\varphi}^{(1)}_{k'}\}$, $k' = 1, \ldots, N_1$, $\{\widetilde{\varphi}^{(2)}_{k''}\}$, $k'' = 1, \ldots, N_2$, and $\{\widetilde{\varphi}_{m,\Gamma}\}$, $m = 1, \ldots, N^{(1)}_\Gamma$.

Remark 2.5.1 In the mortar method the interface matching is achieved through a L^2-interface projection, or, equivalently, by equating first-order moments, thus involving computation of interface integrals. In particular, from equations (2.5.3) we have two different kinds of integrals to evaluate (take, for instance, $i = 2$):

$$I_{12} := \int_\Gamma v^{(1)}_\delta \mu^{(2)}_\delta, \quad I_{22} := \int_\Gamma v^{(2)}_\delta \mu^{(2)}_\delta.$$

The computation of I_{22} raises no special difficulties, because both functions $v^{(2)}_\delta$ and $\mu^{(2)}_\delta$ live on the same mesh, the one inherited from Ω_2. On the contrary, $v^{(1)}_\delta$ and $\mu^{(2)}_\delta$ are functions defined on different domains, and the computation of integrals like I_{12} requires proper quadrature rules. This process needs to be done with special care, especially for three-dimensional problems, for which subdomain interfaces are made up of faces, edges and vertices (see Ben Belgacem and Maday 1997), otherwise the overall accuracy of the mortar approximation could be compromised. This issue is addressed in Cazabeau *et al.* (1997). □

Any substructuring iterative method that decouples the computation in Ω_1 from that in Ω_2 is particularly well suited to solving the coupled problem (2.5.1).

An instance is provided by the Dirichlet–Neumann method, where this time the Neumann step should be applied in the master subdomain and the Dirichlet step (enforcing the matching relations (2.5.3)) in the slave subdomain. With obvious meaning of notations, this iterative procedure will generate new values of $(u_\delta^{(1)})^{k+1}$ and $(u_\delta^{(2)})^{k+1}$ from those available, $(u_\delta^{(1)})^k$ and $(u_\delta^{(2)})^k$, through the solution of the following subproblems:

- Neumann step in Ω_1

$$\text{(2.5.15)}\quad \begin{cases} \text{find } (u_\delta^{(1)})^{k+1} \in Y_{1,\delta} \ : \\ \displaystyle\int_{\Omega_1} \nabla (u_\delta^{(1)})^{k+1} \cdot \nabla \varphi_{k'}^{(1)} = \int_{\Omega_1} f \varphi_{k'}^{(1)} \qquad \forall\ k' = 1, \ldots, N_1 \\ \displaystyle\int_{\Omega_1} \nabla (u_\delta^{(1)})^{k+1} \cdot \nabla \varphi_{m,\Gamma}^{(1)} = \int_{\Omega_1} f \varphi_{m,\Gamma}^{(1)} + \int_{\Omega_2} f \widetilde{\varphi}_{m,\Gamma}^{(2)} \\ \qquad\qquad - \displaystyle\int_{\Omega_2} \nabla (u_\delta^{(2)})^k \cdot \nabla \widetilde{\varphi}_{m,\Gamma}^{(2)} \qquad \forall\ m = 1, \ldots, N_\Gamma^{(1)}; \end{cases}$$

- Dirichlet step in Ω_2

$$\text{(2.5.16)}\quad \begin{cases} \text{find } (u_\delta^{(2)})^{k+1} \in Y_{2,\delta} \ : \\ \displaystyle\int_{\Omega_2} \nabla (u_\delta^{(2)})^{k+1} \cdot \nabla \varphi_{k''}^{(2)} = \int_{\Omega_2} f \varphi_{k''}^{(2)} \qquad \forall\ k'' = 1, \ldots, N_2 \\ \displaystyle\int_{\Gamma} [(u_\delta^{(2)})^{k+1} - (u_\delta^{(1)})^{k+1}] \varphi_{j,\Gamma}^{(2)} = 0 \qquad \forall\ j = 1, \ldots, N_\Gamma^{(2)}, \end{cases}$$

with a possible relaxation applied on the last set of interface equations.

Neumann–Neumann iterations can be defined in a similar manner (see Le Tallec 1993).

For the analysis, see Achdou *et al.* (1996), Seshaiyer and Suri (1997), and Stefanica (1998).

The discrete problem (2.5.1) can also be reformulated as a saddle point problem of the following form:

$$\begin{cases} \text{find } u_\delta \in Y_\delta, \lambda_\delta \in \Lambda_\delta^{(2)} \ : \\ a(u_\delta, v_\delta) + b(v_\delta, \lambda_\delta) = \displaystyle\sum_{i=1}^{2} (f, v_\delta^{(i)})_{\Omega_i} \quad \forall\ v_\delta \in Y_\delta \\ b(u_\delta, \mu_\delta) = 0 \qquad\qquad \forall\ \mu_\delta \in \Lambda_\delta^{(2)}, \end{cases}$$

where

$$a(w_\delta, v_\delta) := \sum_{i=1}^{2} \int_{\Omega_i} \nabla w_\delta^{(i)} \cdot \nabla v_\delta^{(i)}, \quad b(v_\delta, \mu_\delta) := \int_\Gamma (v_\delta^{(1)} - v_\delta^{(2)}) \mu_\delta.$$

In this system, λ_δ plays the role of the Lagrange multiplier associated with the 'constraint' (2.5.3).

Denoting by φ_j, $j = 1, \ldots, N_1 + N_2 + N_\Gamma^{(1)} + N_\Gamma^{(2)}$, a basis of Y_δ and by ψ_l, $l = 1, \ldots, N_\Gamma^{(2)}$, a basis of $\Lambda_\delta^{(2)}$, we introduce the matrices

$$A_{sj} := a(\varphi_j, \varphi_s), \quad B_{ls} := b(\varphi_s, \psi_l).$$

Defining by $\mathbf{u}$ and $\boldsymbol{\lambda}$ the vectors of the nodal values of u_δ and λ_δ, respectively, and by $\mathbf{f}$ the vector whose components are given by $\sum_{i=1}^{2}(f, \varphi_s^{(i)})_{\Omega_i}$, $s = 1, \ldots, N_1 + N_2 + N_\Gamma^{(1)} + N_\Gamma^{(2)}$, we have the linear system

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix}.$$

The matrix A is block-diagonal (with one block per subdomain Ω_i), each block corresponding to a problem for the Laplace operator with a Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega$ and a Neumann boundary condition on $\partial\Omega_i \setminus \partial\Omega$.

After elimination of the degrees of freedom internal to the subdomains, the method leads to the reduced linear system (still of a saddle point type)

$$\begin{pmatrix} S & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_\Gamma \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_\Gamma \\ \mathbf{0} \end{pmatrix},$$

where the matrix S is block-diagonal, C is a jump operator, $\mathbf{u}_\Gamma$ is the set of all nodal values at subdomain interfaces, and $\mathbf{g}_\Gamma$ is a suitable right-hand side.

This system can be regarded as an extension of the Schur complement system (2.3.5) to non-conforming approximation (the Lagrange multiplier $\boldsymbol{\lambda}$ indeed accounts for non-matching discretisation at subdomain interfaces). In fact, the ith block of S is the analogue of $\Sigma_{i,h}$, and corresponds to a discretised Steklov–Poincaré operator on the subdomain Ω_i.

For the efficient numerical solution of the reduced system, Achdou *et al.* (1995) have proposed conjugate gradient iterations that make use of a preconditioner of the following form:

$$P = \begin{pmatrix} \hat{S} & C^T \\ C & 0 \end{pmatrix},$$

$\hat{S}$ being a convenient preconditioner of S. Basing $\hat{S}$ on the local approximation of S can bring the condition number of the preconditioned system down to $O(h^{-1/2})$ for a finite element approximation of uniform mesh size h.

For other approaches involving preconditioned iterations for finite element problems with Lagrange multipliers see Kuznetsov (1995, 1997).

2.5.2 The three-field method at the finite dimensional level

The three-field method in the differential case has been introduced in Section 1.7, for the general symmetric elliptic operator L introduced in (1.4.2). The

finite dimensional approximation of (1.7.1) can be devised by selecting suitable subspaces $V_{i,h}$ of V_i, $\Xi_{i,h}$ of Λ' and Λ_h of Λ, $i = 1, 2$, then introducing the following discrete three-field problem: find $u_{i,h} \in V_{i,h}$, $\sigma_{i,h} \in \Xi_{i,h}$, $\lambda_h \in \Lambda_h$ such that

$$
\text{(2.5.17)} \qquad \begin{cases} a_1^*(u_{1,h}, v_{1,h}) - \langle \sigma_{1,h}, v_{1,h|\Gamma} \rangle_\Gamma = (f, v_{1,h})_{\Omega_1} & \forall\ v_{1,h} \in V_{1,h} \\ \langle \rho_{1,h}, \lambda_h - u_{1,h|\Gamma} \rangle_\Gamma = 0 & \forall\ \rho_{1,h} \in \Xi_{1,h} \\ \langle \sigma_{1,h} + \sigma_{2,h}, \mu_h \rangle_\Gamma = 0 & \forall\ \mu_h \in \Lambda_h \\ \langle \rho_{2,h}, \lambda_h - u_{2,h|\Gamma} \rangle_\Gamma = 0 & \forall\ \rho_{2,h} \in \Xi_{2,h} \\ a_2^*(u_{2,h}, v_{2,h}) - \langle \sigma_{2,h}, v_{2,h|\Gamma} \rangle_\Gamma = (f, v_{2,h})_{\Omega_2} & \forall\ v_{2,h} \in V_{2,h}. \end{cases}
$$

It is worthwhile to point out that, in principle, all the spaces above can be chosen independently, as is suggested from Fig. 2.5.3.

FIG. 2.5.3. Different meshes for the three-field discrete problem (2.5.17).

This arbitrarity should, however, undergo suitable compatibility conditions in order for problem (2.5.17) to be non-singular. Obviously, if all meshes in Fig. 2.5.3 are compatible (i.e. they all coincide on Γ), then the above problem reduces to the standard finite element approximation (2.1.1), where V_h is the space of functions whose restriction to Ω_i coincides with $V_{i,h}$, for $i = 1, 2$.

Regarding the compatibility conditions between the finite dimensional spaces appearing in (2.5.17), we can make the following considerations. Once the space of traces Λ_h has been chosen, the spaces of multipliers $\Xi_{1,h}$ and $\Xi_{2,h}$ must be rich enough to control the traces. This demands a discrete inf–sup condition of the following type: there exists $\beta_{0,h} > 0$ such that for each $\mu_h \in \Lambda_h$ it is possible

to find two multipliers $\rho_{1,h} \in \Xi_{1,h}$ and $\rho_{2,h} \in \Xi_{2,h}$, $\rho_{1,h} \neq 0$, $\rho_{2,h} \neq 0$, such that

$$\sum_{i=1}^{2} \langle \rho_{i,h}, \mu_h \rangle_\Gamma \geq \beta_{0,h} \left(\sum_{i=1}^{2} ||\rho_{i,h}||^2_{\Lambda'} \right)^{1/2} ||\mu_h||_\Lambda. \tag{2.5.18}$$

Afterwards, the finite element functions in the subdomain spaces $V_{1,h}$ and $V_{2,h}$ should suffice to control the multipliers. This means that the following inf–sup conditions need to be satisfied: for each $i = 1, 2$ there exist $\beta_{i,h} > 0$ such that for each $\rho_{i,h} \in \Xi_{i,h}$ it is possible to find $v_{i,h} \in V_{i,h}$, $v_{i,h} \neq 0$, such that

$$\langle \rho_{i,h}, v_{i,h|\Gamma} \rangle_\Gamma \geq \beta_{i,h} ||\rho_{i,h}||_{\Lambda'} \, ||v_{i,h}||_{1,\Omega_i}. \tag{2.5.19}$$

Once (2.5.18) is satisfied, a possible way to satisfy (2.5.19) is based on the use of a stabilisation procedure through enriching the subdomain spaces $V_{1,h}$ and $V_{2,h}$ via suitable bubble functions (Brezzi 1998).

Remark 2.5.2 The mortar method can be regarded as a special case of the three-field method, corresponding to the following choices of the subspaces. The trace space Λ_h is exactly the space of traces of $V_{1,\delta}$, hence there is no need for the space of multipliers $\Xi_{1,h}$ ($\Xi_{1,h} = \emptyset$, and $(2.5.17)_2$ disappears). Instead, the space of Lagrangian multipliers $\Xi_{2,h}$ coincides with the space of traces of $V_{2,\delta}$ (see (2.5.3), which corresponds to $(2.5.17)_4$). □

With obvious choice of notations, problem (2.5.17) can be stated in algebraic form as follows:

$$\begin{cases} A_1 \mathbf{u}_1 - B_1^T \boldsymbol{\sigma}_1 = \mathbf{f}_1 \\ -B_1 \mathbf{u}_1 + C_1^T \boldsymbol{\lambda} = \mathbf{0} \\ C_1 \boldsymbol{\sigma}_1 + C_2 \boldsymbol{\sigma}_2 = \mathbf{0} \\ -B_2 \mathbf{u}_2 + C_2^T \boldsymbol{\lambda} = \mathbf{0} \\ A_2 \mathbf{u}_2 - B_2^T \boldsymbol{\sigma}_2 = \mathbf{f}_2. \end{cases} \tag{2.5.20}$$

Proceeding by block-Gaussian elimination we obtain the following linear system for $\boldsymbol{\lambda}$:

$$\widehat{\Sigma} \boldsymbol{\lambda} = \mathbf{g}, \tag{2.5.21}$$

where $\widehat{\Sigma} := \widehat{\Sigma}_1 + \widehat{\Sigma}_2$, $\mathbf{g} := \mathbf{g}_1 + \mathbf{g}_2$ and

$$\widehat{\Sigma}_i := C_i D_i^{-1} C_i^T, \quad \mathbf{g}_i := C_i D_i^{-1} B_i A_i^{-1} \mathbf{f}_i, \quad D_i := B_i A_i^{-1} B_i^T, \quad i = 1, 2.$$

In the current context, (2.5.21) is a generalised Schur complement system and, indeed, $\widehat{\Sigma}$ is the Schur complement with respect to $\mathbf{u}$ and $\boldsymbol{\sigma}$ of the whole system (2.5.20).

The compatibility conditions on the discrete spaces can be expressed through the requirement that $\ker B_i^T = \ker C_i^T = \mathbf{0}$ for $i = 1, 2$, in which case $\widehat{\Sigma}$ is symmetric and positive definite.

The reformulation (2.5.20), (2.5.21), whose structure resembles that investigated in Section 2.3, suggests the way to solve problem (2.5.17). As $\widehat{\Sigma}$ is symmetric and positive definite, the interface system (2.5.21) can be solved first by preconditioned conjugate gradient iterations (the issue of which kind of preconditioner should be used is still an open problem). Note that, at each step, the calculation of the residual $\mathbf{r}^k = \mathbf{g} - \widehat{\Sigma}\boldsymbol{\lambda}^k$ requires the solution of local elliptic equations in the subdomains.

3

ITERATIVE DOMAIN DECOMPOSITION METHODS AT THE DISCRETE LEVEL

In this chapter the iterative methods introduced in Sections 1.3 and 1.5 will be reformulated at the discrete level for the solution of the finite element problem (2.1.1). Their relation with domain decomposition preconditioners will be pointed out, and several acceleration techniques will be considered for the case where many subdomains are used. The convergence analysis will be carried out in Chapter 4.

We will mainly focus on the Laplace operator $-\Delta$, but the extension of the results in this chapter to the symmetric elliptic operator L introduced in (1.4.2) is straightforward.

3.1 Iterative substructuring methods at the finite element level

Following what was carried out in Section 1.3, to solve the finite element problem we can introduce iterative algorithms that exploit the subdomain partition. Let us begin with the partition of Ω into two disjoint subdomains Ω_1 and Ω_2, as in Sections 2.1 and 2.2.

The Dirichlet–Neumann method for solving the finite element problem (2.1.1) can be defined as follows. Let $\lambda_h^0 \in \Lambda_h$ be an initial guess. For each $k \geq 0$, let us define $u_{1,h}^{k+1}$ and $u_{2,h}^{k+1}$ to be the solution of the following problems:

$$\text{(3.1.1)} \qquad \begin{cases} u_{1,h}^{k+1} \in V_{1,h} \ : \\ a_1(u_{1,h}^{k+1}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \qquad \forall\ v_{1,h} \in V_{1,h}^0 \\ u_{1,h}^{k+1} = \lambda_h^k \qquad \text{on } \Gamma \end{cases}$$

$$\text{(3.1.2)} \qquad \begin{aligned} u_{2,h}^{k+1} \in V_{2,h} \ :\ a_2(u_{2,h}^{k+1}, v_{2,h}) &= (f, v_{2,h})_{\Omega_2} + (f, \mathcal{R}_{1,h} v_{2,h|\Gamma})_{\Omega_1} \\ &\quad - a_1(u_{1,h}^{k+1}, \mathcal{R}_{1,h} v_{2,h|\Gamma}) \ \ \forall\ v_{2,h} \in V_{2,h}, \end{aligned}$$

where notation is as in Sections 1.2 and 2.2, and $\mathcal{R}_{1,h}$ is any extension operator from Λ_h into $V_{1,h}$ (as previously noted, in practice $\mathcal{R}_{1,h}$ is an interpolant operator); then update

$$\text{(3.1.3)} \qquad \lambda_h^{k+1} = \theta u_{2,h|\Gamma}^{k+1} + (1-\theta)\lambda_h^k \qquad \text{on } \Gamma.$$

Note that (3.1.1) is the finite element approximation of the Dirichlet problem (1.3.1), whereas (3.1.2) is the finite element approximation of the mixed problem (1.3.2).

As such, (3.1.2) can be equivalently rewritten as

$$\begin{cases} u_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2(u_{2,h}^{k+1}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \qquad \forall\ v_{2,h} \in V_{2,h}^0 \\ a_2(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\ \qquad\qquad\qquad\qquad -a_1(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) \qquad \forall\ \mu_h \in \Lambda_h, \end{cases} \tag{3.1.4}$$

where $\mathcal{R}_{2,h}$ is any extension operator from Λ_h into $V_{2,h}$.

If the iterative method converges, the limit solutions

$$u_{1,h} := \lim_{k\to\infty} u_{1,h}^k, \quad u_{2,h} := \lim_{k\to\infty} u_{2,h}^k$$

satisfy (2.1.9).

The other iterative methods introduced in Section 1.3 can be similarly reformulated at the finite element level. Let us focus, for instance, on the Neumann–Neumann method. Again, let $\lambda_h^0 \in \Lambda_h$ be an initial guess and for each $k \geq 0$, define $u_{1,h}^{k+1}$ and $u_{2,h}^{k+1}$ to be the solution of the following problems:

$$\begin{cases} \text{find } u_{i,h}^{k+1} \in V_{i,h} \ : \\ a_i(u_{i,h}^{k+1}, v_{i,h}) = (f, v_{i,h})_{\Omega_i} \qquad \forall\ v_{i,h} \in V_{i,h}^0 \\ u_{i,h}^{k+1} = \lambda_h^k \qquad \text{on } \Gamma \end{cases} \tag{3.1.5}$$

for $i = 1, 2$, and

$$\begin{cases} \text{find } \psi_{1,h}^{k+1} \in V_{1,h} \ : \\ a_1(\psi_{1,h}^{k+1}, v_{1,h}) = 0 \qquad \forall\ v_{1,h} \in V_{1,h}^0 \\ a_1(\psi_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) = -(f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} - (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} \\ \qquad\qquad +a_1(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) + a_2(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) \qquad \forall\ \mu_h \in \Lambda_h, \end{cases} \tag{3.1.6}$$

$$
\text{(3.1.7)}\qquad \begin{cases} \text{find } \psi_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2(\psi_{2,h}^{k+1}, v_{2,h}) = 0 \qquad \forall\ v_{2,h} \in V_{2,h}^0 \\ a_2(\psi_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) = (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} + (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} \\ \qquad\qquad -a_1(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) - a_2(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) \qquad \forall\ \mu_h \in \Lambda_h, \end{cases}
$$

and then update

$$
\text{(3.1.8)}\qquad \lambda_h^{k+1} = \lambda_h^k - \theta(\sigma_1 \psi_{1,h|\Gamma}^{k+1} - \sigma_2 \psi_{2,h|\Gamma}^{k+1}) \qquad \text{on } \Gamma.
$$

In the case of convergence of the sequence λ_h^k to a value λ_h, the limit functions $\psi_{1,h}$ and $\psi_{2,h}$ would satisfy

$$
\begin{cases} \Delta\psi_{i,h} = 0 & \text{in } \Omega_i \\ \psi_{i,h} = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ \sigma_1 \psi_{1,h|\Gamma} = \sigma_2 \psi_{2,h|\Gamma} & \text{on } \Gamma \\ \left(\dfrac{\partial\psi_{1,h}}{\partial n}\right)_{|\Gamma} = \left(\dfrac{\partial\psi_{2,h}}{\partial n}\right)_{|\Gamma} & \text{on } \Gamma. \end{cases}
$$

(Both conditions on the interface Γ are satisfied in a weak form.) This yields $\psi_{i,h} = 0$ in Ω_i: in fact, when $\sigma_1 = \sigma_2$, the function

$$
\psi_h := \begin{cases} \psi_{1,h} & \text{in } \Omega_1 \\ \psi_{2,h} & \text{in } \Omega_2 \end{cases}
$$

is harmonic in Ω with a boundary datum equal to 0 on $\partial\Omega$; when $\sigma_1 \neq \sigma_2$, the result follows from the algebraic formulation that we are going to present in Section 3.2.

Since $\psi_{i,h} = 0$ in Ω_i, it follows at once that the limit functions $u_{1,h}$ and $u_{2,h}$ satisfy (2.1.9), and therefore

$$
u_h := \begin{cases} u_{1,h} & \text{in } \Omega_1 \\ u_{2,h} & \text{in } \Omega_2 \end{cases}
$$

is the solution we are looking for.

3.2 The link between the Schur complement system and iterative substructuring methods

In the previous section, we have reformulated the methods introduced in Section 1.3 within a finite element framework. This is accomplished by replacing any

boundary value problem on each subdomain by its finite element approximation. As a consequence, each one of the methods there denoted by 1–4 can be regarded as if we were operating on $\lambda_h = u_{h|\Gamma}$, namely, the restriction of the finite element solution to the interface Γ.

Precisely, since the kth step provides a new value λ_h^{k+1} that depends linearly on the previous one, λ_h^k, each of the schemes 1–4 can be regarded as an iterative procedure applied directly to the discrete Steklov–Poincaré equation (2.2.3).

Identifying λ_h^k with the vector $\boldsymbol{\lambda}^k$ of its values at the finite element nodes on Γ, we obtain a sequence that approximates the solution $\mathbf{u}_\Gamma$ of the Schur complement system (2.3.5).

For instance, approach 1 (Dirichlet–Neumann method) introduced in Section 1.3 and approximated in Section 3.1 yields the following iteration procedure for (2.3.5):

$$P_h(\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k) = \theta(\boldsymbol{\chi}_\Gamma - \Sigma_h \boldsymbol{\lambda}^k), \quad k \geq 0, \tag{3.2.1}$$

with $P_h = \Sigma_{2,h}$. Equation (3.2.1) can be recognised as a Richardson iteration on (2.3.5), using the matrix $\Sigma_{2,h}$ as a *preconditioner* for Σ_h.

As a matter of fact, the Dirichlet–Neumann scheme, carried out at the finite element level (see Section 3.1), corresponds to the following steps. Let $\boldsymbol{\lambda}^k$ be given, and solve

$$A_{11}\mathbf{u}_1^{k+1} = \mathbf{f}_1 - A_{1\Gamma}\boldsymbol{\lambda}^k, \tag{3.2.2}$$

which corresponds to the Dirichlet step in Ω_1 (see (3.1.1)). Now solve

$$\begin{pmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma}^{(2)} \end{pmatrix} \begin{pmatrix} \mathbf{u}_2^{k+1} \\ \boldsymbol{\lambda}^{k+1/2} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_2 \\ \mathbf{f}_\Gamma - A_{\Gamma 1}\mathbf{u}_1^{k+1} - A_{\Gamma\Gamma}^{(1)}\boldsymbol{\lambda}^k \end{pmatrix}, \tag{3.2.3}$$

which in turn corresponds to the Neumann step in Ω_2 (see (3.1.2) or (3.1.4)). Here, we are making use of the split form of the matrix $A_{\Gamma\Gamma}$ introduced in (2.3.8). Finally, we carry out the relaxation step (3.1.3):

$$\boldsymbol{\lambda}^{k+1} = \theta\boldsymbol{\lambda}^{k+1/2} + (1-\theta)\boldsymbol{\lambda}^k. \tag{3.2.4}$$

A straightforward elimination of $\mathbf{u}_2^{k+1}$ carried out in (3.2.3) yields

$$\begin{aligned} (A_{\Gamma\Gamma}^{(2)} &- A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma})\boldsymbol{\lambda}^{k+1/2} \\ &= \mathbf{f}_\Gamma - A_{\Gamma 1}\mathbf{u}_1^{k+1} - A_{\Gamma\Gamma}^{(1)}\boldsymbol{\lambda}^k - A_{\Gamma 2}A_{22}^{-1}\mathbf{f}_2, \end{aligned} \tag{3.2.5}$$

and similarly from (3.2.2) we find that

$$\mathbf{u}_1^{k+1} = A_{11}^{-1}\mathbf{f}_1 - A_{11}^{-1}A_{1\Gamma}\boldsymbol{\lambda}^k. \tag{3.2.6}$$

Therefore, we can conclude with the following recursive formula

$$\Sigma_{2,h}\boldsymbol{\lambda}^{k+1/2} = \boldsymbol{\chi}_\Gamma - \Sigma_{1,h}\boldsymbol{\lambda}^k, \tag{3.2.7}$$

where we have used the definitions (2.3.6) and (2.3.9).

Owing to (3.2.4) we have therefore

$$\boldsymbol{\lambda}^{k+1} = \theta\Sigma_{2,h}^{-1}(\boldsymbol{\chi}_\Gamma - \Sigma_{1,h}\boldsymbol{\lambda}^k) + (1-\theta)\boldsymbol{\lambda}^k. \tag{3.2.8}$$

Written in this form, the Dirichlet–Neumann algorithm can be regarded as a successive under-relaxation method to solve (2.3.5), with the splitting

$$\Sigma_h = \Sigma_{2,h} - (-\Sigma_{1,h}).$$

On the other hand, using the identity $-\Sigma_{1,h} = \Sigma_{2,h} - \Sigma_h$, we can easily derive (3.2.1) from (3.2.8).

At this stage, if θ is properly chosen, in view of (2.3.14) we can infer that the error reduction factor at each iteration is

$$\rho = \frac{\hat{C}-1}{\hat{C}+1}.$$

Thus, the convergence rate of $\boldsymbol{\lambda}^k$ to $\mathbf{u}_\Gamma$ is independent of the grid size h (in the case of two subdomains).

Remark 3.2.1 On the grounds of the previous considerations, the same terminology (Dirichlet–Neumann) is used to identify either the substructuring iterative procedure (3.1.1)–(3.1.3) for the solution of the finite element problem (2.1.1), or the preconditioner $\Sigma_{2,h}$, which can be used in the framework of any iterative method to solve the Steklov–Poincaré equation (2.3.5). □

Remark 3.2.2 In view of (3.2.8) it follows that, under the assumptions made in Remark 2.3.1, the Dirichlet–Neumann method converges exactly in two iterations if we take $\theta = 1/2$. □

A similar property can be established for approach 2 considered in Section 1.3, the Neumann–Neumann method. Indeed, this iteration procedure can be regarded as a preconditioned iterative process like (3.2.1) using now a preconditioner P_h given by

$$P_h = (\sigma_1\Sigma_{1,h}^{-1} + \sigma_2\Sigma_{2,h}^{-1})^{-1}, \tag{3.2.9}$$

for $\sigma_1 > 0$ and $\sigma_2 > 0$. In fact, in the algebraic form, the Neumann–Neumann iteration reads

$$A_{11}\mathbf{u}_1^{k+1} = \mathbf{f}_1 - A_{1\Gamma}\boldsymbol{\lambda}^k, \quad A_{22}\mathbf{u}_2^{k+1} = \mathbf{f}_2 - A_{2\Gamma}\boldsymbol{\lambda}^k \tag{3.2.10}$$

$$\begin{pmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma}^{(1)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_1^{k+1} \\ \boldsymbol{\mu}_1^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\mathbf{f}_\Gamma + A_{\Gamma 1}\mathbf{u}_1^{k+1} + A_{\Gamma\Gamma}^{(1)}\boldsymbol{\lambda}^k + A_{\Gamma 2}\mathbf{u}_2^{k+1} + A_{\Gamma\Gamma}^{(2)}\boldsymbol{\lambda}^k \end{pmatrix} \tag{3.2.11}$$

$$\begin{pmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma}^{(2)} \end{pmatrix} \begin{pmatrix} \boldsymbol{\psi}_2^{k+1} \\ \boldsymbol{\mu}_2^{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f}_\Gamma - A_{\Gamma 1}\mathbf{u}_1^{k+1} - A_{\Gamma\Gamma}^{(1)}\boldsymbol{\lambda}^k - A_{\Gamma 2}\mathbf{u}_2^{k+1} - A_{\Gamma\Gamma}^{(2)}\boldsymbol{\lambda}^k \end{pmatrix} \tag{3.2.12}$$

with

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k - \theta(\sigma_1\boldsymbol{\mu}_1^{k+1} - \sigma_2\boldsymbol{\mu}_2^{k+1}). \tag{3.2.13}$$

Eliminating $\boldsymbol{\psi}_1^{k+1}$ and $\boldsymbol{\psi}_2^{k+1}$ from (3.2.11) and (3.2.12), and computing $\mathbf{u}_1^{k+1}$ and $\mathbf{u}_2^{k+1}$ from (3.2.10), we find that

$$\Sigma_{1,h}\boldsymbol{\mu}_1^{k+1} = -\boldsymbol{\chi}_\Gamma + \Sigma_h\boldsymbol{\lambda}^k \tag{3.2.14}$$

and

$$\Sigma_{2,h}\boldsymbol{\mu}_2^{k+1} = \boldsymbol{\chi}_\Gamma - \Sigma_h\boldsymbol{\lambda}^k, \tag{3.2.15}$$

therefore

$$\begin{aligned} \boldsymbol{\lambda}^{k+1} &= \boldsymbol{\lambda}^k + \theta[\sigma_1\Sigma_{1,h}^{-1}(\boldsymbol{\chi}_\Gamma - \Sigma_h\boldsymbol{\lambda}^k) + \sigma_2\Sigma_{2,h}^{-1}(\boldsymbol{\chi}_\Gamma - \Sigma_h\boldsymbol{\lambda}^k)] \\ &= \boldsymbol{\lambda}^k + \theta(\sigma_1\Sigma_{1,h}^{-1} + \sigma_2\Sigma_{2,h}^{-1})(\boldsymbol{\chi}_\Gamma - \Sigma_h\boldsymbol{\lambda}^k). \end{aligned} \tag{3.2.16}$$

The Neumann–Neumann preconditioner (3.2.9) can still be proved to be optimal in the case of two subdomains (see De Roeck and Le Tallec 1991 and Le Tallec 1994). We will give a proof of this result in Section 4.4.

Remark 3.2.3 The Neumann–Neumann method can also be interpreted as a Schwarz method, see Dryja and Widlund (1990). □

Also the more general Agoshkov–Lebedev method (1.3.9)–(1.3.12) can be reformulated as an interface iterative process, making use of the Steklov–Poincaré operator.

Since P_h is symmetric and positive definite in both the Dirichlet–Neumann and Neumann–Neumann cases, conjugate gradient (rather than Richardson) iterations could be used, based on the same kind of preconditioner. These approaches can still be interpreted as subdomain iteration methods that differ slightly from (1.3.1)–(1.3.3) and (1.3.4)–(1.3.6) (see Quarteroni and Sacchi Landriani 1989 and Bourgat *et al.* 1989).

Similar analogies between substructuring iterations and preconditioned Richardson iterations on the Schur complement system can also be established in the case of many subdomains. This will be pointed out in Section 3.3.2 for both Dirichlet–Neumann and Neumann–Neumann iterations. However, in this case, the spectral condition number is no longer independent of the number of subdomains, demanding the development of a coarse grid mechanism and new preconditioners.

3.3 Schur complement preconditioners

The Schur complement problem (2.3.5) (for decompositions with two subdomains), or (2.4.7) (for the case of many subdomains) allows in principle the direct computation of the vector $\mathbf{u}_\Gamma$ of interface variables. However, the explicit calculation of the matrix Σ_h is expensive, because its components are much denser (albeit smaller) than the original stiffness matrix. Iterative substructuring methods aim at solving the Schur complement system by Krylov-type algorithms, without forming the different components $\Sigma_{i,h}$ of Σ_h (see (2.3.10) and (2.4.11)), and using a parallel preconditioner.

From (2.4.13) we see that the condition number of Σ_h blows up either when the grid size h tends to 0 (for a fixed number of subdomains), or when $H_{\min}$ tends to 0 (that is, the number of subdomains tends to infinity, even for a fixed grid size).

Finding a preconditioner for Σ_h is therefore recommended to obtain a convergence rate that depends as little as possible on the geometrical parameters h and H. In the next sections we will briefly review the most popular preconditioners; we will keep the discussion for the two subdomain case separate from that for many subdomains. In the former case, it is important to investigate the dependence of the condition number on h, while in the latter on H (or M, the number of subdomains). For a more extensive overview and for a discussion of the algorithmic aspects we refer, for example, to J. Xu (1992), Chan and Mathew (1994*a*), Le Tallec (1994), Smith *et al.* (1996), J. Xu and Zou (1998) and the references therein.

3.3.1 Decomposition with two subdomains

In the two subdomain case (2.3.5) the condition number of Σ_h grows proportionally to h^{-1} (see (2.3.13)). In very particular cases, much can be said about Σ_h. For instance, as we noted in Remark 2.3.1, if we consider the Poisson problem (1.1) and use a uniform mesh and two equal-sized subdomains that are mirror images of each other (about the interface Γ), then $\Sigma_{1,h} = \Sigma_{2,h} = \frac{1}{2}\Sigma_h$. Thus, both $\Sigma_{1,h}$ and $\Sigma_{2,h}$ are ideal preconditioners for Σ_h.

Still for the same problem (1.1) on a rectangular domain Ω, discretised with a uniform mesh using piecewise-linear finite elements (or, equivalently, the standard five-point finite difference stencil), the Schur complement Σ_h can be explicitly computed. Suppose that $\Omega = (0,1) \times (0,b)$, $\Omega_1 = (0,1) \times (0,a)$, $\Omega_2 = (0,1) \times (a,b)$, and take the grid size $h = \frac{1}{N_\Gamma+1}$ and $N_\Gamma(m_1 + m_2 + 1)$ grid

points, with $(m_1+1)h = a$, $(m_2+1)h = b-a$. Then the Schur complement is given by (Bjørstad and Widlund 1986)

$$\Sigma_h = F\Lambda F, \tag{3.3.1}$$

where F is the orthogonal sine transform

$$F_{jl} := \sqrt{\frac{2}{N_\Gamma+1}}\sin\left(\frac{jl\pi}{N_\Gamma+1}\right), \quad j,l = 1,\dots,N_\Gamma,$$

while Λ is the diagonal matrix whose elements are

$$\Lambda_{jj} := \left(\frac{1+\tau_j^{m_1+1}}{1-\tau_j^{m_1+1}} + \frac{1+\tau_j^{m_2+1}}{1-\tau_j^{m_2+1}}\right)\sqrt{\sigma_j+\sigma_j^2/4},$$

where

$$\sigma_j := 4\sin^2\left(\frac{j\pi}{2(N_\Gamma+1)}\right), \quad \tau_j := \left(1+\sigma_j/2-\sqrt{\sigma_j+\sigma_j^2/4}\right)^2.$$

The eigenvalues Λ_{jj} of Σ_h satisfy

$$\hat{c}_1 h \le \Lambda_{jj} \le \hat{c}_2,$$

and this estimate is sharp. Since the eigenvalues $\delta_{j,h}$ of the discrete Steklov–Poincaré operator S_h are bounded as follows:

$$\hat{c}_3 h^{1-d}\Lambda_{jj} \le \delta_{j,h} \le \hat{c}_4 h^{1-d}\Lambda_{jj},$$

where d is the dimension of Ω (and $d-1$ the dimension of Γ), it follows that estimate (2.2.8) is sharp.

If m_1 and m_2 are large enough, then two good approximations of Σ_h are

$$P_h^{\mathrm{D}} := FM^{1/2}F \quad \text{and} \quad P_h^{\mathrm{GM}} := F(M+M^2/4)^{1/2}F, \tag{3.3.2}$$

where $M := \mathrm{diag}\,(\sigma_j)$.

The preconditioner P_h^{D} was proposed by Dryja (1982) in the functional setting of the trace space $H^{1/2}(\Gamma)$. Its improvement P_h^{GM} was proposed by Golub and Mayers (1984). Both preconditioners are spectrally equivalent to Σ_h; that is, $\kappa(P_h^{-1}\Sigma_h)$ is bounded independently of h. Moreover, using the Fast Sine Transform algorithm, the solution of the linear system associated with P_h^{D} or P_h^{GM} requires $O(n\log n)$ operations.

The weakness of these preconditioners is that they depend on the specific problem (1.1) and on the geometry of Ω, and besides, the condition number $\kappa(P_h^{-1}\Sigma_h)$ depends on the aspect ratios of the subdomains (namely, on a and b in our example).

The *Probing* preconditioner is a purely algebraic technique proposed by Chan and Resasco (1985), motivated by the observation that the decay of the off-diagonal terms of Σ_h is faster than the decay of the Green function of the original symmetric elliptic operator. Consequently, the idea is to compute efficiently a banded approximation P_h^{PB} (symmetric and tridiagonal) of Σ_h by *probing* the action of Σ_h on a few carefully selected vectors (remember that Σ_h is never computed explicitly). For the effective computation of P_h^{PB} see Keyes and Gropp (1987) and Chan and Mathew (1992).

The analysis shows that $\kappa((P_h^{\mathrm{PB}})^{-1}\Sigma_h) \simeq O(h^{-1/2})$, hence the Probing preconditioner is not spectrally equivalent to Σ_h. However, P_h^{PB} adapts pretty well to the aspect ratios of the subdomains and to the variations of the coefficients.

Finally, let us recall two other preconditioners that are spectrally equivalent to Σ_h, and that have been introduced in Section 3.2: the Dirichlet–Neumann and the Neumann–Neumann preconditioners

$$P_h^{\mathrm{DN}} := \Sigma_{2,h} \quad \text{and} \quad P_h^{\mathrm{NN}} := (\sigma_1 \Sigma_{1,h}^{-1} + \sigma_2 \Sigma_{2,h}^{-1})^{-1}, \tag{3.3.3}$$

σ_1 and σ_2 being two positive parameters.

The preconditioners introduced above, being all related to a single interface, are often referred to as *interface preconditioners*. They can be used as building blocks for the construction of preconditioners for Σ_h in the case of many subdomains.

3.3.2 Decomposition with many subdomains

Consider now the case of a partition of Ω into $M > 2$ non-overlapping subdomains. We retain the notation of Section 2.4, and look for preconditioners for the Schur complement system (2.4.7) (now related to the general symmetric elliptic operator L introduced in (1.4.2)). These preconditioners should have good parallel properties on arbitrary elliptic operators: ideally, they should be *perfectly scaleable*; that is, their performance should be insensitive to the number of subdomains. This, in general, will require us to include a coarse grid problem to allow for fast global communication and enhance the convergence rate dramatically.

Let us start by considering the two-dimensional case. The use of many subdomains introduces *vertex points* and *edges*. The former are isolated points on the interfaces that are shared by more than two subdomains. The edges are the lines (typically made up of several finite element sides) that separate two adjoining subdomains (see Fig. 3.3.1). An edge does not include its end points.

The global interface Γ is partitioned as a union of edges $\{e_k, k = 1, \ldots, m\}$ and vertex points $\{v_j, j = 1, \ldots, n\}$. We reorder the unknowns on the interface Γ, listing first those lying on each edge, then those at vertices, so that we write $\mathbf{u}_\Gamma = (\mathbf{u}_{e_1}, \mathbf{u}_{e_2}, \ldots, \mathbf{u}_{\mathcal{V}})$. Using deliberately a duplicity of notation and denoting by e_k the indices of the nodes lying on the edge e_k, and using $\mathcal{V}$ to denote the indices of the vertices, with the above reordering we obtain the following block partition of Σ_h:

FIG. 3.3.1. Edges and vertices for a many-domain decomposition.

$$\Sigma_h = \begin{pmatrix} \Sigma_{e_1e_1} & \cdots & \Sigma_{e_1e_m} & \Sigma_{e_1\mathcal{V}} \\ \Sigma_{e_1e_2}^T & \cdots & \Sigma_{e_2e_m} & \Sigma_{e_2\mathcal{V}} \\ \cdot & \cdots & \cdot & \cdot \\ \Sigma_{e_1\mathcal{V}}^T & \cdots & \Sigma_{e_m\mathcal{V}}^T & \Sigma_{\mathcal{V}\mathcal{V}} \end{pmatrix}. \tag{3.3.4}$$

Note that $\Sigma_{\mathcal{V}\mathcal{V}}$ is a diagonal matrix, and $\Sigma_{e_ke_l} = 0$ if e_k and e_l are not part of the same subdomain.

With obvious notation we can write

$$\Sigma_h = \begin{pmatrix} \Sigma_{ee} & \Sigma_{e\mathcal{V}} \\ \Sigma_{e\mathcal{V}}^T & \Sigma_{\mathcal{V}\mathcal{V}} \end{pmatrix}. \tag{3.3.5}$$

What follows is a short review of parallel preconditioners that are based on the partition (3.3.5). We follow closely the presentation given by Chan and Mathew (1994*a*) and Smith *et al.* (1996).

A simple *block-Jacobi* preconditioner P_h^{J} is obtained from Σ_h by dropping all the couplings between different edges and between edges and vertex points. The result is a block diagonal preconditioner given by

$$P_h^{\mathrm{J}} := \begin{pmatrix} \hat{\Sigma}_{ee} & 0 \\ 0 & \Sigma_{\mathcal{V}\mathcal{V}} \end{pmatrix}, \quad \hat{\Sigma}_{ee} := \begin{pmatrix} \hat{\Sigma}_{e_1e_1} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & \hat{\Sigma}_{e_me_m} \end{pmatrix}, \tag{3.3.6}$$

where $\hat{\Sigma}_{e_ke_k}$ is either $\Sigma_{e_ke_k}$ or one of the interface preconditioners for the edge e_k

that we have introduced in Section 3.3.1. Also, $\Sigma_{\mathcal{V}\mathcal{V}}$ may be approximated; for instance, by the diagonal matrix $A_{\mathcal{V}\mathcal{V}}$, the principal submatrix of A corresponding to the vertices.

Since P_h^{J} does not involve global coupling between the subdomains, its spectral properties deteriorate as the number of subdomains increases. Indeed, there exists a positive constant C, independent of h and H, but possibly dependent on the coefficients of the operator L, such that

$$\kappa((P_h^{\mathrm{J}})^{-1}\Sigma_h) \leq CH^{-2}\left(1+\log\frac{H}{h}\right)^2 \tag{3.3.7}$$

(see Bramble *et al.* 1986*b*; Widlund 1988; and Dryja and Widlund 1994).

The presence of the H^{-2} term can be heuristically justified as follows. Since the information is exchanged only among neighbouring substructures, the number of steps required by the conjugate gradient method to converge must necessarily be at least equal to the inverse of the diameter of Ω_i.

On the other hand, the presence of $\log(\frac{H}{h})$ can be regarded as being a measure of the maximum number of refinement steps that are required when going from any substructure (i.e. subdomain) made up of big triangles of diameter H to its finest refinement level (the triangles) of diameter h, by repeatedly cutting triangles in four. Another (more involved) argument stems from the consideration that the global preconditioner is made up of local edge preconditioners $\hat{\Sigma}_{e_k e_k}$ and of the vertex contributions $\Sigma_{\mathcal{V}\mathcal{V}}$. The latter yield pointwise values that should be controlled in terms of the energy norm, which is possible at the expense of a logarithmic factor.

Before generalizing P_h^{J} in order to remove the H^{-2} factor in (3.3.7), let us describe it in terms of restriction and extension matrices. For each edge e_k let R_{e_k} denote the pointwise restriction map from Γ onto the nodes on e_k; then $R_{e_k}^T$ denotes the corresponding extension map. Similarly, $R_{\mathcal{V}}$ denotes the pointwise restriction map onto the vertices $\mathcal{V}$, and $R_{\mathcal{V}}^T$ denotes extension by 0 of nodal values on $\mathcal{V}$ to Γ. All these maps are expressed by rectangular matrices.

With this notation, the block-Jacobi preconditioner P_h^{J} satisfies

$$(P_h^{\mathrm{J}})^{-1} = \sum_{k=1}^{m} R_{e_k}^T \hat{\Sigma}_{e_k e_k}^{-1} R_{e_k} + R_{\mathcal{V}}^T \Sigma_{\mathcal{V}\mathcal{V}}^{-1} R_{\mathcal{V}}. \tag{3.3.8}$$

A variant of P_h^{J} is obtained by inserting in (3.3.8) some mechanism for global coupling, through a coarse grid problem based on a coarse triangulation; for instance, the one $\mathcal{T}_H$ induces from the vertex points (here we are assuming that the subdomains define a coarse grid, therefore they are triangles or quadrilaterals), see Fig. 3.3.2.

Let A_H be the associated stiffness matrix, and let R_H^T denote an extension map (for instance, piecewise-linear interpolation) from the nodal values on $\mathcal{V}$ onto all the nodes of Γ. Correspondingly, R_H is a weighted restriction map from Γ onto $\mathcal{V}$. Then the modified preconditioner is defined as

FIG. 3.3.2. The coarse grid.

$$(P_h^{\mathrm{BPS}})^{-1} := \sum_{k=1}^{m} R_{e_k}^T \hat{\Sigma}_{e_k e_k}^{-1} R_{e_k} + R_H^T A_H^{-1} R_H. \tag{3.3.9}$$

For this preconditioner, which was proposed by Bramble *et al.* (1986*b*) with a specific choice of $\hat{\Sigma}_{e_k e_k}$, the estimate (3.3.7) improves as follows:

$$\kappa((P_h^{\mathrm{BPS}})^{-1}\Sigma_h) \leq C\left(1 + \log\frac{H}{h}\right)^2. \tag{3.3.10}$$

This estimate is sharp (in the sense that a similar lower bound holds, at least for $H/h \gg 1$; see Brenner 1998*b*). In three dimensions, the condition number grows faster than H/h.

The constant C, in addition to being independent of h and H, is also independent of the coefficients of the differential operator L if they are constant in each subdomain Ω_i. From (3.3.10) we deduce that the number of iterations of a Krylov subspace method to achieve a given tolerance grows like $\log \frac{H}{h}$.

With the aim of removing this residual (albeit mild) dependence on $\frac{H}{h}$, additional coupling between the edges and vertex points is necessary. The Schur complement Σ_h, introduced in (3.3.4), is not block diagonal in the permutation $\{e_1, e_2, \cdots, e_m, \mathcal{V}\}$, since, in general, $\Sigma_{e_k e_j} \neq 0$ whenever e_k and e_j are edges of the same subdomain. Having ignored this coupling in both preconditioners P_h^{J} and P_h^{BPS} the result is the logarithmic growth factor in the condition number, see (3.3.7) and (3.3.10).

Some overlap in the decomposition of interface $\Gamma = (\cup_{k=1}^m \{e_k\}) \cup (\cup_{j=1}^n \{v_j\})$ is obtained by introducing *vertex regions* $\{r_1, r_2, \cdots, r_n\}$. The jth vertex region

FIG. 3.3.3. The vertex regions.

is defined as the cross-shaped region centred at the jth vertex point v_j containing 'segments' of length δH $(0 < \delta \leq 1)$ of all the edges that emanate from v_j (see Fig. 3.3.3).

Now let R_{r_j} denote the restriction map that associates with full vectors the subvectors corresponding to the indices in the regions r_j. Its transpose $R_{r_j}^T$ denotes the extension by 0 of subvectors with indices r_j to full vectors. The principal submatrix of Σ_h corresponding to the indices r_j will be denoted by Σ_{r_j}; then

$$\Sigma_{r_j} = R_{r_j} \Sigma_h R_{r_j}^T.$$

The *Vertex Space* preconditioner P_h^{VS}, proposed by Smith (1990, 1992) is obtained from (3.3.9) as follows:

$$(P_h^{\mathrm{VS}})^{-1} := (P_h^{\mathrm{BPS}})^{-1} + \sum_{j=1}^{n} R_{r_j}^T \Sigma_{r_j}^{-1} R_{r_j}. \tag{3.3.11}$$

Since Σ_{r_j} is dense and expensive to compute, approximations based on the interface preconditioners of the previous section can still be used to alleviate the computational complexity of (3.3.11).

The vertex space preconditioner satisfies the following estimate:

$$\kappa((P_h^{\mathrm{VS}})^{-1} \Sigma_h) \leq C(1 + \log \delta^{-1})^2, \tag{3.3.12}$$

where C is independent of both h and H, but may depend on the variation of coefficients of the differential operator. If the coefficients of the operator L are constant in each Ω_i, but have large jumps across the subdomain interfaces, the estimate becomes

$$\kappa((P_h^{\rm VS})^{-1}\Sigma_h) \leq C\left(1+\log\frac{H}{h}\right), \tag{3.3.13}$$

and now C is a constant, possibly depending on δ. For the proof of these results, see Smith (1992) and Dryja *et al.* (1994).

The preconditioner $P_h^{\rm VS}$ can be interpreted as if it were generated by a decomposition of the reference finite element space V_h in overlapping subspaces (edge spaces and vertex spaces). As such, it can be regarded as a Schwarz preconditioner, and it was earlier used by Nepomnyaschikh (1984, 1986).

In the three-dimensional case, it is possible to develop the extension of the vertex space preconditioner (see Dryja *et al.* 1994), or the so-called *Wire-Basket* preconditioner (see Bramble *et al.* 1989 and Smith 1991). In the latter case, the condition number satisfies

$$\kappa((P_h^{\rm WB})^{-1}\Sigma_h) \leq C\left(1+\log\frac{H}{h}\right)^2, \tag{3.3.14}$$

no matter how large is the coefficient variation between subdomains.

We conclude this section by generalizing to the case of many subdomains the two interface preconditioners of Neumann–Neumann and Dirichlet–Neumann type, for both the two- and three-dimensional cases. Since the Schur complement is the sum of local elements (see (2.4.12)), and every local Schur complement $\Sigma_{i,h}$ is easy to invert, then it is natural to precondition Σ_h by a weighted sum of inverses $\Sigma_{i,h}^{-1}$:

$$(P_h^{\rm NN})^{-1} := \sum_{i=1}^{M} D_i (R_{\Gamma_i}^T \Sigma_{i,h}^{-1} R_{\Gamma_i}) D_i, \tag{3.3.15}$$

where D_i is a diagonal weighting matrix with positive entries, which is often chosen in order to satisfy the partition of unity property $\sum_{i=1}^{M} D_i = I$, and R_{Γ_i} is the restriction map from Γ onto $\Gamma_i := \Gamma \cap \partial\Omega_i$. The matrix $P_h^{\rm NN}$ is called the *Neumann–Neumann* preconditioner and generalises to the case of many subdomains the one considered in Section 3.2 (see (3.2.9)). Note that, in the two-domain case, the restriction operator R_{Γ_i} coincides with the identity, because $\Gamma = \Gamma_i$, $i = 1, 2$.

The action of $\Sigma_{i,h}^{-1}$ can be calculated without explicitly forming $\Sigma_{i,h}$, because

$$\Sigma_{i,h}^{-1}\mathbf{q} = \begin{pmatrix} 0 & I \end{pmatrix} A_i^{-1} \begin{pmatrix} 0 \\ I \end{pmatrix} \mathbf{q}, \tag{3.3.16}$$

which corresponds to solving a discrete harmonic problem in Ω_i with a Neumann boundary condition on the interior interface Γ_i (and the homogeneous Dirichlet boundary condition on $\partial\Omega \cap \partial\Omega_i$).

If Ω_i is an interior subdomain and the coefficient a_0 of the operator L is equal to zero, it is known that $\Sigma_{i,h}$ is a singular matrix (because the pure Neumann

problem requires a compatibility condition and lacks uniqueness of the solution). We have therefore to use in (3.3.15) a regularised inverse instead of $\Sigma_{i,h}^{-1}$. This can be the Schur complement of the modified local stiffness matrix $A_i + \alpha I$, or the (more expensive to compute) Moore–Penrose pseudo-inverse. Otherwise, we can solve a linear system in the complement of the null space (see Le Tallec 1994).

Concerning the spectral properties of P_h^{NN} one has (in both two and three dimensions)

$$\kappa((P_h^{\mathrm{NN}})^{-1}\Sigma_h) \leq CH^{-2}\left(1+\log\frac{H}{h}\right)^2, \tag{3.3.17}$$

which is not better than (3.3.7) (see De Roeck and Le Tallec 1991). However, this preconditioner is much less sensitive to coefficient variations.

The presence of the H^{-2} term is the symptom of the need for a global coupling mechanism. The *balancing Neumann–Neumann* preconditioner proposed by Mandel (1993) is a Neumann–Neumann preconditioner with the addition of a simple coarse grid correction constructed by using a piecewise-constant coarse grid space. The corresponding operator A_H involves one unknown per subdomain in the scalar case. With obvious notation the balanced preconditioner is defined as follows:

$$\begin{aligned}(P_h^{\mathrm{NN},b})^{-1} := {} & [(I - R_\Gamma^T A_H^{-1} R_\Gamma \Sigma_h)(P_h^{\mathrm{NN}})^{-1} + R_\Gamma^T A_H^{-1} R_\Gamma] \\ & \times (I - \Sigma_h R_\Gamma^T A_H^{-1} R_\Gamma) + R_\Gamma^T A_H^{-1} R_\Gamma.\end{aligned} \tag{3.3.18}$$

This new preconditioner satisfies a spectral estimate of the same form as (3.3.10) in both two and three dimensions.

A nice feature of the balancing Neumann–Neumann approach is that it allows the use of completely unstructured subdomains. Indeed, using piecewise-constant solvers, the subdomains need not form a coarse triangulation and may have an arbitrary form (for example, as in Fig. 3.3.1).

This method is called balancing because the corresponding iterative algorithm is a three-step method

$$\begin{aligned}\boldsymbol{\lambda}^{n+1/3} &= \boldsymbol{\lambda}^n + R_\Gamma^T A_H^{-1} R_\Gamma(\boldsymbol{\chi} - \Sigma_h \boldsymbol{\lambda}^n) \\ \boldsymbol{\lambda}^{n+2/3} &= \boldsymbol{\lambda}^{n+1/3} + (P_h^{\mathrm{NN}})^{-1}(\boldsymbol{\chi} - \Sigma_h \boldsymbol{\lambda}^{n+1/3}) \\ \boldsymbol{\lambda}^{n+1} &= \boldsymbol{\lambda}^{n+2/3} + R_\Gamma^T A_H^{-1} R_\Gamma(\boldsymbol{\chi} - \Sigma_h \boldsymbol{\lambda}^{n+2/3}),\end{aligned}$$

and the calculation of $\boldsymbol{\lambda}^{n+1/3}$ and $\boldsymbol{\lambda}^{n+1}$ balances the average of $\boldsymbol{\lambda}$ on each subdomain.

Concerning the Dirichlet–Neumann iterative substructuring method introduced in Section 1.3 for the differential boundary value problem, it can be formulated at the algebraic level as follows. We use a superscript B (black) and W (white) to denote the colour of the subdomain. Then the Schur complement

matrix (2.4.12) can be split as

$$\Sigma_h = \sum_{i\in I_B} (R_{\Gamma_i}^{(B)})^T \Sigma_{i,h}^{(B)} R_{\Gamma_i}^{(B)} + \sum_{i\in I_W} (R_{\Gamma_i}^{(W)})^T \Sigma_{i,h}^{(W)} R_{\Gamma_i}^{(W)},$$

and the *Dirichlet–Neumann* preconditioner is defined through

$$(P_h^{\mathrm{DN}})^{-1} := \sum_{i\in I_W} (R_{\Gamma_i}^{(W)})^T (\Sigma_{i,h}^{(W)})^{-1} R_{\Gamma_i}^{(W)}. \tag{3.3.19}$$

Also for this preconditioner it is possible to include a global coarse problem, see Dryja (1988).

As we have seen, a critical issue is how to obtain optimality of preconditioners when partitioning the domain into many substructures. At this stage, a multilevel approach (that makes use also of a coarse domain partition) needs to be pursued, in order to ensure a fast propagation of information among subdomains, even in the case of local grid refinement. This feature is shared by iterative substructuring methods and Schwarz methods, and will be further addressed in Section 3.6.

3.4 The Schwarz method for finite elements

The Schwarz iterative methods introduced in Section 1.5 can be straightforwardly adapted to the solution of the discrete problem (2.1.1), rewritten for the general symmetric bilinear form

$$a^*(w,v) := \int_\Omega \left(\sum_{l,j=1}^{d} a_{lj} D_j w D_l v + a_0\, w\, v \right).$$

Using the notations of Section 2.2 and following (1.5.8), the alternating (multiplicative) Schwarz method at the discrete level reads as follows.

Given $u_h^0 \in V_h$, solve for each $k \geq 0$

$$\begin{aligned}
&w_{1,h}^k \in V_{1,h}^0 \ : \ a_1^*(w_{1,h}^k, v_{1,h}) = (f, v_{1,h})_{\Omega_1} - a_1^*(u_h^k, v_{1,h}) \quad \forall\ v_{1,h} \in V_{1,h}^0\\
&u_h^{k+1/2} = u_h^k + \widetilde{w_{1,h}^k}\\
&w_{2,h}^k \in V_{2,h}^0 \ : \ a_2^*(w_{2,h}^k, v_{2,h}) = (f, v_{2,h})_{\Omega_2} - a_2^*(u_h^{k+1/2}, v_{2,h}) \quad \forall\ v_{2,h} \in V_{2,h}^0\\
&u_h^{k+1} = u_h^{k+1/2} + \widetilde{w_{2,h}^k},
\end{aligned}$$

where $\widetilde{w_{i,h}^k}$ is the finite element function that extends $w_{i,h}^k$ by 0 in $\Omega \setminus \Omega_i$, and a_i^* denotes the restriction of a^* to Ω_i.

Similarly, we can define the additive Schwarz method by restating (1.5.9) at the finite element level.

FIG. 3.4.1. Overlapping partitioning of the domain Ω with more than one layer of overlap.

FIG. 3.4.2. Overlapping partitioning of the domain Ω with only one layer of overlap.

We now reformulate these methods in algebraic terms. As we have seen in the previous chapter, the finite element approximation (2.1.1) (with $a^*(\cdot,\cdot)$ instead of $a(\cdot,\cdot)$) of the original boundary value problem (1.4.1) yields the algebraic system (2.1.13), where $A_{lj} := a^*(\varphi_j, \varphi_l)$.

Corresponding to the overlapping subregions Ω_1 and Ω_2 (see Fig. 3.4.1), let I_1 and I_2 denote the indices of the nodes in the interior of Ω_1 and Ω_2, respectively. Clearly, if N_h denotes the number of internal nodes of Ω, and I the set of all indices from 1 to N_h, then we have that I_1 and I_2 form an overlapping partition of I, i.e. $I_1 \cup I_2 = I$, $I_1 \cap I_2 \neq \emptyset$. If n_1 and n_2 indicate the number of indices in I_1 and I_2, respectively, due to overlap $n_1 + n_2 > N_h$. However, note that for partitions with only one layer of overlap (see Fig. 3.4.2), $I_1 \cap I_2 = \emptyset$ and $n_1 + n_2 = N_h$.

Order now the indices in such a way that those corresponding to the nodes

FIG. 3.4.3. Partitioning of the stiffness matrix.

internal to Ω_1 but not internal to Ω_2 are first, followed by those corresponding to the nodes internal to $\Omega_1 \cap \Omega_2$, and finally we take the remaining ones. Let A_1 and A_2 denote the principal submatrices of A formed by the first n_1 rows and columns and the last n_2 rows and columns, respectively (see Fig. 3.4.3).

Then A_1 is the stiffness matrix for the subdomain Ω_1, and A_2 that of Ω_2. These matrices are clearly symmetric and positive definite. Moreover, they are related to the global stiffness matrix A by the algebraic relations

$$A_1 = R_1 A R_1^T, \quad A_2 = R_2 A R_2^T. \tag{3.4.1}$$

Here R_i^T and R_i, $i = 1, 2$, are extension and restriction matrices, respectively.

More precisely, R_i^T is a $N_h \times n_i$ rectangular matrix whose action extends by 0 a vector of nodal values in Ω_i. Therefore, given a subvector $\mathbf{v}^i$ of length n_i of nodal values of a function $v_{i,h} \in V_{i,h}^0$, we have

$$(R_i^T \mathbf{v}^i)_j = \begin{cases} v_j^i & \text{for } j \in I_i \\ 0 & \text{for } j \in I \setminus I_i \end{cases}.$$

In other words, R_1^T is the matrix whose first n_1 rows and columns form the identity matrix, whereas the entries of the last $N_h - n_1$ rows are all 0 (see Fig. 3.4.4).

The transpose R_i of R_i^T is a restriction matrix whose action restricts a full vector $\mathbf{v} \in \mathbf{R}^{N_h}$ to a vector of length n_i by preserving the entries with indices belonging to I_i. Thus, $R_i \mathbf{v}$ is the subvector of nodal values of $\mathbf{v}$ in the interior of Ω_i.

The algebraic form of the alternating (multiplicative) Schwarz method follows immediately from (1.5.8) upon replacing the operators with the corresponding matrices:

FIG. 3.4.4. The extension matrices.

$$\begin{aligned} \mathbf{u}^{k+1/2} &= \mathbf{u}^k + R_1^T A_1^{-1} R_1(\mathbf{f} - A\mathbf{u}^k) \\ \mathbf{u}^{k+1} &= \mathbf{u}^{k+1/2} + R_2^T A_2^{-1} R_2(\mathbf{f} - A\mathbf{u}^{k+1/2}). \end{aligned} \tag{3.4.2}$$

Replacing $\mathbf{f}$ with $A\mathbf{u}$, these equations can be rewritten as

$$\begin{aligned} \mathbf{u}^{k+1/2} &= \mathbf{u}^k + P_1(\mathbf{u} - \mathbf{u}^k) = (I - P_1)\mathbf{u}^k + P_1\mathbf{u} \\ \mathbf{u}^{k+1} &= \mathbf{u}^{k+1/2} + P_2(\mathbf{u} - \mathbf{u}^{k+1/2}) = (I - P_2)\mathbf{u}^{k+1/2} + P_2\mathbf{u}, \end{aligned} \tag{3.4.3}$$

where we have introduced the discrete projection operators

$$P_i := R_i^T A_i^{-1} R_i A, \quad i = 1, 2. \tag{3.4.4}$$

Note the formal analogy of (3.4.2), (3.4.3) and (3.4.4) with (1.5.15)–(1.5.16), (1.5.12) and (1.5.31), respectively.

Lemma 3.4.1 *The matrices P_i are symmetric and non-negative definite with respect to the A-scalar product*

$$(\mathbf{w}, \mathbf{v})_A := (A\mathbf{w}, \mathbf{v}) \qquad \forall\ \mathbf{w}, \mathbf{v} \in \mathbf{R}^{N_h}, \tag{3.4.5}$$

which is induced by the symmetric and positive definite stiffness matrix A. Moreover, P_i is the orthogonal projection in the A-scalar product onto the subspace spanned by the rows of R_i^T.

Proof We have

$$\begin{aligned} (P_i\mathbf{w}, \mathbf{v})_A &= (AP_i\mathbf{w}, \mathbf{v}) \\ &= (R_i^T A_i^{-1} R_i A\mathbf{w}, A\mathbf{v}) \\ &= (A\mathbf{w}, R_i^T A_i^{-1} R_i A\mathbf{v}) \\ &= (A\mathbf{w}, P_i\mathbf{v}) = (\mathbf{w}, P_i\mathbf{v})_A \qquad \forall\ \mathbf{v}, \mathbf{w} \in \mathbf{R}^{N_h} \end{aligned} \tag{3.4.6}$$

and

$$
\begin{aligned}
(P_i\mathbf{v},\mathbf{v})_A &= (AP_i\mathbf{v},\mathbf{v}) \\
&= (R_i^T A_i^{-1} R_i A\mathbf{v}, A\mathbf{v}) \\
&= (A_i^{-1} R_i A\mathbf{v}, R_i A\mathbf{v}) \geq 0 \qquad \forall\ \mathbf{v} \in \mathbf{R}^{N_h}.
\end{aligned} \tag{3.4.7}
$$

Moreover,

$$
\begin{aligned}
(P_i\mathbf{v}, R_i^T\mathbf{w})_A &= (AP_i\mathbf{v}, R_i^T\mathbf{w}) \\
&= (A\mathbf{v}, R_i^T A_i^{-1} R_i A R_i^T\mathbf{w}) \\
&= (A\mathbf{v}, R_i^T\mathbf{w}) \\
&= (\mathbf{v}, R_i^T\mathbf{w})_A \qquad \forall\ \mathbf{v},\mathbf{w} \in \mathbf{R}^{N_h},
\end{aligned} \tag{3.4.8}
$$

which completes the proof. □

Note also that, if we set

$$
Q_i := R_i^T A_i^{-1} R_i = P_i A^{-1}, \quad i = 1, 2, \tag{3.4.9}
$$

then the correction

$$
\mathbf{e}_i^k := Q_i(\mathbf{f} - A\mathbf{u}^k)
$$

is such that $P(\mathbf{u} - \mathbf{u}^k) = \mathbf{e}_i^k$; that is, $\mathbf{e}_i^k$ is the closest vector to the error $\mathbf{u} - \mathbf{u}^k$ in the subspace spanned by the rows of R_i^T.

In a compact form the multiplicative Schwarz method reads

$$
\begin{aligned}
\mathbf{u}^{k+1} &= \mathbf{u}^k + (Q_1 + Q_2 - Q_2 A Q_1)(\mathbf{f} - A\mathbf{u}^k) \\
&= \mathbf{u}^k + [I - (I - P_2)(I - P_1)]A^{-1}(\mathbf{f} - A\mathbf{u}^k).
\end{aligned} \tag{3.4.10}
$$

Similarly, the additive Schwarz method becomes

$$
\mathbf{U}^{k+1} = \mathbf{U}^k + (Q_1 + Q_2)(\mathbf{f} - A\mathbf{U}^k). \tag{3.4.11}
$$

The generalisation to M subdomains, $M > 2$, is straightforward. Setting

$$
P_i := R_i^T A_i^{-1} R_i A, \quad Q_i := P_i A^{-1}, \quad i = 1, \ldots, M,
$$

the multiplicative Schwarz method becomes

$$
\begin{aligned}
\mathbf{u}^{k+\frac{i}{M}} &= (I - P_i)\mathbf{u}^{k+\frac{i-1}{M}} + P_i\mathbf{u} \\
&= \mathbf{u}^{k+\frac{i-1}{M}} + R_i^T A_i^{-1} R_i(\mathbf{f} - A\mathbf{u}^{k+\frac{i-1}{M}}), \quad i = 1, \ldots, M,
\end{aligned} \tag{3.4.12}
$$

and the additive Schwarz method reads

$$
\begin{aligned}
\mathbf{U}^{k+1} &= \left(I - \sum_{i=1}^{M} P_i\right)\mathbf{U}^k + \sum_{i=1}^{M} R_i^T A_i^{-1} R_i \mathbf{f} \\
&= \mathbf{U}^k + \left(\sum_{i=1}^{M} Q_i\right)(\mathbf{f} - A\mathbf{U}^k).
\end{aligned} \tag{3.4.13}
$$

The error equation for the multiplicative case takes the form

$$\mathbf{u} - \mathbf{u}^{k+1} = (I - P_M) \cdots (I - P_1)(\mathbf{u} - \mathbf{u}^k), \tag{3.4.14}$$

and

$$\mathbf{u} - \mathbf{U}^{k+1} = \left(I - \sum_{i=1}^{M} P_i \right) (\mathbf{u} - \mathbf{U}^k) \tag{3.4.15}$$

for the additive case, in analogy with the differential case (see Section 1.5.6).

Note that the matrix $Q_{\mathrm{a}} := \sum_{i=1}^{M} P_i$ is symmetric and positive definite with respect to the same scalar product (3.4.5), because

$$(Q_{\mathrm{a}}\mathbf{v}, \mathbf{v})_A = (AQ_{\mathrm{a}}\mathbf{v}, \mathbf{v}) = \sum_{i=1}^{M} (A_i^{-1} R_i A\mathbf{v}, R_i A\mathbf{v}) \geq 0 \qquad \forall\ \mathbf{v} \in \mathbf{R}^{N_h},$$

and the equality is achieved if and only if $R_i A\mathbf{v} = \mathbf{0}$ for each $i = 1, \ldots, M$; that is, if and only if $\mathbf{v} = \mathbf{0}$.

On the other hand, each matrix Q_i is clearly symmetric and, by following the argument just employed, the matrix $\sum_i Q_i$ is symmetric and positive definite (with respect to the Euclidean scalar product).

Remark 3.4.2 (Schwarz iterations and parallelism) In contrast with the additive Schwarz method, the multiplicative method has very little potential for parallel implementation. However, since we may expect that many subdomains do not share any common grid point, a strategy of *subdomain colouring* can be adopted to allow a simultaneous and independent update of subsets of equations by (3.4.12) (see Fig. 3.4.5 for an example, and Smith *et al.* 1996, pp. 19–24, for a thorough discussion of this technique). □

Remark 3.4.3 In special situations there is a close relationship between the additive Schwarz method and the Dirichlet–Neumann method (see, for example, Smith *et al.* 1996, pp. 115–6). □

3.5 Acceleration of the Schwarz method

We note that the multiplicative Schwarz method (3.4.2) coincides with a generalised block-Gauss–Seidel iteration on system (2.1.13) (using overlapping blocks).

To prove it, we generalise the notation of Section 2.4 to the case of overlapping subdomains. For $i = 1, 2$ we define the rectangular matrix

$$\mathcal{D}_i^0 = \begin{pmatrix} \widehat{A}_i & B_i \end{pmatrix}, \tag{3.5.1}$$

which represents the discrete form of the differential operator L restricted to Ω_i. The component $\widehat{A}_i$ concerns the coupling between the interior nodes, while B_i

FIG. 3.4.5. Subdomain colouring (with four colours).

represents the coupling between the interior nodes and the nodes that lie on the artificial boundary Γ_i. The matrix $\mathcal{D}_i^0$ has one row for each interior node and a column for every node (including those on Γ_i). Correspondingly, let us denote by $\mathbf{u}_i$ and $\mathbf{u}_{\Gamma_i}$ the values of the unknown solution at the interior nodes of Ω_i and at the nodes on Γ_i, respectively.

The multiplicative Schwarz method (3.4.2) can be written as

$$\begin{pmatrix} \widehat{A}_1 & B_1 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^{k+1} \\ \mathbf{u}_{\Gamma_1}^{k+1} \end{pmatrix} = \mathbf{f}_1, \quad \mathbf{u}_{\Gamma_1}^{k+1} := I_{\Gamma_1}^{(2)} \mathbf{u}_2^k \tag{3.5.2}$$

and

$$\begin{pmatrix} \widehat{A}_2 & B_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_2^{k+1} \\ \mathbf{u}_{\Gamma_2}^{k+1} \end{pmatrix} = \mathbf{f}_2, \quad \mathbf{u}_{\Gamma_2}^{k+1} := I_{\Gamma_2}^{(1)} \mathbf{u}_1^{k+1}, \tag{3.5.3}$$

where $I_{\Gamma_i}^{(j)}$ denotes a discrete linear operator that interpolates values from the nodes in the interior of Ω_j to the nodes on the interface Γ_i, for $i = 1$, $j = 2$ or $i = 2$, $j = 1$.

Remark 3.5.1 There are cases where the discretisation of Ω is non-conforming; that is, the triangulations of the two subdomains, say $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$, do not coincide in the overlapping region. In this situation, the Schwarz method can still be formulated through (3.5.2) and (3.5.3). Since this time the nodes of $\mathcal{T}_h^1$ on Γ_1 are not necessarily nodes of $\mathcal{T}_h^2$, $I_{\Gamma_1}^{(2)}$ is now a discrete operator that interpolates values from the nodes of $\mathcal{T}_h^2$ to the nodes of $\mathcal{T}_h^1$ on Γ_1. The operator $I_{\Gamma_2}^{(1)}$ is defined similarly. □

From (3.5.2) and (3.5.3) we obtain

$$\begin{aligned} \widehat{A}_1 \mathbf{u}_1^{k+1} &= \mathbf{f}_1 - B_1 I_{\Gamma_1}^{(2)} \mathbf{u}_2^k \\ \widehat{A}_2 \mathbf{u}_2^{k+1} &= \mathbf{f}_2 - B_2 I_{\Gamma_2}^{(1)} \mathbf{u}_1^{k+1}. \end{aligned} \tag{3.5.4}$$

This is precisely a block-Gauss–Seidel iteration for the linear system

$$\begin{pmatrix} \widehat{A}_1 & B_1 I_{\Gamma_1}^{(2)} \\ B_2 I_{\Gamma_2}^{(1)} & \widehat{A}_2 \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}. \tag{3.5.5}$$

We also note that (3.5.4) coincides with the classical block-Gauss–Seidel method in the case of a one-layer overlap (in such a way that the internal nodes of each subdomain Ω_i are not internal nodes of any other subdomain, see Fig. 3.4.2).

System (3.5.5) is not symmetric even if the two submatrices $\widehat{A}_1$ and $\widehat{A}_2$ are symmetric; in fact, in general, $B_1 I_{\Gamma_1}^{(2)} \neq (B_2 I_{\Gamma_2}^{(1)})^T$. However, the multiplicative Schwarz method can be easily symmetrised by including a third step

$$\begin{aligned} \mathbf{u}^{k+1/3} &= \mathbf{u}^k + R_1^T A_1^{-1} R_1 (\mathbf{f} - A\mathbf{u}^k) \\ \mathbf{u}^{k+2/3} &= \mathbf{u}^{k+1/3} + R_2^T A_2^{-1} R_2 (\mathbf{f} - A\mathbf{u}^{k+1/3}) \\ \mathbf{u}^{k+1} &= \mathbf{u}^{k+2/3} + R_1^T A_1^{-1} R_1 (\mathbf{f} - A\mathbf{u}^{k+2/3}). \end{aligned} \tag{3.5.6}$$

This can also be written as a one-step method:

$$\begin{aligned} \mathbf{u}^{k+1} = \mathbf{u}^k + (2Q_1 &+ Q_2 - Q_2 A Q_1 - Q_1 A Q_2 - Q_1 A Q_1 \\ &+ Q_1 A Q_2 A Q_1)(\mathbf{f} - A\mathbf{u}^k). \end{aligned} \tag{3.5.7}$$

Similar equivalence can be stated between the additive Schwarz method and a generalised block-Jacobi iteration method (see, for example, Smith *et al.* 1996, p. 14).

Let us now introduce the matrix

$$P_{\text{as}} := (R_1^T A_1^{-1} R_1 + R_2^T A_2^{-1} R_2)^{-1} = (Q_1 + Q_2)^{-1}, \tag{3.5.8}$$

and note that

$$P_{\text{as}}^{-1} A = P_1 + P_2. \tag{3.5.9}$$

Then the additive Schwarz method can be regarded as a fixed-point method for the following system (equivalent to (2.1.13)):

$$P_{\text{as}}^{-1} A\mathbf{u} = P_{\text{as}}^{-1} \mathbf{f}. \tag{3.5.10}$$

Clearly, P_{as} plays the role of a *preconditioner*, and it is given the name of *additive Schwarz* preconditioner.

When the Schwarz method is used as a preconditioner, each application of the preconditioner is one iteration of the alternating Schwarz method with a vanishing initial guess.

In the case of M subdomains, the preconditioner takes the following form:

$$P_{\text{as}} := \left(\sum_{i=1}^{M} R_i^T A_i^{-1} R_i \right)^{-1} = \left(\sum_{i=1}^{M} Q_i \right)^{-1}, \tag{3.5.11}$$

and the preconditioned matrix becomes

$$P_{\text{as}}^{-1} A = \sum_{i=1}^{M} P_i = Q_{\text{a}}. \tag{3.5.12}$$

As suggested by (3.4.13), the additive Schwarz method is simply a *Richardson method* for (2.1.13) with preconditioner P_{as} for A. However, since P_{as} is symmetric and positive definite, the preconditioned system can be more effectively accelerated by the conjugate gradient (CG) method, which converges faster than the Richardson method does.

The CG iteration for system (2.1.13) preconditioned by P_{as} reads: assign $\mathbf{u}^0$, set $\mathbf{r}^0 := \mathbf{f} - A\mathbf{u}^0$, $\mathbf{p}^0 := \mathbf{z}^0 := P_{\text{as}}^{-1}\mathbf{r}^0$, and for $k \geq 0$ solve

$$\begin{aligned} \alpha_k &:= \frac{(\mathbf{z}^k, \mathbf{r}^k)}{(\mathbf{p}^k, A\mathbf{p}^k)} \\ \mathbf{u}^{k+1} &:= \mathbf{u}^k + \alpha_k \mathbf{p}^k \\ \mathbf{r}^{k+1} &:= \mathbf{r}^k - \alpha_k A\mathbf{p}^k \\ \mathbf{z}^{k+1} &:= P_{\text{as}}^{-1}\mathbf{r}^{k+1} \\ \beta_{k+1} &:= \frac{(\mathbf{z}^{k+1}, \mathbf{r}^{k+1})}{(\mathbf{z}^k, \mathbf{r}^k)} \\ \mathbf{p}^{k+1} &:= \mathbf{z}^{k+1} + \beta_{k+1}\mathbf{p}^k. \end{aligned} \tag{3.5.13}$$

This method involves the same matrix operations as the original Schwarz method, namely the solution of the local subproblems associated with the local submatrices A_i, for $i = 1, \ldots, M$. Clearly, each subdomain solve can be carried out concurrently.

The convergence rate of the new sequence is as follows (see, for example, Golub and Van Loan 1989)

$$||\mathbf{U}^k - \mathbf{U}||_A \leq 2 \left(\frac{\sqrt{\kappa(P_{\text{as}}^{-1}A)} - 1}{\sqrt{\kappa(P_{\text{as}}^{-1}A)} + 1} \right)^k ||\mathbf{U}^0 - \mathbf{U}||_A, \tag{3.5.14}$$

where $||\mathbf{v}||_A = \sqrt{(\mathbf{v}, \mathbf{v})_A}$ is the norm associated with the scalar product $(\mathbf{v}, \mathbf{w})_A$ introduced in (3.4.5).

As for the condition number $\kappa(P_{\text{as}}^{-1}A)$, denoting as usual by H the maximum diameter of the subdomains Ω_i, $i = 1, \ldots, M$, and by δH the linear measure

of the overlapping region between two adjacent subdomains ($0 < \delta \leq 1$), the following estimate holds:

$$\kappa(P_{\text{as}}^{-1}A) \leq C\frac{1}{\delta^2 H^2},$$

where the constant C possibly depends on the coefficients of the symmetric elliptic operator L (see Dryja and Widlund 1989, 1992).

If a coarse grid correction is added to (3.5.11) (see (3.6.11) below), the estimate on the condition number is improved to

$$\kappa(P_{\text{cas}}^{-1}A) \leq C\delta^{-1}.$$

This estimate is sharp, in the sense that the condition number also satisfies $\kappa(P_{\text{cas}}^{-1}A) \geq C\delta^{-1}$ for $\delta = O(h)$, for both two- and three-dimensional cases (see Brenner 1998*b*).

When the coefficients of the operator L are constant in each coarse grid element, but have large jumps across interfaces, the estimate above has to be substituted by

$$\begin{aligned} &\kappa(P_{\text{cas}}^{-1}A) \leq C\left(1 + \log\frac{H}{h}\right) && \text{(in two dimensions)} \\ &\kappa(P_{\text{cas}}^{-1}A) \leq C\frac{H}{h} && \text{(in three dimensions),} \end{aligned}$$

where C does not depend on the coefficients, but may possibly depend on the parameter δ (see Dryja *et al.* 1994).

A similar interpretation can be given for the multiplicative Schwarz method. This time, the *multiplicative Schwarz* preconditioner takes the following form:

$$P_{\text{ms}} := A[I - (I - P_M)\cdots(I - P_1)]^{-1}, \tag{3.5.15}$$

which can also be written as

$$P_{\text{ms}} = \left[\sum_{i=1}^{M} Q_i - \sum_{j>i} Q_j A\, Q_i + \sum_{l>j>i} Q_l A Q_j A Q_i \right. \\ \left. \cdots + (-1)^{M-1} Q_M A \cdots Q_2 A Q_1\right]^{-1}.$$

Since this is no longer symmetric, rather than using CG iterations we should use GMRES or other Krylov procedures that are suitable for non-symmetric matrices (see, for example, Saad 1996).

Remark 3.5.2 The restriction operator R_i can be regarded as being the sum of two terms $R_i = R_i^0 + R_i^{\text{c}}$, where R_i^0 accounts only for the nodes inside the domain Ω_i, excluding the overlapping region, while R_i^{c} for those in the overlapping region.

A new preconditioner has been introduced recently by Cai and Sarkis (1997), which is defined as follows:

$$P_{\rm as}^{\rm r} = \left(\sum_{i=1}^{M} (R_i^0)^T A_i^{-1} R_i \right)^{-1},$$

and is called the *restricted additive Schwarz* preconditioner.

The main motivation for designing $P_{\rm as}^{\rm r}$ is that, in a parallel implementation, substantial communication cost can be saved because computing $R_i^0 \mathbf{v}$ does not involve any data exchange with the neighbouring processors. □

3.5.1 Inexact solvers

If the Schwarz method is used as a preconditioner of an acceleration scheme, it is not necessary for the local subproblems to be solved exactly. In other words, in the preconditioning step of the CG iterations (3.5.13), $P_{\rm as}^{-1}$ could be replaced by the following approximation:

$$\widetilde{P}_{\rm as}^{-1} := \sum_{i=1}^{M} R_i^T \mathcal{A}_i^{-1} R_i,$$

where $\mathcal{A}_i$ is a convenient approximation of A_i on the domain Ω_i. A practical implementation of this idea consists of replacing the bilinear form $a_i^*(\cdot,\cdot)$ in Ω_i by a simpler one $\widehat{a}_i^*(\cdot,\cdot)$ (for example, freezing to a constant the coefficients in Ω_i if the original elliptic problem has variable coefficients, or else by approximating A_i by an inexact factorisation).

In any case, if $a_i^*(\cdot,\cdot)$ and $\widehat{a}_i^*(\cdot,\cdot)$ are *spectrally equivalent*; that is, there exist two positive constants K_1 and K_2 such that

$$K_1 \widehat{a}_i^*(v_i, v_i) \le a_i^*(v_i, v_i) \le K_2 \widehat{a}_i^*(v_i, v_i) \qquad \forall\, v_i \in V_i,$$

then the corresponding preconditioned system $\widetilde{P}_{\rm as}^{-1} A$ retains the same spectral properties of $P_{\rm as}^{-1} A$, and the 'inexact' Schwarz algorithm converges with the same rate of the exact one.

3.6 Two-level methods

The convergence rate of the preconditioned iterative domain decomposition methods deteriorates when the number of subdomains becomes large. As has already been pointed out, this is due to the fact that in Schwarz algorithms, as well as in substructuring iterations, information is exchanged only between neighbouring subdomains. This weakness of the method has been overcome in previous sections; for instance, by introducing a coarse global problem set over the whole domain in order to guarantee a mechanism of global communication among all subdomains.

This is not the only way, though; indeed, the coarse grid solver can be regarded as a special case of a two-level method, which is based on the simultaneous use of a 'fine' problem (the original one) and an auxiliary 'coarse' problem.

3.6.1 Abstract setting of two-level methods

Two-level methods involve the smoothing of the original problem and the solution (or preconditioning) of an auxiliary problem on a related mesh that is coarser than the original one.

They can be thought of as an additive version of the general two-level multigrid algorithm. On a given grid, the preconditioner for the original problem is obtained as a superposition of the solution of an auxiliary problem on a related grid and a smoother on the original grid. The auxiliary problem should be simpler to solve, which is the case if it originates from a coarser grid or a numerical method simpler than the one on the original grid (for example, it uses piecewise polynomials of lower degree, or involves averaging the coefficients of the original operator).

To formulate our problem in an abstract setting, we consider two finite dimensional problems, the original one

$$\text{find } u_h \in V_h \ : \ a_h(u_h, v_h) = \mathcal{F}_h(v_h) \qquad \forall\, v_h \in V_h, \tag{3.6.1}$$

and the auxiliary one

$$\text{find } u_H \in V_H \ : \ a_H(u_H, v_H) = \mathcal{F}_H(v_H) \qquad \forall\, v_H \in V_H, \tag{3.6.2}$$

where $a_h(\cdot,\cdot)$ and $a_H(\cdot,\cdot)$ denote suitable bilinear forms on V_h and V_H, respectively.

Given a basis φ_j, $j = 1, \ldots, N_h$, of V_h and a basis ψ_l, $l = 1, \ldots, N_H$, of V_H, we construct the matrices

$$\begin{aligned} (A_h)_{sj} &:= a_h(\varphi_j, \varphi_s) \\ (A_H)_{ml} &:= a_H(\psi_l, \psi_m). \end{aligned} \tag{3.6.3}$$

The goal is to find a preconditioner for A_h using either A_H or a good preconditioner of A_H.

The two spaces V_h and V_H are related by an operator $I_h : V_H \to V_h$. Typically, I_h is the linear interpolation from the coarse grid to the fine grid and its matrix representation is R^T (see Section 3.4). In these cases, h and H are the maximum diameter of the elements of the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively.

Two different adjoint operators can be associated with I_h:

$$\begin{cases} I_h^T : V_h \to V_H \ : \\ (I_h^T w_h, v_H)_{V_H} = (w_h, I_h v_H)_{V_h} \qquad \forall\, w_h \in V_h, v_H \in V_H \end{cases} \tag{3.6.4}$$

and

$$\begin{cases} J_h^T : V_h \to V_H \ : \\ \\ a_H(J_h^T w_h, v_H) = a_h(w_h, J_h v_H) \qquad \forall\ w_h \in V_h, v_H \in V_H. \end{cases} \tag{3.6.5}$$

In matrix form, keeping the same notation, this last definition reads $A_H J_h^T = I_h^T A_h$, yielding $J_h^T = A_H^{-1} I_h^T A_h$.

A preconditioner P_h for A_h can be constructed in the following way:

$$P_h^{-1} := Q_H^{-1} + Q_h^{-1}, \quad Q_H^{-1} := I_h A_H^{-1} I_h^T, \tag{3.6.6}$$

where Q_h is any convenient symmetric positive definite matrix, much simpler than A_h itself. The preconditioned matrix becomes

$$P_h^{-1} A_h = I_h J_h^T + Q_h^{-1} A_h.$$

If Q_h is a good preconditioner for A_h, P_h remains a good preconditioner for A_h even replacing in (3.6.6) A_H^{-1} with a good preconditioner on the coarse grid.

In a two-level multi-grid context, Q_h is the smoothing operator on the fine grid (in general, a few Jacobi iterations to damp the high frequencies of the error), while A_H^{-1} is a coarse grid solution operator.

In general, it is not enough to use Q_H^{-1} alone as the preconditioner, because it has a large null space. As a matter of fact, the rank of Q_H^{-1} is equal to the dimension of A_H, which is lower than the dimension of A_h. Therefore, any component of the residual lying in the null space of Q_H^{-1} would never be corrected. For this reason, Q_h^{-1} must have full rank.

For an elliptic partial differential equation, Q_H is designed to account for the long-range effects, Q_h for the local ones. At the lowest level, Q_h coincides with the diagonal part of the stiffness matrix A_h. In the domain decomposition context, the application of Q_h^{-1} will involve subdomain solves.

3.6.2 Multiplicative and additive two-level preconditioners

Define the vector $\mathbf{f}$ as $f_j := \mathcal{F}(\varphi_j)$; a two-step preconditioner containing both the long-range and local-range components, Q_H and Q_h, respectively, is

$$\begin{aligned} Q_H(\mathbf{u}^{n+1/2} - \mathbf{u}^n) &= \mathbf{f} - A_h \mathbf{u}^n \\ Q_h(\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}) &= \mathbf{f} - A_h \mathbf{u}^{n+1/2}. \end{aligned} \tag{3.6.7}$$

It can be written as a one-step method:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (Q_H^{-1} + Q_h^{-1} - Q_h^{-1} A_h Q_H^{-1})(\mathbf{f} - A\mathbf{u}^n). \tag{3.6.8}$$

This is the *multiplicative two-level method*. Its corresponding additive form, the *additive two-level method*, replaces (3.6.8) with the following equation:

$$\mathbf{u}^{n+1} = \mathbf{u}^n + (Q_H^{-1} + Q_h^{-1})(\mathbf{f} - A\mathbf{u}^n). \tag{3.6.9}$$

This is one step of the Richardson method for the fine grid problem $A_h\mathbf{u} = \mathbf{f}$ with preconditioner P_h (see (3.6.6)). It can be made more efficient when it is used as part of a Krylov subspace accelerator; namely, if the same preconditioner P_h is used within a GMRES or a conjugate gradient iterative procedure.

Parallel multi-level preconditioners have been developed by Bramble *et al.* (1990); see also Zhang (1992) and Griebel (1994).

3.6.3 The case of the Schwarz method

When Q_h is either a multiplicative or an additive overlapping Schwarz preconditioner ($Q_h = P_{\rm ms}$ or $Q_h = P_{\rm as}$, see Section 3.5), we say that (3.6.8) or (3.6.9) are a two-level Schwarz methods.

The generation of a coarse grid for the Schwarz method requires extra care with respect to the case of non-overlapping partitions. A commonly used technique for constructing an overlapping decomposition of Ω into M subdomains $\Omega_1, \cdots, \Omega_M$ assumes that a non-overlapping partition $\omega_1, \cdots, \omega_M$ of Ω is available. One possibility is to choose each subregion ω_i as an element from a coarse finite element triangulation $\mathcal{T}_H$ of Ω of size H. Next, each ω_i is extended to a larger domain Ω_i, consisting of all points in Ω at a distance not larger than δH from ω_i, with $0 < \delta \leq 1$. The restriction and extension maps, R_i and R_i^T, as well as the local matrices A_i, are defined accordingly.

Assume now that the finite element triangulation $\mathcal{T}_h$ is a refinement of the coarse grid partition $\mathcal{T}_H$. Accordingly, let us denote by R_H^T the interpolation map of coarse grid functions to fine grid functions. When using piecewise-linear elements, R_H^T interpolates the nodal values from the coarse grid vertices (of ω_i) to all the vertices of the fine grid. Its transpose R_H is a weighted restriction map. Correspondingly, let A_H denote the stiffness matrix of our elliptic problem on the coarse mesh $\mathcal{T}_H$, i.e. $A_H = R_H A R_H^T$.

In the additive case, we have

$$Q_h^{-1} = \sum_{i=1}^{M} R_i^T A_i^{-1} R_i, \quad Q_H^{-1} = R_H^T A_H^{-1} R_H, \tag{3.6.10}$$

where the index i refers to the ith subdomain, $i = 1, \ldots, M$, and the index H refers to the coarse grid (where subdomains play the role of elements). The resulting preconditioner is therefore

$$P_h = P_{\rm cas} := \left(\sum_{i=0}^{M} R_i^T A_i^{-1} R_i \right)^{-1}, \tag{3.6.11}$$

where we have set, for notational convenience, $R_0 := R_H$ and $A_0 := A_H$. The multiplicative preconditioner is obtained similarly.

3.6.4 Convergence of two-level methods

Let us assume that the two grids underlying problems (3.6.1) and (3.6.2), say $\mathcal{T}_h$ and $\mathcal{T}_H$, are *comparable*, so that there exist two positive constants C_0 and C_1

such that for any $K_h \in \mathcal{T}_h$ and $K_H \in \mathcal{T}_H$ with $K_h \cap K_H \neq \emptyset$ it holds that

$$C_0 \operatorname{diam} K_H \leq \operatorname{diam} K_h \leq C_1 \operatorname{diam} K_H. \tag{3.6.12}$$

If the two grids are quasi-uniform and of comparable size (i.e. $C_0, C_1 \simeq 1$), using either a block-Jacobi or a symmetric block-Gauss–Seidel iteration as a smoother, the preconditioner P_h is uniformly optimal for the operator A_h (the spectrum of $P_h^{-1}A_h$ is uniformly bounded with respect to h).

If, instead, $\mathcal{T}_H$ is genuinely coarser than $\mathcal{T}_h$, which means that there are triangles K_h and K_H, $K_h \cap K_H \neq \emptyset$, for which (3.6.12) holds only for extremely small C_0, then the above smoothers are no longer sufficient to guarantee that P_h is uniformly optimal for A_h. However, if the smoothing operator is based on the overlapping Schwarz methods (Q_h given by the symmetric multiplicative Schwarz preconditioner, see (3.5.7), or $Q_h = P_{\text{as}}$, see (3.5.8)), then P_h^{-1} is uniformly optimal, provided the coarse grid $\mathcal{T}_H$ is 'comparable' with the subdomain partition. This means that there should exist positive constants $\hat{C}_0$ and $\hat{C}_1$ such that for each $K_H \in \mathcal{T}_H$ such that $K_H \cap \Omega_i \neq \emptyset$ it holds that

$$\hat{C}_0 \operatorname{diam} K_H \leq \operatorname{diam} \Omega_i \leq \hat{C}_1 \operatorname{diam} K_H.$$

For the proofs of these results, see Bramble *et al.* (1996), and the references therein. In particular, we point out that the spectrum is bounded independently of h, H and the linear measure of the overlapping region, provided the latter is kept proportional to H, say, given by δH. The convergence of the iterative procedures is poor for very small values of δ, but improves rapidly as the overlap increases. Finally, it is worthwhile mentioning that the number of iterations for the symmetric multiplicative Schwarz method is roughly half of that needed for the additive Schwarz method. For numerical evidence see Smith *et al.* (1996); we refer to the same monograph for the analysis of multi-level methods with more than two levels.

Remark 3.6.1 The generation of coarse grid preconditioners on unstructured grids is a difficult task, especially for three-dimensional problems. An algebraic way, inspired by agglomeration multi-grid techniques, is often adopted. A discussion can be found in Chan and Mathew (1994*a*); Bank and J. Xu (1995); and Chan and Smith (1995). See also Smith *et al.* (1996), Section 2.6 and the references therein. □

3.7 Direct Galerkin approximation of the Steklov–Poincaré equation

All the methods described in Section 1.3 share the property of being derived by a suitable splitting of the given boundary value problem over the subdomains. Thus, they are amenable to iterative procedures that, at each step, require the solution of at least as many independent boundary value problems as the number of subdomains. On the other hand, from a merely speculative point of view,

these iterative processes on the *primitive variables* $u_i = u_{|\Omega_i}$ (u being the solution of the given boundary value problem) can be interpreted as suitable iterative schemes for the *dual variable* $\lambda = u_{|\Gamma}$, which is the solution of the interface equation (1.1.7). More precisely, the iterative procedure on the primitive variables u_i induces a preconditioned iterative procedure on λ.

The same interpretation clearly holds at the finite dimensional level, which has been dealt with in Section 3.1. A suitable approximation is introduced first for the given boundary value problem (for example, a Galerkin finite element approximation), then an appropriate domain decomposition iterative procedure is chosen for the discrete primitive variables $u_{i,h} = u_{h|\Omega_i}$, and the latter can be regarded as an iterative scheme for the dual variable $\lambda_h = u_{h|\Gamma}$, which in turn is the solution of the discrete interface equation (2.2.3).

It has to be pointed out, however, that the driving mechanism of the domain decomposition procedure is the subdomain iterative method acting on the discrete primitive variables.

A different approach consists of attacking *directly* the interface equation (1.1.7) by a Galerkin method on a suitable subspace Λ_h of the space of traces Λ. More precisely, we start by reformulating (1.1.7) in a variational way as follows:

$$\text{find } \lambda \in \Lambda \ : \ \mathcal{S}(\lambda, \mu) = \langle \chi, \mu \rangle \qquad \forall\, \mu \in \Lambda, \tag{3.7.1}$$

where $\mathcal{S}(\eta, \mu) := \langle S\eta, \mu \rangle$ is the bilinear form associated with S, which is symmetric and coercive in Λ. Then we associate with (3.7.1) the following internal Galerkin approximation

$$\text{find } \lambda_h \in \Lambda_h \ : \ \mathcal{S}(\lambda_h, \mu_h) = \langle \chi_h, \mu_h \rangle \qquad \forall\, \mu_h \in \Lambda_h, \tag{3.7.2}$$

where χ_h is a convenient approximation to the right-hand side χ of (1.1.7). The latter problem is completely defined after having chosen the finite dimensional subspace Λ_h of Λ. Obviously, (3.7.2) yields an algebraic problem that is symmetric and positive definite.

This approach has been introduced by Agoshkov and Ovtchinnikov (1994), and has been given the name of *Projection Decomposition method* (PDM). Its interest relies mainly on the possibility of constructing well-conditioned, piecewise-polynomial bases for the space Λ_h, an option that has been successively pursued by Ovtchinnikov. In a series of papers (Ovtchinnikov 1993, 1995; Gervasio *et al.* 1997; and Xantis and Ovtchinnikov 1994) it has been proved how to obtain these bases in a fast and accurate way for both Laplace and Stokes operators (through a Gram–Schmidt orthogonalisation procedure).

The good conditioning of these bases makes it possible to solve the algebraic problem by a *non-preconditioned* conjugate gradient method. At each step, independent boundary value problems have to be solved in each subdomain.

The accuracy with which these piecewise-polynomial functions are obtained allows us to maintain the order of accuracy of the solution of local problems in the subdomains, even when high-order methods are adopted therein. In several

instances, the PDM has been seen to be very effective compared with other domain decomposition methods.

4

CONVERGENCE ANALYSIS FOR ITERATIVE DOMAIN DECOMPOSITION ALGORITHMS

In this chapter we present some abstract theorems that are useful for proving the convergence of iteration-by-subdomain methods. The main part of our analysis will cover the case of substructuring procedures (for disjoint subdomains). The convergence analysis of the Schwarz method for overlapping partitions will be addressed in the last section of this chapter. For a complete analysis of Schwarz methods we refer to Dryja and Widlund (1990, 1995), J. Xu (1992) and Smith *et al.* (1996).

The chapter is organised as follows. We begin in Section 4.1 by providing some extension theorems in different function spaces ($H^1(\Omega_i)$, $H(\mathrm{div};\Omega_i)$, $H(\mathrm{rot};\Omega_i)$), as well as in the corresponding finite element subspaces.

As outlined in the previous chapters, the local extension operators represent the basic mathematical ingredients of the Steklov–Poincaré operator. On the other hand, iterative substructuring methods are amenable to preconditioned iterative algorithms on the Steklov–Poincaré equation. For this reason, in Section 4.2 we focus on operators in Hilbert spaces, given in split form, and consider preconditioners whose inverses are made up of suitable combinations of inverses of suboperators. Then we provide some abstract convergence theorems for iterative methods of a preconditioned Krylov type in this framework. The analysis is also particularised to the algebraic finite dimensional counterpart of our equations.

As of our paradigmatic elliptic boundary value problem, we address its convergence analysis in this chapter, immediately after having carried out the abstract analysis of Section 4.2.

Since we also cover the non-symmetric and complex cases, our abstract results will be applied (through the following chapters) to a wide class of boundary value problems.

In particular, Theorems 4.2.2 and 4.2.5 can be applied to Dirichlet–Neumann and Neumann–Neumann iterations, respectively, for a wide variety of situations, including general elliptic problems, the elasticity operator, the Stokes problem, and its generalisations, as well as for advection–diffusion equations. The first theorem will also be useful for some of the heterogeneous domain decomposition procedures that we will describe in Chapter 8.

Theorems 4.2.10 and 4.2.13 (and Corollaries 4.2.11 and 4.2.14) are concerned with iterations for complex matrices, with a non-symmetric preconditioner, and can be applied to the Dirichlet–Neumann and Neumann–Neumann iterative scheme, respectively, for situations like those arising with the time-harmonic Maxwell equations.

The proof of the convergence of the Robin iterative substructuring method is based on an ad hoc argument and is presented in Section 4.5.

4.1 Extension theorems and spectrally equivalent operators

As anticipated in the previous chapters, typical domain decomposition preconditioners are expressed in terms of the sum of operators, each one related to a certain subdomain Ω_i of the (bounded) computational domain $\Omega \subset \mathbf{R}^d$, $d = 2, 3$. This is the case for the Steklov–Poincaré operator S for the Laplace operator (or, similarly, for the symmetric elliptic operator L introduced in (1.4.2)), which is defined as

$$\begin{aligned} &\langle S\eta, \mu\rangle = \sum_{i=1}^{M} \langle S_i\eta, \mu\rangle \\ &\langle S_i\eta, \mu\rangle = a_i(H_i\eta, H_i\mu) = \int_{\Omega_i} \nabla H_i\eta \cdot \nabla H_i\mu, \end{aligned} \tag{4.1.1}$$

where η and μ are trace functions on the interface Γ, and H_i denotes the harmonic extension operator from Γ into Ω_i (see Section 1.2). The preconditioners are constructed by assembling the local operators S_i.

To analyse the spectral properties of a preconditioner based on substructuring; namely, based on the local operators S_i, it is necessary to show that the extension operators H_i induce an equivalent norm on Λ, the space of traces on Γ.

In the next section we will focus on the equivalence of these extension operators. Then in Sections 4.1.2 and 4.1.3 we will consider extension operators in different function spaces, that will then be used in Chapter 5 for the analysis of many other boundary value problems.

As we have already mentioned in the Preface, we restrict our attention to the case of a two-domain decomposition of the domain Ω.

4.1.1 Extension theorems in $H^1(\Omega_i)$

The first result we are interested in concerns the harmonic extension.

Proposition 4.1.1 *Let the space Λ be defined in* (1.2.4) *and the extension operators H_i in* (1.2.14)*. Then there exist two positive constants C_1 and C_2 such that*

$$C_1||\eta||_\Lambda \leq ||H_i\eta||_{1,\Omega_i} \leq C_2||\eta||_\Lambda \qquad \forall\ \eta \in \Lambda, \quad i = 1, 2. \tag{4.1.2}$$

Proof The left-hand inequality follows from the trace inequality (1.2.5), while the right-hand one follows from the a priori estimate (1.2.11) for the Dirichlet boundary value problem for (coercive) elliptic equations (see, for example, J.-L. Lions and Magenes 1972). □

An easy consequence is the following result:

Proposition 4.1.2 *The operators S_i, $i = 1, 2$, defined in* (4.1.1) *are continuous and coercive in Λ. Moreover, they are spectrally equivalent; that is, there exist two positive constants C_3 and C_4 such that*

$$C_3\langle S_1\eta, \eta\rangle \le \langle S_2\eta, \eta\rangle \le C_4\langle S_1\eta, \eta\rangle \qquad \forall\ \eta \in \Lambda. \tag{4.1.3}$$

Consequently, the operator $S = S_1 + S_2$ is continuous and coercive in Λ, and both S_1 and S_2 are spectrally equivalent to S.

Proof The bilinear forms a_i are continuous and coercive in $H^1(\Omega_i)$, hence from Proposition 4.1.1 we see that for each η, $\mu \in \Lambda$ it holds that

$$\langle S_i\eta, \mu\rangle \le ||H_i\eta||_{1,\Omega_i}||H_i\mu||_{1,\Omega_i} \le C_2^2||\eta||_\Lambda||\mu||_\Lambda$$

and

$$\begin{aligned}\langle S_i\eta, \eta\rangle &= ||\nabla H_i\eta||^2_{0,\Omega_i} \ge (1 + C_\Omega)^{-1}||H_i\eta||^2_{1,\Omega_i}\\ &\ge C_1^2(1 + C_\Omega)^{-1}||\eta||^2_\Lambda,\end{aligned}$$

for $i = 1, 2$, where C_Ω is the constant in the Poincaré inequality (1.2.2).

As a consequence,

$$\langle S_1\eta, \eta\rangle \le C_2^2||\eta||^2_\Lambda \le C_2^2(1 + C_\Omega)C_1^{-2}\langle S_2\eta, \eta\rangle,$$

and the same is true by interchanging the roles of S_1 and S_2. □

Let us consider now the finite dimensional case. The finite element approximation of the homogeneous Dirichlet problem for the Laplace operator has been introduced in (2.1.1). Following (2.2.4), the discrete Steklov–Poincaré operators are defined as

$$\langle S_{i,h}\eta_h, \mu_h\rangle := a_i(H_{i,h}\eta_h, H_{i,h}\mu_h) \qquad \forall\ \eta_h, \mu_h \in \Lambda_h \tag{4.1.4}$$

(for notation see (2.1.7) and (2.2.1)).

We prove the following theorem, known as the *finite element uniform extension theorem* (see, for example, Bramble *et al.* 1986*a*; Bjørstad and Widlund 1986; or Marini and Quarteroni 1988, 1989). It provides a finite element counterpart of Proposition 4.1.1, and we have already stated it as inequality (2.2.5). We point out that different constructions of uniform extension operators have also been given by Brenner (1998*a*) and Bernardi and Girault (1998).

Theorem 4.1.3 (Uniform extension theorem) *Let Ω, Ω_1 and Ω_2 be Lipschitz polygonal domains. Let the space V_h be defined as in* (2.1.3)*, with X_h^r introduced in* (2.1.2) *or* (2.1.4)*. Assume that the family of triangulations $\mathcal{T}_h$ is regular, and that the family of triangulations $\mathcal{M}_h$, induced by $\mathcal{T}_h$ on the interface Γ, is quasi-uniform. There exist two positive constants $\hat{C}_1$ and $\hat{C}_2$, which depend on the relative sizes of Ω_1 and Ω_2 but are independent of h, such that*

$$\hat{C}_1||\eta_h||_\Lambda \le ||H_{i,h}\eta_h||_{1,\Omega_i} \le \hat{C}_2||\eta_h||_\Lambda \qquad \forall\ \eta_h \in \Lambda_h, \quad i = 1, 2. \tag{4.1.5}$$

Proof Since η_h is the trace on the interface Γ of both $H_{1,h}\eta_h$ and $H_{2,h}\eta_h$, the *trace inequality* (1.2.5) states that there exist constants $C_i^* > 0$ such that

$$||\eta_h||_\Lambda \leq C_i^* ||H_{i,h}\eta_h||_{1,\Omega_i} \qquad \forall\ \eta_h \in \Lambda_h, \quad i = 1, 2.$$

Therefore, the left-hand inequality in (4.1.5) follows by choosing

$$\hat{C}_1 := \min(1/C_1^*, 1/C_2^*).$$

On the other hand, we have

$$||H_{i,h}\eta_h||_{1,\Omega_i} \leq ||H_{i,h}\eta_h - H_i\eta_h||_{1,\Omega_i} + ||H_i\eta_h||_{1,\Omega_i}.$$

From (4.1.2), it follows that

$$||H_i\eta_h||_{1,\Omega_i} \leq C_2||\eta_h||_\Lambda, \quad i = 1, 2.$$

Since η_h is a piecewise-polynomial continuous function on Γ and Ω_i is a Lipschitz polygonal domain, the solution $H_i\eta_h$ belongs to the Sobolev space $H^{1+s^*}(\Omega_i)$ for a suitable $s^* > 1/2$ (see Dauge 1988, Corollary 18.15). Denoting by $||\cdot||_{\sigma,\Omega_i}$ and $||\cdot||_{\sigma,\Gamma}$, respectively, the norm of the Sobolev spaces $H^\sigma(\Omega_i)$ and $H^\sigma(\Gamma)$, for each $\sigma \in \mathbf{R}$, the following *regularity* estimate also holds:

$$||H_i\eta_h||_{1+s^*,\Omega_i} \leq C||\eta_h||_{1/2+s^*,\Gamma}.$$

Moreover, we have the finite element error estimate

$$||H_{i,h}\eta_h - H_i\eta_h||_{1,\Omega_i} \leq Ch^{s^*}||H_i\eta_h||_{1+s^*,\Omega_i},$$

which is a consequence of the *continuity* and *coerciveness* of the bilinear forms $a_i(\cdot,\cdot)$ and of the *interpolation error* estimate (see, for example, Ciarlet 1978). The *inverse inequality* (see again Ciarlet 1978) gives

$$h^{s^*}||\eta_h||_{1/2+s^*,\Gamma} \leq C||\eta_h||_\Lambda \qquad \forall\ \eta_h \in \Lambda_h,$$

with a constant C independent of h. Therefore, for a suitable constant $\hat{C}_2$ independent of h we obtain

$$||H_{i,h}\eta_h||_{1,\Omega_i} \leq \hat{C}_2||\eta_h||_\Lambda,$$

and the right-hand inequality in (4.1.5) is proved. □

Proceeding as in Proposition 4.1.2, we easily find

Theorem 4.1.4 *Under the assumptions of Theorem 4.1.3, the discrete Steklov–Poincaré operators $S_{1,h}$, $S_{2,h}$ and $S_h = S_{1,h} + S_{2,h}$ are continuous and coercive in Λ_h, uniformly with respect to h. Moreover, they are uniformly spectrally equivalent; that is, the double inequality of spectral equivalence holds with constants independent of h.*

This result has important consequences at the algebraic level. In fact, let us introduce the matrices $\Sigma_{i,h}$ associated with the Steklov–Poincaré operators $S_{i,h}$, defined as

$$[\Sigma_{i,h}\boldsymbol{\eta}, \boldsymbol{\mu}] := \langle S_{i,h}\eta_h, \mu_h\rangle \quad \forall\ \eta_h, \mu_h \in \Lambda_h, \tag{4.1.6}$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbf{R}^{N_\Gamma}$, and for each $\mu_h \in \Lambda_h$, $\boldsymbol{\mu}$ denotes the set of its values at the N_Γ nodes on Γ. We obtain at once that both $\Sigma_{i,h}$ are symmetric and positive definite. In particular, $[\Sigma_{i,h}\boldsymbol{\eta}, \boldsymbol{\eta}] \geq \alpha_i ||\eta_h||^2_\Lambda$ for $i = 1, 2$, having denoted by α_i the coerciveness constant of $S_{i,h}$.

Therefore, since $\Sigma_h = \Sigma_{1,h} + \Sigma_{2,h}$, we have

$$[\Sigma_{2,h}\boldsymbol{\eta}, \boldsymbol{\eta}] \leq [\Sigma_h\boldsymbol{\eta}, \boldsymbol{\eta}], \tag{4.1.7}$$

moreover

$$\begin{aligned} [\Sigma_h\boldsymbol{\eta}, \boldsymbol{\eta}] &= [\Sigma_{1,h}\boldsymbol{\eta}, \boldsymbol{\eta}] + [\Sigma_{2,h}\boldsymbol{\eta}, \boldsymbol{\eta}] \\ &\leq \beta_1 ||\eta_h||^2_\Lambda + [\Sigma_{2,h}\boldsymbol{\eta}, \boldsymbol{\eta}] \\ &\leq \left(\frac{\beta_1}{\alpha_2} + 1\right) [\Sigma_{2,h}\boldsymbol{\eta}, \boldsymbol{\eta}], \end{aligned} \tag{4.1.8}$$

where β_i is the continuity constant of $S_{i,h}$, $i = 1, 2$.

Let us recall the following result (see, for example, Quarteroni and Valli 1994, p. 54).

Theorem 4.1.5 *Let P and Q be two symmetric and positive definite $M \times M$ (real) matrices. Assume that there exist constants $K_1 > 0$ and $K_2 > 0$ such that*

$$K_1[P\boldsymbol{\eta}, \boldsymbol{\eta}] \leq [Q\boldsymbol{\eta}, \boldsymbol{\eta}] \leq K_2[P\boldsymbol{\eta}, \boldsymbol{\eta}] \qquad \forall\ \boldsymbol{\eta} \in \mathbf{R}^M, \tag{4.1.9}$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbf{R}^M$. Then the eigenvalues of the preconditioned matrix $P^{-1}Q$ satisfy the inequality

$$K_1 \leq \nu_{\min} \leq \nu_{\max} \leq K_2,$$

and the spectral condition number $\kappa(P^{-1}Q) := \frac{\nu_{\max}}{\nu_{\min}}$ satisfies

$$\kappa(P^{-1}Q) \leq \frac{K_2}{K_1}.$$

As a consequence of (4.1.7) and (4.1.8) we have thus proved that the condition number of the preconditioned matrix $\Sigma^{-1}_{2,h}\Sigma_h$ is bounded, uniformly with respect to h, hence the Dirichlet–Neumann preconditioner $\Sigma_{2,h}$ is an optimal preconditioner for Σ_h. Clearly, the same is true for $\Sigma^{-1}_{1,h}\Sigma_h$.

We are also in a position to analyse another preconditioner, the Neumann–Neumann preconditioner. First, let us point out that if a linear operator $\mathcal{L}$:

$\Lambda \to \Lambda'$ is continuous and coercive, with continuity constant given by β and coerciveness constant given by α, then its inverse $\mathcal{L}^{-1} : \Lambda' \to \Lambda$ exists (by the Lax–Milgram lemma), it is continuous with constant α^{-1}, and coercive with constant α/β^2. In fact, for any $G \in \Lambda'$ the solution $\lambda = \mathcal{L}^{-1}G \in \Lambda$ of $\mathcal{L}\lambda = G$ satisfies

$$\alpha||\lambda||_\Lambda^2 \leq \langle \mathcal{L}\lambda, \lambda\rangle = \langle G, \lambda\rangle \leq ||G||_{\Lambda'}||\lambda||_\Lambda,$$

hence

$$||\mathcal{L}^{-1}G||_\Lambda \leq \frac{1}{\alpha}||G||_{\Lambda'}.$$

Moreover,

$$\begin{aligned} ||G||_{\Lambda'}^2 &= ||\mathcal{L}\lambda||_{\Lambda'}^2 \leq \beta^2||\lambda||_\Lambda^2 \\ &\leq \frac{\beta^2}{\alpha}\langle \mathcal{L}\lambda, \lambda\rangle = \frac{\beta^2}{\alpha}\langle G, \mathcal{L}^{-1}G\rangle. \end{aligned}$$

Consequently, for each $\sigma_1 > 0$ and $\sigma_2 > 0$ the Neumann–Neumann operator $\mathcal{N}_h := (\sigma_1 S_{1,h}^{-1} + \sigma_2 S_{2,h}^{-1})^{-1}$ is symmetric, continuous, and coercive in Λ. Let us denote by $\beta_\mathcal{N}$ and $\alpha_\mathcal{N}$ its continuity and coerciveness constant, respectively; by a straightforward computation, we can see that they are respectively given by

$$\begin{aligned} \beta_\mathcal{N} &:= \frac{\beta_1^2\beta_2^2}{\sigma_1\alpha_1\beta_2^2 + \sigma_2\alpha_2\beta_1^2} \\ \alpha_\mathcal{N} &:= \frac{(\sigma_1\alpha_1\beta_2^2 + \sigma_2\alpha_2\beta_1^2)\alpha_1^2\alpha_2^2}{\beta_1^2\beta_2^2(\alpha_1\sigma_2 + \alpha_2\sigma_1)^2}, \end{aligned}$$

and are independent of h.

Introducing the Neumann–Neumann preconditioner

$$[N_h\boldsymbol{\eta}, \boldsymbol{\mu}] := \langle \mathcal{N}_h\eta_h, \mu_h\rangle,$$

it can be seen that

$$N_h = (\sigma_1\Sigma_{1,h}^{-1} + \sigma_2\Sigma_{2,h}^{-1})^{-1}. \tag{4.1.10}$$

We have

$$\begin{aligned} [N_h\boldsymbol{\eta}, \boldsymbol{\eta}] &= \langle \mathcal{N}_h\eta_h, \eta_h\rangle \leq \beta_\mathcal{N}||\eta_h||_\Lambda^2 \\ &\leq \frac{\beta_\mathcal{N}}{\alpha_1 + \alpha_2}[\Sigma_h\boldsymbol{\eta}, \boldsymbol{\eta}] \qquad \forall\ \boldsymbol{\eta} \in \mathbf{R}^{N_\Gamma}, \end{aligned}$$

moreover

$$\begin{aligned} [\Sigma_h\boldsymbol{\eta}, \boldsymbol{\eta}] &= \langle S_h\eta_h, \eta_h\rangle \leq (\beta_1 + \beta_2)||\eta_h||_\Lambda^2 \\ &\leq \frac{\beta_1 + \beta_2}{\alpha_\mathcal{N}}[N_h\boldsymbol{\eta}, \boldsymbol{\eta}] \qquad \forall\ \boldsymbol{\eta} \in \mathbf{R}^{N_\Gamma}. \end{aligned}$$

Altogether, the two previous inequalities allow us to conclude that the matrix N_h is an optimal preconditioner of the matrix Σ_h, owing to Theorem 4.1.5.

Similar considerations are possible if we focus now on the homogeneous Dirichlet boundary value problem for the *general* second-order symmetric elliptic operator

$$Lw := -\sum_{l,j=1}^{d} D_l(a_{lj} D_j w) + a_0 w \tag{4.1.11}$$

introduced in (1.4.2), as well as its approximation by means of the finite elements.

The local bilinear forms

$$a_i^*(w_i, v_i) := \int_{\Omega_i} \left(\sum_{l,j=1}^{d} a_{lj} D_j w_i D_l v_i + a_0\, w_i\, v_i \right), \quad i = 1, 2, \tag{4.1.12}$$

associated with L are symmetric and continuous; besides, they are coercive if, for example, $a_0 \geq 0$.

For $i = 1, 2$ we introduce the operators E_i^*, which extend in Ω_i a trace η defined on Γ to the function $E_i^* \eta$, which solves the following problem

$$\begin{cases} E_i^* \eta \in V_i \ : \\ a_i^*(E_i^* \eta, v_i) = 0 \qquad \forall\ v_i \in V_i^0 \\ E_i^* \eta = \eta \qquad \text{on } \Gamma. \end{cases} \tag{4.1.13}$$

The Steklov–Poincaré operators associated with the operator L are defined as

$$\langle S_i \eta, \mu \rangle := a_i^*(E_i^* \eta, E_i^* \mu) \qquad \forall\ \eta, \mu \in \Lambda. \tag{4.1.14}$$

For the finite element approximation the definition is similar:

$$\langle S_{i,h} \eta_h, \mu_h \rangle := a_i^*(E_{i,h}^* \eta_h, E_{i,h}^* \mu_h) \qquad \forall\ \eta_h, \mu_h \in \Lambda_h, \tag{4.1.15}$$

where now $E_{i,h}^* \eta_h$ is a finite element extension of η_h in Ω_i and is given by

$$\begin{cases} E_{i,h}^* \eta_h \in V_{i,h} \ : \\ a_i^*(E_{i,h}^* \eta_h, v_{i,h}) = 0 \qquad \forall\ v_{i,h} \in V_{i,h}^0 \\ E_{i,h}^* \eta_h = \eta_h \qquad \text{on } \Gamma. \end{cases} \tag{4.1.16}$$

The infinite dimensional Steklov–Poincaré operators S_1 and S_2, introduced in (4.1.14), as well as $S = S_1 + S_2$, are symmetric, and the same is true for their discrete counterparts $S_{1,h}$ and $S_{2,h}$, defined in (4.1.15), and for $S_h = S_{1,h} + S_{2,h}$. Moreover, we have

Theorem 4.1.6 *The Steklov–Poincaré operators S_1, S_2 and $S = S_1 + S_2$ introduced in* (4.1.14) *are continuous and coercive in Λ; hence, they are spectrally equivalent. If the assumptions of Theorem 4.1.3 are satisfied, the same is true for the Steklov–Poincaré operators $S_{1,h}$, $S_{2,h}$ and S_h introduced in* (4.1.15)*, and the result holds uniformly with respect to h.*

Proof We only consider the discrete case, the proof for the infinite dimensional case being easier. Coerciveness of $S_{i,h}$ is a direct consequence of the coerciveness of the forms $a_i^*(\cdot,\cdot)$ and of the trace inequality (1.2.5), which yields

$$||\eta_h||_\Lambda \leq C_i^* ||E_{i,h}^* \eta_h||_{1,\Omega_i} \qquad \forall\ \eta_h \in \Lambda_h.$$

Hence, the result holds uniformly with respect to h.

To prove the uniform continuity of $S_{i,h}$ we proceed as follows. Since we are allowed to take $v_{i,h} = E_{i,h}^* \eta_h - H_{i,h}\eta_h$ in (4.1.16), we find that

$$a_i^*(E_{i,h}^*\eta_h, E_{i,h}^*\eta_h) = a_i^*(E_{i,h}^*\eta_h, H_{i,h}\eta_h).$$

From the continuity and coerciveness of $a_i^*(\cdot,\cdot)$ we deduce that

$$||E_{i,h}^*\eta_h||_{1,\Omega_i} \leq K||H_{i,h}\eta_h||_{1,\Omega_i} \leq K\hat{C}_2||\eta_h||_\Lambda. \tag{4.1.17}$$

The constant K is the ratio between the continuity constant and the coerciveness constant of a_i^*, while $\hat{C}_2$ is the constant in (4.1.5); both K and $\hat{C}_2$ are independent of h. The result then follows from the continuity of the forms $a_i^*(\cdot,\cdot)$. □

In particular, we have proved that both operators $E_{i,h}^*$, $i = 1, 2$, are uniform extension operators from Λ to $H^1(\Omega_i)$.

As in the preceding case, from Theorem 4.1.6 we obtain at once that the matrix $\Sigma_{2,h}$, associated with the discrete operator $S_{2,h}$, is an optimal preconditioner of the Schur complement matrix Σ_h, associated with the Steklov–Poincaré operator S_h. The same is true for $\Sigma_{1,h}$, and for the Neumann–Neumann preconditioner $N_h = (\sigma_1\Sigma_{1,h}^{-1} + \sigma_2\Sigma_{2,h}^{-1})^{-1}$.

Remark 4.1.7 Let us analyse the main properties that are used in the proof given in Theorem 4.1.3. We can identify the following essential ingredients:

1. The bilinear forms describing the multi-domain formulation are continuous and coercive in a suitable Hilbert space X.
2. The trace theorem from X to a space of functions defined on the interface Γ holds.
3. The solution in Ω_i of the continuous problem, having a homogeneous right-hand side and a boundary datum on $\partial\Omega \setminus \Gamma$, and being equal to a discrete function on Γ, belongs to a space of functions that are more regular than the functions in X.
4. For a suitable choice of finite elements, the interpolation error estimate in X holds with an optimal order of convergence.

5. The inverse inequality holds with an optimal exponent for the finite elements restricted on Γ.

More precisely, properties 1, 3, 4 and 5 are used to prove the uniform continuity of the Steklov–Poincaré operators, whereas properties 1 and 2 give their uniform coerciveness. One can note that 1 and 3 are related to the specific boundary value problem at hand, 2 is a general theorem for the trace space, and 4 and 5 concern the family of finite elements that are being used in the approximation. □

4.1.2 Extension theorems in $H(\mathrm{div};\Omega_i)$

The analysis that follows is concerned with function spaces that are associated with the weak formulation of fluid dynamics problems (see Section 5.5), or with the mixed-type formulation of elliptic problems (see, for example, Brezzi and Fortin 1991).

The Hilbert space $H(\mathrm{div};\Omega)$ is defined as

$$H(\mathrm{div};\Omega) := \{\mathbf{v} \in (L^2(\Omega))^d \mid \mathrm{div}\,\mathbf{v} \in L^2(\Omega)\}, \tag{4.1.18}$$

endowed with the norm

$$||\mathbf{v}||_{H(\mathrm{div};\Omega)} := (||\mathbf{v}||^2_{0,\Omega} + ||\,\mathrm{div}\,\mathbf{v}||^2_{0,\Omega})^{1/2},$$

and we set

$$H_0(\mathrm{div};\Omega) := \{\mathbf{v} \in H(\mathrm{div};\Omega) \,|\, (\mathbf{v}\cdot\mathbf{n}^*)_{|\partial\Omega} = 0\}, \tag{4.1.19}$$

where $\mathbf{n}^*$ denotes the unit outward normal vector on $\partial\Omega$ (see also Chapter 9).

Considering, as usual, the domain Ω split into two non-overlapping subdomains Ω_1 and Ω_2, with interface Γ, for $i = 1, 2$ we introduce the local spaces and trace space

$$\begin{aligned} W_i &:= \{\mathbf{v}_i \in H(\mathrm{div};\Omega_i) \mid (\mathbf{v}_i\cdot\mathbf{n}^*)_{|\partial\Omega\cap\partial\Omega_i} = 0\} \\ W_i^0 &:= H_0(\mathrm{div};\Omega_i) \\ \Psi &:= \{\psi : \Gamma \to \mathbf{R} \mid \psi = (\mathbf{v}\cdot\mathbf{n})_{|\Gamma},\ \mathbf{v} \in H_0(\mathrm{div};\Omega)\}. \end{aligned} \tag{4.1.20}$$

The trace space Ψ coincides with the dual space of $H^{1/2}(\Gamma)$ (see Girault and Raviart 1986, p. 27–9), and therefore Ψ is the dual of Λ if $\Gamma \cap \partial\Omega = \emptyset$, as in Fig. 1.1 (right), whereas Ψ is strictly included in the dual space of Λ if $\Gamma \cap \partial\Omega \neq \emptyset$, as in Fig. 1.1 (left). The norm in Ψ will be denoted by $||\cdot||_\Psi$.

For each $\psi \in \Psi$, the vector function $\mathbf{Q}_i\psi \in W_i$, $i = 1, 2$, is the solution to

$$\begin{cases} (\mathbf{Q}_i\psi, \mathbf{v}_i)_{\Omega_i} + (\mathrm{div}(\mathbf{Q}_i\psi), \mathrm{div}\,\mathbf{v}_i)_{\Omega_i} = 0 \qquad \forall\ \mathbf{v}_i \in W_i^0 \\ (\mathbf{Q}_i\psi\cdot\mathbf{n})_{|\Gamma} = \psi \qquad \text{on } \Gamma, \end{cases} \tag{4.1.21}$$

where $(\cdot,\cdot)_{\Omega_i}$ denotes the $L^2(\Omega_i)$-scalar product. In differential form, $\mathbf{Q}_i\psi$ satisfies the following problem in the sense of distributions

$$\begin{cases} \mathbf{Q}_i\psi - \nabla \operatorname{div} \mathbf{Q}_i\psi = \mathbf{0} & \text{in } Om_i \\ (\mathbf{Q}_i\psi \cdot \mathbf{n}^*)_{|\partial\Omega_i \cap \partial\Omega} = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ (\mathbf{Q}_i\psi \cdot \mathbf{n})_{|\Gamma} = \psi & \text{on } \Gamma. \end{cases}$$

In other words, $\mathbf{Q}_i\psi$ is an extension of the normal trace ψ on Γ from Ψ to W_i.

Proposition 4.1.8 *There exist two positive constants C_1 and C_2 such that*

$$C_1||\psi||_\Psi \le ||\mathbf{Q}_i\psi||_{H(\operatorname{div};\Omega_i)} \le C_2||\psi||_\Psi \qquad \forall\ \psi \in \Psi,\quad i = 1, 2. \tag{4.1.22}$$

Proof The left-hand side inequality follows from the trace inequality

$$||(\mathbf{v}_i \cdot \mathbf{n})_{|\Gamma}||_\Psi \le C_i^*||\mathbf{v}_i||_{H(\operatorname{div};\Omega_i)} \qquad \forall\ \mathbf{v}_i \in W_i \tag{4.1.23}$$

(see, for example, Girault and Raviart 1986, pp. 27–8).

On the other hand, we remark that $\mathbf{Q}_i\psi = \nabla M_i\psi$, where $M_i\psi \in H^1(\Omega_i)$ is the distributional solution of

$$\begin{cases} M_i\psi - \Delta M_i\psi = 0 & \text{in } \Omega \\ \dfrac{\partial M_i\psi^*}{\partial n} = 0 & \text{on } \partial\Omega_i \cap \partial\Omega \\ \dfrac{\partial M_i\psi}{\partial n} = \psi & \text{on } \Gamma\ . \end{cases}$$

Now the right-hand side inequality in (4.1.22) follows from well known a priori estimates for the Neumann problem. □

Let us introduce the finite dimensional analogue of the operators $\mathbf{Q}_i$, defined on the div-*conforming* Nédélec finite elements (see Nédélec 1980; and also Raviart and Thomas 1977 for the two-dimensional case).

Let us set for $r \ge 1$

$$\mathcal{D}_r := (\mathbf{P}_{r-1})^d \oplus \{p(\mathbf{x})\,\mathbf{x} \,|\, p \in \tilde{\mathbf{P}}_{r-1}\}, \tag{4.1.24}$$

where $\tilde{\mathbf{P}}_{r-1}$ denotes the set of homogeneous polynomials of degree $r-1$. The div-conforming Nédélec finite element space is given by

$$N_h^r := \{\mathbf{v}_h \in H(\operatorname{div};\Omega) \,|\, \mathbf{v}_{h|K} \in \mathcal{D}_r(K)\ \forall\ K \in \mathcal{T}_h\}, \tag{4.1.25}$$

where $\mathcal{D}_r(K)$ denotes the restriction of $\mathcal{D}_r$ to K.

Other possible choices that we might consider are the Nédélec finite element spaces of the second type, proposed in Nédélec (1986), or, for $k \ge 2$, those introduced in Brezzi *et al.* (1985).

Let us define now the finite element space on Ω, as well as the local spaces and trace space:

$$
\begin{aligned}
&W_h := N_h^r \cap H_0(\mathrm{div};\Omega) \\
&W_{i,h} := \{\mathbf{v}_{h|\Omega_i} \mid \mathbf{v}_h \in W_h\}, \quad i = 1,2 \\
&W_{i,h}^0 := W_{i,h} \cap H_0(\mathrm{div};\Omega_i), \quad i = 1,2 \\
&\Psi_h := \{(\mathbf{v}_h \cdot \mathbf{n})_{|\Gamma} \mid \mathbf{v}_h \in W_h\}.
\end{aligned} \tag{4.1.26}
$$

For each $\psi_h \in \Psi_h$, denote by $\mathbf{Q}_{i,h}\psi_h \in W_{i,h}$ the solution of

$$
\begin{cases}
(\mathbf{Q}_{i,h}\psi_h, \mathbf{v}_{i,h})_{\Omega_i} + (\mathrm{div}(\mathbf{Q}_{i,h}\psi_h), \mathrm{div}\,\mathbf{v}_{i,h})_{\Omega_i} = 0 \qquad \forall\ \mathbf{v}_{i,h} \in W_{i,h}^0 \\
(\mathbf{Q}_{i,h}\psi_h \cdot \mathbf{n})_{|\Gamma} = \psi_h \qquad \text{on } \Gamma.
\end{cases} \tag{4.1.27}
$$

The discrete extension operator $\mathbf{Q}_{i,h}$ satisfies the two-sided bound stated in the next theorem.

Theorem 4.1.9 (Uniform extension theorem) *Let Ω, Ω_1 and Ω_2 be Lipschitz polygonal domains. Assume that the family of triangulations $\mathcal{T}_h$ is regular, and that the family of triangulations $\mathcal{M}_h$, induced by $\mathcal{T}_h$ on the interface Γ, is quasi-uniform. There exist two positive constants $\hat{C}_1$ and $\hat{C}_2$, which depend on the relative sizes of Ω_1 and Ω_2 but are independent of h, such that*

$$
\hat{C}_1||\psi_h||_\Psi \le ||\mathbf{Q}_{i,h}\psi_h||_{H(\mathrm{div};\Omega_i)} \le \hat{C}_2||\psi_h||_\Psi \qquad \forall\ \psi_h \in \Psi_h, \quad i = 1,2. \tag{4.1.28}
$$

Proof We want to show that the conditions 1–5 presented in Remark 4.1.7 are satisfied. Condition 1 trivially holds for the space $X = W_i$, because the local bilinear forms are the scalar products in $H(\mathrm{div};\Omega_i)$, $i = 1,2$. Condition 2 is the trace inequality (4.1.23). Since ψ_h is a piecewise-polynomial function on Γ and Ω_i is a polygonal domain, the regularity of $\mathbf{Q}_i\psi_h = \nabla M_i\psi_h$ (namely, the proof of 3) is given in Dauge (1988), Corollary 23.5, where it is shown that $M_i\psi_h \in H^{1+s^*}(\Omega_i)$ for a suitable $s^* > 1/2$, and that

$$
||M_i\psi_h||_{1+s^*,\Omega_i} \le C||\psi_h||_{s^*-1/2,\Gamma}.
$$

The proofs of 4 and 5 are rather technical, and we omit them, referring to Quarteroni *et al.* (1991). The results read

$$
||\mathbf{v}_i - \boldsymbol{\pi}_{i,h}^W\mathbf{v}_i||_{H(\mathrm{div};\Omega_i)} \le Ch^s(||\mathbf{v}_i||_{s,\Omega_i} + ||\,\mathrm{div}\,\mathbf{v}_i||_{s,\Omega_i}),
$$

where $\boldsymbol{\pi}_{i,h}^W\mathbf{v}_i \in W_{i,h}$ is the interpolant of $\mathbf{v}_i$ in Ω_i, and

$$
||\psi_h||_{s-1/2,\Gamma} \le Ch^{-s}||\psi_h||_\Psi,
$$

for $0 < s \le 1$.

Now the proof can be completed by following the same guidelines as for Theorem 4.1.3. In fact, the left-hand inequality in (4.1.28) is the trace inequality. The right-hand inequality in (4.1.28) can be proved as follows. We have

$$||\mathbf{Q}_{i,h}\psi_h||_{H(\mathrm{div};\Omega_i)} \leq ||\mathbf{Q}_{i,h}\psi_h - \mathbf{Q}_i\psi_h||_{H(\mathrm{div};\Omega_i)} + ||\mathbf{Q}_i\psi_h||_{H(\mathrm{div};\Omega_i)}.$$

From (4.1.22) it follows that

$$||\mathbf{Q}_i\psi_h||_{H(\mathrm{div};\Omega_i)} \leq C_2||\psi_h||_{\Psi}.$$

Moreover,

$$\begin{aligned} ||\mathbf{Q}_{i,h}\psi_h - \mathbf{Q}_i\psi_h||_{H(\mathrm{div};\Omega_i)} &\leq ||\mathbf{Q}_i\psi_h - \boldsymbol{\pi}_h^r(\mathbf{Q}_i\psi_h)||_{H(\mathrm{div};\Omega_i)} \\ &\leq Ch^{s^*}(||\mathbf{Q}_i\psi_h||_{s^*,\Omega_i} + ||\,\mathrm{div}\,\mathbf{Q}_i\psi_h||_{s^*,\Omega_i} \\ &\leq Ch^{s^*}||M_i\psi_h||_{1+s^*,\Omega_i} \leq Ch^{s^*}||\psi_h||_{s^*-1/2,\Gamma} \\ &\leq C||\psi_h||_{\Psi}, \end{aligned}$$

and (4.1.28) follows. □

4.1.3 Extension theorems in $H(\mathrm{rot};\Omega_i)$

The analysis that follows is concerned with function spaces that are associated with the weak formulation of the Maxwell equations (see Section 5.7).

The Hilbert space $H(\mathrm{rot};\Omega)$ is defined as

$$H(\mathrm{rot};\Omega) := \{\mathbf{v} \in (L^2(\Omega))^d \mid \mathrm{rot}\,\mathbf{v} \in (L^2(\Omega))^d\}, \tag{4.1.29}$$

endowed with the norm

$$||\mathbf{v}||_{H(\mathrm{rot};\Omega)} := (||\mathbf{v}||_{0,\Omega}^2 + ||\,\mathrm{rot}\,\mathbf{v}||_{0,\Omega}^2)^{1/2},$$

and we set

$$H_0(\mathrm{rot};\Omega) := \{\mathbf{v} \in H(\mathrm{rot};\Omega)\,|\,(\mathbf{n}^* \times \mathbf{v})_{|\partial\Omega} = \mathbf{0}\}, \tag{4.1.30}$$

where $\mathbf{n}^*$ denotes the unit outward normal vector on $\partial\Omega$ (see also Chapter 9).

For $i = 1, 2$ the local and trace spaces are defined as

$$\begin{aligned} Z_i &:= \{\mathbf{v}_i \in H(\mathrm{rot};\Omega_i) \mid (\mathbf{n}^* \times \mathbf{v}_i)_{|\partial\Omega\cap\partial\Omega_i} = \mathbf{0}\} \\ Z_i^0 &:= H_0(\mathrm{rot};\Omega_i) \\ \mathcal{X}_\Gamma &:= \{\boldsymbol{\psi} : \Gamma \to \mathbf{R}^d \mid \boldsymbol{\psi} = (\mathbf{n} \times \mathbf{v})_{|\Gamma},\ \mathbf{v} \in H_0(\mathrm{rot};\Omega)\}. \end{aligned} \tag{4.1.31}$$

A characterisation of the space of tangential traces $\mathcal{X}_\Gamma$ has been given in Alonso and Valli (1996), under suitable geometrical conditions on the interface Γ. The norm in this space will be denoted by $||\cdot||_{\mathcal{X}_\Gamma}$.

For each $\boldsymbol{\psi} \in \mathcal{X}_\Gamma$, the vector function $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi} \in Z_i$, $i = 1, 2$, is the solution to

$$\text{(4.1.32)} \qquad \begin{cases} (\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}, \mathbf{v}_i)_{\Omega_i} + (\mathrm{rot}(\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}), \mathrm{rot}\,\mathbf{v}_i)_{\Omega_i} = 0 \qquad \forall\ \mathbf{v}_i \in Z_i^0 \\ (\mathbf{n} \times \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi})_{|\Gamma} = \boldsymbol{\psi} \qquad \text{on } \Gamma. \end{cases}$$

In differential form, $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}$ satisfies (in the sense of distributions) the problem

$$\begin{cases} \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi} + \mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi} = \mathbf{0} & \text{in } \Omega_i \\ (\mathbf{n}^* \times \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi})_{|\partial\Omega_i \cap \partial\Omega} = \mathbf{0} & \text{on } \partial\Omega_i \cap \partial\Omega \\ (\mathbf{n} \times \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi})_{|\Gamma} = \boldsymbol{\psi} & \text{on } \Gamma. \end{cases}$$

The vector function $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}$ is an extension of the tangential trace $\boldsymbol{\psi}$ on Γ from $\mathcal{X}_\Gamma$ to Z_i.

Proposition 4.1.10 *There exists a positive constant C_1 such that*

$$\text{(4.1.33)} \qquad C_1||\boldsymbol{\psi}||_{\mathcal{X}_\Gamma} \le ||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)} \qquad \forall\ \boldsymbol{\psi} \in \mathcal{X}_\Gamma, \quad i = 1, 2.$$

Moreover, if Γ is a convex portion of the boundary of the Lipschitz polygonal domain Ω_i, there exists a positive constant C_2 such that

$$\text{(4.1.34)} \qquad ||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)} \le C_2||\boldsymbol{\psi}||_{\mathcal{X}_\Gamma} \qquad \forall\ \boldsymbol{\psi} \in \mathcal{X}_\Gamma, \quad i = 1, 2.$$

Proof Inequality (4.1.33) follows from the trace inequality

$$\text{(4.1.35)} \qquad ||(\mathbf{n} \times \mathbf{v}_i)_{|\Gamma}||_{\mathcal{X}_\Gamma} \le C_i^*||\mathbf{v}_i||_{H(\mathrm{rot};\Omega_i)},$$

that is proved in Alonso and Valli (1996).

In the same paper, under the assumption that Γ is regular enough or that Γ is a convex portion of the boundary $\partial\Omega_i$, a continuous extension operator $\boldsymbol{\mathcal{R}}_{i,\Gamma}$ from $\mathcal{X}_\Gamma$ to W_i has been constructed, satisfying $(\mathbf{n} \times \boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi})_{|\Gamma} = \boldsymbol{\psi}$. Therefore, the solution $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}$ can be written as

$$\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi} = \mathbf{z}_i + \boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi},$$

where $\mathbf{z}_i \in Z_i^0$ is the solution to

$$\begin{aligned} (\mathbf{z}_i, \mathbf{v}_i)_{\Omega_i} &+ (\mathrm{rot}\,\mathbf{z}_i), \mathrm{rot}\,\mathbf{v}_i)_{\Omega_i} \\ &= -(\boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi}, \mathbf{v}_i)_{\Omega_i} - (\mathrm{rot}(\boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi}), \mathrm{rot}\,\mathbf{v}_i)_{\Omega_i} \quad \forall\ \mathbf{v}_i \in Z_i^0. \end{aligned}$$

Taking $\mathbf{v}_i = \mathbf{z}_i$ and using the Cauchy–Schwarz inequality for the scalar product in $H(\mathrm{rot};\Omega_i)$, it follows that

$$||\mathbf{z}_i||_{H(\mathrm{rot};\Omega_i)} \le ||\boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)}.$$

Hence

$$\begin{aligned}||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)} &\leq ||\mathbf{z}_i||_{H(\mathrm{rot};\Omega_i)} + ||\boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)} \\ &\leq 2||\boldsymbol{\mathcal{R}}_{i,\Gamma}\boldsymbol{\psi}||_{H(\mathrm{rot};\Omega_i)} \leq C_2||\boldsymbol{\psi}||_{\mathcal{X}_\Gamma},\end{aligned}$$

owing to the fact that $\boldsymbol{\mathcal{R}}_{i,\Gamma}$ is continuous. □

We construct a finite dimensional approximation of the operator $\mathbf{F}_i$ in the so-called rot-*conforming* Nédélec finite element space (see Nédélec 1980).

First of all we need to define, for $r \geq 1$,

$$\mathcal{R}_r := (\mathbf{P}_{r-1})^d \oplus \{\mathbf{p} \in (\tilde{\mathbf{P}}_r)^d \,|\, \mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0\}, \tag{4.1.36}$$

where $\tilde{\mathbf{P}}_r$ denotes the set of homogeneous polynomials of degree r. The rot-conforming Nédélec finite element space is given by

$$M_h^r := \{\mathbf{v}_h \in H(\mathrm{rot};\Omega) \,|\, \mathbf{v}_{h|K} \in \mathcal{R}_r(K) \;\forall\; K \in \mathcal{T}_h\}, \tag{4.1.37}$$

where $\mathcal{R}_r(K)$ indicates the restriction of $\mathcal{R}_r$ to K.

Another possible choice relies on the Nédélec finite elements of the second type, see Nédélec (1986).

Let us introduce now the finite element space on Ω, as well as the local and trace spaces:

$$\begin{aligned}Z_h &:= M_h^r \cap H_0(\mathrm{rot};\Omega) \\ Z_{i,h} &:= \{\mathbf{v}_{h|\Omega_i} \mid \mathbf{v}_h \in Z_h\}, \quad i = 1,2 \\ Z_{i,h}^0 &:= Z_{i,h} \cap H_0(\mathrm{rot};\Omega_i), \quad i = 1,2 \\ \mathcal{X}_{\Gamma,h} &:= \{(\mathbf{n} \times \mathbf{v}_h)_{|\Gamma} \mid \mathbf{v}_h \in Z_h\}.\end{aligned} \tag{4.1.38}$$

For each $\boldsymbol{\psi}_h \in \mathcal{X}_{\Gamma,h}$, denote by $\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h \in Z_{i,h}$ the finite element solution of

$$\begin{cases} (\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h, \mathbf{v}_{i,h})_{\Omega_i} + (\mathrm{rot}(\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h), \mathrm{rot}\,\mathbf{v}_{i,h})_{\Omega_i} = 0 & \forall\; \mathbf{v}_{i,h} \in Z_{i,h}^0 \\ (\mathbf{n} \times \boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h)_{|\Gamma} = \boldsymbol{\psi}_h & \text{on } \Gamma. \end{cases} \tag{4.1.39}$$

The discrete extension operator $\boldsymbol{\mathcal{F}}_{i,h}$ satisfies the bounds stated in the next theorem.

Theorem 4.1.11 (Uniform extension theorem) *Let Ω, Ω_1 and Ω_2 be Lipschitz polygonal domains. Assume that the family of triangulations $\mathcal{T}_h$ is regular, and that the family of triangulations $\mathcal{M}_h$, induced by $\mathcal{T}_h$ on the interface Γ, is quasi-uniform. There exists a positive constant $\hat{C}_1$, which depends on the relative sizes of Ω_1 and Ω_2 but is independent of h, such that*

$$\hat{C}_1||\boldsymbol{\psi}_h||_{\mathcal{X}_\Gamma} \leq ||\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} \qquad \forall\; \boldsymbol{\psi}_h \in \mathcal{X}_{\Gamma,h}, \quad i = 1,2. \tag{4.1.40}$$

Moreover, if Γ is a convex portion of the boundary of Ω_i, there exists a positive constant $\hat{C}_2$, which depends on the relative sizes of Ω_1 and Ω_2 but is independent of h, such that

$$||\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} \le \hat{C}_2||\boldsymbol{\psi}_h||_{\mathcal{X}_\Gamma} \qquad \forall\ \boldsymbol{\psi}_h \in \mathcal{X}_{\Gamma,h}, \quad i=1,2. \tag{4.1.41}$$

Proof We want to show that the conditions 1–5 presented in Remark 4.1.7 are satisfied. Condition 1 trivially holds, because the local bilinear forms are the scalar products in $H(\mathrm{rot};\Omega_i)$, $i=1,2$. Condition 2 is the trace inequality (4.1.35). The proof of 3, namely, of the regularity of $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h$, is given in Alonso and Valli (1999), where it is proved that $\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h \in H^{1/2+\delta^*}(\Omega_i)$ and that $\mathrm{rot}\,\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h \in H^{1/2+\delta^*}(\Omega_i)$, for $\delta^* > 0$ small enough. Moreover, the following a priori estimate holds

$$||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{1/2+\delta^*,\Omega_i} + ||\,\mathrm{rot}\,\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{1/2+\delta^*,\Omega_i} \le C(||\boldsymbol{\psi}_h^\diamond||_{\delta^*,\partial\Omega_i} + ||\mathrm{div}_\tau\boldsymbol{\psi}_h^\diamond||_{\delta^*,\partial\Omega_i}),$$

where $\boldsymbol{\psi}_h^\diamond$ denotes the extension of $\boldsymbol{\psi}_h$ by $\mathbf{0}$ on $\partial\Omega_i \setminus \Gamma$, and $\mathrm{div}_\tau\,\boldsymbol{\psi}_h^\diamond$ is the tangential divergence of $\boldsymbol{\psi}_h^\diamond$ on $\partial\Omega_i$. Also, the proofs of 4 and 5 are rather technical, and we omit them, referring to Alonso and Valli (1999). The results read

$$||\mathbf{v}_i - \boldsymbol{\pi}_{i,h}^Z\mathbf{v}_i||_{H(\mathrm{rot};\Omega_i)} \le Ch^{1/2+\delta}(||\mathbf{v}_i||_{1/2+\delta,\Omega_i} + ||\,\mathrm{rot}\,\mathbf{v}_i||_{1/2+\delta,\Omega_i}),$$

where $\boldsymbol{\pi}_{i,h}^Z\mathbf{v}_i \in Z_{i,h}$ is the interpolant of $\mathbf{v}_i$ in Ω_i, and

$$||\boldsymbol{\psi}_h^\diamond||_{\delta,\partial\Omega_i} + ||\mathrm{div}_\tau\boldsymbol{\psi}_h^\diamond||_{\delta,\partial\Omega_i} \le Ch^{-1/2-\delta}||\boldsymbol{\psi}_h||_{\mathcal{X}_\Gamma},$$

for $0 < \delta < 1/2$.

The proof now follows as in Theorem 4.1.3. In fact, inequality (4.1.40) is the trace inequality. Inequality (4.1.41) can be proved as follows. We have

$$||\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} \le ||\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h - \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} + ||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)}.$$

From (4.1.34) it follows that

$$||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} \le C_2||\boldsymbol{\psi}_h||_{\mathcal{X}_\Gamma}.$$

Moreover,

$$\begin{aligned}
||\boldsymbol{\mathcal{F}}_{i,h}\boldsymbol{\psi}_h - \boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{H(\mathrm{rot};\Omega_i)} &\le ||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h - \boldsymbol{\pi}_h^r(\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h)||_{H(\mathrm{rot};\Omega_i)} \\
&\le Ch^{1/2+\delta^*}(||\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{1/2+\delta^*,\Omega_i} + ||\,\mathrm{rot}\,\boldsymbol{\mathcal{F}}_i\boldsymbol{\psi}_h||_{1/2+\delta^*,\Omega_i}) \\
&\le Ch^{1/2+\delta^*}(||\boldsymbol{\psi}_h^\diamond||_{\delta^*,\partial\Omega_i} + ||\mathrm{div}_\tau\boldsymbol{\psi}_h^\diamond||_{\delta^*,\partial\Omega_i}) \\
&\le C||\boldsymbol{\psi}_h||_{\mathcal{X}_\Gamma},
\end{aligned}$$

which completes the proof. □

4.2 Splitting of operators and preconditioned iterative methods

Let X be a (real) Hilbert space, and X' its dual space, with the duality pairing between X' and X denoted by $\langle\cdot,\cdot\rangle$. We consider a linear invertible continuous operator $\mathcal{Q} : X \to X'$, which can be split as

$$\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2, \tag{4.2.1}$$

where both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are linear operators, and they will be associated, in our application, with the subdomains Ω_1 and Ω_2 in a way that will be made precise from case to case.

We will address the solution of the problem

$$\mathcal{Q}\lambda = G, \tag{4.2.2}$$

with $G \in X'$ given and $\lambda \in X$ to be determined. To solve problem (4.2.2) we consider iterative methods based on a preconditioner $\mathcal{P}$ for $\mathcal{Q}$ that is given by either $\mathcal{Q}_2$ or $(\sigma_1\mathcal{Q}_1^{-1} + \sigma_2\mathcal{Q}_2^{-1})^{-1}$, σ_1 and σ_2 being positive parameters.

Remark 4.2.1 In our application, we will take $\mathcal{Q} = S$, the Steklov–Poincaré operator introduced in Chapter 1 for infinite dimensional boundary value problems, but also $\mathcal{Q} = S_h$, which represents the associated finite element approximation (see Chapter 2). In fact, we have already pointed out that both S and S_h can be regarded as the sum of contributions coming from the subdomains Ω_1 and Ω_2. In this framework, the choice $\mathcal{P} = \mathcal{Q}_2$ is the one underlying the Dirichlet–Neumann method, whereas $\mathcal{P} = (\sigma_1\mathcal{Q}_1^{-1} + \sigma_2\mathcal{Q}_2^{-1})^{-1}$ is that originating from the Neumann–Neumann method (see Sections 1.3 and 3.2). □

The simplest possible example of a preconditioned iterative method for the solution of (4.2.2) is the preconditioned Richardson method:

$$\mathcal{P}(\lambda^{k+1} - \lambda^k) = \theta(G - \mathcal{Q}\lambda^k), \ k \geq 0. \tag{4.2.3}$$

If both $\mathcal{P}$ and $\mathcal{Q}$ are symmetric and positive definite matrices, a more efficient method is instead the preconditioned conjugate gradient method.

An estimate of the rate of convergence of these iterative methods can be derived in terms of the spectral properties of the preconditioned operator $\mathcal{P}^{-1}\mathcal{Q}$. Equivalently, one can estimate the norm of the iteration operator $T_\theta := I - \theta\mathcal{P}^{-1}\mathcal{Q}$ associated with (4.2.3). The present section is therefore devoted to the analysis of the spectrum of $\mathcal{P}^{-1}\mathcal{Q}$.

We start by presenting an abstract convergence theorem (Alonso *et al.* 1998; see also Quarteroni and Valli 1991*b*) concerning the Richardson iterations for equation (4.2.2) preconditioned by $\mathcal{P} = \mathcal{Q}_2$.

Theorem 4.2.2 *Suppose that*

(a) $\mathcal{Q}_2$ *is continuous and coercive; that is,*

(a)$_1$ *there exists* $\beta_2 > 0$ *such that*

$$\langle \mathcal{Q}_2\eta, \mu\rangle \leq \beta_2 \, ||\eta||_X \, ||\mu||_X \qquad \forall \ \eta, \mu \in X;$$

(a)$_2$ *there exists* $\alpha_2 > 0$ *such that*

$$\langle \mathcal{Q}_2\eta, \eta\rangle \geq \alpha_2 \, ||\eta||_X^2 \qquad \forall \ \eta \in X;$$

(b) $\mathcal{Q}_1$ *is continuous; that is, there exists* $\beta_1 > 0$ *such that*

$$\langle \mathcal{Q}_1\eta, \mu\rangle \le \beta_1 \, ||\eta||_X \, ||\mu||_X \qquad \forall\ \eta, \mu \in X.$$

(c) there exists a constant $\kappa^* > 0$ *such that*

$$\langle \mathcal{Q}_2\eta, \mathcal{Q}_2^{-1}\mathcal{Q}\eta\rangle + \langle \mathcal{Q}\eta, \eta\rangle \ge \kappa^* ||\eta||_X^2 \qquad \forall\ \eta \in X.$$

Then for any given λ^0 *in* X *and for any* θ *satisfying* $0 < \theta < \theta_{\max}$*, with*

$$\theta_{\max} := \frac{\kappa^* \alpha_2^2}{\beta_2(\beta_1 + \beta_2)^2},$$

the sequence

$$\lambda^{k+1} = \lambda^k + \theta\mathcal{Q}_2^{-1}(G - \mathcal{Q}\lambda^k) \tag{4.2.4}$$

converges in X *to the solution of problem* (4.2.2)*.*

Proof First of all, note that the operator $\mathcal{Q}_2$ is invertible as a consequence of assumption (a) and of the Lax–Milgram lemma.

Let us introduce the $\mathcal{Q}_2$-scalar product

$$(\eta, \mu)_{\mathcal{Q}_2} := \frac{1}{2}(\langle \mathcal{Q}_2\eta, \mu\rangle + \langle \mathcal{Q}_2\mu, \eta\rangle).$$

The corresponding $\mathcal{Q}_2$-norm $||\eta||_{\mathcal{Q}_2} := (\eta, \eta)_{\mathcal{Q}_2}^{1/2} = \langle \mathcal{Q}_2\eta, \eta\rangle^{1/2}$ is equivalent to the norm $||\eta||_X$; actually, it satisfies the two-sided inequality

$$\alpha_2 ||\eta||_X^2 \le ||\eta||_{\mathcal{Q}_2}^2 \le \beta_2 ||\eta||_X^2 \qquad \forall\ \eta \in X.$$

To prove the convergence of the sequence $\{\lambda^k\}$ it is sufficient to show that the mapping

$$T_\theta : X \to X, \quad T_\theta\eta := \eta - \theta\mathcal{Q}_2^{-1}\mathcal{Q}\eta$$

is a contraction with respect to the $\mathcal{Q}_2$-norm. With this aim, assuming that $\theta \ge 0$, we have

$$\begin{aligned} ||T_\theta\eta||_{\mathcal{Q}_2}^2 &= ||\eta||_{\mathcal{Q}_2}^2 + \theta^2\langle \mathcal{Q}\eta, \mathcal{Q}_2^{-1}\mathcal{Q}\eta\rangle - \theta(\langle \mathcal{Q}_2\eta, \mathcal{Q}_2^{-1}\mathcal{Q}\eta\rangle + \langle \mathcal{Q}\eta, \eta\rangle) \\ &\le ||\eta||_{\mathcal{Q}_2}^2 + \theta^2 \frac{(\beta_1 + \beta_2)^2}{\alpha_2} ||\eta||_X^2 - \theta\kappa^* ||\eta||_X^2. \end{aligned}$$

Setting

$$K_\theta = 1 + \theta^2 \frac{(\beta_1 + \beta_2)^2}{\alpha_2^2} - \theta\frac{\kappa^*}{\beta_2}$$

we obtain

$$||T_\theta\eta||_{\mathcal{Q}_2}^2 \le K_\theta ||\eta||_{\mathcal{Q}_2}^2.$$

The bound $K_\theta < 1$ is guaranteed if $0 < \theta < \theta_{\max}$. □

Remark 4.2.3 In Theorem 4.2.2, the upper bound $\theta_{\max}$, as well as the contraction constant $K_\theta^{1/2}$, depends only on α_2, β_1, β_2 and κ^*; consequently, the rate of convergence of the preconditioned Richardson iterative procedure in the $\mathcal{Q}_2$-norm depends only on these same parameters. □

Remark 4.2.4 If the operator $\mathcal{Q}_2$ is symmetric, then assumption (c) reduces to the coerciveness of $\mathcal{Q}$; that is,

$$\langle \mathcal{Q}\eta, \eta \rangle \geq \frac{\kappa^*}{2} ||\eta||_X^2.$$

Since the coerciveness of $\mathcal{Q}_1$ is not required, we shall be able to apply Theorem 4.2.2 even to the analysis of heterogeneous domain decomposition algorithms (see Chapter 8). □

We present now another abstract theorem, which concerns the preconditioned Richardson iteration for equation (4.2.2) based on the preconditioner $\mathcal{P} = (\sigma_1 \mathcal{Q}_1^{-1} + \sigma_2 \mathcal{Q}_2^{-1})^{-1}$.

Theorem 4.2.5 *Assume that both $\mathcal{Q}_i$ are continuous and coercive; that is, for $i = 1, 2$ we have that*

(a) there exists $\beta_i > 0$ such that

$$\langle \mathcal{Q}_i \eta, \mu \rangle \leq \beta_i \, ||\eta||_X \, ||\mu||_X \qquad \forall \ \eta, \mu \in X;$$

(b) there exists $\alpha_i > 0$ such that

$$\langle \mathcal{Q}_i \eta, \eta \rangle \geq \alpha_i \, ||\eta||_X^2 \qquad \forall \ \eta \in X.$$

Assume, moreover, that, for any choice of the averaging parameters $\sigma_1 > 0$ and $\sigma_2 > 0$, the operator $\mathcal{N} := (\sigma_1 \mathcal{Q}_1^{-1} + \sigma_2 \mathcal{Q}_2^{-1})^{-1}$ satisfies the condition

(c) there exists $\kappa^* > 0$ such that

$$\langle \mathcal{N}\eta, \mathcal{N}^{-1}\mathcal{Q}\eta \rangle + \langle \mathcal{Q}\eta, \eta \rangle \geq \kappa^* ||\eta||_X^2 \qquad \forall \ \eta \in X.$$

Then there exists $\theta^0 > 0$ such that for each $\theta \in (0, \theta^0)$ and for any given $\lambda^0 \in X$ the sequence

$$\lambda^{k+1} = \lambda^k + \theta \mathcal{N}^{-1}(G - \mathcal{Q}\lambda^k) \tag{4.2.5}$$

converges in X to the solution of problem (4.2.2).

Proof We have already pointed out in Section 4.1 that, due to assumptions (a) and (b), the operator $\mathcal{N} : X \to X'$ is continuous and coercive. Let us denote by $\beta_{\mathcal{N}}$ and $\alpha_{\mathcal{N}}$ its continuity and coerciveness constants. We introduce the $\mathcal{N}$-scalar product

$$(\eta, \mu)_{\mathcal{N}} := \frac{1}{2}(\langle \mathcal{N}\eta, \mu \rangle + \langle \mathcal{N}\mu, \eta \rangle),$$

and the corresponding $\mathcal{N}$-norm $||\eta||_{\mathcal{N}} := (\eta,\eta)_{\mathcal{N}}^{1/2} = \langle \mathcal{N}\eta,\eta\rangle^{1/2}$, which is equivalent to the norm $||\eta||_X$, i.e.

$$\alpha_{\mathcal{N}}||\eta||_X^2 \le ||\eta||_{\mathcal{N}}^2 \le \beta_{\mathcal{N}}||\eta||_X^2.$$

To prove the convergence of λ^k we show that the following map

$$T^{(\theta)} : X \to X, \quad T^{(\theta)}\eta := \eta - \theta\mathcal{N}^{-1}\mathcal{Q}\eta$$

is a contraction with respect to the norm $||\cdot||_{\mathcal{N}}$. Assuming that $\theta \ge 0$, for $\eta \in X$ we have

$$\begin{aligned} ||T_\theta\eta||_{\mathcal{N}}^2 &= ||\eta||_{\mathcal{N}}^2 + \theta^2\langle \mathcal{Q}\eta, \mathcal{N}^{-1}\mathcal{Q}\eta\rangle - \theta(\langle \mathcal{N}\eta, \mathcal{N}^{-1}\mathcal{Q}\eta\rangle + \langle \mathcal{Q}\eta,\eta\rangle) \\ &\le ||\eta||_{\mathcal{N}}^2 + \theta^2\langle \mathcal{Q}\eta, \mathcal{N}^{-1}\mathcal{Q}\eta\rangle - \theta\kappa^*||\eta||_X^2. \end{aligned}$$

The operator $\mathcal{N}^{-1} = \sigma_1\mathcal{Q}_1^{-1} + \sigma_2\mathcal{Q}_2^{-1}$ is continuous and satisfies

$$\langle \psi, \mathcal{N}^{-1}\psi\rangle \le \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right)||\psi||_{X'}^2 \qquad \forall\ \psi \in X',$$

therefore

$$\langle \mathcal{Q}\eta, \mathcal{N}^{-1}\mathcal{Q}\eta\rangle \le \left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right)(\beta_1+\beta_2)^2||\eta||_X^2.$$

Setting

$$K_\theta = 1 + \theta^2\left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right)\frac{(\beta_1+\beta_2)^2}{\alpha_{\mathcal{N}}} - \theta\frac{\kappa^*}{\beta_{\mathcal{N}}}$$

we obtain

$$||T_\theta\eta||_{\mathcal{N}}^2 \le K_\theta||\eta||_{\mathcal{N}}^2.$$

Setting

$$\theta^0 := \frac{\kappa^*\alpha_{\mathcal{N}}}{\beta_{\mathcal{N}}\left(\frac{\sigma_1}{\alpha_1} + \frac{\sigma_2}{\alpha_2}\right)(\beta_1+\beta_2)^2},$$

we conclude that $T^{(\theta)}$ is a contraction for all $\theta \in (0,\theta^0)$. □

Again, it can be noted that the rate of convergence in the $\mathcal{N}$-norm of the preconditioned Richardson iterative procedure (4.2.5) depends only on the constants σ_1, σ_2, α_1, α_2, β_1, β_2 and κ^*.

Moreover, if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are symmetric, assumption (c) is equivalent to the coerciveness of $\mathcal{Q}$. On the other hand, this property follows directly from (b); hence, in the symmetric case, assumption (c) is not necessary.

4.2.1 The finite dimensional case

In this section we analyse the case of finite dimensional operators, and point out the conditions under which the rate of convergence in Theorems 4.2.2 and 4.2.5 is independent of the discretisation parameter h.

As noted in Remark 4.2.3, the upper bound $\theta_{\max}$, as well as the contraction constant $K_\theta^{1/2}$ in Theorem 4.2.2, depends only on α_2, β_1, β_2 and κ^*; consequently, should these constants be independent of h, the rate of convergence of the Richardson method would also be independent of h. A similar remark holds for the method analysed in Theorem 4.2.5.

It is worthwhile to note that in the finite dimensional case, if conditions (a), (b) and (c) in Theorem 4.2.2 are satisfied uniformly with respect to h, other preconditioned iterative procedures different from the Richardson method (such as, for example, GCR or GMRES, see Saad 1996) are convergent with a rate independent of h.

Consider, for instance, the case of GMRES iterations. Let us denote the finite dimensional Hilbert space by X_h, and the finite dimensional operators by $\mathcal{Q}_h$, $\mathcal{Q}_{1,h}$ and $\mathcal{Q}_{2,h}$. The dimension of X_h will be indicated by N_h, and its norm by $||\cdot||_X$. Moreover, introduce the matrix Q_h as

$$[Q_h\boldsymbol{\eta},\boldsymbol{\mu}] := \langle \mathcal{Q}_h\eta_h, \mu_h\rangle \qquad \forall\ \eta_h, \mu_h \in X_h,$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbf{R}^{N_h}$, and for each $\mu_h \in X_h$, the vector $\boldsymbol{\mu}$ denotes the components of μ_h with respect to a suitable basis of X_h. The matrices $Q_{1,h}$ and $Q_{2,h}$ are defined in a similar way, so that $Q_h = Q_{1,h} + Q_{2,h}$.

Denote by B the symmetric part of $Q_{2,h}$, i.e.

$$B := \frac{Q_{2,h} + Q_{2,h}^T}{2}.$$

As a consequence of condition (a), this matrix satisfies

$$\begin{aligned}[B\boldsymbol{\eta},\boldsymbol{\eta}] &= \frac{1}{2}[(Q_{2,h} + Q_{2,h}^T)\boldsymbol{\eta},\boldsymbol{\eta}] = [Q_{2,h}\boldsymbol{\eta},\boldsymbol{\eta}] \\ &= \langle Q_{2,h}\eta_h, \eta_h\rangle \geq \alpha_2||\eta_h||_X^2\end{aligned}$$

and

$$[B\boldsymbol{\eta},\boldsymbol{\eta}] \leq \beta_2||\eta_h||_X^2.$$

Hence, B is symmetric and positive definite, and the B-scalar product $(\boldsymbol{\eta},\boldsymbol{\mu})_B := [B\boldsymbol{\eta},\boldsymbol{\mu}]$ induces a vector norm $|\cdot|_B$ that is equivalent to the norm $||\cdot||_X$; namely,

$$\alpha_2||\eta_h||_X^2 \leq |\boldsymbol{\eta}|_B^2 \leq \beta_2||\eta_h||_X^2. \tag{4.2.6}$$

The matrix $Q_{2,h}^{-1}Q_h$ is positive definite with respect to the B-scalar product, as from condition (c) it follows that

$$
\begin{aligned}
(Q_{2,h}^{-1}Q_h\boldsymbol{\eta},\boldsymbol{\eta})_B &= \frac{1}{2}[(Q_{2,h}+Q_{2,h}^T)Q_{2,h}^{-1}Q_h\boldsymbol{\eta},\boldsymbol{\eta}] \\
&= \frac{1}{2}[Q_h\boldsymbol{\eta},\boldsymbol{\eta}] + \frac{1}{2}[Q_{2,h}^{-1}Q_h\boldsymbol{\eta}, Q_{2,h}\boldsymbol{\eta}] \\
&= \frac{1}{2}\langle \mathcal{Q}_h\eta_h,\eta_h\rangle + \frac{1}{2}\langle \mathcal{Q}_{2,h}\eta_h, \mathcal{Q}_{2,h}^{-1}\mathcal{Q}_h\eta_h\rangle \\
&\geq \frac{\kappa^*}{2}||\eta_h||_X^2 \geq \frac{\kappa^*}{2\beta_2}|\boldsymbol{\eta}|_B^2,
\end{aligned}
\tag{4.2.7}
$$

having used (4.2.6).

Therefore, the reduction factor ρ of the GMRES methods applied to the preconditioned matrix $Q_{2,h}^{-1}Q_h$ is given by

$$
\rho := \sqrt{1-\frac{\lambda_{\min}^2}{\lambda_{\max}}}
$$

(see, for example, Quarteroni and Valli 1994, Section 2.6), where $\lambda_{\min}$ is the minimum eigenvalue of the matrix $[Q_{2,h}^{-1}Q_h+(Q_{2,h}^{-1}Q_h)^{T_B}]/2$ and $\lambda_{\max}$ the maximum eigenvalue of the matrix $(Q_{2,h}^{-1}Q_h)^{T_B}Q_{2,h}^{-1}Q_h$, the symbol E^{T_B} denoting the transpose of a matrix E with respect to the B-scalar product.

We have

$$
\begin{aligned}
\lambda_{\min} &= \frac{1}{2}\min_{\boldsymbol{\eta}\neq\mathbf{0}}\frac{(Q_{2,h}^{-1}Q_h\boldsymbol{\eta},\boldsymbol{\eta})_B + ((Q_{2,h}^{-1}Q_h)^{T_B}\boldsymbol{\eta},\boldsymbol{\eta})_B}{|\boldsymbol{\eta}|_B^2} \\
&= \min_{\boldsymbol{\eta}\neq\mathbf{0}}\frac{(Q_{2,h}^{-1}Q_h\boldsymbol{\eta},\boldsymbol{\eta})_B}{|\boldsymbol{\eta}|_B^2} \geq \frac{\kappa^*}{2\beta_2},
\end{aligned}
$$

due to (4.2.7).

Moreover

$$
\begin{aligned}
\lambda_{\max} &= \max_{\boldsymbol{\eta}\neq\mathbf{0}}\frac{\big((Q_{2,h}^{-1}Q_h)^{T_B}Q_{2,h}^{-1}Q_h\boldsymbol{\eta},\boldsymbol{\eta}\big)_B}{|\boldsymbol{\eta}|_B^2} \\
&= \max_{\boldsymbol{\eta}\neq\mathbf{0}}\frac{(Q_{2,h}^{-1}Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta})_B}{|\boldsymbol{\eta}|_B^2},
\end{aligned}
$$

and from conditions (a) and (b)

$$
\begin{aligned}
(Q_{2,h}^{-1}Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta})_B &= \frac{1}{2}[(Q_{2,h}+Q_{2,h}^T)Q_{2,h}^{-1}Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta}] \\
&= \frac{1}{2}[Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta}] + \frac{1}{2}[Q_{2,h}^TQ_{2,h}^{-1}Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta}] \\
&= [Q_h\boldsymbol{\eta},Q_{2,h}^{-1}Q_h\boldsymbol{\eta}] = \langle \mathcal{Q}_h,\mathcal{Q}_{2,h}^{-1}\mathcal{Q}_h\eta_h\rangle \\
&\leq \frac{(\beta_1+\beta_2)^2}{\alpha_2}||\eta_h||_X^2.
\end{aligned}
$$

Hence, using (4.2.6) we conclude that

$$\lambda_{\max} \leq \frac{(\beta_1 + \beta_2)^2}{\alpha_2^2},$$

and the GMRES method converges at a rate independent of h.

The argument above can also be applied to prove the convergence of the GMRES iterations based on the Neumann–Neumann preconditioner $N_h := (\sigma_1 Q_{1,h}^{-1} + \sigma_2 Q_{2,h}^{-1})^{-1}$. In this case, denoting respectively by $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{N}}$ the coerciveness and continuity constants of N_h and by B its symmetric part, we still obtain (4.2.6), with $\alpha_{\mathcal{N}}$ and $\beta_{\mathcal{N}}$ instead of α_2 and β_2. The rest of the proof can be repeated without modification, substituting $Q_{2,h}$ with N_h. The final result reads

$$\lambda_{\min} \geq \frac{\kappa^*}{2\beta_{\mathcal{N}}}, \quad \lambda_{\max} \leq \frac{(\beta_1 + \beta_2)^2}{\alpha_{\mathcal{N}}^2},$$

therefore, if the assumptions of Theorem 4.2.5 are satisfied uniformly with respect to h, the rate of convergence is also independent of h.

Remark 4.2.6 The convergence of the GMRES method can be proved also by resorting to a general result. In fact, let $(\cdot,\cdot)_*$ be a scalar product in $\mathbf{R}^M$, $|\cdot|_*$ the associated vector norm and $||\cdot||_*$ the corresponding natural matrix norm. Consider the linear system associated with the matrix $P^{-1}Q$, and suppose that the Richardson iterative method applied to it is convergent with respect to the norm $|\cdot|_*$ for a suitable $\theta_* > 0$; namely, there exists a constant $K \in (0,1)$ such that

$$||I - \theta_* P^{-1}Q||_* \leq K. \tag{4.2.8}$$

Then also the GMRES method converges with respect to the norm $|\cdot|_*$, and its rate of convergence only depends on K.

This is proved as follows. As we have already noted, the reduction factor of the GMRES method in the norm $|\cdot|_*$ is

$$\rho := \sqrt{1 - \frac{\lambda_{\min}^2}{\lambda_{\max}}},$$

where $\lambda_{\min}$ is the minimum eigenvalue of the matrix $[P^{-1}Q + (P^{-1}Q)^{T_*}]/2$ and $\lambda_{\max}$ the maximum eigenvalue of the matrix $(P^{-1}Q)^{T_*}P^{-1}Q$, the symbol E^{T_*} denoting the transpose of a matrix E with respect to the scalar product $(\cdot,\cdot)_*$. Hence

$$\lambda_{\max} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{((P^{-1}Q)^{T_*}P^{-1}Q\mathbf{x},\mathbf{x})_*}{|\mathbf{x}|_*^2} = \max_{\mathbf{x}\neq\mathbf{0}} \frac{|P^{-1}Q\mathbf{x}|_*^2}{|\mathbf{x}|_*^2},$$

and

$$\lambda_{\min} = \frac{1}{2}\min_{\mathbf{x}\neq\mathbf{0}} \frac{((P^{-1}Q + (P^{-1}Q)^{T_*})\mathbf{x},\mathbf{x})_*}{|\mathbf{x}|_*^2} = \min_{\mathbf{x}\neq\mathbf{0}} \frac{(P^{-1}Q\mathbf{x},\mathbf{x})_*}{|\mathbf{x}|_*^2}.$$

From (4.2.8) we have

$$|\mathbf{x}|_*^2 - 2\theta_*(P^{-1}Q\mathbf{x},\mathbf{x})_* + \theta_*^2|P^{-1}Q\mathbf{x}|_*^2 \le K^2|\mathbf{x}|_*^2 \qquad \forall\, \mathbf{x}\in\mathbf{R}^M,$$

hence

$$(1-K^2)|\mathbf{x}|_*^2 + \theta_*^2|P^{-1}Q\mathbf{x}|_*^2 \le 2\theta_*(P^{-1}Q\mathbf{x},\mathbf{x})_* \le 2\theta_*|P^{-1}Q\mathbf{x}|_*\,|\mathbf{x}|_*.$$

As a consequence, we obtain

$$|P^{-1}Q\mathbf{x}|_* \le \frac{2}{\theta_*}|\mathbf{x}|_*, \quad (P^{-1}Q\mathbf{x},\mathbf{x})_* \ge \frac{1-K^2}{2\theta_*}|\mathbf{x}|_*^2,$$

thus

$$\rho \le \sqrt{1-\frac{(1-K^2)^2}{16}},$$

which proves the asserted result. □

4.2.2 The case of symmetric matrices

A different point of view can be adopted in the finite dimensional symmetric case. Let us denote by P the matrix associated with the preconditioner $\mathcal{P}$ and by Q the matrix associated with $\mathcal{Q}$, and assume that both P and Q are symmetric and positive definite. From now on, our analysis concerns a general preconditioner P, not necessarily Q_2 (Dirichlet–Neumann) or $(\sigma_1 Q_1^{-1} + \sigma_2 Q_2^{-1})^{-1}$ (Neumann–Neumann). First, we report a general result:

Theorem 4.2.7 *Let P and Q be two symmetric and positive definite $M\times M$ (real) matrices. Then the preconditioned Richardson iterations*

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta P^{-1}(\mathbf{G} - Q\boldsymbol{\lambda}^k) \tag{4.2.9}$$

converge to the solution of $Q\boldsymbol{\lambda} = \mathbf{G}$ if, and only if, the parameter θ satisfies

$$0 < \theta < \frac{2}{\nu_{\max}},$$

where $\nu_{\max}$ is the maximum eigenvalue of $P^{-1}Q$.

Proof Let us proceed for a while considering two general non-singular complex matrices P and Q, as this will be useful in what follows (see Theorem 4.2.9). Setting $\mathbf{e}^k := \boldsymbol{\lambda}^k - \boldsymbol{\lambda}$, the error equation associated with (4.2.9) is

$$\mathbf{e}^{k+1} = (I - \theta P^{-1}Q)\mathbf{e}^k,$$

where $T_\theta := I - \theta P^{-1}Q$ is the iteration matrix. Let τ_s, $s = 1,\dots,M$ be the eigenvalues of T_θ and ν_s those of $P^{-1}Q$. Since $\tau_s = 1 - \theta\nu_s$, we have

$$\begin{aligned}|\tau_s|^2 &= (1-\theta\,\mathrm{Re}\,\nu_s)^2 + \theta^2(\mathrm{Im}\,\nu_s)^2\\ &= 1 - \theta^2|\nu_s|^2\left(\frac{2\,\mathrm{Re}\,\nu_s}{\theta|\nu_s|^2} - 1\right).\end{aligned}$$

When P and Q are symmetric and positive definite real matrices, the matrix $P^{-1}Q$ is symmetric and positive definite with respect to the scalar product $(\boldsymbol{\eta}, \boldsymbol{\mu})_P := [P\boldsymbol{\eta}, \boldsymbol{\mu}]$, therefore its eigenvalues ν_s are real and positive. Hence the spectral radius $\rho_\theta := \max_s |\tau_s|$ of the iteration matrix T_θ is less than 1 if, and only if,

$$1 < \frac{2}{\theta \nu_s} \qquad \forall\ s = 1, \ldots, M,$$

and this proves our result. □

Let us recall that the spectral radius ρ_θ of the iteration matrix T_θ satisfies $\rho_\theta = ||T_\theta||_P$ (see, for example, Quarteroni and Valli 1994, Lemma 2.4.1), and therefore

$$||\boldsymbol{\lambda}^k - \boldsymbol{\lambda}||_P \leq \rho_\theta^k ||\boldsymbol{\lambda}^0 - \boldsymbol{\lambda}||_P.$$

The value of ρ_θ is given by

$$\rho_\theta = \begin{cases} 1 - \theta\nu_{\min} & \text{for } 0 < \theta \leq \dfrac{2}{\nu_{\max} + \nu_{\min}} \\ \theta\nu_{\max} - 1 & \text{for } \dfrac{2}{\nu_{\max} + \nu_{\min}} \leq \theta < \dfrac{2}{\nu_{\max}} \end{cases}, \tag{4.2.10}$$

where $\nu_{\min}$ is the minimum eigenvalue of $P^{-1}Q$ (see Fig. 4.2.1).

FIG. 4.2.1. The value of the spectral radius ρ_θ.

Now we want to analyse other iterative methods for the solution of the linear system associated with the preconditioned matrix $P^{-1}Q$. Assume that P and Q are symmetric and positive definite real matrices, and that the spectral condition number $\kappa(P^{-1}Q) = \frac{\nu_{\max}}{\nu_{\min}}$ satisfies the bound

$$\kappa(P^{-1}Q) \leq K^*, \tag{4.2.11}$$

for some $K^* \geq 1$ independent of the dimension of P and Q (P is therefore an optimal preconditioner for Q). Then we can prove the following convergence result.

1. For the *stationary Richardson method* (4.2.9) with $\theta = \theta_{\text{opt}} := \frac{2}{\nu_{\max}+\nu_{\min}}$,

$$|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}|_P \leq \left(\frac{K^*-1}{K^*+1}\right)^k |\boldsymbol{\lambda}^0 - \boldsymbol{\lambda}|_P, \quad k \geq 0.$$

The preconditioned Richardson iterative method with a dynamical choice of the parameter θ reads:

$$\begin{cases} P\mathbf{z}^k = \mathbf{r}^k \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta_k \mathbf{z}^k \\ \mathbf{r}^{k+1} = \mathbf{r}^k - \theta_k Q\mathbf{z}^k, \end{cases} \tag{4.2.12}$$

where $\mathbf{r}^k := \mathbf{G} - Q\boldsymbol{\lambda}^k$. For this iteration we have

2. For the *minimum-residual Richardson method*, choosing as parameter θ_k that minimising the P-norm of the preconditioned residual $\mathbf{z}^k$; that is, $\theta_k = \frac{(\mathbf{z}^k, Q\mathbf{z}^k)}{(Q\mathbf{z}^k, P^{-1}Q\mathbf{z}^k)}$,

$$|\mathbf{z}^k|_P \leq \left(\frac{K^*-1}{K^*+1}\right)^k |\mathbf{z}^0|_P, \quad k \geq 0.$$

3. For the *steepest-descent Richardson method* (or gradient method), choosing as parameter θ_k that minimising the Q-norm of the error $\boldsymbol{\lambda}^k - \boldsymbol{\lambda}$; namely, $\theta_k = \frac{(\mathbf{z}^k, \mathbf{r}^k)}{(\mathbf{z}^k, Q\mathbf{z}^k)}$,

$$|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}|_Q \leq \left(\frac{K^*-1}{K^*+1}\right)^k |\boldsymbol{\lambda}^0 - \boldsymbol{\lambda}|_Q, \quad k \geq 0.$$

The same type of information, namely, $\kappa(P^{-1}Q) \leq K^*$, can be effectively exploited also in the context of different iterative methods, notably for the preconditioned *conjugate gradient method* (see (3.5.13), with the obvious change of notation). For the latter, we can still infer an error estimate of the form

$$|\boldsymbol{\lambda}^k - \boldsymbol{\lambda}|_Q \leq 2\left(\frac{\sqrt{K^*}-1}{\sqrt{K^*}+1}\right)^k |\boldsymbol{\lambda}^0 - \boldsymbol{\lambda}|_Q, \quad k \geq 0.$$

Note that, in all the previous iterative schemes, the rate of convergence only depends on K^*, therefore it is independent of the dimension M of P and Q.

In turn, estimate (4.2.11) can be obtained when the bounds

$$K_1 \leq \nu_{\min} \leq \nu_{\max} \leq K_2 \tag{4.2.13}$$

are available, with K_1 and K_2 independent of M (see Theorem 4.1.5). From these estimates, one can prove the convergence of Richardson iterations even

for a non-optimal choice of the parameter θ; namely, for θ chosen in a suitable interval of the real line (a motivation for this choice will be made clear in Remark 4.2.8). In fact, using (4.2.13) in (4.2.10) we conclude that for each $\theta \in (0, 1/K_2)$ the spectral radius ρ_θ of the iteration matrix $I - \theta P^{-1}Q$ satisfies

$$0 < \rho_\theta \leq 1 - K_1\theta < 1.$$

Remark 4.2.8 (Cost per iteration) We have already shown in Section 1.3 that the Dirichlet–Neumann algorithm is equivalent to a preconditioned Richardson iterative scheme for the Schur complement matrix $\Sigma_h = \Sigma_{1,h} + \Sigma_{2,h}$.

The implementation of this algorithm in the general case has been reported in (4.2.12). In the present case, we have $P = \Sigma_{2,h}$, and the solution of the system $\Sigma_{2,h}\mathbf{z}^k = \mathbf{r}^k$ yields the solution of a Neumann problem in Ω_2. Next, when $\mathbf{z}^k$ is available, the calculation of $\Sigma_h\mathbf{z}^k$ requires the solution of two Dirichlet problems, one in Ω_1 and one in Ω_2. Finally, in the case of the minimum-residual method, the computation of θ_k requires another Neumann solve in Ω_2, while, for the steepest-descent method, θ_k can be computed without additional solves. Therefore, the global cost of each iteration amounts to solving three boundary value problems for the steepest-descent method, and four boundary value problems for the minimum-residual method.

Should we take θ without using an optimal strategy, we can reformulate the kth iteration step as

$$\boldsymbol{\lambda}^{k+1} = (1-\theta)\boldsymbol{\lambda}^k + \theta\Sigma_{2,h}^{-1}(\boldsymbol{\chi} - \Sigma_{1,h}\boldsymbol{\lambda}^k),$$

which only requires the solution of two boundary value problems, a Dirichlet problem in Ω_1 and a Neumann problem in Ω_2.

Whether the extra cost (per iteration) of (4.2.12) would be compensated by a quicker convergence is a matter that depends on the specific problem at hand.

If the preconditioner employed is the Neumann–Neumann preconditioner $N_h = (\sigma_1\Sigma_{1,h}^{-1} + \sigma_2\Sigma_{2,h}^{-1})^{-1}$, the solution of $N_h\mathbf{z}^k = \mathbf{r}^k$ requires the solution of two Neumann problems, one in Ω_1 and one in Ω_2. Therefore, the global cost of each iteration amounts to solving four boundary value problems for the steepest-descent method, and six boundary value problems for the minimum-residual method.

Similar considerations can be carried out for the conjugate gradient iterations, which have the same computational cost as the steepest-descent iterations. □

4.2.3 The case of complex matrices

As we are envisaging the application of subdomain iterations to non-symmetric boundary value problems for complex functions, we start by formulating the complex counterpart of Theorem 4.2.7; the proof is similar and will be omitted.

Theorem 4.2.9 *Let P and Q be two $M \times M$ non-singular complex matrices. Then the preconditioned Richardson iterations*

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta P^{-1}(\mathbf{G} - Q\boldsymbol{\lambda}^k)$$

converge to the solution $\boldsymbol{\lambda}$ *of* $Q\boldsymbol{\lambda} = \mathbf{G}$ *if, and only if,*

$$|\nu_s|^2 < \frac{2}{\theta}\,\mathrm{Re}\,\nu_s$$

for each eigenvalue ν_s *of* $P^{-1}Q$, $s = 1, \ldots, M$.

As a consequence of this theorem, we can prove the following results, in which the preconditioner for the matrix $Q = Q_1 + Q_2$ is given by Q_2 itself or else by $(\sigma_1 Q_1^{-1} + \sigma_2 Q_2^{-1})^{-1}$. Corollaries 4.2.11 and 4.2.14 below will be used for the convergence analysis of the time-harmonic Maxwell problem.

Theorem 4.2.10 *Suppose that*

$$\begin{aligned}\mathcal{N}(\boldsymbol{\eta}) := \mathrm{Re}\,[Q\boldsymbol{\eta},\boldsymbol{\eta}]\,\mathrm{Re}[Q_2\boldsymbol{\eta},\boldsymbol{\eta}] \\ +\mathrm{Im}[Q\boldsymbol{\eta},\boldsymbol{\eta}]\,\mathrm{Im}[Q_2\boldsymbol{\eta},\boldsymbol{\eta}] > 0 \qquad \forall\ \boldsymbol{\eta} \in \mathbf{C}^M \setminus \mathbf{0},\end{aligned} \tag{4.2.14}$$

where $[\cdot,\cdot]$ *denotes now the Euclidean scalar product in* $\mathbf{C}^M$. *Moreover, set*

$$C_0 := \min_{\boldsymbol{\eta}\in\mathbf{C}^M\setminus\mathbf{0}} \frac{\mathcal{N}(\boldsymbol{\eta})}{|[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]|^2} > 0 \tag{4.2.15}$$

and

$$\widehat{C}_0 := \max_{\boldsymbol{\eta}\in\mathbf{C}^M\setminus\mathbf{0}} \frac{|[Q\boldsymbol{\eta},\boldsymbol{\eta}]|}{|[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]|} > 0. \tag{4.2.16}$$

Then each eigenvalue ν_s *of* $Q_2^{-1}Q$ *satisfies*

$$C_* := \frac{2\,C_0}{\widehat{C}_0^2} \le 2\,\frac{\mathrm{Re}\,\nu_s}{|\nu_s|^2} \qquad \forall\ s = 1, \ldots, M. \tag{4.2.17}$$

Proof Since (4.2.14) implies that $[Q\boldsymbol{\eta},\boldsymbol{\eta}] \neq 0$ and $[Q_2\boldsymbol{\eta},\boldsymbol{\eta}] \neq 0$ for each $\boldsymbol{\eta} \in \mathbf{C}^M \setminus \mathbf{0}$, the ratios in (4.2.15) and (4.2.16) make sense.

Let ν be an eigenvalue of $Q_2^{-1}Q$. Its associated eigenvector $\boldsymbol{\omega} \in \mathbf{C}^{M_h}$, $\boldsymbol{\omega} \neq \mathbf{0}$, satisfies $Q\boldsymbol{\omega} = \nu Q_2\boldsymbol{\omega}$, therefore

$$[Q\boldsymbol{\omega},\boldsymbol{\omega}] = \nu[Q_2\boldsymbol{\omega},\boldsymbol{\omega}],$$

or, equivalently,

$$\begin{aligned}\mathrm{Re}[Q\boldsymbol{\omega},\boldsymbol{\omega}] &= \mathrm{Re}\,\nu\mathrm{Re}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}] - \mathrm{Im}\,\nu\mathrm{Im}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}] \\ \mathrm{Im}[Q\boldsymbol{\omega},\boldsymbol{\omega}] &= \mathrm{Re}\,\nu\mathrm{Im}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}] + \mathrm{Im}\,\nu\mathrm{Re}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}].\end{aligned}$$

If we multiply these equations by $\mathrm{Re}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}]$ and $\mathrm{Im}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}]$, respectively, and sum up the results we find

$$(\mathrm{Re}\,\nu)\,[(\mathrm{Re}\,[Q_2\boldsymbol{\omega},\boldsymbol{\omega}])^2+(\mathrm{Im}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}])^2]$$
$$=\mathrm{Re}[Q\boldsymbol{\omega},\boldsymbol{\omega}]\,\mathrm{Re}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}]+\mathrm{Im}[Q\boldsymbol{\omega},\boldsymbol{\omega}]\,\mathrm{Im}[Q_2\boldsymbol{\omega},\boldsymbol{\omega}]=\mathcal{N}(\boldsymbol{\omega}).$$

On the other hand,

$$|\nu|^2=\frac{|[Q\boldsymbol{\omega},\boldsymbol{\omega}]|^2}{|[Q_2\boldsymbol{\omega},\boldsymbol{\omega}]|^2}\le\widehat{C}_0^2,$$

so (4.2.17) follows. □

Corollary 4.2.11 *Under the assumption of Theorem 4.2.10, the Richardson iterations applied to the preconditioned system*

$$Q_2^{-1}Q\boldsymbol{\lambda}=Q_2^{-1}\boldsymbol{\chi} \tag{4.2.18}$$

converge for each $\theta\in(0,C_*)$, and the spectral radius of the iteration matrix $I-\theta Q_2^{-1}Q$ is bounded by a positive constant $\kappa_\theta<1$, which depends on C_0 and $\widehat{C}_0$, but not on the dimension M.

Proof From (4.2.17) we have

$$|\nu_s|^2<\frac{2}{\theta}\,\mathrm{Re}\,\nu_s\qquad\forall\ s=1,\dots,M,\quad\forall\ \theta\in(0,C_*), \tag{4.2.19}$$

therefore convergence follows from Theorem 4.2.9. Moreover, any eigenvalue of the iteration matrix $I-\theta Q_2^{-1}Q$ has the form $\tau=1-\theta\nu$, where ν is an eigenvalue of $Q_2^{-1}Q$. From (4.2.15) and (4.2.16), for each $\theta\in(0,C_*)$ we find that

$$|\tau|^2=1-2\theta\,\mathrm{Re}\,\nu+\theta^2|\nu|^2$$
$$\le 1-\theta(2\,C_0-\theta\widehat{C}_0^2).$$

The result follows upon setting $\kappa_\theta:=[1-\theta(2\,C_0-\theta\widehat{C}_0^2)]^{1/2}$. □

Remark 4.2.12 Note that the conclusion in Corollary 4.2.11 does not have the same strength as that in Theorem 4.2.2. Actually, by Corollary 4.2.11, we have only proved that the spectral radius of the (non-symmetric) iteration matrix is smaller than 1, whereas in Theorem 4.2.2 the convergence of the Richardson scheme is achieved with respect to a specific norm; namely, the norm induced by the symmetric part of the preconditioner $\mathcal{Q}_2$. □

The analysis of the convergence of the preconditioned Richardson iterations with the Neumann–Neumann preconditioner $(\sigma_1Q_1^{-1}+\sigma_2Q_2^{-1})^{-1}$ is based on the following result, due to Berselli (1997).

Theorem 4.2.13 *Suppose that*

$$[Q_i\boldsymbol{\eta},\boldsymbol{\eta}]\neq 0\qquad\forall\ \boldsymbol{\eta}\in\mathbf{C}^M\setminus\mathbf{0},\quad i=1,2, \tag{4.2.20}$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbf{C}^M$, and that

$$\begin{aligned}(4.2.21)\qquad &\mathrm{Re}[Q_1\boldsymbol{\eta}\,,\boldsymbol{\eta}]\,\mathrm{Re}[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]\\ &\quad+\mathrm{Im}[Q_1\boldsymbol{\eta},\boldsymbol{\eta}]\,\mathrm{Im}[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]\geq 0 \qquad \forall\ \boldsymbol{\eta}\in\mathbf{C}^M.\end{aligned}$$

Moreover set

$$(4.2.22)\qquad C_1:=\sup_{\boldsymbol{\eta}\in\mathbf{C}^M\setminus\mathbf{0}}\frac{|[Q_1\boldsymbol{\eta},\boldsymbol{\eta}]|}{|[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]|},\quad C_2:=\sup_{\boldsymbol{\eta}\in\mathbf{C}^M\setminus\mathbf{0}}\frac{|[Q_2\boldsymbol{\eta},\boldsymbol{\eta}]|}{|[Q_1\boldsymbol{\eta},\boldsymbol{\eta}]|}.$$

Then, for any choice of the averaging parameters $\sigma_1>0$ and $\sigma_2>0$, every eigenvalue ν_s of $(\sigma_1Q_1^{-1}+\sigma_2Q_2^{-1})Q$ satisfies

$$(4.2.23)\qquad \hat{C}^*\leq 2\,\frac{\mathrm{Re}\,\nu_s}{|\nu_s|^2}\qquad\forall\ s=1,\dots,M,$$

where

$$(4.2.24)\qquad \hat{C}^*:=\min\left(\frac{1}{\sigma_1+\sigma_2},\frac{2(\sigma_1+\sigma_2)}{(\sigma_1+\sigma_2)^2+H^*}\right),\ H^*:=(\sigma_2C_1+\sigma_1C_2)^2\,.$$

Proof Since

$$(\sigma_1Q_1^{-1}+\sigma_2Q_2^{-1})Q=(\sigma_1+\sigma_2)I+\sigma_2Q_2^{-1}Q_1+\sigma_1Q_1^{-1}Q_2,$$

any eigenvalue ν of $(\sigma_1Q_1^{-1}+\sigma_2Q_2^{-1})Q$ can be written as $\nu=\sigma_1+\sigma_2+\hat{\zeta}$, where $\hat{\zeta}$ is an eigenvalue of $\sigma_2Q_2^{-1}Q_1+\sigma_1Q_1^{-1}Q_2$.

Let us transform $Q_2^{-1}Q_1$ into an upper-triangular matrix U with diagonal elements given by the eigenvalues ζ_s of $Q_2^{-1}Q_1$; namely, write

$$ZQ_2^{-1}Q_1Z^{-1}=U,$$

for a suitable transformation matrix Z. The matrix

$$U^{-1}=ZQ_1^{-1}Q_2Z^{-1}$$

is also upper-triangular, and features the eigenvalues ζ_s^{-1} of $Q_1^{-1}Q_2$ on the diagonal. In conclusion,

$$Z(\sigma_2Q_2^{-1}Q_1+\sigma_1Q_1^{-1}Q_2)Z^{-1}$$

is upper-triangular, with the terms $\sigma_2\zeta_s+\sigma_1\zeta_s^{-1}$ on the diagonal; therefore, the eigenvalues of $\sigma_2Q_2^{-1}Q_1+\sigma_1Q_1^{-1}Q_2$ are given by $\sigma_2\zeta_s+\sigma_1\zeta_s^{-1}$.

We have thus proved that any eigenvalue ν of $(\sigma_1Q_1^{-1}+\sigma_2Q_2^{-1})Q$ can be written as $\nu=\sigma_1+\sigma_2+\sigma_2\zeta+\sigma_1\zeta^{-1}$, where ζ is an eigenvalue of $Q_2^{-1}Q_1$. Proceeding as in Theorem 4.2.10, we find $\mathrm{Re}\,\zeta\geq 0$ and $|\zeta|\leq C_1$, and repeating the same argument for the matrix $Q_2^{-1}Q_1$ we also obtain $\mathrm{Re}\,(\zeta^{-1})\geq 0$ and $|\zeta^{-1}|\leq C_2$. Hence

$$2\,\frac{\mathrm{Re}\,\nu}{|\nu|^2} = 2\,\frac{\sigma_1+\sigma_2+\sigma_2\,\mathrm{Re}\,\zeta+\sigma_1\,\mathrm{Re}\,(\zeta^{-1})}{(\sigma_1+\sigma_2)^2+2(\sigma_1+\sigma_2)[\sigma_2\,\mathrm{Re}\,\zeta+\sigma_1\,\mathrm{Re}\,(\zeta^{-1})]+|\sigma_2\zeta+\sigma_1\zeta^{-1}|^2}$$
$$\geq 2\,\frac{\sigma_1+\sigma_2+\sigma_2\,\mathrm{Re}\,\zeta+\sigma_1\,\mathrm{Re}\,(\zeta^{-1})}{(\sigma_1+\sigma_2)^2+2(\sigma_1+\sigma_2)[\sigma_2\,\mathrm{Re}\,\zeta+\sigma_1\,\mathrm{Re}\,(\zeta^{-1})]+H^*}.$$

Note now that the function

$$F^*(\xi) := 2\,\frac{\sigma_1+\sigma_2+\xi}{(\sigma_1+\sigma_2)^2+2(\sigma_1+\sigma_2)\xi+H^*}$$

is strictly increasing when $H^* > (\sigma_1+\sigma_2)^2$, strictly decreasing when $H^* < (\sigma_1+\sigma_2)^2$, and equal to $1/(\sigma_1+\sigma_2)$ when $H^* = (\sigma_1+\sigma_2)^2$. Moreover,

$$F^*(0) = \frac{2(\sigma_1+\sigma_2)}{(\sigma_1+\sigma_2)^2+H^*}, \quad \lim_{\xi\to\infty} F^*(\xi) = \frac{1}{\sigma_1+\sigma_2},$$

hence $\hat{C}^*$ is the infimum of F^* for $\xi \geq 0$. □

Corollary 4.2.14 *Under the assumption of Theorem 4.2.13, the Richardson iterations applied to the preconditioned system*

$$(\sigma_1 Q_1^{-1}+\sigma_2 Q_2^{-1})(Q\boldsymbol{\lambda}-\boldsymbol{\chi}) = \mathbf{0}$$

converge for each $\theta \in (0,\hat{C}^*)$, and the spectral radius of the iteration matrix $I-\theta(\sigma_1 Q_1^{-1}+\sigma_2 Q_2^{-1})Q$ is bounded by a positive constant $\hat{\kappa}_\theta < 1$, which depends on C_1 and C_2, but not on the dimension M.

Proof From (4.2.23) we have

$$|\nu_s|^2 < \frac{2}{\theta}\,\mathrm{Re}\,\nu_s \qquad \forall\ s=1,\ldots,M,\ \ \forall\ \theta\in(0,\hat{C}^*),$$

therefore convergence follows from Theorem 4.2.9. Moreover, any eigenvalue of the iteration matrix $I-\theta(\sigma_1 Q_1^{-1}+\sigma_2 Q_2^{-1})Q$ can be written as $\tau = 1-\theta\nu$, where ν is an eigenvalue of $(\sigma_1 Q_1^{-1}+\sigma_2 Q_2^{-1})Q$. From (4.2.23) we find that, for each $\theta\in(0,\hat{C}^*)$,

$$|\tau|^2 = 1-2\theta\,\mathrm{Re}\,\nu+\theta^2|\nu|^2$$
$$\leq 1-\theta(\hat{C}^*-\theta)|\nu|^2 \leq 1-\theta(\hat{C}^*-\theta)(\sigma_1+\sigma_2)^2,$$

because in the proof of Theorem 4.2.13 we have shown that

$$|\nu| \geq \mathrm{Re}\,\nu \geq \sigma_1+\sigma_2.$$

The result follows on setting $\hat{\kappa}_\theta := [1-\theta(\hat{C}^*-\theta)(\sigma_1+\sigma_2)^2]^{1/2}$. □

4.3 Convergence of the Dirichlet–Neumann iterative method

The analysis that we have developed in the previous sections can be applied to proving the convergence of the iterative substructuring procedures introduced in Sections 1.3 and 3.1. We start by considering the homogeneous Dirichlet problem for the Laplace operator (1.2.1) and its finite element counterpart (2.1.1). In Chapter 5 we will generalise our results to other families of boundary value problems. In what follows, we will confine ourselves to the case of a domain partitioned in two subdomains Ω_1 and Ω_2 that do not overlap, as in Fig. 1.1.

We have already noted in Section 1.3 that the Dirichlet–Neumann iterative method (1.3.1)–(1.3.3) can be interpreted as a preconditioned Richardson procedure for the Steklov–Poincaré equation (1.1.7). The local Steklov–Poincaré operators S_i, $i = 1, 2$, defined in (4.1.1), are symmetric. Moreover, in view of Proposition 4.1.2 they are continuous and coercive. We can therefore apply Theorem 4.2.2 taking $X = \Lambda$, $\mathcal{Q}_1 = S_1$ and $\mathcal{Q}_2 = S_2$, and conclude that the Dirichlet–Neumann iterative procedure (1.3.1)–(1.3.3) converges.

Let us consider now the finite element approximation (2.1.1) of the same problem. Owing to (4.1.4), the discrete Steklov–Poincaré operators $S_{i,h}$ are symmetric, coercive (with constant α_i), and continuous (with constant β_i). The constants α_i and β_i are independent of h, since they only depend on the coerciveness and continuity constants of the bilinear forms a_i, as well as on $\hat{C}_1$ and $\hat{C}_2$ introduced in Theorem 4.1.3. If we apply Theorem 4.2.2 with $X = \Lambda_h$, $\mathcal{Q}_1 = S_{1,h}$ and $\mathcal{Q}_2 = S_{2,h}$, the rate of convergence of the iteration-by-subdomain method, being driven only by α_i and β_i (see Remark 4.2.3), is independent of h in the norm associated with the preconditioner $\Sigma_{2,h}$, or, equivalently, in the norm of the trace space Λ.

The same conclusion holds for the general second-order symmetric elliptic operator L introduced in (1.4.2), as well as for its conforming finite element approximation.

4.3.1 An alternative way to prove convergence

The convergence analysis of Dirichlet–Neumann iterations can be carried out by a more direct argument that does not exploit the analogy with Steklov–Poincaré operators. This is derived proving that

$$M_\theta := \sup_{k\geq 0} \frac{||e^{k+1}||_\Lambda}{||e^k||_\Lambda} < 1, \tag{4.3.1}$$

where e^k denotes the interface error $e^k := u^k_{1|\Gamma} - u_{|\Gamma}$.

Clearly, e^{k+1} is related to e^k through the iteration equation $e^{k+1} = T_\theta e^k$, where $T_\theta := I - \theta S_2^{-1} S$. However, the proof of (4.3.1) can be based on a functional analysis argument for partial differential equations, and does not necessarily require the construction of T_θ, and not even the reinterpretation of our problem in terms of Steklov–Poincaré operators. The interested reader can refer to Marini and Quarteroni (1989), where such an approach is adopted for elliptic equations.

The constant M_θ in (4.3.1) is a lower bound for

$$||T_\theta||_{\mathcal{L}(\Lambda)} := \sup_{\mu\in\Lambda} \frac{||T_\theta\mu||_\Lambda}{||\mu||_\Lambda}.$$

This alternative approach can be particularly helpful for convergence analysis in those cases in which a Steklov–Poincaré interpretation is not easily available.

It is worthwhile to point out that the same argument can be developed at the finite dimensional level; in which case a relation between the convergence rates that are inherent in either approach can be established. As a matter of fact, let us suppose that the finite dimensional Steklov–Poincaré operators $S_{1,h}$ and $S_{2,h}$ are both symmetric, and, moreover, that $S_{2,h}$ is coercive. In the finite dimensional space Λ_h let us choose the following scalar product

$$(\mu_h, \eta_h)_{\Lambda_h} := \langle S_{2,h}\mu_h, \eta_h\rangle;$$

and denote by $||\cdot||_{\Lambda_h}$ the associated norm. Similarly, in $\mathbf{R}^{N_\Gamma}$ we choose the scalar product

$$[\boldsymbol{\mu}, \boldsymbol{\eta}]_{\Sigma_{2,h}} := [\Sigma_{2,h}\boldsymbol{\mu}, \boldsymbol{\eta}],$$

where $[\cdot,\cdot]$ is the Euclidean scalar product in $\mathbf{R}^{N_\Gamma}$. The associated vector norm is denoted by $|\cdot|_{\Sigma_{2,h}}$. Clearly, we have

$$(\mu_h, \eta_h)_{\Lambda_h} = [\boldsymbol{\mu}, \boldsymbol{\eta}]_{\Sigma_{2,h}}, \quad ||T_\theta^{(h)}\mu_h||_{\Lambda_h} = |(I - \theta\Sigma_{2,h}^{-1}\Sigma_h)\boldsymbol{\mu}|_{\Sigma_{2,h}},$$

where $T_\theta^{(h)} := I - \theta S_{2,h}^{-1}S_h$, and $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ denote the set of the values of μ_h and η_h at the N_Γ nodes on Γ, respectively. Hence

$$\begin{aligned} ||T_\theta^{(h)}||_{\mathcal{L}(\Lambda_h)} &= \sup_{\mu_h\neq 0} \frac{||T_\theta^{(h)}\mu_h||_{\Lambda_h}}{||\mu_h||_{\Lambda_h}} = \sup_{\boldsymbol{\mu}\neq\mathbf{0}} \frac{|(I - \theta\Sigma_{2,h}^{-1}\Sigma_h)\boldsymbol{\mu}|_{\Sigma_{2,h}}}{|\boldsymbol{\mu}|_{\Sigma_{2,h}}} \\ &= ||I - \theta\Sigma_{2,h}^{-1}\Sigma_h||_{\Sigma_{2,h}}, \end{aligned}$$

where we have denoted by $||\cdot||_{\Sigma_{2,h}}$ the matrix norm associated with the vector norm $|\cdot|_{\Sigma_{2,h}}$. Since the iteration matrix $I - \theta\Sigma_{2,h}^{-1}\Sigma_h$ is symmetric with respect to $[\cdot,\cdot]_{\Sigma_{2,h}}$, we finally have $||I - \theta\Sigma_{2,h}^{-1}\Sigma_h||_{\Sigma_{2,h}} = \rho(I - \theta\Sigma_{2,h}^{-1}\Sigma_h)$, the spectral radius of $I - \theta\Sigma_{2,h}^{-1}\Sigma_h$. Therefore,

$$M_\theta^{(h)} := \sup_{k\geq 0} \frac{||e_h^{k+1}||_{\Lambda_h}}{||e_h^k||_{\Lambda_h}} \leq \rho(I - \theta\Sigma_{2,h}^{-1}\Sigma_h).$$

Hence, the convergence of the iterative scheme is proved if we show that $M_\theta^{(h)} < 1$, which can be easier than proving that $\rho(I - \theta\Sigma_{2,h}^{-1}\Sigma_h) < 1$.

4.4 Convergence of the Neumann–Neumann iterative method

As we have already noted in Section 3.2, the Neumann–Neumann algorithm (3.1.5)–(3.1.8) is a Richardson iterative scheme for the Steklov–Poincaré equation (2.2.3), with the operator

$$\mathcal{N}_h := (\sigma_1 S_{1,h}^{-1} + \sigma_2 S_{2,h}^{-1})^{-1}$$

acting as a preconditioner. The same is true in the infinite dimensional case.

Since the Steklov–Poincaré operators S_1 and S_2 are symmetric, continuous, and coercive, using Theorem 4.2.5 instead of Theorem 4.2.2 we can prove the convergence of the Neumann–Neumann method for the homogeneous Dirichlet boundary value problem associated with the symmetric elliptic operator L.

The same result holds for the discrete approximation by means of continuous piecewise-polynomial finite elements, provided that the assumptions of Theorem 4.2.2 are satisfied. The rate of convergence is independent of h when measured in the norm associated with the preconditioner N_h (see (4.1.10)), or, equivalently, in the norm of the trace space Λ.

Remark 4.4.1 The results of the last two sections show that, for the two-subdomain case, both Dirichlet–Neumann and Neumann–Neumann iterations converge at a rate independent of h. The Neumann–Neumann algorithm has no particular advantage over the Dirichlet–Neumann algorithm (and actually the former requires more subdomain solves per iteration than the latter). The situation can, however, be different in the case of many subdomains (see, for example, Dryja and Widlund 1995). □

4.5 Convergence of the Robin iterative method

We present the proof of the convergence of the Robin iterative method introduced in Section 1.3 in the simplified situation $\gamma_1 = \gamma_2 = \gamma$ (we refer to P.-L. Lions 1990 for the general case). For the sake of simplicity, we will write the problem in its differential form; however, one could proceed as we will do in Theorem 6.4.2 (setting there the convective field $\mathbf{b}$ equal to zero), obtaining the proof of convergence for the variational formulation of the Robin method, as well as for its finite element discretisation.

The local errors $e_i^k := u_i^k - u_{|\Omega_i}$, $i = 1, 2$, satisfy the following boundary value problems

$$\begin{cases} -\Delta e_1^{k+1} = 0 & \text{in } \Omega_1 \\ e_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \dfrac{\partial e_1^{k+1}}{\partial n} + \gamma e_1^{k+1} = \dfrac{\partial e_2^k}{\partial n} + \gamma e_2^k & \text{on } \Gamma, \end{cases} \tag{4.5.1}$$

and

$$
\begin{cases} -\Delta e_2^{k+1} = 0 & \text{in } \Omega_2 \\ e_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \dfrac{\partial e_2^{k+1}}{\partial n} - \gamma e_2^{k+1} = \dfrac{\partial e_1^{k+1}}{\partial n} - \gamma e_1^{k+1} & \text{on } \Gamma, \end{cases} \tag{4.5.2}
$$

having denoted, as usual, by $\mathbf{n}$ the unit normal vector on Γ, directed from Ω_1 to Ω_2. Multiplying $(4.5.2)_1$ by e_2^{k+1} and integrating by parts we find that

$$
||\nabla e_2^{k+1}||^2_{0,\Omega_2} = -\int_\Gamma \frac{\partial e_2^{k+1}}{\partial n}\, e_2^{k+1}. \tag{4.5.3}
$$

Using the identity

$$
AB = \frac{1}{4\gamma}[(A+\gamma B)^2 - (A-\gamma B)^2]
$$

we can write

$$
\begin{aligned}
-\int_\Gamma \frac{\partial e_2^{k+1}}{\partial n}\, e_2^{k+1} = \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial n} - \gamma e_2^{k+1}\right)^2 \\
-\frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial n} + \gamma e_2^{k+1}\right)^2.
\end{aligned}
$$

Using the boundary condition on Γ for e_2^{k+1}, we are left with

$$
\begin{aligned}
||\nabla e_2^{k+1}||^2_{0,\Omega_2} + \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial n} + \gamma e_2^{k+1}\right)^2 \\
= \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_1^{k+1}}{\partial n} - \gamma e_1^{k+1}\right)^2.
\end{aligned} \tag{4.5.4}
$$

Repeating the same argument for e_1^{k+1} we find that

$$
\begin{aligned}
||\nabla e_1^{k+1}||^2_{0,\Omega_1} + \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_1^{k+1}}{\partial n} - \gamma e_1^{k+1}\right)^2 \\
= \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k}}{\partial n} + \gamma e_2^{k}\right)^2.
\end{aligned} \tag{4.5.5}
$$

Adding (4.5.4) and (4.5.5) we obtain

$$
\begin{aligned}
||\nabla e_1^{k+1}||^2_{0,\Omega_1} + ||\nabla e_2^{k+1}||^2_{0,\Omega_2} + &\frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k+1}}{\partial n} + \gamma e_2^{k+1}\right)^2 \\
&= \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{k}}{\partial n} + \gamma e_2^{k}\right)^2 .
\end{aligned} \tag{4.5.6}
$$

Now, summing over k from $k = 0$ to $k = M - 1$, it follows that

$$
\begin{aligned}
\sum_{k=1}^{M}(||\nabla e_1^k||^2_{0,\Omega_1} + ||\nabla e_2^k||^2_{0,\Omega_2}) + &\frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{M}}{\partial n} + \gamma e_2^{M}\right)^2 \\
&= \frac{1}{4\gamma}\int_\Gamma \left(\frac{\partial e_2^{0}}{\partial n} + \gamma e_2^{0}\right)^2 .
\end{aligned} \tag{4.5.7}
$$

Consequently, the series

$$
\sum_{k=1}^{\infty}(||\nabla e_1^k||^2_{0,\Omega_1} + ||\nabla e_2^k||^2_{0,\Omega_2})
$$

is convergent, and e_i^k tends to 0 in $H^1(\Omega_i)$, $i = 1, 2$.

In contrast with the Dirichlet–Neumann and Neumann–Neumann methods, for the Robin substructuring iterative method we have no estimates of the error reduction factor at each iteration, nor do we have any information about the rate of convergence.

Remark 4.5.1 The convergence of the Robin method in the multi-domain case has been proved by P.-L. Lions (1990). □

4.6 Convergence of the alternating Schwarz method

The alternating Schwarz method presented in Section 1.5 furnishes a couple of sequences u^k and $u^{k+1/2}$ that converge to the solution u of the Dirichlet boundary value problem (1.4.1). Following P.-L. Lions (1988), we present here the proof based on a variational approach and not on the maximum principle, like the classical one.

Recalling the results of Section 1.5, the sequences u^k and $u^{k+1/2}$ generated by the Schwarz method satisfy

$$
\begin{aligned}
u - u^{k+1/2} &= (I - \mathcal{P}_1)(u - u^k) \\
u - u^{k+1} &= (I - \mathcal{P}_2)(u - u^{k+1/2}),
\end{aligned} \tag{4.6.1}
$$

where, for $i = 1, 2$, $\mathcal{P}_i = \mathcal{I}_i\mathcal{P}_i^*$, $\mathcal{I}_i$ is the immersion of

$$
V_i^* := \{v \in V \,|\, v = 0 \text{ in } \Omega \setminus \overline{\Omega_i}\}
$$

into $V := H_0^1(\Omega)$, and $\mathcal{P}_i^*$ is the orthogonal projection of V onto V_i^* with respect to the scalar product induced by the bilinear form

$$a^*(w,v) := \int_\Omega \left(\sum_{l,j=1}^{d} a_{lj} D_j w D_l v + a_0\, w\, v \right).$$

Let us denote by $||\cdot||_*$ the norm associated with the form $a^*(\cdot,\cdot)$, which is equivalent to the usual norm of $H^1(\Omega)$. Moreover, for any couple of vector spaces W_1 and W_2, denote by $W_1 \oplus W_2$ their direct sum; namely, the vector space of all elements w that can be written in a unique way as the sum of an element in W_1 and an element in W_2.

Theorem 4.6.1 *Assume that Ω is a bounded domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$. If $\overline{V_1^* \oplus V_2^*} = V$, then both u^k and $u^{k+1/2}$ converge to u in $H_0^1(\Omega)$ as $k \to \infty$.*

Proof Denoting the errors by $e^k := u - u^k$ and $e^{k+1/2} := u - u^{k+1/2}$, it is sufficient to prove the convergence to 0 of the sequence $v^k := e^{k/2}$. Note that $v^k \in (V_2^*)^\perp$ for k even and $v^k \in (V_1^*)^\perp$ for k odd.

We have

$$||v^{k+1}||_*^2 + ||v^{k+1} - v^k||_*^2 = ||v^k||_*^2,$$

because $v^{k+1} - v^k$ is orthogonal to v^{k+1}. Therefore, the sequence $||v^k||_*$ is non-increasing and convergent, and $||v^{k+1} - v^k||_*$ tends to 0.

Since v^k is bounded in $H^1(\Omega)$, by well known properties of Hilbert spaces (see, for example, Yosida 1974, p. 126) we can extract from each subsequence another subsequence v^{k_m}, which is weakly convergent to some limit v in $H^1(\Omega)$, and, moreover, such that also v^{k_m+1} is weakly convergent to another limit, say w, in $H^1(\Omega)$. Furthermore, owing to the properties of weakly convergent sequences (see Section 9.1), we find that

$$||w - v||_* \le \liminf_{m\to\infty} ||v^{k_m+1} - v^{k_m}||_* = 0,$$

and, consequently, $w = v$. It follows that $v \in (V_1^*)^\perp \cap (V_2^*)^\perp$, and, recalling that the assumption of the theorem is equivalent to $(V_1^*)^\perp \cap (V_2^*)^\perp = \{0\}$, we have $v = 0$. Therefore, we conclude that the whole sequence v^k converges weakly to 0 in $H^1(\Omega)$.

Take now $k = 2l$, $l \ge 0$. Then

$$\begin{aligned}
||v^k||_*^2 &= a^*(e^l, e^l) = a^*((I - \mathcal{P}_2)(I - \mathcal{P}_1)e^{l-1}, e^l) \\
&= a^*(e^{l-1}, (I - \mathcal{P}_1)(I - \mathcal{P}_2)e^l) = a^*(e^{l-1}, e^{l+1/2}) \\
&= a^*(e^{l-2}, (I - \mathcal{P}_1)(I - \mathcal{P}_2)e^{l+1/2}) = a^*(e^{l-2}, e^{l+3/2}) \\
&= a^*(e^0, e^{2l-1/2}) = a^*(e^0, e^{k-1/2}) = a^*(e^0, v^{2k-1}).
\end{aligned}$$

The same result holds for $k = 2l + 1$ as well, therefore the weak convergence of v^k implies the strong convergence to 0. □

It is worthwhile to note that the assumption of Theorem 4.6.1 is always satisfied, provided that $\Omega = \Omega_1 \cup \Omega_2$ (see P.-L. Lions 1988). Under the slightly more

restrictive assumption $V_1^* \oplus V_2^* = V$, which is satisfied for a large class of subdomains Ω_1 and Ω_2, it is possible to give an estimate of the rate of convergence. Let us first prove a technical result.

Lemma 4.6.2 *If $V_1^* \oplus V_2^* = V$, then there exists a constant $C_0 \geq 1$ such that*

$$||v||_* \leq C_0(||\mathcal{P}_1 v||^2 + ||\mathcal{P}_2 v||^2)^{1/2} \qquad \forall\ v \in V. \tag{4.6.2}$$

Proof An application of the Open Mapping Theorem (see, for example, Yosida 1974, p. 75) to the surjective map $(v_1, v_2) \to v_1 + v_2$ from $V_1^* \times V_2^*$ to $V = V_1^* \oplus V_2^*$ yields the existence of $C_0 \geq 1$ such that for each $v \in V$ it is possible to choose $v_1 \in V_1^*$ and $v_2 \in V_2^*$ such that $v = v_1 + v_2$ and

$$(||v_1||_*^2 + ||v_2||_*^2)^{1/2} \leq C_0 ||v||_*.$$

Then by the Cauchy–Schwarz inequality we have

$$\begin{aligned} ||v||_*^2 &= a^*(v_1, v) + a^*(v_2, v) = a^*(v_1, \mathcal{P}_1 v) + a^*(v_2, \mathcal{P}_2 v) \\ &\leq (||v_1||_*^2 + ||v_2||_*^2)^{1/2} (||\mathcal{P}_1 v||_*^2 + ||\mathcal{P}_2 v||_*^2)^{1/2} \\ &\leq C_0 ||v||_* (||\mathcal{P}_1 v||_*^2 + ||\mathcal{P}_2 v||_*^2)^{1/2}, \end{aligned}$$

and the result follows. □

We are now in a position to prove a more precise convergence result.

Theorem 4.6.3 *If $V_1^* \oplus V_2^* = V$, then the iteration operators $(I - \mathcal{P}_1)(I - \mathcal{P}_2)$ and $(I - \mathcal{P}_1)(I - \mathcal{P}_2)$ are contractions in $H_0^1(\Omega)$ with respect to the norm induced by the bilinear form $a^*(\cdot, \cdot)$.*

Proof From Lemma 4.6.2 we have

$$||(I - \mathcal{P}_1)v||_* \leq C_0 ||\mathcal{P}_2 (I - \mathcal{P}_1)v||_*.$$

Therefore, since $\mathcal{P}_2$ is an orthogonal projection,

$$\begin{aligned} ||(I - \mathcal{P}_1)v||_*^2 &= ||(I - \mathcal{P}_2)(I - \mathcal{P}_1)v||_*^2 + ||\mathcal{P}_2 (I - \mathcal{P}_1)v||_*^2 \\ &\geq ||(I - \mathcal{P}_2)(I - \mathcal{P}_1)v||_*^2 + C_0^{-2} ||(I - \mathcal{P}_1)v||_*^2, \end{aligned}$$

hence

$$||(I - \mathcal{P}_2)(I - \mathcal{P}_1)v||_*^2 \leq (1 - C_0^{-2}) ||(I - \mathcal{P}_1)v||_*^2 \leq (1 - C_0^{-2}) ||v||_*^2.$$

Interchanging the roles of the indices we find the same result, and the proof is complete. □

As a consequence of this theorem, the convergence of the alternating Schwarz method takes place with a geometric rate. In fact, setting $K_0 := (1 - C_0^{-2})^{1/2}$, we have

$$||e^{k+1}||_* = ||(I-\mathcal{P}_2)(I-\mathcal{P}_1)e^k||_* \\ \leq K_0||e^k||_* \leq K_0^{k+1}||e^0||_*,$$

and analogously

$$||e^{k+3/2}||_* = ||(I-\mathcal{P}_1)(I-\mathcal{P}_2)e^{k+1/2}||_* \\ \leq K_0^{k+1}||e^{1/2}||_* = K_0^{k+1}||(I-\mathcal{P}_1)e^0||_* \leq K_0^{k+1}||e^0||_*.$$

Similar convergence results hold for a decomposition with M subdomains, and in the finite dimensional case (see Bramble *et al.* 1991; see also Matsokin and Nepomnyaschikh 1985; and Dryja and Widlund 1989). The convergence rate turns out to be independent of the mesh parameter h, but it deteriorates as H^{-2}, H being the maximum diameter of the subdomains Ω_i, $i = 1, \ldots, M$.

The introduction of a coarse grid correction makes the rate of convergence independent of both h and H, provided that the linear measure of the overlapping region between two subdomains is δH, for $0 < \delta \leq 1$. In the latter case, the rate of convergence depends only on δ and on the coefficients a_{lj} and a_0 of the operator L.

5

OTHER BOUNDARY VALUE PROBLEMS

The theory that was developed in the previous chapters for the Poisson equation and for symmetric elliptic equations can be extended to the case of a more general differential problem of the form

$$\mathcal{L}u = f \qquad \text{in } \Omega, \tag{5.1}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz boundary, $\mathcal{L}$ is a partial differential operator, f is a given data, and u is the unknown solution. More precisely, we will consider: non-symmetric elliptic problems; the linear elasticity problem; the Stokes and Oseen problems for incompressible flows; a Stokes-like problem that arises from the analysis of compressible flows; the advection equation; and the time-harmonic Maxwell system.

Our presentation will be developed according to the guidelines we have followed in the previous chapters. After having partitioned Ω into two non-overlapping subdomains Ω_1 and Ω_2, and denoted by $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ the interface (see Fig. 1.1), for each problem we derive its multi-domain variational formulation and introduce the associated Steklov–Poincaré problem. Next, we illustrate iterative substructuring methods, and analyse their convergence on the basis of the abstract theorems of Chapter 4. The same presentation is then repeated for the finite dimensional approximation of each problem.

Other stationary problems of a special nature may require an ad hoc interface treatment, and will be addressed in the following chapters. Precisely, convection–diffusion equations when convection is dominant are discussed in Chapter 6, while the coupling between equations of different types is considered in Chapter 8.

5.1 Non-symmetric elliptic operators

We consider a non-symmetric elliptic operator of the following form:

$$L^\flat w := -\sum_{l,j=1}^{d} D_l(a_{lj} D_j w) + \sum_{j=1}^{d} D_j(b_j w) + a_0 w, \tag{5.1.1}$$

where the coefficients a_{lj} are assumed to satisfy the ellipticity condition

$$\sum_{l,j=1}^{d} a_{lj}(\mathbf{x})\xi_j\xi_l \geq \alpha_0 |\boldsymbol{\xi}|^2 \qquad \text{for all } \boldsymbol{\xi} \in \mathbf{R}^d, \text{ for almost every } \mathbf{x} \in \Omega$$

for a suitable constant $\alpha_0 > 0$.

The homogeneous Dirichlet boundary value problem associated with the operator $L^\flat$ reads

$$\begin{cases} L^\flat u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases}$$

Denoting by u_i the restriction of u to Ω_i, $i = 1, 2$, the interface conditions on Γ are the same as in the symmetric case (see (1.4.5)):

$$\begin{aligned} & u_1 = u_2 && \text{on } \Gamma \\ & \frac{\partial u_1}{\partial n_L} = \frac{\partial u_2}{\partial n_L} && \text{on } \Gamma. \end{aligned} \tag{5.1.2}$$

Clearly, the second interface condition in (5.1.2) can be replaced by

$$\frac{\partial u_1}{\partial n_L} + \gamma u_1 = \frac{\partial u_2}{\partial n_L} + \gamma u_2 \qquad \text{on } \Gamma,$$

for any function γ defined on Γ.

5.1.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

Let us assume that a_{lj}, b_j and a_0 belong to $L^\infty(\Omega)$ for each $l, j = 1, \ldots, d$, and that $\operatorname{div} \mathbf{b} \in L^\infty(\Omega)$. Then we can introduce in $H^1(\Omega)$ the continuous bilinear form

$$\begin{aligned} a^\flat(w, v) := & \int_\Omega \left[\sum_{l,j=1}^{d} a_{lj}\, D_j w\, D_l v + \left(\frac{1}{2} \operatorname{div} \mathbf{b} + a_0 \right) w\, v \right] \\ & + \frac{1}{2} \int_\Omega (v\, \mathbf{b} \cdot \nabla w - w\, \mathbf{b} \cdot \nabla v), \end{aligned} \tag{5.1.3}$$

associated with the operator $L^\flat$.

Under the further assumption that

$$\frac{1}{2} \operatorname{div} \mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq 0 \qquad \text{for almost every } \mathbf{x} \in \Omega, \tag{5.1.4}$$

the bilinear form $a^\flat(\cdot, \cdot)$ is coercive in $V = H_0^1(\Omega)$.

Therefore, the homogeneous Dirichlet boundary value problem

$$\text{find } u \in H_0^1(\Omega)\ :\ a^\flat(u, v) = (f, v) \qquad \forall\ v \in H_0^1(\Omega) \tag{5.1.5}$$

has a unique solution.

Defining in V_i (see (1.2.4) for notation) the local bilinear forms

$$
\begin{aligned}
a_i^\flat(w_i, v_i) := & \int_{\Omega_i} \left[\sum_{l,j=1}^{d} a_{lj}\, D_j w_i\, D_l v_i + \left(\frac{1}{2}\, \mathrm{div}\, \mathbf{b} + a_0 \right) w_i\, v_i \right] \\
& + \frac{1}{2} \int_{\Omega} (v_i\, \mathbf{b} \cdot \nabla w_i - w_i\, \mathbf{b} \cdot \nabla v_i),
\end{aligned} \tag{5.1.6}
$$

$i = 1, 2$, the corresponding two-domain weak formulation is: find $u_1 \in V_1$, $u_2 \in V_2$ such that

$$
\begin{cases}
a_1^\flat(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\ v_1 \in V_1^0 \\
\\
u_1 = u_2 \qquad \text{on } \Gamma \\
\\
a_2^\flat(u_2, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\ v_2 \in V_2^0 \\
\\
a_2^\flat(u_2, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} - a_1^\flat(u_1, \mathcal{R}_1\mu) \qquad \forall\ \mu \in \Lambda,
\end{cases} \tag{5.1.7}
$$

where $\mathcal{R}_i$ denotes any extension operator from Λ to V_i, and notation is the same as in Section 1.2.

Note that, due to the choice of the bilinear forms $a_i^\flat(\cdot, \cdot)$, the last equation in (5.1.7) is the weak form of the interface condition

$$
\frac{\partial u_2}{\partial n_L} - \frac{1}{2}\, \mathbf{b} \cdot \mathbf{n}\, u_2 = \frac{\partial u_1}{\partial n_L} - \frac{1}{2}\, \mathbf{b} \cdot \mathbf{n}\, u_1 \qquad \text{on } \Gamma. \tag{5.1.8}
$$

We introduce the Steklov–Poincaré operators S_i defined on the trace space Λ as

$$
\langle S_i \eta, \mu \rangle := a_i^\flat(E_i^\flat \eta, E_i^\flat \mu), \quad \eta, \mu \in \Lambda, \tag{5.1.9}
$$

where $E_i^\flat \eta$ is the solution of the Dirichlet boundary value problem

$$
\begin{cases}
E_i^\flat \eta \in V_i \ : \\
\\
a_i^\flat(E_i^\flat \eta, v_i) = 0 \qquad \forall\ v_i \in V_i^0 \\
\\
E_i^\flat \eta = \eta \qquad \text{on } \Gamma.
\end{cases} \tag{5.1.10}
$$

Moreover, for $i = 1, 2$, define $\chi_i \in \Lambda'$ as

$$
\langle \chi_i, \mu \rangle := (f, E_i^\flat \mu)_{\Omega_i} - a_i^\flat(u_i^\flat, E_i^\flat \mu) \qquad \forall\ \mu \in \Lambda, \tag{5.1.11}
$$

where

$$
u_i^\flat \in V_i^0 \ : \ a_i^\flat(u_i^\flat, v_i) = (f, v_i)_{\Omega_i} \qquad \forall\ v_i \in V_i^0. \tag{5.1.12}
$$

Then, denoting with $\lambda \in \Lambda$ the common value of $u_{1|\Gamma}$ and $u_{2|\Gamma}$, λ turns out to be the solution of the following Steklov–Poincaré equation

$$S\lambda = \chi, \tag{5.1.13}$$

having set $S := S_1 + S_2$ and $\chi := \chi_1 + \chi_2$.

By following the proof given in Section 4.1.1 for the symmetric elliptic equations, it can be shown that the Steklov–Poincaré operators S, S_1 and S_2 are continuous and coercive. In fact, the solution $E_i^\flat$ satisfies the well known elliptic a priori estimate

$$||E_i^\flat \eta||_{1,\Omega_i} \leq C||\eta||_\Lambda$$

and the trace inequality (1.2.5), i.e.

$$||E_i^\flat \eta||_{1,\Omega_i} \geq C||\eta||_\Lambda.$$

Besides, we have already noted that the bilinear forms $a_i^\flat(\cdot,\cdot)$ are continuous and coercive in V_i.

Moreover, if the skew-symmetric part of S_2 is small enough, we can prove that condition (c) in Theorem 4.2.2 is satisfied; namely

$$\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \geq \kappa^*||\eta||_\Lambda^2 \qquad \forall\ \eta \in \Lambda.$$

With this aim, let us introduce the symmetric and skew-symmetric parts of the bilinear form $a^\flat(\cdot,\cdot)$, which are given respectively by

$$\begin{aligned} a^{\rm s}(w,v) &:= \int_\Omega \left[\sum_{l,j=1}^{d} \frac{1}{2}(a_{lj} + a_{jl})\, D_j w\, D_l v \right. \\ &\qquad \left. + \left(\frac{1}{2}\ \mathrm{div}\,\mathbf{b} + a_0 \right) w\, v \right] \\ a^{\rm ss}(w,v) &:= \int_\Omega \left[\sum_{l,j=1}^{d} \frac{1}{2}(a_{lj} - a_{jl})\, D_j w\, D_l v \right. \\ &\qquad \left. + \frac{1}{2}(v\,\mathbf{b}\cdot\nabla w - w\,\mathbf{b}\cdot\nabla v) \right]. \end{aligned} \tag{5.1.14}$$

Clearly, $a^\flat = a^{\rm s} + a^{\rm ss}$. In a similar way we can define the local symmetric and skew-symmetric parts $a_i^{\rm s}(\cdot,\cdot)$ and $a_i^{\rm ss}(\cdot,\cdot)$, $i = 1, 2$, at the subdomain level. Setting $\mu := S_2^{-1}S\eta$, from (5.1.9) we have

$$\begin{aligned} \langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle &= \langle S_2\eta,\mu\rangle - \langle S\eta,\eta\rangle + 2\langle S\eta,\eta\rangle \\ &= \langle S_2\eta,\mu\rangle - \langle S_2\mu,\eta\rangle + 2\langle S\eta,\eta\rangle \\ &= a_2^{\rm s}(E_2^\flat\eta, E_2^\flat\mu) + a_2^{\rm ss}(E_2^\flat\eta, E_2^\flat\mu) \\ &\qquad - a_2^{\rm s}(E_2^\flat\mu, E_2^\flat\eta) - a_2^{\rm ss}(E_2^\flat\mu, E_2^\flat\eta) + 2\langle S\eta,\eta\rangle \\ &= 2a_2^{\rm ss}(E_2^\flat\eta, E_2^\flat S_2^{-1}S\eta) + 2\langle S\eta,\eta\rangle. \end{aligned}$$

Denoting by α the coerciveness constant of S, we have

$$\langle S_2\eta, S_2^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \geq 2\alpha||\eta||^2_\Lambda - 2|a_2^{\rm ss}(E_2^\flat\eta, E_2^\flat S_2^{-1}S\eta)|.$$

Therefore, we only have to show that

$$|a_2^{\rm ss}(E_2^\flat\eta, E_2^\flat S_2^{-1}S\eta)| \leq \rho||\eta||^2_\Lambda, \quad 0<\rho<\alpha. \tag{5.1.15}$$

Setting

$$\kappa_i^{\rm ss} := \max_{l,j} ||a_{lj}-a_{jl}||_{L^\infty(\Omega_i)} + ||\mathbf{b}||_{L^\infty(\Omega_i)}, \quad i=1,2,$$

we have

$$\begin{aligned}|a_2^{\rm ss}(E_2^\flat\eta, E_2^\flat S_2^{-1}S\eta)| &\leq C_1\kappa_2^{\rm ss}\,||E_2^\flat\eta||_{1,\Omega_2}||E_2^\flat S_2^{-1}S\eta||_{1,\Omega_2}\\ &\leq C_2\kappa_2^{\rm ss}\,||\eta||_\Lambda||S_2^{-1}S\eta||_\Lambda\\ &\leq C_3\kappa_2^{\rm ss}\,||\eta||^2_\Lambda,\end{aligned}$$

and (5.1.15) follows if $C_3\kappa_2^{\rm ss}<\alpha$, which is a smallness assumption on the skew-symmetric part of the operator $L^\flat$ in Ω_2.

If the skew-symmetric part of the operator $L^\flat$ is small in both Ω_1 and Ω_2, we can also prove that condition (c) in Theorem 4.2.5 is satisfied; namely, that

$$\langle\mathcal{N}\eta,\mathcal{N}^{-1}S\eta\rangle + \langle S\eta,\eta\rangle \geq \kappa^*||\eta||^2_\Lambda \qquad \forall\ \eta\in\Lambda,$$

where $\mathcal{N}=(\sigma_1S_1^{-1}+\sigma_2S_2^{-1})^{-1}$. In fact, setting $\mu:=\mathcal{N}^{-1}S\eta$, we have

$$\begin{aligned}\langle\mathcal{N}\eta,\mathcal{N}^{-1}S\eta\rangle + \langle S\eta,\eta\rangle &= \langle\mathcal{N}\eta,\mu\rangle - \langle S\eta,\eta\rangle + 2\langle S\eta,\eta\rangle\\ &= \langle\mathcal{N}\eta,\mu\rangle - \langle\mathcal{N}\mu,\eta\rangle + 2\langle S\eta,\eta\rangle.\end{aligned}$$

Moreover, setting $\rho_i := S_i^{-1}\mathcal{N}\eta$ and $\xi_i := S_i^{-1}\mathcal{N}\mu$ for $i=1,2$, we have $\mathcal{N}\eta = S_1\rho_1 = S_2\rho_2$, $\mathcal{N}\mu = S_1\xi_1 = S_2\xi_2$,

$$\eta = \mathcal{N}^{-1}\mathcal{N}\eta = \sigma_1\rho_1+\sigma_2\rho_2$$

and similarly

$$\mu = \sigma_1\xi_1+\sigma_2\xi_2.$$

Therefore

$$\begin{aligned}\langle\mathcal{N}\eta,\mu\rangle\ &-\langle\mathcal{N}\mu,\eta\rangle\\ &= \langle\mathcal{N}\eta,\sigma_1\xi_1+\sigma_2\xi_2\rangle - \langle\mathcal{N}\mu,\sigma_1\rho_1+\sigma_2\rho_2\rangle\\ &= \sigma_1(\langle S_1\rho_1,\xi_1\rangle - \langle S_1\xi_1,\rho_1\rangle) + \sigma_2(\langle S_2\rho_2,\xi_2\rangle - \langle S_2\xi_2,\rho_2\rangle)\\ &= 2\sum_{i=1}^2\sigma_i a_i^{\rm ss}(E_i^\flat\rho_i, E_i^\flat\xi_i)\\ &= 2\sum_{i=1}^2\sigma_i a_i^{\rm ss}(E_i^\flat S_i^{-1}\mathcal{N}\eta, E_i^\flat S_i^{-1}S\eta).\end{aligned}$$

Proceeding as before, we obtain that condition (c) in Theorem 4.2.5 is satisfied, provided that for both $i = 1, 2$ the constant κ_i^{ss} is small enough.

5.1.2 Substructuring iterative methods

The Dirichlet–Neumann iterative scheme for (5.1.7) reads

$$\begin{cases} \text{find } u_1^{k+1} \in V_1 \ : \\ a_1^\flat(u_1^{k+1}, v_1) = (f, v_1)_{\Omega_1} \qquad \forall\ v_1 \in V_1^0 \\ u_1^{k+1} = \lambda^k \qquad \text{on } \Gamma \end{cases} \tag{5.1.16}$$

$$\begin{cases} \text{find } u_2^{k+1} \in V_2 \ : \\ a_2^\flat(u_2^{k+1}, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\ v_2 \in V_2^0 \\ a_2^\flat(u_2^{k+1}, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} \\ \qquad\qquad -a_1^\flat(u_1^{k+1}, \mathcal{R}_1\mu) \qquad \forall\ \mu \in \Lambda \end{cases} \tag{5.1.17}$$

$$\lambda^{k+1} = \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma. \tag{5.1.18}$$

Again, following the proof given in Section 1.3 for the Laplace operator, it can be shown that the Dirichlet–Neumann scheme (5.1.16)–(5.1.18) is equivalent to a preconditioned Richardson method for (5.1.13) where the preconditioner is provided by S_2:

$$\lambda^{k+1} = \lambda^k + S_2^{-1}(\chi - S\lambda^k).$$

The convergence in Λ of these iterations is a consequence of Theorem 4.2.2, because we have already verified that all the assumptions are satisfied. Moreover, it is easily checked that the rate of convergence depends only on the continuity and coerciveness constants of S_1 and S_2, as well as on the continuity constant of $E_2^\flat$.

A similar convergence result can be also found in Marini and Quarteroni (1989).

Following the guidelines of what was carried out in Section 1.3 for the Laplace operator, it is also easy to introduce the Neumann–Neumann iteration scheme applied to (5.1.5). It corresponds to a Richardson method for the Steklov–Poincaré equation having $(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ as a preconditioner, therefore its convergence is a straightforward consequence of Theorem 4.2.5. The rate of convergence depends on the continuity and coerciveness constants of S_1 and S_2, as well as on the continuity constants of $E_1^\flat$ and $E_2^\flat$.

5.1.3 The finite dimensional approximation

The same arguments as in the previous sections can be used for the finite element approximation of the homogeneous Dirichlet boundary value problem associated with the operator $L^\flat$, taking finite element spaces $V_{i,h} \subset H^1(\Omega_i)$ as in Section 2.1.1.

Although the problem is of the advection–diffusion type, a standard Galerkin approximation can be successfully used (without resorting to any stabilisation approach) if κ_2^{ss} is small, because in this case the problem is dominated by the diffusion.

The discrete Steklov–Poincaré operators $S_{i,h}$, which are the finite dimensional counterparts of those introduced in (5.1.9), are coercive and their coerciveness constants are independent of h. Actually, they depend only on the coerciveness constants of the bilinear forms $a_i^\flat(\cdot,\cdot)$, as well as on the constant in the trace inequality (1.2.5). Moreover, their continuity constants can be estimated in terms of those of the forms $a_i^\flat$, as well as those of the finite element extension operators $E_{i,h}^\flat$, which are defined as

$$\begin{cases} E_{i,h}^\flat \eta_h \in V_{i,h} \ : \\ a_i^\flat(E_{i,h}^\flat \eta_h, v_{i,h}) = 0 \qquad \forall\ v_{i,h} \in V_{i,h}^0 \\ E_{i,h}^\flat \eta_h = \eta_h \qquad \text{on } \Gamma. \end{cases} \tag{5.1.19}$$

Setting $v_{i,h} = E_{i,h}^\flat \eta_h - H_{i,h}\eta_h$, where $H_{i,h}$ are the finite element harmonic extension operators introduced in (2.2.1), we have

$$a_i^\flat(E_{i,h}^\flat \eta_h, E_{i,h}^\flat \eta_h) = a_i^\flat(E_{i,h}^\flat \eta_h, H_{i,h}\eta_h),$$

hence

$$||E_{i,h}^\flat \eta_h||_{1,\Omega_i} \leq C||H_{i,h}\eta_h||_{1,\Omega_i},$$

where $C > 0$ is independent of h. Assuming that the family of triangulations $\mathcal{T}_h$ of Ω is regular and the family of triangulations $\mathcal{M}_h$ of Γ is quasi-uniform, we can apply the uniform extension Theorem 4.1.3, and find that also $E_{i,h}^\flat$, $i = 1, 2$ are uniformly continuous operators.

All the assumptions of Theorem 4.2.2 are therefore satisfied, with constants independent of h. As a consequence, we can conclude that the Dirichlet–Neumann iterations (namely, the equivalent preconditioned Richardson iterations applied to the Steklov–Poincaré equation) converge, at a rate independent of h.

Applying Theorem 4.2.5, a similar result holds for the Neumann–Neumann iterative scheme.

5.2 The problem of linear elasticity

For linear materials, the stress tensor $\boldsymbol{\sigma}(\mathbf{w})$ of a vector function $\mathbf{w}$ is defined through the constitutive equations

$$\sigma_{lj}(\mathbf{w}) := \hat{\mu}(D_l w_j + D_j w_l) + \hat{\lambda}\,\mathrm{div}\,\mathbf{w}\,\delta_{lj}, \quad l,j = 1,\ldots,d,$$

where $\hat{\mu} > 0$, $\hat{\lambda} \geq 0$ are the Lamé's constants and δ_{lj} is the Kronecker's tensor. Let us define

$$(\mathbf{L}\,\mathbf{w})_l := -\sum_{j=1}^{d} D_j \sigma_{lj}(\mathbf{w}), \quad l = 1,\ldots,d,$$

and note that

$$\mathbf{L}\,\mathbf{w} = -\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{w}) = -\hat{\mu}\Delta\mathbf{w} - (\hat{\mu}+\hat{\lambda})\nabla\,\mathrm{div}\,\mathbf{w}.$$

Then we consider the following boundary value problem: seek $\mathbf{u} : \Omega \to \mathbf{R}^d$ such that

$$\text{(5.2.1)} \qquad \begin{cases} (\mathbf{L}\,\mathbf{u})_l = f_l & \text{in } \Omega, \quad l = 1,\ldots,d \\ \mathbf{u} = \boldsymbol{\varphi}_D & \text{on } \Gamma_D \\ \displaystyle\sum_{j=1}^{d} \sigma_{lj}(\mathbf{u}) n_j^* = (\boldsymbol{\varphi}_N)_l & \text{on } \Gamma_N, \quad l = 1,\ldots,d, \end{cases}$$

where $\mathbf{f}$, $\boldsymbol{\varphi}_D$ and $\boldsymbol{\varphi}_N$ are given vector functions, $\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$, $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $\mathbf{n}^*$ denotes the unit outward normal vector on $\partial\Omega$. Here $\mathbf{u}$ is the vector of the displacements of an elastic body that occupies at rest the region Ω, $\mathbf{f}$ is the vector of volume forces acting on the body, $\boldsymbol{\varphi}_D$ is the given displacement on the portion of the domain boundary Γ_D, while $\boldsymbol{\varphi}_N$ are the tractions applied on the complementary part Γ_N, and the first equations of (5.2.1) are the *equilibrium equations*.

Indicating by $\mathbf{u}_i$ the restriction of the solution $\mathbf{u}$ to the domain Ω_i, the transmission conditions on Γ are given in this case by

$$\text{(5.2.2)} \qquad \begin{aligned} &\mathbf{u}_1 = \mathbf{u}_2 && \text{on } \Gamma \\ &\sum_{j=1}^{d} \sigma_{lj}(\mathbf{u}_1) n_j = \sum_{j=1}^{d} \sigma_{lj}(\mathbf{u}_2) n_j && \text{on } \Gamma, \quad l = 1,\ldots,d, \end{aligned}$$

and they express the continuity of displacements and normal stresses on Γ.

5.2.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

We introduce the bilinear form

$$\text{(5.2.3)} \quad e(\mathbf{w},\mathbf{v}) := \frac{\hat{\mu}}{2} \int_\Omega \sum_{j,l=1}^{d} (D_l w_j + D_j w_l)\,(D_l v_j + D_j v_l) + \hat{\lambda} \int_\Omega \mathrm{div}\,\mathbf{w}\ \mathrm{div}\,\mathbf{v},$$

and set, as in (1.4.11), (1.4.12),

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \,|\, v_{|\Gamma_D} = 0\}$$
$$(\boldsymbol{\phi}, \boldsymbol{\psi})_{\Gamma_N} := \int_{\Gamma_N} \boldsymbol{\phi} \cdot \boldsymbol{\psi}.$$

Assume that $\mathbf{f} \in (L^2(\Omega))^d$, $\boldsymbol{\varphi}_D \in (H^{1/2}(\Gamma_D))^d$ and $\boldsymbol{\varphi}_N \in (L^2(\Gamma_N))^d$. Then the weak form of (5.2.1) reads

$$\begin{aligned} \text{find } \mathbf{W} \in (H^1_{\Gamma_D}(\Omega))^d \;:\; & e(\mathbf{W}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + (\boldsymbol{\varphi}_N, \mathbf{v}_{|\Gamma_N})_{\Gamma_N} \\ & -e(\widehat{\boldsymbol{\varphi}}_D, \mathbf{v}) \qquad \forall\, \mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d, \end{aligned} \tag{5.2.4}$$

where $\widehat{\boldsymbol{\varphi}}_D \in (H^1(\Omega))^d$ denotes any extension in Ω of the non-homogeneous Dirichlet datum $\boldsymbol{\varphi}_D$. The solution $\mathbf{u}$ is obtained by adding $\mathbf{W}$ to $\widehat{\boldsymbol{\varphi}}_D$.

The existence of a unique solution to (5.2.4) is a consequence of the Korn inequality

$$\int_{\Omega_i} \sum_{j,l=1}^{d} (D_l v_j + D_j v_l)^2 \geq K^* ||\mathbf{v}||^2_{1,\Omega} \qquad \forall\, \mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d. \tag{5.2.5}$$

This result is proved upon combining the inequality (see, for example, Fichera (1972), p. 382)

$$\int_{\Omega} \sum_{j,l=1}^{d} (D_l v_j + D_j v_l)^2 + \int_{\Omega} |v|^2 \geq K_* ||\mathbf{v}||^2_{1,\Omega} \qquad \forall\, \mathbf{v} \in (H^1(\Omega))^d$$

together with the generalised Poincaré inequality

$$\int_{\Omega} |v|^2 \leq C \int_{\Omega} \sum_{j,l=1}^{d} (D_l v_j + D_j v_l)^2 \qquad \forall\, \mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d,$$

which can be obtained using a standard contradiction argument.

In the case of the pure Neumann problem, i.e. $\Gamma_D = \emptyset$, compatibility conditions on the data $\mathbf{f}$ and $\boldsymbol{\varphi}_N$ are necessary to prove the existence and uniqueness of the solution. We refer the interested reader to Section 1.4 (where similar considerations for general second-order elliptic equations are presented) and to Fichera (1972).

The multi-domain weak formulation of (5.2.4) has the following form. Take, for simplicity, $\Gamma_N = \emptyset$ and $\boldsymbol{\varphi}_D = \mathbf{0}$, and for each subdomain Ω_i, $i = 1, 2$, let us introduce the bilinear form

$$e_i(\mathbf{w}_i,\mathbf{v}_i) := \frac{\hat{\mu}}{2}\int_{\Omega_i}\sum_{j,l=1}^{d}(D_l w_{i,j}+D_j w_{i,l})\,(D_l v_{i,j}+D_j v_{i,l}) + \hat{\lambda}\int_{\Omega_i}\operatorname{div}\mathbf{w}_i\,\operatorname{div}\mathbf{v}_i. \tag{5.2.6}$$

Then problem (5.2.4) can be written as: find $\mathbf{u}_1 \in (V_1)^d$, $\mathbf{u}_2 \in (V_2)^d$ such that

$$\begin{cases} e_1(\mathbf{u}_1,\mathbf{v}_1) = (\mathbf{f},\mathbf{v}_1)_{\Omega_1} \quad \forall\ \mathbf{v}_1 \in (V_1^0)^d \\ \mathbf{u}_1 = \mathbf{u}_2 \qquad \text{on } \Gamma \\ e_2(\mathbf{u}_2,\mathbf{v}_2) = (\mathbf{f},\mathbf{v}_2)_{\Omega_2} \qquad \forall\ \mathbf{v}_2 \in (V_2^0)^d \\ e_2(\mathbf{u}_2,\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) = (\mathbf{f},\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})_{\Omega_2} + (\mathbf{f},\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})_{\Omega_1} \\ \qquad\qquad -e_1(\mathbf{u}_1,\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d, \end{cases} \tag{5.2.7}$$

where $\mathbf{u}_i$ is the restriction of $\mathbf{u}$ to Ω_i, $\boldsymbol{\mathcal{R}}_i$ denotes any extension operator from $(\Lambda)^d$ to $(V_i)^d$; the spaces V_i, V_i^0 and Λ have been introduced in Section 1.2.

Problem (5.2.7) can be transformed into an equation set on the interface Γ. In fact, for $i = 1, 2$ and for each $\boldsymbol{\eta} \in (\Lambda)^d$ introduce the solution $\boldsymbol{\mathcal{E}}_i\boldsymbol{\eta}$ of the Dirichlet boundary value problem

$$\begin{cases} \boldsymbol{\mathcal{E}}_i\boldsymbol{\eta} \in (V_i)^d \ : \\ e_i(\boldsymbol{\mathcal{E}}_i\boldsymbol{\eta},\mathbf{v}_i) = 0 \qquad \forall\ \mathbf{v}_i \in (V_i^0)^d \\ \boldsymbol{\mathcal{E}}_i\boldsymbol{\eta} = \boldsymbol{\eta} \qquad \text{on } \Gamma, \end{cases} \tag{5.2.8}$$

and the solution $\mathbf{u}_i^*$ of

$$\mathbf{u}_i^* \in (V_i^0)^d \ :\ e_i(\mathbf{u}_i^*,\mathbf{v}_i) = (\mathbf{f},\mathbf{v}_i)_{\Omega_i} \qquad \forall\ \mathbf{v}_i \in (V_i^0)^d. \tag{5.2.9}$$

The Steklov–Poincaré operators are defined as

$$\langle S_i\boldsymbol{\eta},\boldsymbol{\mu}\rangle := e_i(\boldsymbol{\mathcal{E}}_i\boldsymbol{\eta},\boldsymbol{\mathcal{E}}_i\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\eta},\boldsymbol{\mu} \in (\Lambda)^d. \tag{5.2.10}$$

It is easily seen that $\mathbf{u}_{|\Omega_i} = \boldsymbol{\mathcal{E}}_i\mathbf{u}_{|\Gamma} + \mathbf{u}_i^*$; therefore, setting $\boldsymbol{\lambda} := \mathbf{u}_{|\Gamma}$, the stress continuity condition (the last equation in (5.2.7)) can be rewritten as the following interface equation on Γ:

$$S\boldsymbol{\lambda} = \boldsymbol{\chi}, \tag{5.2.11}$$

where $S := S_1 + S_2$ and $\boldsymbol{\chi} := \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2 \in (\Lambda')^d$, with

$$\langle\boldsymbol{\chi}_i,\boldsymbol{\mu}\rangle := (\mathbf{f},\boldsymbol{\mathcal{E}}_i\boldsymbol{\mu})_{\Omega_i} - e_i(\mathbf{u}_i^*,\boldsymbol{\mathcal{E}}_i\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d. \tag{5.2.12}$$

The Steklov–Poincaré operators S_1 and S_2 are symmetric, while their coerciveness is a consequence of the Korn inequality (5.2.5) stated in Ω_i and the trace inequality (1.2.5). Consequently, S is also a coercive operator. Finally, continuity of S_1 and S_2 follows from well known a priori estimates for the solution $\boldsymbol{\mathcal{E}}_i\boldsymbol{\eta}$ of (5.2.8).

5.2.2 Substructuring iterative methods

The Dirichlet–Neumann iterative procedure for solving (5.2.7) reads: given $\boldsymbol{\lambda}^0 \in (\Lambda)^d$, for each $k \geq 0$ solve

$$\begin{cases} \text{find } \mathbf{u}_1^{k+1} \in (V_1)^d \ : \\ e_1(\mathbf{u}_1^{k+1}, \mathbf{v}_1) = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1} \quad \forall\ \mathbf{v}_1 \in (V_1^0)^d \\ \mathbf{u}_1^{k+1} = \boldsymbol{\lambda}^k \qquad \text{on } \Gamma, \end{cases} \tag{5.2.13}$$

then

$$\begin{cases} \text{find } \mathbf{u}_2^{k+1} \in (V_2)^d \ : \\ e_2(\mathbf{u}_2^{k+1}, \mathbf{v}_2) = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \quad \forall\ \mathbf{v}_2 \in (V_2^0)^d \\ e_2(\mathbf{u}_2^{k+1}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) = (\mathbf{f}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})_{\Omega_2} + (\mathbf{f}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})_{\Omega_1} \\ \qquad\qquad\qquad -e_1(\mathbf{u}_1^{k+1}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d, \end{cases} \tag{5.2.14}$$

and finally update the interface value

$$\boldsymbol{\lambda}^{k+1} = \theta \mathbf{u}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k \qquad \text{on } \Gamma. \tag{5.2.15}$$

By a straightforward computation, similar to that used in Section 1.3 for the Laplace operator, we can verify that this iterative scheme on $\boldsymbol{\lambda}^k$ reduces to the preconditioned Richardson method

$$\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta S_2^{-1}(\boldsymbol{\chi} - S\boldsymbol{\lambda}^k), \quad k \geq 0. \tag{5.2.16}$$

Convergence is therefore proved by applying Theorem 4.2.2.

Similarly, it is also easy to introduce the Neumann–Neumann iteration scheme applied to (5.2.4), and to show that it corresponds to a Richardson method for the Steklov–Poincaré equation having $(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ as a preconditioner. Its convergence is thus a consequence of Theorem 4.2.5.

5.2.3 The finite dimensional approximation

The finite element approximation of the problem at hand can be based upon the usual piecewise-polynomial finite element spaces $(V_h)^d$, $(V_{i,h})^d$, $(V_{i,h}^0)^d$ and $(\Lambda_h)^d$

defined in (2.1.3), (2.1.7) and (2.1.8). If we make, for simplicity, the assumption that $\Gamma_N = \emptyset$ and $\boldsymbol{\varphi}_D = \mathbf{0}$ on $\partial\Omega$, then the finite element problem corresponding to (5.2.4) is

$$\text{find } \mathbf{u}_h \in (V_h)^d \ : \ e(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall\ \mathbf{v}_h \in (V_h)^d. \tag{5.2.17}$$

The finite dimensional multi-domain weak formulation can be obtained in a similar way from (5.2.7), by projecting the first equation on $(V_{1,h}^0)^d$, the third one on $(V_{2,h}^0)^d$ and the last one on $(\Lambda_h)^d$.

Introducing the solution $\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h$ to the Dirichlet boundary value problem

$$\begin{cases} \boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h \in (V_{i,h})^d \ : \\ \\ e_i(\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h, \mathbf{v}_{i,h}) = 0 \qquad \forall\ \mathbf{v}_{i,h} \in (V_{i,h}^0)^d \\ \\ \boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h = \boldsymbol{\eta}_h \qquad \text{on } \Gamma, \end{cases} \tag{5.2.18}$$

the discrete Steklov–Poincaré operators are defined through

$$\langle S_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h\rangle := e_i(\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\mu}_h) \qquad \forall\ \boldsymbol{\eta}_h, \boldsymbol{\mu}_h \in (\Lambda_h)^d, \quad i = 1, 2, \tag{5.2.19}$$

and the Steklov–Poincaré equation by

$$S_h\boldsymbol{\lambda}_h = \boldsymbol{\chi}_h, \tag{5.2.20}$$

where $\boldsymbol{\chi}_h$ is the finite element counterpart of $\boldsymbol{\chi}$ defined in (5.2.12) and $S_h := S_{1,h} + S_{2,h}$. The solution $\boldsymbol{\lambda}_h$ of (5.2.20) is given by $\mathbf{u}_{h|\Gamma}$, where $\mathbf{u}_h$ is the solution to (5.2.17).

We have already pointed out that the bilinear forms $e_i(\cdot,\cdot)$ associated with the operator $\mathbf{L}$ are symmetric, continuous, and coercive in $(V_i)^d$, therefore the Steklov–Poincaré operators are symmetric, and, as a consequence of the trace inequality (1.2.5), coercive, uniformly with respect to h.

Their uniform continuity can be proved as follows. Taking in (5.2.18) $\mathbf{v}_{i,h} = \boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h - \mathbf{H}_{i,h}\boldsymbol{\eta}_h$, where $\boldsymbol{\eta}_h = (\eta_{h,1}, \cdots, \eta_{h,d})$,

$$\mathbf{H}_{i,h}\boldsymbol{\eta}_h := (H_{i,h}\eta_{h,1}, \cdots, H_{i,h}\eta_{h,d}) \tag{5.2.21}$$

and $H_{i,h}\eta_h$ is the solution to (2.2.1), we have

$$e_i(\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h) = e_i(\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h, \mathbf{H}_{i,h}\boldsymbol{\eta}_h).$$

Hence, if the family of triangulations $\mathcal{T}_h$ of Ω is regular and the family of triangulations $\mathcal{M}_h$, induced on Γ by $\mathcal{T}_h$, is quasi-uniform, we can use the uniform extension Theorem 4.1.3 and find

$$||\boldsymbol{\mathcal{E}}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i} \le K||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i} \le KC_2||\boldsymbol{\eta}_h||_{\Lambda}$$

where $K > 0$ is independent of h. Using this inequality in (5.2.19) shows that the Steklov–Poincaré operators are uniformly continuous.

If we now consider the preconditioned Richardson method

$$\boldsymbol{\lambda}_h^{k+1} = \boldsymbol{\lambda}_h^k + \theta S_{2,h}^{-1}(\boldsymbol{\chi}_h - S_h \boldsymbol{\lambda}_h^k), \tag{5.2.22}$$

by the usual arguments we can show that it is equivalent to the Dirichlet–Neumann iteration-by-subdomain procedure:

$$\begin{cases} \text{find } \mathbf{u}_{1,h}^{k+1} \in (V_{1,h})^d \ : \\ e_1(\mathbf{u}_{1,h}^{k+1}, \mathbf{v}_{1,h}) = (\mathbf{f}, \mathbf{v}_{1,h})_{\Omega_1} \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ \mathbf{u}_{1,h}^{k+1} = \boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma \end{cases} \tag{5.2.23}$$

$$\begin{cases} \text{find } \mathbf{u}_{2,h}^{k+1} \in (V_2)^d \ : \\ e_2(\mathbf{u}_{2,h}^{k+1}, \mathbf{v}_{2,h}) = (\mathbf{f}, \mathbf{v}_{2,h})_{\Omega_2} \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\ e_2(\mathbf{u}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) = (\mathbf{f}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)_{\Omega_2} + (\mathbf{f}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)_{\Omega_1} \\ \qquad\qquad -e_1(\mathbf{u}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) \qquad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d, \end{cases} \tag{5.2.24}$$

with

$$\boldsymbol{\lambda}_h^{k+1} = \theta \mathbf{u}_{2,h|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma. \tag{5.2.25}$$

As usual, $\boldsymbol{\mathcal{R}}_{i,h}$ denotes any extension operator from $(\Lambda_h)^d$ to $(V_{i,h})^d$, $i = 1, 2$.

The convergence of the Dirichlet–Neumann algorithm can be proved by applying Theorem 4.2.2, and we also conclude that the rate of convergence is independent of the grid size h.

A similar result holds for the Neumann–Neumann iterative scheme, making use of Theorem 4.2.5.

5.3 The Stokes problem

Another example we consider is a classical problem in the theory of incompressible fluid dynamics. It involves the linear Stokes operator, and in this case the stress tensor $\mathbf{T}(\mathbf{w}, q)$ is given by

$$T_{lj}(\mathbf{w}, q) := \nu(D_l w_j + D_j w_l) - q\,\delta_{lj}, \quad l, j = 1, \ldots, d, \tag{5.3.1}$$

where $\nu > 0$ is a given constant. The Stokes problem reads: find $\mathbf{u} : \Omega \subset \mathbf{R}^d \to \mathbf{R}^d$, $d = 2, 3$, and $p : \Omega \to \mathbf{R}$ such that

FIG. 5.3.1. Boundary decomposition into Γ_N and Γ_D.

$$\text{(5.3.2)} \qquad \begin{cases} -\mathbf{divT}(\mathbf{u},p) = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \boldsymbol{\varphi}_D & \text{on } \Gamma_D \\ \mathbf{T}(\mathbf{u},p)\cdot\mathbf{n}^* = \boldsymbol{\varphi}_N & \text{on } \Gamma_N, \end{cases}$$

where $\mathbf{f}$, $\boldsymbol{\varphi}_D$ and $\boldsymbol{\varphi}_N$ are given vector-valued functions. As usual, $\mathbf{n}^*$ denotes the unit outward normal vector on $\partial\Omega$, $\overline{\Gamma_D}\cup\overline{\Gamma_N} = \partial\Omega$, $\Gamma_D\cap\Gamma_N = \emptyset$, and either Γ_D or Γ_N could be empty (see Fig. 5.3.1).

The unknowns $\mathbf{u}$ and p denote, respectively, the velocity and the pressure of a viscous incompressible fluid that is confined to the region Ω, and is subjected to a body force $\mathbf{f}$, to a surface normal stress $\boldsymbol{\varphi}_N$ on Γ_N, and has a prescribed velocity $\boldsymbol{\varphi}_D$ on Γ_D.

Note that in the momentum equation $-\mathbf{divT}(\mathbf{u},p) = \mathbf{f}$ the pressure p occurs only through its gradient. Therefore, when $\Gamma_N = \emptyset$ (namely, when we are imposing the Dirichlet boundary condition for the velocity on the whole boundary $\partial\Omega$) the pressure is defined up to an additive constant. To recover uniqueness, in this case it is usually assumed that the additional condition $\int_\Omega p = 0$ holds. This request is not necessary when $\Gamma_N \neq \emptyset$, because the pressure p is present in the boundary condition on Γ_N. This different behaviour motivates the development of two alternative formulations, one for the case where $\Gamma_N = \emptyset$, the other for the case where $\Gamma_N \neq \emptyset$.

A more explicit form of the boundary condition on Γ_N reads

$$\nu\sum_{j=1}^{d}(D_l u_j + D_j u_l)n_j^* - pn_l^* = (\boldsymbol{\varphi}_N)_l, \quad l = 1,\dots,d.$$

As is well known, other boundary conditions are admissible for the Stokes

problem (see, for example, Pironneau 1988, pp. 123–7; Quarteroni and Valli 1994, Section 10.1.1), which, however, will not be considered here.

The incompressibility equation $\operatorname{div}\mathbf{u} = 0$ yields the more familiar identity

$$-\mathbf{div}\mathbf{T}(\mathbf{u}, p) = -\nu\Delta\mathbf{u} + \nabla p,$$

which gives rise to the most common form of the conservation of momentum for an incompressible viscous flow.

The interface conditions for the Stokes problem are given by the continuity of the velocity field and that of the normal stress vector $\mathbf{T}(\mathbf{u}, p)\cdot\mathbf{n}$; precisely, denoting by $\mathbf{u}_i = \mathbf{u}_{|\Omega_i}$ and $p_i = p_{|\Omega_i}$, $i = 1, 2$, the interface conditions take the form

$$\begin{aligned} &\mathbf{u}_1 = \mathbf{u}_2 \qquad \text{on } \Gamma \\ &\nu\sum_{j=1}^{d}(D_l u_{1,j} + D_j u_{1,l})n_j - p_1 n_l \\ &\qquad = \nu\sum_{j=1}^{d}(D_l u_{2,j} + D_j u_{2,l})n_j - p_2 n_l \qquad \text{on } \Gamma, \quad l = 1, \dots, d. \end{aligned} \tag{5.3.3}$$

We do not, therefore, have to fulfil the continuity of the pressure field.

Let us note that in the two-dimensional case, if the interface Γ is parallel to a coordinate axis, say the x_2-axis, the interface conditions on the normal stress split into

$$\begin{aligned} &2\nu\frac{\partial u_{1,n}}{\partial n} - p_1 = 2\nu\frac{\partial u_{2,n}}{\partial n} - p_2 \qquad \text{on } \Gamma \\ &\nu\frac{\partial u_{1,\tau}}{\partial n} + \nu\frac{\partial u_{1,n}}{\partial \tau} = \nu\frac{\partial u_{2,\tau}}{\partial n} + \nu\frac{\partial u_{2,n}}{\partial \tau} \qquad \text{on } \Gamma. \end{aligned}$$

Note, moreover, that, by virtue of the condition $\mathbf{u}_1 = \mathbf{u}_2$ on Γ, the latter condition can be reduced to

$$\nu\frac{\partial u_{1,\tau}}{\partial n} = \nu\frac{\partial u_{2,\tau}}{\partial n} \qquad \text{on } \Gamma.$$

A generalisation of the Stokes problem is the so-called *Oseen problem*

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \boldsymbol{\varphi}_D & \text{on } \partial\Omega\ , \end{cases}$$

where $\mathbf{b}$ is a given vector field (which could be the velocity field itself at a preceding time step, when considering a time-advancing scheme for the discretisation of the non-stationary, non-linear Navier–Stokes equations). The problem is non-symmetric, due to the presence of the first-order term $(\mathbf{b}\cdot\nabla)\mathbf{u}$. If $\mathbf{b}$ is continuous

across Γ, then the interface conditions for the Oseen problem are the same as for the Stokes problem. However, due to the lack of symmetry, the convergence of substructuring iterative methods can be studied by means of the same technique that we are going to present in Chapter 6 for scalar advection–diffusion equations.

Remark 5.3.1 The incompressibility constraint $\operatorname{div}\mathbf{u} = 0$ has several consequences, concerning both the existence theory and the numerical approximation of the solution. In particular, it induces compatibility conditions on the data. In fact, as a consequence of the divergence theorem we have

$$\int_{\partial\mathcal{O}} \mathbf{u}_{|\partial\mathcal{O}} \cdot \mathbf{n} = \int_{\mathcal{O}} \operatorname{div}\mathbf{u} = 0,$$

for each set $\mathcal{O} \subset \Omega$, having denoted by $\mathbf{n}$ the unit outward normal vector on $\partial\mathcal{O}$. Therefore, in the case where $\Gamma_N = \emptyset$ (namely, for the pure Dirichlet boundary value problem), the boundary datum has to satisfy the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{\varphi}_D \cdot \mathbf{n}^* = 0. \tag{5.3.4}$$

Moreover, if Γ_N intersects only one of the two subdomains Ω_1 or Ω_2, say $\Gamma_N \subset \partial\Omega_2$, a compatibility condition on $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ has to be satisfied; namely

$$\begin{aligned}\int_{\Gamma} \mathbf{u}_{|\Gamma} \cdot \mathbf{n} &= \int_{\Gamma} \mathbf{u}_{|\Gamma} \cdot \mathbf{n} + \int_{\partial\Omega_1\setminus\Gamma} \mathbf{u}_{|\partial\Omega_1\setminus\Gamma} \cdot \mathbf{n}^* - \int_{\partial\Omega_1\setminus\Gamma} \boldsymbol{\varphi}_D \cdot \mathbf{n}^* \\ &= \int_{\Omega_1} \operatorname{div}\mathbf{u} - \int_{\partial\Omega_1\setminus\Gamma} \boldsymbol{\varphi}_D \cdot \mathbf{n}^* = -\int_{\partial\Omega_1\setminus\Gamma} \boldsymbol{\varphi}_D \cdot \mathbf{n}^*,\end{aligned}$$

having chosen on Γ the unit vector $\mathbf{n}$ directed towards Ω_2. This condition cannot be avoided if $\Gamma_N = \emptyset$; on the contrary, when $\Gamma_N \neq \emptyset$ the decomposition of Ω will be chosen in such a way that this compatibility condition on Γ is not necessary (see Remark 5.3.6). □

5.3.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

We warn the reader that our analysis will present some technical subtleties, due to the special nature of the problem, particularly:

- The problem data should obey special compatibility conditions (see Remarks 5.3.1 and 5.3.3).
- The functional spaces for admissible velocities and pressures should satisfy a suitable relation (see (5.3.7) and (5.3.8)), which in the finite dimensional case will be expressed by the well known inf–sup condition (see (5.3.43) and (5.3.45)).
- The issue of the uniqueness of the pressure, which depends upon the type of boundary conditions, is reflected by a special choice of test functions for

the continuity equation, for both the one-domain and the multi-domain formulation of the Stokes problem.

In order to derive the variational formulation of (5.3.2), and discuss its well-posedness, let us assume that $\mathbf{f} \in (L^2(\Omega))^d$, $\boldsymbol{\varphi}_D \in (H^{1/2}(\Gamma_D))^d$ and $\boldsymbol{\varphi}_N \in (L^2(\Gamma_N))^d$. Then introduce the bilinear forms

$$s(\mathbf{w}, \mathbf{v}) := \int_\Omega \frac{\nu}{2} \sum_{l,m=1}^{d} (D_l w_m + D_m w_l)(D_l v_m + D_m v_l) \tag{5.3.5}$$

and

$$b(\mathbf{w}, q) := -\int_\Omega q \ \mathrm{div}\, \mathbf{w}, \tag{5.3.6}$$

for all $\mathbf{w}, \mathbf{v} \in (H^1(\Omega))^d$ and $q \in L^2(\Omega)$.

First of all, we remark that there exists a constant $\beta^0 > 0$ (which depends on Ω) such that

$$\forall\ q \in L^2_0(\Omega)\ \exists\ \mathbf{v} \in (H^1_0(\Omega))^d,\ \mathbf{v} \neq \mathbf{0}\ :\ b(\mathbf{v}, q) \geq \beta^0 ||\mathbf{v}||_{1,\Omega} ||q||_{0,\Omega}, \tag{5.3.7}$$

where the space $L^2_0(\Omega)$ has been introduced in (1.4.17). Indeed, this inequality is easily proved, taking

$$\mathbf{v} \in (H^1_0(\Omega))^d\ :\ \mathrm{div}\, \mathbf{v} = -q \qquad \text{in } \Omega,$$

which satisfies $||\mathbf{v}||_{1,\Omega} \leq C||q||_{0,\Omega}$ (see Girault and Raviart 1986, p. 24).

Similarly, we have

Proposition 5.3.2 *Assume that $\Gamma_D \neq \partial\Omega$, or, equivalently, $\Gamma_N \neq \emptyset$. There exists a constant $\beta^* > 0$ such that*

$$\forall\ q \in L^2(\Omega)\ \exists\ \mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d,\ \mathbf{v} \neq \mathbf{0}\ :\ b(\mathbf{v}, q) \geq \beta^* ||\mathbf{v}||_{1,\Omega} ||q||_{0,\Omega}, \tag{5.3.8}$$

where $H^1_{\Gamma_D}(\Omega)$ is the space introduced in (1.4.11).

Proof This inequality can be proved by means of (5.3.7). In fact, given a function $q \in L^2(\Omega)$, $q \neq 0$, denote by $\widetilde{q}$ its extension by zero in a Lipschitz domain $\Omega' \supset \Omega$, such that $\Gamma_D \subset \partial\Omega'$ (see Fig. 5.3.2).

Define, moreover, $\widetilde{q}_{\Omega'} := (\mathrm{meas}\ \Omega')^{-1} \int_{\Omega'} \widetilde{q}$. In correspondence to the function $\widetilde{q} - \widetilde{q}_{\Omega'} \in L^2_0(\Omega')$ there exists $\mathbf{v}' \in (H^1_0(\Omega'))^d$, $\mathbf{v}' \neq \mathbf{0}$, such that

$$-\int_{\Omega'} (\widetilde{q} - \widetilde{q}_{\Omega'})\ \mathrm{div}\, \mathbf{v}' \geq \beta^0 ||\mathbf{v}'||_{1,\Omega'} ||\widetilde{q} - \widetilde{q}_{\Omega'}||_{0,\Omega'}.$$

Setting $\mathbf{v} := \mathbf{v}'_{|\Omega} \in (H^1_{\Gamma_D}(\Omega))^d$, and taking into account that $\int_{\Omega'} \mathrm{div}\, \mathbf{v}' = 0$ (as $\mathbf{v}'_{|\partial\Omega'} = \mathbf{0}$), we have

FIG. 5.3.2. The extended domain Ω'.

$$b(\mathbf{v},q) = -\int_{\Omega'} \operatorname{div}\mathbf{v}'\,\widetilde{q} = -\int_{\Omega'} \operatorname{div}\mathbf{v}'\,(\widetilde{q}-\widetilde{q}_{\Omega'})$$
$$\geq \beta^0||\mathbf{v}'||_{1,\Omega'}||\widetilde{q}-\widetilde{q}_{\Omega'}||_{0,\Omega'} \geq \beta^0||\mathbf{v}||_{1,\Omega}||q-\widetilde{q}_{\Omega'}||_{0,\Omega}.$$

In particular, it follows that $\mathbf{v} \neq \mathbf{0}$. Moreover,

$$||q||_{0,\Omega} \leq ||q-\widetilde{q}_{\Omega'}||_{0,\Omega} + ||\widetilde{q}_{\Omega'}||_{0,\Omega},$$

and

$$||\widetilde{q}_{\Omega'}||^2_{0,\Omega} = \frac{\text{meas }\Omega}{(\text{meas }\Omega')^2}\left(\int_{\Omega'}\widetilde{q}\right)^2 = \frac{\text{meas }\Omega}{(\text{meas }\Omega')^2}\left(\int_{\Omega} q\right)^2$$
$$\leq \frac{(\text{meas }\Omega)^2}{(\text{meas }\Omega')^2}||q||^2_{0,\Omega}.$$

Therefore

$$\left(1-\frac{\text{meas }\Omega}{\text{meas }\Omega'}\right)||q||_{0,\Omega} \leq ||q-\widetilde{q}_{\Omega'}||_{0,\Omega},$$

and taking

$$\beta^* := \left(1-\frac{\text{meas }\Omega}{\text{meas }\Omega'}\right)\beta^0$$

inequality (5.3.8) follows. □

The inequalities (5.3.7) and (5.3.8) yield the well-posedness of the Stokes problem in the case where $\Gamma_N = \emptyset$ (condition (5.3.7)) or where $\Gamma_N \neq \emptyset$ (condition (5.3.8)); see also Remark 5.3.4. Their finite element counterpart is known as the *inf–sup condition*, and is crucial for the analysis of the finite element approximation of the Stokes problem.

Since $H^{1/2}(\Gamma_D)$ is the trace space on Γ_D of $H^1(\Omega)$, we can construct a vector function $\boldsymbol{\Phi} \in (H^1(\Omega))^d$ such that $\boldsymbol{\Phi}_{|\Gamma_D} = \boldsymbol{\varphi}_D$. We also construct a vector function $\boldsymbol{\Psi} \in (H^1_{\Gamma_D}(\Omega))^d$ such that

$$\operatorname{div}\boldsymbol{\Psi} = -\operatorname{div}\boldsymbol{\Phi} \qquad \text{in } \Omega. \tag{5.3.9}$$

The existence of $\boldsymbol{\Psi}$ is guaranteed by the inf–sup condition (5.3.7) or (5.3.8) (see, for example, Brezzi 1974; and Brezzi and Fortin 1991). Note that, in the case where $\Gamma_N = \emptyset$, from (5.3.4) the necessary solvability condition

$$\int_\Omega \operatorname{div} \boldsymbol{\Phi} = 0$$

is satisfied.

The case where $\Gamma_N \neq \emptyset$

We start by presenting the variational formulation of the Stokes problem (5.3.2) in the case where $\Gamma_N \neq \emptyset$. It reads

$$\begin{cases} \text{find } \mathbf{W} \in (H^1_{\Gamma_D}(\Omega))^d,\ p \in L^2(\Omega)\ : \\ s(\mathbf{W}, \mathbf{v}) + b(\mathbf{v}, p) = \mathcal{F}(\mathbf{v}) \qquad \forall\ \mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d \\ b(\mathbf{W}, q) = 0 \qquad \forall\ q \in L^2(\Omega), \end{cases} \tag{5.3.10}$$

where $\mathbf{W} = \mathbf{u} - \boldsymbol{\Phi} - \boldsymbol{\Psi}$ and

$$\mathcal{F}(\mathbf{v}) := (\mathbf{f}, \mathbf{v}) + (\boldsymbol{\varphi}_N, \mathbf{v}_{|\Gamma_N})_{\Gamma_N} - s(\boldsymbol{\Phi} + \boldsymbol{\Psi}, \mathbf{v}).$$

The second equation in (5.3.10) clearly implies $\operatorname{div} \mathbf{W} = 0$ in Ω, because we can take $q = \operatorname{div} \mathbf{W}$ as a test function.

Remark 5.3.3 In the case of the pure Neumann boundary value problem, i.e. when $\Gamma_N = \partial\Omega$ or, equivalently, $\Gamma_D = \emptyset$, a different kind of compatibility condition between $\mathbf{f}$ and $\boldsymbol{\varphi}_N$ arises; namely,

$$\int_\Omega \mathbf{f} + \int_{\partial\Omega} \boldsymbol{\varphi}_N = 0. \tag{5.3.11}$$

We will not deal with this last case in what follows, and will always assume that $\Gamma_D \neq \emptyset$; however, we refer to Section 1.4, where the interested reader can find how the pure Neumann boundary value problem associated with the general second-order symmetric elliptic operator is analysed. □

Remark 5.3.4 The existence and uniqueness of the solution to problem (5.3.10) is a consequence of well known abstract theorems; in particular, on the coerciveness of $s(\cdot,\cdot)$ in $(H^1_{\Gamma_D}(\Omega))^d$ (which follows from Korn inequality (5.2.5)) and the validity of the inf–sup condition (5.3.8). □

The variational formulation of problem (5.3.2) can be equivalently rewritten in a two-domain fashion. First of all, for all $\mathbf{w}_i, \mathbf{v}_i \in (H^1(\Omega_i))^d$ and $q_i \in L^2(\Omega_i)$ let us define

$$s_i(\mathbf{w}_i, \mathbf{v}_i) := \int_{\Omega_i} \frac{\nu}{2} \sum_{l,m=1}^{d} (D_l w_{i,m} + D_m w_{i,l})(D_l v_{i,m} + D_m v_{i,l}) \tag{5.3.12}$$

and

$$b_i(\mathbf{w}_i, q_i) := -\int_{\Omega_i} q_i \ \mathrm{div}\, \mathbf{w}_i. \tag{5.3.13}$$

The two-domain formulation of (5.3.10) reads: find $(\mathbf{W}_1, p_1) \in (V_1)^d \times L^2(\Omega_1)$, $(\mathbf{W}_2, p_2) \in (V_2)^d \times L^2(\Omega_2)$ such that

$$\begin{cases} s_1(\mathbf{W}_1, \mathbf{v}_1) + b_1(\mathbf{v}_1, p_1) = \mathcal{F}_1(\mathbf{v}_1) \qquad \forall\ \mathbf{v}_1 \in (V_1^0)^d \\ b_1(\mathbf{W}_1, q_1) = 0 \qquad \forall\ q_1 \in L^2(\Omega_1) \\ \mathbf{W}_1 = \mathbf{W}_2 \qquad \text{on } \Gamma \\ s_2(\mathbf{W}_2, \mathbf{v}_2) + b_2(\mathbf{v}_2, p_2) = \mathcal{F}_2(\mathbf{v}_2) \qquad \forall\ \mathbf{v}_2 \in (V_2^0)^d \\ b_2(\mathbf{W}_2, q_2) = 0 \qquad \forall\ q_2 \in L^2(\Omega_2) \\ s_2(\mathbf{W}_2, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + b_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}, p_2) = \mathcal{F}_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + \mathcal{F}_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) \\ \qquad\qquad -s_1(\mathbf{W}_1, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) - b_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}, p_1) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d, \end{cases} \tag{5.3.14}$$

where

$$\begin{aligned} &\mathcal{F}_i(\mathbf{v}_i) := (\mathbf{f}, \mathbf{v}_i)_{\Omega_i} + (\boldsymbol{\varphi}_N, \mathbf{v}_{i|\Gamma_N})_{\Gamma_N \cap \partial\Omega_i} - s_i(\boldsymbol{\Phi}_{|\Omega_i} + \boldsymbol{\Psi}_{|\Omega_i}, \mathbf{v}_i) \\ &V_i := \{v \in H^1(\Omega_i) \,|\, v_{|\Gamma_D \cap \partial\Omega_i} = 0\},\ V_i^0 := \{v \in H^1(\Omega_i) \,|\, v_{|\partial\Omega_i \setminus \Gamma_N} = 0\} \\ &\Lambda := \{\eta \in H^{1/2}(\Gamma) \,|\, \eta = v_{|\Gamma} \text{ for a suitable } v \in H^1_{\Gamma_D}(\Omega)\}, \end{aligned}$$

and $\boldsymbol{\mathcal{R}}_i$ denotes any extension operator from $(\Lambda)^d$ to $(V_i)^d$, $i = 1, 2$.

For the sake of completeness, let us show the equivalence between (5.3.10) and (5.3.14).

Lemma 5.3.5 *The Stokes problem* (5.3.10) *is equivalent to* (5.3.14)*, in the sense that* $\mathbf{W}_{|\Omega_i} = \mathbf{W}_i$ *and* $p_{|\Omega_i} = p_i$*,* $i = 1, 2$.

Proof Let us start by proving that a solution $(\mathbf{W}, p)$ to (5.3.10) yields a solution to (5.3.14). Let us set $\mathbf{W}_i := \mathbf{W}_{|\Omega_i}$, $p_i := p_{|\Omega_i}$. We have that $\mathbf{W}_i \in V_i$, $p_i \in L^2(\Omega_i)$ and all the equations in (5.3.14), except the last one, are trivially satisfied simply by extending by zero in the complementary domain any local test function. To verify the last equation, for each $\boldsymbol{\mu} \in (\Lambda)^d$ let us define the vector function

$$\boldsymbol{\mathcal{R}}\boldsymbol{\mu} := \begin{cases} \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu} & \text{in } \Omega_1 \\ \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu} & \text{in } \Omega_2 \end{cases}.$$

Then the last equation in (5.3.14) can be equivalently rewritten as

$$s(\mathbf{W}, \boldsymbol{\mathcal{R}}\boldsymbol{\mu}) + b(\boldsymbol{\mathcal{R}}\boldsymbol{\mu}, p) = \mathcal{F}(\boldsymbol{\mathcal{R}}\boldsymbol{\mu}).$$

Since $\boldsymbol{\mathcal{R}\mu} \in (H^1_{\Gamma_D}(\Omega))^d$, this equation is a special case of (5.3.10) when we take $\mathbf{v} = \boldsymbol{\mathcal{R}\mu}$.

Conversely, let us show that, if $(\mathbf{W}_i, p_i)$ is a solution to (5.3.14) and we set

$$\mathbf{W} := \begin{cases} \mathbf{W}_1 & \text{in } \Omega_1 \\ \mathbf{W}_2 & \text{in } \Omega_2 \end{cases}, \quad p := \begin{cases} p_1 & \text{in } \Omega_1 \\ p_2 & \text{in } \Omega_2 \end{cases},$$

then $(\mathbf{W}, p)$ is a solution of (5.3.10). Clearly, $\mathbf{W} \in (H^1_{\Gamma_D}(\Omega))^d$ and $p \in L^2(\Omega)$. Then, for every $\mathbf{v} \in (H^1_{\Gamma_D}(\Omega))^d$ we set $\boldsymbol{\mu} := \mathbf{v}_{|\Gamma} \in (\Lambda)^d$. For $i = 1, 2$ we have $(\mathbf{v}_{|\Omega_i} - \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) \in (V_i^0)^d$ and thus from (5.3.14) we obtain

$$\begin{aligned} s(\mathbf{W}, \mathbf{v}) + b(\mathbf{v}, p) &= \sum_{i=1}^{2} [s_i(\mathbf{W}_i, \mathbf{v}_{|\Omega_i} - \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + s_i(\mathbf{W}_i, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})] \\ &\quad + \sum_{i=1}^{2} [b_i(\mathbf{v}_{|\Omega_i} - \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, p_i) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, p_i)] \\ &= \sum_{i=1}^{2} [\mathcal{F}_i(\mathbf{v}_{|\Omega_i} - \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + \mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})] = \mathcal{F}(\mathbf{v}). \end{aligned}$$

The set of momentum equations of (5.3.10) is therefore satisfied. Now we consider the continuity equation. For any $q \in L^2(\Omega)$, both restrictions $q_{|\Omega_i}$ belong to $L^2(\Omega_i)$ and can be used as test functions in (5.3.14), therefore

$$b(\mathbf{W}, q) = \sum_{i=1}^{2} b_i(\mathbf{W}_i, q_{|\Omega_i}) = 0,$$

yielding the desired continuity equation. □

Remark 5.3.6 If we split the domain Ω in such a way that either $\partial\Omega_1 \cap \Gamma_N = \emptyset$ or $\partial\Omega_2 \cap \Gamma_N = \emptyset$ (clearly, this is always the case if $\Gamma_N = \emptyset$; namely, for the pure Dirichlet boundary condition that we will consider later on), due to the divergence theorem the normal component of the solution $\mathbf{W}$ has to satisfy a compatibility condition on Γ; namely,

$$\int_\Gamma \mathbf{W}_{|\Gamma} \cdot \mathbf{n} = 0. \tag{5.3.15}$$

This constraint for the space of traces on Γ introduces some technical problems, which, when it is possible, we prefer to avoid. Therefore, in the case where $\Gamma_N \neq \emptyset$ we choose Γ in such a way that $\partial\Omega_i \cap \Gamma_N \neq \emptyset$ for each $i = 1, 2$ (see Fig. 5.3.3), and consequently there is no need for (5.3.15) to be satisfied. □

We transform now the multi-domain problem (5.3.14) into an equivalent problem set on the interface Γ, for the associated Steklov–Poincaré operator. For each

FIG. 5.3.3. Two-domain decomposition.

$\boldsymbol{\eta} \in (\Lambda)^d$ consider now the solution $(\mathbf{U}_i\boldsymbol{\eta}, P_i\boldsymbol{\eta}) \in (V_i)^d \times L^2(\Omega_i)$, $i = 1, 2$, of the problem

$$\begin{cases} s_i(\mathbf{U}_i\boldsymbol{\eta}, \mathbf{v}_i) + b_i(\mathbf{v}_i, P_i\boldsymbol{\eta}) = 0 \quad \forall\ \mathbf{v}_i \in (V_i^0)^d \\ \\ b_i(\mathbf{U}_i\boldsymbol{\eta}, q_i) = 0 \quad \forall\ q_i \in L^2(\Omega_i) \\ \\ \mathbf{U}_i\boldsymbol{\eta} = \boldsymbol{\eta} \qquad \text{on } \Gamma, \end{cases} \tag{5.3.16}$$

which is a Stokes problem in the domain Ω_i with a non-homogeneous Dirichlet boundary condition on Γ. The couple $(\mathbf{U}_i\boldsymbol{\eta}, P_i\boldsymbol{\eta})$ is called the *Stokes extension* in Ω_i of the interface value $\boldsymbol{\eta}$ on Γ.

Introduce also the solution $(\mathbf{U}_i^*, P_i^*) \in (V_i^0)^d \times L^2(\Omega_i)$ of the problem

$$\begin{cases} s_i(\mathbf{U}_i^*, \mathbf{v}_i) + b_i(\mathbf{v}_i, P_i^*) = \mathcal{F}_i(\mathbf{v}_i) \quad \forall\ \mathbf{v}_i \in (V_i^0)^d \\ \\ b_i(\mathbf{U}_i^*, q_i) = 0 \quad \forall\ q_i \in L^2(\Omega_i), \end{cases} \tag{5.3.17}$$

for $i = 1, 2$, which is a Stokes problem in Ω_i with homogeneous Dirichlet boundary data on Γ.

The local Steklov–Poincaré operators $S_i : (\Lambda)^d \to (\Lambda')^d$ are defined as:

$$\begin{aligned} \langle S_i\boldsymbol{\eta}, \boldsymbol{\mu}\rangle : = & \ s_i(\mathbf{U}_i\boldsymbol{\eta}, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) \\ & + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i\boldsymbol{\eta}) \qquad \forall\ \boldsymbol{\eta}, \boldsymbol{\mu} \in (\Lambda)^d \end{aligned} \tag{5.3.18}$$

and the continuous linear functionals $\boldsymbol{\chi}_i \in (\Lambda')^d$ as

$$\begin{aligned} \langle \boldsymbol{\chi}_i, \boldsymbol{\mu}\rangle : = & \ \mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) - s_i(\mathbf{U}_i^*, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) \\ & - b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i^*) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d. \end{aligned} \tag{5.3.19}$$

Introducing the Steklov–Poincaré operator $S := S_1 + S_2$ and setting $\boldsymbol{\chi} := \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2$, we are in a position to prove the following equivalence result:

Lemma 5.3.7 *If we set $\boldsymbol{\lambda} := \mathbf{W}_{|\Gamma}$, where $(\mathbf{W}, p)$ is the solution to* (5.3.10)*, then $\boldsymbol{\lambda}$ satisfies the following Steklov–Poincaré equation on Γ:*

$$\boldsymbol{\lambda} \in (\Lambda)^d \ : \ \langle S\boldsymbol{\lambda}, \boldsymbol{\mu}\rangle = \langle \boldsymbol{\chi}, \boldsymbol{\mu}\rangle \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d. \tag{5.3.20}$$

Conversely, from the solution $\boldsymbol{\lambda}$ of (5.3.20) *we obtain the solution of* (5.3.10) *by setting*

$$\begin{aligned} \mathbf{W}_{|\Omega_i} &:= \mathbf{U}_i\boldsymbol{\lambda} + \mathbf{U}_i^* \\ p_{|\Omega_i} &:= P_i\boldsymbol{\lambda} + P_i^*. \end{aligned} \tag{5.3.21}$$

Proof Let $(\mathbf{W}, p)$ be the solution to (5.3.10). By Lemma 5.3.5 $\mathbf{W}_{|\Omega_i}$ and $p_{|\Omega_i}$, $i = 1, 2$, are the solutions to the two-domain problem (5.3.14), hence

$$\begin{aligned} \mathbf{W}_{|\Omega_i} &= \mathbf{U}_i\mathbf{W}_{|\Gamma} + \mathbf{U}_i^* \\ p_{|\Omega_i} &= P_i\mathbf{W}_{|\Gamma} + P_i^*. \end{aligned}$$

Therefore, the stress interface condition $(5.3.14)_6$ can be rewritten as

$$\sum_{i=1}^{2}[s_i(\mathbf{U}_i\mathbf{W}_{|\Gamma} + \mathbf{U}_i^*, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i\mathbf{W}_{|\Gamma} + P_i^*)] = \sum_{i=1}^{2}\mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d,$$

which coincides with (5.3.20) if $\boldsymbol{\lambda} = \mathbf{W}_{|\Gamma}$.

Conversely, taking the solution $\boldsymbol{\lambda}$ to (5.3.20) and setting for $i = 1, 2$

$$\begin{aligned} \mathbf{W}_i &:= \mathbf{U}_i\boldsymbol{\lambda} + \mathbf{U}_i^* \\ p_i &:= P_i\boldsymbol{\lambda} + P_i^*, \end{aligned}$$

the same argument shows that $(\mathbf{W}_i, p_i)$ are the solutions of (5.3.14). □

If we take the extension operator $\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu} = \mathbf{U}_i\boldsymbol{\mu}$ introduced in (5.3.16), then from (5.3.18) and (5.3.19) we obtain

$$\langle S_i\boldsymbol{\eta}, \boldsymbol{\mu}\rangle = s_i(\mathbf{U}_i\boldsymbol{\eta}, \mathbf{U}_i\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\eta}, \boldsymbol{\mu} \in (\Lambda)^d \tag{5.3.22}$$

and

$$\langle \boldsymbol{\chi}_i, \boldsymbol{\mu}\rangle = \mathcal{F}_i(\mathbf{U}_i\boldsymbol{\mu}) - s_i(\mathbf{U}_i^*, \mathbf{U}_i\boldsymbol{\mu}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d. \tag{5.3.23}$$

Therefore, both operators S_i are symmetric in $(\Lambda)^d$, and S is symmetric too. Moreover, using the Korn inequality (5.2.5) in Ω_i and the trace inequality (1.2.5),

FIG. 5.3.4. Possible decompositions: both S_1 and S_2 are coercive (left), or only S_2 is coercive, while S_1 is non-negative (right).

FIG. 5.3.5. Alternative decomposition: S_1 is coercive (hence, it could be used as a preconditioner in place of S_2), while S_2 is non-negative.

we can show that the operator S_i is coercive in $(\Lambda)^d$, provided that we have chosen Γ in such a way that $\partial\Omega_i \cap \Gamma_D \neq \emptyset$. Having assumed that $\Gamma_D \neq \emptyset$, this choice is possible for at least one subdomain: for definiteness, in what follows we will always suppose that $\partial\Omega_2 \cap \Gamma_D \neq \emptyset$, so that S_2 is coercive. Since we have already assumed that $\partial\Omega_i \cap \Gamma_N \neq \emptyset$ for every $i = 1, 2$ (see Remark 5.3.6), the geometric decomposition is as described in Fig. 5.3.4 (instead, the decomposition of Ω in Fig. 5.3.5 is such that $\partial\Omega_2 \cap \Gamma_D = \emptyset$, the case that we are not going to consider in the following). The operator S_2 is coercive and therefore invertible, and can be used as a preconditioner for the global Steklov–Poincaré operator S (see Section 5.3.2). Finally, the operator S_1 is non-negative (or coercive, if the decomposition of Ω is as in Fig. 5.3.4, left), therefore the global operator S is coercive.

Finally, the continuity of the Steklov–Poincaré operators S_i in $(\Lambda)^d$ is a consequence of well known a priori estimates for the solution $\mathbf{U}_i\boldsymbol{\eta}$ of (5.3.16).

The case where $\Gamma_N = \emptyset$

We shall highlight only those points of the procedure which are different than in the preceding case. As we have already noted, for the pure Dirichlet boundary value problem the pressure p is defined up to an additive constant; therefore, to have uniqueness we seek p in the space $L^2_0(\Omega)$. The variational formulation reads

$$\text{(5.3.24)} \qquad \begin{cases} \text{find } \mathbf{W} \in (H^1_0(\Omega))^d,\ p \in L^2_0(\Omega) \ : \\ s(\mathbf{W}, \mathbf{v}) + b(\mathbf{v}, p) = \mathcal{F}(\mathbf{v}) \qquad \forall\ \mathbf{v} \in (H^1_0(\Omega))^d \\ b(\mathbf{W}, q) = 0 \qquad \forall\ q \in L^2_0(\Omega), \end{cases}$$

where $\mathbf{W} = \mathbf{u} - \boldsymbol{\Phi} - \boldsymbol{\Psi}$,

$$\mathcal{F}(\mathbf{v}) := (\mathbf{f}, \mathbf{v}) - s(\boldsymbol{\Phi} + \boldsymbol{\Psi}, \mathbf{v}).$$

Again, the second equation in (5.3.24) implies $\operatorname{div} \mathbf{W} = 0$ in Ω, because we can take $q = \operatorname{div} \mathbf{W} \in L^2_0(\Omega)$ as a test function. Therefore, also in the case where $\Gamma_N = \emptyset$ we have $\int_\Omega \operatorname{div} \mathbf{W}\, \psi = 0$ for every $\psi \in L^2(\Omega)$, and not only for $\psi \in L^2_0(\Omega)$.

The existence and uniqueness of the solution to (5.3.24) follow from the coerciveness of $s(\cdot,\cdot)$ in $H^1_0(\Omega)$ and the inf–sup condition (5.3.7).

The two-domain formulation is given by: find $(\mathbf{W}_1, p_1) \in (V_1)^d \times L^2(\Omega_1)$, $(\mathbf{W}_2, p_2) \in (V_2)^d \times L^2(\Omega_2)$ such that

$$\text{(5.3.25)} \qquad \begin{cases} s_1(\mathbf{W}_1, \mathbf{v}_1) + b_1(\mathbf{v}_1, p_1) = \mathcal{F}_1(\mathbf{v}_1) \quad \forall\ \mathbf{v}_1 \in (V_1^0)^d \\ b_1(\mathbf{W}_1, q_1) = 0 \quad \forall\ q_1 \in L^2_0(\Omega_1) \\ \mathbf{W}_1 = \mathbf{W}_2 \qquad \text{on } \Gamma \\ s_2(\mathbf{W}_2, \mathbf{v}_2) + b_2(\mathbf{v}_2, p_2) = \mathcal{F}_2(\mathbf{v}_2) \quad \forall\ \mathbf{v}_2 \in (V_2^0)^d \\ b_2(\mathbf{W}_2, q_2) = 0 \quad \forall\ q_2 \in L^2(\Omega_2) \\ s_2(\mathbf{W}_2, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + b_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}, p_2) = \mathcal{F}_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + \mathcal{F}_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) \\ \qquad\qquad -s_1(\mathbf{W}_1, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) - b_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}, p_1) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d \\ \displaystyle\int_{\Omega_1} p_1 = -\int_{\Omega_2} p_2, \end{cases}$$

where

$$\mathcal{F}_i(\mathbf{v}_i) := (\mathbf{f}, \mathbf{v}_i)_{\Omega_i} - s_i(\boldsymbol{\Phi}_{|\Omega_i} + \boldsymbol{\Psi}_{|\Omega_i}, \mathbf{v}_i),$$

the spaces V_i, V_i^0 and Λ are as in Section 1.2 and $\boldsymbol{\mathcal{R}}_i$ denotes any extension operator from $(\Lambda)^d$ to $(V_i)^d$, $i = 1, 2$.

The further condition on the pressure field

$$\int_{\Omega_1} p_1 = -\int_{\Omega_2} p_2$$

in (5.3.25) guarantees that $\int_\Omega p = 0$.

Remark 5.3.8 In the two-domain formulation (5.3.25) the role of the spaces of test functions for Ω_1 and Ω_2 is not the same. Other two-domain formulations equivalent to (5.3.24) could be defined, with different choices of these spaces. Each specific formulation induces a natural way of devising substructuring iterative methods, because the latter are driven by the way the interface conditions have been split between the subdomains. For instance, (5.3.25) induces a Dirichlet–Neumann procedure in which $(5.3.25)_3$ is used as a Dirichlet boundary condition in Ω_1, and $(5.3.25)_6$ as a Neumann condition in Ω_2. The condition of zero-average pressure $(5.3.25)_7$ has to be imposed in Ω_1 (taking p_2 from the previous iteration) to ensure uniqueness of the pressure p_1 (see Section 5.3.2). □

The following equivalence result is the counterpart of Lemma 5.3.5.

Lemma 5.3.9 *The Stokes problem* (5.3.24) *is equivalent to* (5.3.25)*, in the sense that* $\mathbf{W}_{|\Omega_i} = \mathbf{W}_i$ *and* $p_{|\Omega_i} = p_i$*,* $i = 1, 2$.

Proof The proof follows that of Lemma 5.3.5, and can be repeated with the following changes. In the first part, the only difference arises from the fact that the extension by zero of a test function $q_2 \in L^2(\Omega_2)$ belongs to $L^2(\Omega)$ but not to $L^2_0(\Omega)$, the space of pressure test functions in Ω. However, we have already noted that $b(\mathbf{W}, q) = 0$ not only for any $q \in L^2_0(\Omega)$, but also for every $q \in L^2(\Omega)$, therefore we can proceed in Ω_2 as in Ω_1, extending by zero the local test functions and obtaining that $b_2(\mathbf{W}_2, q_2) = 0$ for every $q_2 \in L^2(\Omega_2)$.

In the second part, one notes that, for a test function $q \in L^2_0(\Omega)$, in general, its restriction $q_{|\Omega_1}$ does not belong to $L^2_0(\Omega_1)$, but only to $L^2(\Omega_1)$. Consequently, the result $b(\mathbf{W}, q) = 0$ is true for all $q \in L^2_0(\Omega)$ such that $q_{|\Omega_1} \in L^2_0(\Omega_1)$. For a general $q \in L^2_0(\Omega)$, setting

$$q_{\Omega_1} := \frac{1}{\operatorname{meas}\Omega_1} \int_{\Omega_1} q_{|\Omega_1},$$

one has

$$\begin{aligned} b(\mathbf{W}, q) &= b(\mathbf{W}, q - q_{\Omega_1}) + q_{\Omega_1} b(\mathbf{W}, 1) \\ &= -q_{\Omega_1} \int_{\partial\Omega} \mathbf{W}_{|\partial\Omega} \cdot \mathbf{n}^* = 0. \end{aligned}$$

The last equality holds since $\mathbf{W}_{|\partial\Omega} = \mathbf{0}$. □

To state the Steklov–Poincaré equation corresponding to (5.3.24), we have to introduce the space

$$\widehat{\mathbf{\Lambda}} := \left\{ \widehat{\boldsymbol{\eta}} \in (\Lambda)^d \mid \int_\Gamma \widehat{\boldsymbol{\eta}} \cdot \mathbf{n} = 0 \right\}, \tag{5.3.26}$$

which is the trace space on Γ of *divergence-free* functions belonging to $(H_0^1(\Omega))^d$ (see Remark 5.3.6).

Moreover, for each $\boldsymbol{\eta} \in (\Lambda)^d$ denote by $\mathbf{U}_i\boldsymbol{\eta} \in (V_i)^d$, $P_i\boldsymbol{\eta} \in L_0^2(\Omega_i)$, $\mathbf{U}_i^* \in (V_i^0)^d$ and $P_i^* \in L_0^2(\Omega_i)$ the solutions of the problems (5.3.16), (5.3.17), where $L^2(\Omega_i)$ has to be substituted by $L_0^2(\Omega_i)$, $i = 1, 2$. The local Steklov–Poincaré operators S_i and the functionals $\boldsymbol{\chi}_i$ are defined as in (5.3.18), (5.3.19), respectively.

Lemma 5.3.10 *If we set* $\widehat{\boldsymbol{\lambda}} := \mathbf{W}_{|\Gamma}$, *where* $(\mathbf{W}, p)$ *is the solution to* (5.3.24), *then* $\widehat{\boldsymbol{\lambda}}$ *satisfies the following Steklov–Poincaré equation on* Γ

$$\widehat{\boldsymbol{\lambda}} \in \widehat{\mathbf{\Lambda}} \ : \ \langle S\widehat{\boldsymbol{\lambda}}, \widehat{\boldsymbol{\mu}} \rangle = \langle \boldsymbol{\chi}, \widehat{\boldsymbol{\mu}} \rangle \qquad \forall\, \widehat{\boldsymbol{\mu}} \in \widehat{\mathbf{\Lambda}}. \tag{5.3.27}$$

Conversely, assuming that Γ *is regular enough, from the solution* $\widehat{\boldsymbol{\lambda}}$ *of* (5.3.27) *we recover the solution of* (5.3.24) *by setting*

$$\begin{aligned} \mathbf{W}_{|\Omega_i} &:= \mathbf{U}_i\boldsymbol{\lambda} + \mathbf{U}_i^* \\ p_{|\Omega_i} &:= P_i\boldsymbol{\lambda} + P_i^* + \hat{p}_i, \end{aligned} \tag{5.3.28}$$

where $\hat{p}_1$ *and* $\hat{p}_2$ *are obtained by solving the linear system*

$$\begin{cases} \hat{p}_1 - \hat{p}_2 = \dfrac{1}{\operatorname{meas}\Gamma} \langle S\widehat{\boldsymbol{\lambda}} - \boldsymbol{\chi}, \mathbf{n} \rangle \\ \hat{p}_1 \operatorname{meas}\Omega_1 + \hat{p}_2 \operatorname{meas}\Omega_2 = 0. \end{cases} \tag{5.3.29}$$

Proof Let $(\mathbf{W}, p)$ be the solution to (5.3.24), and denote by

$$p_{\Omega_i} := \frac{1}{\operatorname{meas}\Omega_i} \int_{\Omega_i} p_{|\Omega_i}$$

the mean value of p on Ω_i. By the equivalence Lemma 5.3.9 the restrictions $\mathbf{W}_{|\Omega_i}$ and $p_{|\Omega_i}$, $i = 1, 2$, are the solutions to the two-domain problem (5.3.25), hence

$$\begin{aligned} \mathbf{W}_{|\Omega_i} &= \mathbf{U}_i\mathbf{W}_{|\Gamma} + \mathbf{U}_i^* \\ p_{|\Omega_i} &= P_i\mathbf{W}_{|\Gamma} + P_i^* + p_{\Omega_i}. \end{aligned}$$

We have added p_{Ω_i} to $P_i\mathbf{W}_{|\Gamma} + P_i^*$ in order to restore the correct mean value of p in Ω_i (which, in general, is not zero in Ω_i, but only in Ω).

The stress interface condition $(5.3.25)_6$ can be rewritten as

$$\begin{aligned} \sum_{i=1}^{2} [s_i(\mathbf{U}_i\mathbf{W}_{|\Gamma} + \mathbf{U}_i^*, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i\mathbf{W}_{|\Gamma} + P_i^*)] \\ = \sum_{i=1}^{2} [\mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) - b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, p_{\Omega_i})] \qquad \forall\, \boldsymbol{\mu} \in (\Lambda)^d. \end{aligned}$$

Moreover,

$$-\sum_{i=1}^{2} b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, p_{\Omega_i}) = \sum_{i=1}^{2} p_{\Omega_i} \int_\Gamma \boldsymbol{\mu}\cdot\mathbf{n}^i = (p_{\Omega_1} - p_{\Omega_2}) \int_\Gamma \boldsymbol{\mu}\cdot\mathbf{n},$$

hence taking as test function $\widehat{\boldsymbol{\mu}} \in \widehat{\boldsymbol{\Lambda}} \subset (\Lambda)^d$, we have that $\widehat{\boldsymbol{\lambda}} = \mathbf{W}_{|\Gamma}$ satisfies (5.3.27).

Conversely, take the solution $\widehat{\boldsymbol{\lambda}}$ to (5.3.27) and set, for $i = 1, 2$,

$$\begin{aligned} \mathbf{W}_i &:= \mathbf{U}_i\widehat{\boldsymbol{\lambda}} + \mathbf{U}_i^* \\ p_i &:= P_i\widehat{\boldsymbol{\lambda}} + P_i^* + \hat{p}_i. \end{aligned}$$

To verify that these functions satisfy (5.3.25) we have to check only the last three equations; namely, the continuity equation in Ω_2, the interface equation on Γ, and the equation for the integrals of p_1 and p_2. The last one is given by $(5.3.29)_2$. Concerning the continuity equation in Ω_2, denoting by $(q_2)_{\Omega_2}$ the mean value of a function $q_2 \in L^2(\Omega_2)$ we obtain

$$\begin{aligned} b_2(\mathbf{W}_2, q_2) &= b_2(\mathbf{W}_2, q_2 - (q_2)_{\Omega_2}) + b_2(\mathbf{W}_2, (q_2)_{\Omega_2}) \\ &= (q_2)_{\Omega_2} b_2(\mathbf{W}_2, 1) = -(q_2)_{\Omega_2} \int_\Gamma \widehat{\boldsymbol{\lambda}}\cdot\mathbf{n}^2 = 0. \end{aligned}$$

Finally, using (5.3.18) and (5.3.19), the interface equation on Γ reads

$$\begin{aligned} &\sum_{i=1}^{2} [s_i(\mathbf{W}_i, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, p_i)] \\ &\quad= \sum_{i=1}^{2} [s_i(\mathbf{U}_i\widehat{\boldsymbol{\lambda}}, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i\widehat{\boldsymbol{\lambda}}) \\ &\qquad + s_i(\mathbf{U}_i^*, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) + b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, P_i^*) + \hat{p}_i b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}, 1)] \\ &\quad= \sum_{i=1}^{2} [\langle S_i\widehat{\boldsymbol{\lambda}}, \boldsymbol{\mu}\rangle + \mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) - \langle \boldsymbol{\chi}_i, \boldsymbol{\mu}\rangle] - \sum_{i=1}^{2} \hat{p}_i \int_\Gamma \boldsymbol{\mu}\cdot\mathbf{n}^i \\ &\quad= \langle S\widehat{\boldsymbol{\lambda}} - \boldsymbol{\chi}, \boldsymbol{\mu}\rangle + \sum_{i=1}^{2} \mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) - (\hat{p}_1 - \hat{p}_2) \int_\Gamma \boldsymbol{\mu}\cdot\mathbf{n}. \end{aligned}$$

Define, for each $\boldsymbol{\mu} \in (\Lambda)^d$, the mean value μ_Γ as

$$\mu_\Gamma := \frac{1}{\text{meas } \Gamma} \int_\Gamma \boldsymbol{\mu}\cdot\mathbf{n}.$$

The $L^2(\Gamma)$-orthogonal projection of $\boldsymbol{\mu}$ on $\widehat{\boldsymbol{\Lambda}}$ is given by $\boldsymbol{\mu} - \mu_\Gamma\mathbf{n}$ (due to the regularity assumption on the interface Γ, the normal vector $\mathbf{n}$ belongs to $(\Lambda)^d$). From (5.3.27) we have

$$\begin{aligned}\langle S\widehat{\boldsymbol{\lambda}}-\boldsymbol{\chi},\boldsymbol{\mu}\rangle &= \langle S\widehat{\boldsymbol{\lambda}}-\boldsymbol{\chi},\boldsymbol{\mu}-\mu_\Gamma\mathbf{n}\rangle+\mu_\Gamma\langle S\widehat{\boldsymbol{\lambda}}-\boldsymbol{\chi},\mathbf{n}\rangle\\ &= \frac{1}{\operatorname{meas}\Gamma}\left(\int_\Gamma\boldsymbol{\mu}\cdot\mathbf{n}\right)\,\langle S\widehat{\boldsymbol{\lambda}}-\boldsymbol{\chi},\mathbf{n}\rangle.\end{aligned}\tag{5.3.30}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{2}\left[s_i(\mathbf{W}_i,\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})+b_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu},p_i)\right]\\ &\quad=\sum_{i=1}^{2}\mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})+\left(\int_\Gamma\boldsymbol{\mu}\cdot\mathbf{n}\right)\,\left(\frac{1}{\operatorname{meas}\Gamma}\langle S\widehat{\boldsymbol{\lambda}}-\boldsymbol{\chi},\mathbf{n}\rangle-\hat{p}_1+\hat{p}_2\right)\\ &\quad=\sum_{i=1}^{2}\mathcal{F}_i(\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}),\end{aligned}$$

having used $(5.3.29)_1$. □

The Steklov–Poincaré operators enjoy the same properties as in the case where $\Gamma_N\neq\emptyset$ (with Ω decomposed as in Fig. 5.3.4, left): S_i, $i=1,2$, and S are symmetric, continuous, and coercive in $(\Lambda)^d$.

It is worthwhile to remark that from equation (5.3.27) we cannot deduce that $S\widehat{\boldsymbol{\lambda}}=\boldsymbol{\chi}$, due to the fact that the space of test functions is not large enough (each function $\widehat{\boldsymbol{\mu}}\in\widehat{\boldsymbol{\Lambda}}$ is subjected to the constraint $\int_\Gamma\widehat{\boldsymbol{\mu}}\cdot\mathbf{n}=0$). Indeed, from (5.3.30) the equation on Γ is

$$S\widehat{\boldsymbol{\lambda}}-\frac{1}{\operatorname{meas}\Gamma}\langle S\widehat{\boldsymbol{\lambda}},\mathbf{n}\rangle\,\mathbf{n}=\boldsymbol{\chi}-\frac{1}{\operatorname{meas}\Gamma}\langle\boldsymbol{\chi},\mathbf{n}\rangle\,\mathbf{n}.\tag{5.3.31}$$

Let us introduce the operator

$$\widetilde{S}\boldsymbol{\eta}:=S\boldsymbol{\eta}-\frac{1}{\operatorname{meas}\Gamma}\langle S\boldsymbol{\eta},\mathbf{n}\rangle\,\mathbf{n},\quad\boldsymbol{\eta}\in(\Lambda)^d$$

and the functional

$$\widetilde{\boldsymbol{\chi}}:=\boldsymbol{\chi}-\frac{1}{\operatorname{meas}\Gamma}\langle\boldsymbol{\chi},\mathbf{n}\rangle\,\mathbf{n}.$$

The trace $\widehat{\boldsymbol{\lambda}}=\mathbf{W}_{|\Gamma}$ of the solution to (5.3.24) thus satisfies

$$\langle\widetilde{S}\widehat{\boldsymbol{\lambda}},\boldsymbol{\mu}\rangle=\langle\widetilde{\boldsymbol{\chi}},\boldsymbol{\mu}\rangle\qquad\forall\,\boldsymbol{\mu}\in(\Lambda)^d,$$

which is the Steklov–Poincaré equation in the space $(\Lambda)^d$.

However, the operator $\widetilde{S}:(\Lambda)^d\to(\Lambda')^d$ is not surjective, because it satisfies

$$\langle\widetilde{S}\boldsymbol{\eta},\mathbf{n}\rangle=0\qquad\forall\,\boldsymbol{\eta}\in(\Lambda)^d.$$

In particular, $\langle\widetilde{S}\mathbf{n},\mathbf{n}\rangle=0$, hence $\widetilde{S}$ is not coercive in $(\Lambda)^d$. We have encountered a similar situation for the Neumann boundary value problem for a symmetric elliptic operator (see Section 1.4.1). Furthermore, $\widetilde{S}$ is not symmetric.

On the other hand, for each $\boldsymbol{\eta} \in (\Lambda)^d$, the functionals $\widetilde{S}\boldsymbol{\eta}$ and $S\boldsymbol{\eta}$ coincide when acting on $\widehat{\boldsymbol{\Lambda}}$; that is,

$$\langle \widetilde{S}\boldsymbol{\eta}, \widehat{\boldsymbol{\mu}} \rangle = \langle S\boldsymbol{\eta}, \widehat{\boldsymbol{\mu}} \rangle \qquad \forall\, \boldsymbol{\eta} \in (\Lambda)^d, \widehat{\boldsymbol{\mu}} \in \widehat{\boldsymbol{\Lambda}},$$

therefore $\widetilde{S}$ is symmetric and coercive in $\widehat{\boldsymbol{\Lambda}}$.

Since the trace $\mathbf{W}_{|\Gamma}$ indeed belongs to $\widehat{\boldsymbol{\Lambda}}$, we have been led to write the Steklov–Poincaré equation on Γ in terms of $\widehat{\boldsymbol{\Lambda}}$ and S, instead of $(\Lambda)^d$ and $\widetilde{S}$.

5.3.2 Substructuring iterative methods

Let us assume that $\Gamma_N \neq \emptyset$. The Dirichlet–Neumann iteration-by-subdomain procedure for the multi-domain weak formulation (5.3.14) reads: given $\boldsymbol{\lambda}^0 \in (\Lambda)^d$, for each $k \geq 0$ solve

$$\begin{cases} \text{find } (\mathbf{W}_1^{k+1}, p_1^{k+1}) \in (V_1)^d \times L^2(\Omega_1) \ : \\ s_1(\mathbf{W}_1^{k+1}, \mathbf{v}_1) + b_1(\mathbf{v}_1, p_1^{k+1}) = \mathcal{F}_1(\mathbf{v}_1) \quad \forall\, \mathbf{v}_1 \in (V_1^0)^d \\ b_1(\mathbf{W}_1^{k+1}, q_1) = 0 \quad \forall\, q_1 \in L^2(\Omega_1) \\ \mathbf{W}_1^{k+1} = \boldsymbol{\lambda}^k \qquad \text{on } \Gamma, \end{cases} \tag{5.3.32}$$

$$\begin{cases} \text{find } (\mathbf{W}_2^{k+1}, p_2^{k+1}) \in (V_2)^d \times L^2(\Omega_2) \ : \\ s_2(\mathbf{W}_2^{k+1}, \mathbf{v}_2) + b_2(\mathbf{v}_2, p_2^{k+1}) = \mathcal{F}_2(\mathbf{v}_2) \quad \forall\, \mathbf{v}_2 \in (V_2^0)^d \\ b_2(\mathbf{W}_2^{k+1}, q_2) = 0 \quad \forall\, q_2 \in L^2(\Omega_2) \\ s_2(\mathbf{W}_2^{k+1}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + b_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}, p_2^{k+1}) = \mathcal{F}_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + \mathcal{F}_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) \\ \qquad\qquad -s_1(\mathbf{W}_1^{k+1}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) - b_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}, p_1^{k+1}) \qquad \forall\, \boldsymbol{\mu} \in (\Lambda)^d, \end{cases} \tag{5.3.33}$$

and finally update the interface value as follows:

$$\boldsymbol{\lambda}^{k+1} = \theta \mathbf{W}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k \qquad \text{on } \Gamma. \tag{5.3.34}$$

A 'parallel' version of this iterative scheme is obtained by replacing $\mathbf{W}_1^{k+1}$ by $\mathbf{W}_1^k$ and p_1^{k+1} by p_1^k in the last set of equations (5.3.33).

The Dirichlet–Neumann iterations (5.3.32)–(5.3.34) can be rewritten as a preconditioned Richardson procedure for the Steklov–Poincaré equation (5.3.20). Noting that $\mathbf{U}_i(\mathbf{W}_{i|\Gamma}^{k+1} - \mathbf{U}_{i|\Gamma}^*) = \mathbf{W}_i^{k+1} - \mathbf{U}_i^*$, $i = 1, 2$, for each $\boldsymbol{\mu} \in (\Lambda)^d$ we have

$$\begin{aligned}\langle S_2\mathbf{W}_{2|\Gamma}^{k+1},\boldsymbol{\mu}\rangle &= \langle S_2(\mathbf{W}_{2|\Gamma}^{k+1}-\mathbf{U}_{2|\Gamma}^{*}),\boldsymbol{\mu}\rangle = s_2(\mathbf{W}_2^{k+1}-\mathbf{U}_2^{*},\mathbf{U}_2\boldsymbol{\mu})\\ &= -b_2(\mathbf{U}_2\boldsymbol{\mu},p_2^{k+1})+\mathcal{F}_2(\mathbf{U}_2\boldsymbol{\mu})+\mathcal{F}_1(\mathbf{U}_1\boldsymbol{\mu})\\ &\quad -s_1(\mathbf{W}_1^{k+1},\mathbf{U}_1\boldsymbol{\mu})-b_1(\mathbf{U}_1\boldsymbol{\mu},p_1^{k+1})-s_2(\mathbf{U}_2^{*},\mathbf{U}_2\boldsymbol{\mu}),\end{aligned}$$

having used the last equation in (5.3.33). Moreover,

$$b_i(\mathbf{U}_i\boldsymbol{\mu},p_i^{k+1})=0 \tag{5.3.35}$$

for each $i=1,2$, and

$$\begin{aligned}s_1(\mathbf{W}_1^{k+1},\mathbf{U}_1\boldsymbol{\mu}) &= s_1(\mathbf{U}_1(\mathbf{W}_{1|\Gamma}^{k+1}-\mathbf{U}_{1|\Gamma}^{*}),\mathbf{U}_1\boldsymbol{\mu})+s_1(\mathbf{U}_1^{*},\mathbf{U}_1\boldsymbol{\mu})\\ &= \langle S_1\boldsymbol{\lambda}^k,\boldsymbol{\mu}\rangle + s_1(\mathbf{U}_1^{*},\mathbf{U}_1\boldsymbol{\mu}).\end{aligned}$$

Hence

$$\langle S_2\mathbf{W}_{2|\Gamma}^{k+1},\boldsymbol{\mu}\rangle = \langle\boldsymbol{\chi},\boldsymbol{\mu}\rangle - \langle S_1\boldsymbol{\lambda}^k,\boldsymbol{\mu}\rangle. \tag{5.3.36}$$

Therefore, we derive from (5.3.34) that the Dirichlet–Neumann iterations can be written in $(\Lambda)^d$ as

$$\begin{cases}\boldsymbol{\lambda}^0\in(\Lambda)^d\\ \boldsymbol{\lambda}^{k+1}=\boldsymbol{\lambda}^k+\theta S_2^{-1}(\boldsymbol{\chi}-S\boldsymbol{\lambda}^k),\quad k\geq 0.\end{cases} \tag{5.3.37}$$

The convergence of this algorithm has been proved at the finite dimensional level by Marini and Quarteroni (1989) for finite elements and by Quarteroni (1989) for spectral approximation, provided that $0<\theta<\theta^*$ for a suitable θ^*. We can now see that convergence in $(\Lambda)^d$ is a direct consequence of Theorem 4.2.2; in fact, as we have already pointed out, both S_1 and S_2 are symmetric and continuous in $(\Lambda)^d$, S_2 is coercive in $(\Lambda)^d$ and S_1 is non-negative (or coercive, if $\partial\Omega_1\cap\Gamma_D\neq\emptyset$), so that S is also coercive.

The Dirichlet–Neumann iterative scheme in the case where $\Gamma_N=\emptyset$ is quite similar. Given $\boldsymbol{\lambda}^0\in(\Lambda)^d$, for each $k\geq 0$ solve

$$\begin{cases}\text{find } (\mathbf{W}_1^{k+1},p_1^{k+1})\in(V_1)^d\times L^2(\Omega_1)\ :\\ s_1(\mathbf{W}_1^{k+1},\mathbf{v}_1)+b_1(\mathbf{v}_1,p_1^{k+1})=\mathcal{F}_1(\mathbf{v}_1)\quad\forall\,\mathbf{v}_1\in(V_1^0)^d\\ b_1(\mathbf{W}_1^{k+1},q_1)=0\quad\forall\,q_1\in L_0^2(\Omega_1)\\ \mathbf{W}_1^{k+1}=\boldsymbol{\lambda}^k\qquad\text{on }\Gamma\\ \displaystyle\int_{\Omega_1}p_1^{k+1}=-\int_{\Omega_2}p_2^{k},\end{cases} \tag{5.3.38}$$

and then proceed as in (5.3.33), (5.3.34).

The interpretation as a Richardson iterative scheme is still valid. Actually, set for each $k \geq 0$ and $i = 1, 2$

$$p_{\Omega_i}^k := \frac{1}{\text{meas}\,\Omega_i} \int_{\Omega_i} p_i^k.$$

Since p_i^{k+1} does not belong to $L_0^2(\Omega_i)$, equation (5.3.35) does not hold, but instead we have

$$\begin{aligned}\sum_{i=1}^{2} b_i(\mathbf{U}_i\boldsymbol{\mu}, p_i^{k+1}) &= \sum_{i=1}^{2} b_i(\mathbf{U}_i\boldsymbol{\mu}, p_i^{k+1} - p_{\Omega_i}^{k+1}) + \sum_{i=1}^{2} p_{\Omega_i}^{k+1} b_i(\mathbf{U}_i\boldsymbol{\mu}, 1) \\ &= p_{\Omega_i}^{k+1} \sum_{i=1}^{2} b_i(\mathbf{U}_i\boldsymbol{\mu}, 1) = -(p_{\Omega_1}^{k+1} - p_{\Omega_2}^{k+1}) \int_{\Gamma} \boldsymbol{\mu} \cdot \mathbf{n},\end{aligned}$$

so that (5.3.36) becomes

$$\langle S_2 \mathbf{W}_{2|\Gamma}^{k+1}, \boldsymbol{\mu}\rangle = \langle \boldsymbol{\chi}, \boldsymbol{\mu}\rangle - \langle S_1 \boldsymbol{\lambda}^k, \boldsymbol{\mu}\rangle + (p_{\Omega_1}^{k+1} - p_{\Omega_2}^{k+1}) \int_{\Gamma} \boldsymbol{\mu} \cdot \mathbf{n}.$$

In other words,

$$\langle S_2 \mathbf{W}_{2|\Gamma}^{k+1}, \widehat{\boldsymbol{\mu}}\rangle = \langle \boldsymbol{\chi}, \widehat{\boldsymbol{\mu}}\rangle - \langle S_1 \boldsymbol{\lambda}^k, \widehat{\boldsymbol{\mu}}\rangle \qquad \forall\, \widehat{\boldsymbol{\mu}} \in \widehat{\boldsymbol{\Lambda}}. \tag{5.3.39}$$

In the case where $\Gamma_N = \emptyset$, i.e. when $V_2 = H^1_{\partial\Omega_2\setminus\Gamma}(\Omega_2)$, the solution $\mathbf{W}_2^{k+1}$ of (5.3.33) satisfies

$$\begin{aligned}\int_{\Gamma} \mathbf{W}_{2|\Gamma}^{k+1} \cdot \mathbf{n} &= -\int_{\Gamma} \mathbf{W}_{2|\Gamma}^{k+1} \cdot \mathbf{n}^2 = -\int_{\Gamma} \mathbf{W}_{2|\Gamma}^{k+1} \cdot \mathbf{n}^2 - \int_{\partial\Omega_2\setminus\Gamma} \mathbf{W}_{2|\partial\Omega_2\setminus\Gamma}^{k+1} \cdot \mathbf{n}^* \\ &= -\int_{\Omega_2} \text{div}\, \mathbf{W}_2^{k+1} = b_2(\mathbf{W}_2^{k+1}, 1) = 0,\end{aligned}$$

having used the continuity equation in (5.3.33).

Hence, taking the initial guess $\boldsymbol{\lambda}^0$ in $\widehat{\boldsymbol{\Lambda}}$, it can be shown by an induction argument that (5.3.34) yields $\boldsymbol{\lambda}^k \in \widehat{\boldsymbol{\Lambda}}$ for each $k \geq 0$. Therefore, the Dirichlet–Neumann iterations can be written as

$$\begin{cases} \boldsymbol{\lambda}^0 \in \widehat{\boldsymbol{\Lambda}} \\ \boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta S_2^{-1}(\boldsymbol{\chi} - S\boldsymbol{\lambda}^k), \quad k \geq 0, \end{cases} \tag{5.3.40}$$

the equality being valid in $\widehat{\boldsymbol{\Lambda}}$.

All the assumptions in Theorem 4.2.2 are now satisfied in the space $\widehat{\boldsymbol{\Lambda}}$, and convergence of these iterations follows as before. Note that the convergence of the sequence $\boldsymbol{\lambda}^k$ does not follow if we take the initial guess $\boldsymbol{\lambda}^0$ in $(\Lambda)^d$.

In both cases where $\Gamma_N \neq \emptyset$ and $\Gamma_N = \emptyset$, we can also consider the Neumann–Neumann iterative scheme, which can be defined following the guidelines of what

has been carried out in Section 1.3 for the Laplace operator. Since, as usual, this algorithm can be interpreted as a Richardson method for solving the Steklov–Poincaré equation (5.3.20) or (5.3.27), with $(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$ as a preconditioner, its convergence is a direct consequence of Theorem 4.2.5.

5.3.3 Finite dimensional approximation: the case of discontinuous pressure

Let us consider finite element approximations of the Stokes equations that make use of *discontinuous* discrete pressures across inter-element boundaries. The case of continuous pressures will be considered in the next section. We therefore consider finite element spaces $V_h \subset H^1_{\Gamma_D}(\Omega)$ for each velocity component and $Q_h \subset L^2(\Omega)$ for the pressure, as well as their subdomain restrictions:

$$\begin{aligned} V_{i,h} &:= \{v_{h|\Omega_i} \,|\, v_h \in V_h\} \\ V^0_{i,h} &:= V_{i,h} \cap V^0_i \\ \Lambda_h &:= \{v_{h|\Gamma} \,|\, v_h \in V_h\} \\ Q_{i,h} &:= \{q_{h|\Omega_i} \,|\, q_h \in Q_h\}, \end{aligned} \tag{5.3.41}$$

for $i = 1, 2$. We suppose that

$$X^r_h \cap H^1_{\Gamma_D}(\Omega) \subset V_h, \quad \Lambda_h = \{v_{h|\Gamma} \,|\, v_h \in X^r_h \cap H^1_{\Gamma_D}(\Omega)\} \tag{5.3.42}$$

for some $r \geq 1$ (X^r_h has been introduced in (2.1.2) or (2.1.4)).

We assume that the *inf–sup condition* is satisfied. When $\Gamma_N = \emptyset$, we have $H^1_{\Gamma_D}(\Omega) = H^1_0(\Omega)$, and the inf–sup condition takes the form: there exists a constant $\beta > 0$, independent of h, such that

$$\begin{aligned} \forall\, q_h \in Q_h \cap L^2_0(\Omega)\ \exists\, \mathbf{v}_h \in (V_h)^d \subset (H^1_0(\Omega))^d,\ \mathbf{v}_h \neq \mathbf{0}\ : \\ b(\mathbf{v}_h, q_h) \geq \beta ||\mathbf{v}_h||_{1,\Omega}\, ||q_h||_{0,\Omega}. \end{aligned} \tag{5.3.43}$$

Moreover, we assume also that the *local* inf–sup condition holds: for each $i = 1, 2$, there exists a constant $\beta_i > 0$, independent of h, such that

$$\begin{aligned} \forall\, q_{i,h} \in Q_{i,h} \cap L^2_0(\Omega_i)\ \exists\, \mathbf{v}_{i,h} \in (V^0_{i,h})^d \subset (H^1_0(\Omega_i))^d,\ \mathbf{v}_{i,h} \neq \mathbf{0}\ : \\ b_i(\mathbf{v}_{i,h}, q_{i,h}) \geq \beta_i ||\mathbf{v}_{i,h}||_{1,\Omega_i}\, ||q_{i,h}||_{0,\Omega_i}. \end{aligned} \tag{5.3.44}$$

When $\Gamma_N \neq \emptyset$, the inf–sup condition in Ω takes the following form: there exists $\beta > 0$, independent of h, such that

$$\begin{aligned} \forall\, q_h \in Q_h\ \exists\, \mathbf{v}_h \in (V_h)^d \subset (H^1_{\Gamma_D}(\Omega))^d,\ \mathbf{v}_h \neq \mathbf{0}\ : \\ b(\mathbf{v}_h, q_h) \geq \beta ||\mathbf{v}_h||_{1,\Omega}\, ||q_h||_{0,\Omega}. \end{aligned} \tag{5.3.45}$$

Furthermore, we also require that the following *local* inf–sup condition is satisfied: for each $i = 1, 2$ there exists $\beta_i > 0$, independent of h, such that

$$\forall\ q_{i,h} \in Q_{i,h}\ \exists\ \mathbf{v}_{i,h} \in (V^0_{i,h})^d \subset (H^1_{\Gamma_D \cup \Gamma}(\Omega_i))^d,\quad \mathbf{v}_{i,h} \neq \mathbf{0}: \qquad b_i(\mathbf{v}_{i,h}, q_{i,h}) \geq \beta_i ||\mathbf{v}_{i,h}||_{1,\Omega_i}\, ||q_{i,h}||_{0,\Omega_i}. \tag{5.3.46}$$

It is worthwhile to note that conditions (5.3.44) and (5.3.46) are very often a consequence of (5.3.43) and (5.3.45), respectively. However, if the number of triangles of $\mathcal{T}_h$ in Ω_i is too small, it can happen that (5.3.44) does not hold, even if (5.3.43) does, and similarly for (5.3.45) and (5.3.46) (see, for example, Boffi 1994, where a counterexample is provided in the case of continuous pressures).

We now introduce the finite element counterparts of the functions $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ defined in Section 5.3.1. Let us assume that the Dirichlet boundary datum $\boldsymbol{\varphi}_D \in (H^{1/2}(\Gamma_D))^d$ is also a continuous function on Γ_D, and denote by $\boldsymbol{\Phi}_h \in (X^r_h)^d$ the interpolant of the nodal values of $\boldsymbol{\varphi}_D$ on Γ_D. Moreover, let $\boldsymbol{\Psi}_h$ be the solution of the following problem (which is the finite dimensional counterpart of (5.3.9))

$$\boldsymbol{\Psi}_h \in (V_h)^d\ :\ b(\boldsymbol{\Psi}_h, q_h) = -b(\boldsymbol{\Phi}_h, q_h) \qquad \forall\ q_h \in \widehat{Q}_h, \tag{5.3.47}$$

where $\widehat{Q}_h := Q_h \cap L^2_0(\Omega)$ if $\Gamma_N = \emptyset$, and $\widehat{Q}_h := Q_h$ if $\Gamma_N \neq \emptyset$. The solvability of the latter problem is guaranteed by the inf–sup conditions (5.3.43) or (5.3.45) (see, for example, Quarteroni and Valli 1994, p. 248). Being an overdetermined system, its effective solution can be attained either by QR factorisation, or by the singular value decomposition (see, for example, Golub and van Loan 1989).

Let us start again with the case where $\Gamma_N \neq \emptyset$. The finite element variational approximation of the Stokes problem (5.3.10) reads:

$$\begin{cases} \text{find}\ \ \mathbf{W}_h \in (V_h)^d \subset (H^1_{\Gamma_D}(\Omega))^d,\ p_h \in Q_h\ : \\ s(\mathbf{W}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \mathcal{F}_h(\mathbf{v}_h) \qquad \forall\ \mathbf{v}_h \in (V_h)^d \\ b(\mathbf{W}_h, q_h) = 0 \qquad \forall\ q_h \in Q_h, \end{cases} \tag{5.3.48}$$

where

$$\mathcal{F}_h(\mathbf{v}_h) := (\mathbf{f}, \mathbf{v}_h)_\Omega + (\boldsymbol{\varphi}_N, \mathbf{v}_{h|\Gamma_N})_{\Gamma_N} - s(\boldsymbol{\Phi}_h + \boldsymbol{\Psi}_h, \mathbf{v}_h).$$

The corresponding two-domain formulation (which is the finite element counterpart of (5.3.14)) reads: find $(\mathbf{W}_{1,h}, p_{1,h}) \in (V_{1,h})^d \times Q_{1,h}$, $(\mathbf{W}_{2,h}, p_{2,h}) \in (V_{2,h})^d \times Q_{2,h}$ such that

$$
\left\{
\begin{aligned}
&s_1(\mathbf{W}_{1,h},\mathbf{v}_{1,h})+b_1(\mathbf{v}_{1,h},p_{1,h})=\mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h}\in (V_{1,h}^0)^d\\
&b_1(\mathbf{W}_{1,h},q_{1,h})=0 \quad \forall\ q_{1,h}\in Q_{1,h}\\
&\mathbf{W}_{1,h}=\mathbf{W}_{2,h} \qquad \text{on } \Gamma\\
&s_2(\mathbf{W}_{2,h},\mathbf{v}_{2,h})+b_2(\mathbf{v}_{2,h},p_{2,h})=\mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h}\in (V_{2,h}^0)^d\\
&b_2(\mathbf{W}_{2,h},q_{2,h})=0 \quad \forall\ q_{2,h}\in Q_{2,h}\\
&s_2(\mathbf{W}_{2,h},\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h,p_{2,h})\\
&\qquad =\mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+\mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)\\
&\qquad\quad -s_1(\mathbf{W}_{1,h},\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)-b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h,p_{1,h}) \quad \forall\ \boldsymbol{\mu}_h\in(\Lambda_h)^d,
\end{aligned}
\right. \tag{5.3.49}
$$

where

$$
\mathcal{F}_{i,h}(\mathbf{v}_{i,h}) := (\mathbf{f},\mathbf{v}_{i,h})_{\Omega_i}+(\boldsymbol{\varphi}_N,\mathbf{v}_{i,h|\Gamma_N})_{\Gamma_N\cap\partial\Omega_i}-s_i(\boldsymbol{\Phi}_{h|\Omega_i}+\boldsymbol{\Psi}_{h|\Omega_i},\mathbf{v}_{i,h}),
$$

and $\boldsymbol{\mathcal{R}}_{i,h}$ denotes any extension operator from $(\Lambda_h)^d$ to $(V_{i,h})^d$, $i=1,2$. As we have already noted in Section 2.1.1, in practical implementation $\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h$ is chosen to be the finite element interpolant $\boldsymbol{\pi}_{i,h}\boldsymbol{\mu}_h$, which belongs to $(V_{i,h})^d$, coincides with $\boldsymbol{\mu}_h$ at the nodes on the interface Γ, and vanishes at the internal nodes in Ω_i.

Following the same guidelines as the proof of Lemma 5.3.5, we can prove the following result.

Lemma 5.3.11 *The discrete Stokes problem* (5.3.48) *is equivalent to the two-domain formulation* (5.3.49)*, in the usual sense that* $\mathbf{W}_{h|\Omega_i}=\mathbf{W}_{i,h}$*,* $p_{h|\Omega_i}=p_{i,h}$ *for* $i=1,2$.

For each $\boldsymbol{\eta}_h\in(\Lambda_h)^d$ we introduce now the discrete local Steklov–Poincaré operators $S_{i,h}:(\Lambda_h)^d\to(\Lambda_h')^d$ defined as

$$
\begin{aligned}
\langle S_{i,h}\boldsymbol{\eta}_h,\boldsymbol{\mu}_h\rangle :&= s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h,\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h)\\
&\quad +b_i(\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h,P_{i,h}\boldsymbol{\eta}_h) \qquad \forall\ \boldsymbol{\eta}_h,\boldsymbol{\mu}_h\in(\Lambda_h)^d.
\end{aligned} \tag{5.3.50}
$$

We also introduce the continuous linear functionals $\boldsymbol{\chi}_{i,h}\in(\Lambda_h')^d$:

$$
\begin{aligned}
\langle \boldsymbol{\chi}_{i,h},\boldsymbol{\mu}_h\rangle :&= \mathcal{F}_{i,h}(\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h)-s_i(\mathbf{U}_{i,h}^*,\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h)\\
&\quad -b_i(\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h,P_{i,h}^*) \qquad \forall\ \boldsymbol{\mu}_h\in(\Lambda_h)^d.
\end{aligned} \tag{5.3.51}
$$

For $i=1,2$ we have denoted by $(\mathbf{U}_{i,h}\boldsymbol{\eta}_h,P_{i,h}\boldsymbol{\eta}_h)\in(V_{i,h})^d\times Q_{i,h}$ the solution to

$$\begin{cases} s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{v}_{i,h}) + b_i(\mathbf{v}_{i,h}, P_{i,h}\boldsymbol{\eta}_h) = 0 \quad \forall\ \mathbf{v}_{i,h} \in (V_{i,h}^0)^d \\ b_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, q_{i,h}) = 0 \quad \forall\ q_{i,h} \in Q_{i,h} \\ \mathbf{U}_{i,h}\boldsymbol{\eta}_h = \boldsymbol{\eta}_h \qquad \text{on } \Gamma, \end{cases} \tag{5.3.52}$$

that is, the *finite element Stokes extension* of $\boldsymbol{\eta}_h$ to Ω_i. Similarly, $(\mathbf{U}_{i,h}^*, P_{i,h}^*) \in (V_{i,h}^0)^d \times Q_{i,h}$ is the finite element solution of the discrete problem

$$\begin{cases} s_i(\mathbf{U}_{i,h}^*, \mathbf{v}_{i,h}) + b_i(\mathbf{v}_{i,h}, P_{i,h}^*) = \mathcal{F}_{i,h}(\mathbf{v}_{i,h}) \quad \forall\ \mathbf{v}_{i,h} \in (V_{i,h}^0)^d \\ b_i(\mathbf{U}_{i,h}^*, q_{i,h}) = 0 \quad \forall\ q_{i,h} \in Q_{i,h}. \end{cases} \tag{5.3.53}$$

Note that the definition of both $S_{i,h}$ and $\boldsymbol{\chi}_{i,h}$ is independent of the choice of the extension operators $\boldsymbol{\mathcal{R}}_{i,h}$.

Proceeding as in Lemma 5.3.7, the vector function $\boldsymbol{\lambda}_h := \mathbf{W}_{h|\Gamma}$, where $\mathbf{W}_h$ is the discrete velocity, is the solution of the Steklov–Poincaré equation

$$\boldsymbol{\lambda}_h \in (\Lambda_h)^d \ :\ \langle S_h\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle = \langle \boldsymbol{\chi}_h, \boldsymbol{\mu}_h\rangle \qquad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d, \tag{5.3.54}$$

having set $S_h := S_{1,h} + S_{2,h}$ and $\boldsymbol{\chi}_h := \boldsymbol{\chi}_{1,h} + \boldsymbol{\chi}_{2,h}$.

Conversely, from the solution $\boldsymbol{\lambda}_h$ of (5.3.54) we obtain the solution of (5.3.48) by setting

$$\begin{aligned} \mathbf{W}_{h|\Omega_i} &:= \mathbf{U}_{i,h}\boldsymbol{\lambda}_h + \mathbf{U}_{i,h}^* \\ p_{h|\Omega_i} &:= P_{i,h}\boldsymbol{\lambda}_h + P_{i,h}^*. \end{aligned} \tag{5.3.55}$$

Both of the discrete operators $S_{1,h}$ and $S_{2,h}$ is linear, hence continuous in $(\Lambda_h)^d$ (their continuity constants could, in principle, depend on h).

Moreover, if we choose $\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h = \mathbf{U}_{i,h}\boldsymbol{\mu}_h$ in (5.3.50), owing to (5.3.52) we obtain the identity

$$\langle S_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h\rangle = s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\mu}_h) \qquad \forall\ \boldsymbol{\eta}_h, \boldsymbol{\mu}_h \in (\Lambda_h)^d.$$

Consequently, both operators $S_{1,h}$ and $S_{2,h}$ are also symmetric.

Finally, due to the Korn inequality (5.2.5) in Ω_i and the trace inequality (1.2.5), the operator $S_{2,h}$ can be shown to be coercive in $(\Lambda_h)^d$, uniformly with respect to h (recall that we have chosen a decomposition of Ω as in Fig. 5.3.4). In particular, $S_{2,h}$ is invertible from $(\Lambda_h')^d$ to $(\Lambda_h)^d$. When $\partial\Omega_1 \cap \Gamma_D \neq \emptyset$, coerciveness also holds for the operator $S_{1,h}$, whereas, when $\partial\Omega_1 \cap \Gamma_D = \emptyset$, we can only infer that $S_{1,h}$ is non-negative. In any case, the operator S_h is coercive in $(\Lambda_h)^d$, uniformly with respect to h.

If the family of triangulations $\mathcal{T}_h$ of Ω is regular and the family of triangulations $\mathcal{M}_h$ of Γ is quasi-uniform, then both of the discrete operators $S_{i,h}$ is uniformly continuous in $(\Lambda_h)^d$. This can be proved as follows. Due to the assump-

tion (5.3.42), we can take in (5.3.52) the test function $\mathbf{v}_{i,h} = \mathbf{U}_{i,h}\boldsymbol{\eta}_h - \mathbf{H}_{i,h}\boldsymbol{\eta}_h$, where $\boldsymbol{\eta}_h \in (\Lambda_h)^d$ and $\mathbf{H}_{i,h}\boldsymbol{\eta}_h$ is defined as in (5.2.21). It holds that

$$\begin{aligned} s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\eta}_h) &= s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{H}_{i,h}\boldsymbol{\eta}_h) \\ &\quad + b_i(\mathbf{H}_{i,h}\boldsymbol{\eta}_h, P_{i,h}\boldsymbol{\eta}_h) \\ &\leq [s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\eta}_h)]^{1/2}[s_i(\mathbf{H}_{i,h}\boldsymbol{\eta}_h, \mathbf{H}_{i,h}\boldsymbol{\eta}_h)]^{1/2} \\ &\quad + C||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i}||P_{i,h}\boldsymbol{\eta}_h||_{0,\Omega_i}, \end{aligned}$$

for a constant $C > 0$ independent of h. Using the inf–sup condition (5.3.44) it follows that

$$\begin{aligned} ||P_{i,h}\boldsymbol{\eta}_h||_{0,\Omega_i} &\leq \frac{1}{\beta_i} \sup_{\mathbf{v}_{i,h} \in (V^0_{i,h})^d \setminus \mathbf{0}} \frac{b_i(\mathbf{v}_{i,h}, P_{i,h}\boldsymbol{\eta}_h)}{||\mathbf{v}_{i,h}||_{1,\Omega_i}} \\ &= \frac{1}{\beta_i} \sup_{\mathbf{v}_{i,h} \in (V^0_{i,h})^d \setminus \mathbf{0}} \frac{s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{v}_{i,h})}{||\mathbf{v}_{i,h}||_{1,\Omega_i}} \\ &\leq C[s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\eta}_h)]^{1/2}, \end{aligned}$$

therefore

$$[s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\eta}_h)]^{1/2} \leq C||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i}.$$

Finally

$$\begin{aligned} \langle S_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h \rangle &\leq [s_i(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}\boldsymbol{\eta}_h)]^{1/2}[s_i(\mathbf{U}_{i,h}\boldsymbol{\mu}_h, \mathbf{U}_{i,h}\boldsymbol{\mu}_h)]^{1/2} \\ &\leq C||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i}||\mathbf{H}_{i,h}\boldsymbol{\mu}_h||_{1,\Omega_i}. \end{aligned}$$

Consequently, the uniform continuity of $S_{i,h}$ follows from Theorem 4.1.3.

Consider now the Dirichlet–Neumann iterative scheme for solving the multi-domain problem (5.3.49): given $\boldsymbol{\lambda}_h^0 \in (\Lambda_h)^d$, for $k \geq 0$

$$\text{(5.3.56)} \quad \begin{cases} \text{find } (\mathbf{W}_{1,h}^{k+1}, p_{1,h}^{k+1}) \in (V_{1,h})^d \times Q_{1,h} \ : \\ s_1(\mathbf{W}_{1,h}^{k+1}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, p_{1,h}^{k+1}) = \mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\mathbf{W}_{1,h}^{k+1}, q_{1,h}) = 0 \quad \forall\ q_{1,h} \in Q_{1,h} \\ \mathbf{W}_{1,h}^{k+1} = \boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma, \end{cases}$$

then

$$
\left\{
\begin{array}{l}
\text{find } (\mathbf{W}_{2,h}^{k+1}, p_{2,h}^{k+1}) \in (V_{2,h})^d \times Q_{2,h} \ : \\
s_2(\mathbf{W}_{2,h}^{k+1}, \mathbf{v}_{2,h}) + b_2(\mathbf{v}_{2,h}, p_{2,h}^{k+1}) = \mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\
b_2(\mathbf{W}_{2,h}^{k+1}, q_{2,h}) = 0 \quad \forall\ q_{2,h} \in Q_{2,h} \\
s_2(\mathbf{W}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}^{k+1}) \\
\qquad = \mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + \mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) \\
\qquad\quad -s_1(\mathbf{W}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}^{k+1}) \quad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d
\end{array}
\right. \tag{5.3.57}
$$

and

$$
\boldsymbol{\lambda}_h^{k+1} = \theta \mathbf{W}_{2,h|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma. \tag{5.3.58}
$$

Problem (5.3.57) has a unique solution provided that the inf–sup condition (5.3.44) is satisfied for the discrete spaces $(V_{2,h})^d$ and $Q_{2,h}$. However, by proceeding as in the proof of Proposition 5.3.2, it can be shown that this is a consequence of (5.3.43), because we are dealing with discontinuous finite element pressures.

Proceeding as in Section 5.3.1, it can be proved that the interface sequence $\{\boldsymbol{\lambda}_h^k\}$ obtained in (5.3.58) is actually equivalent to that produced by the preconditioned Richardson method

$$
\left\{
\begin{array}{l}
\boldsymbol{\lambda}_h^0 \in (\Lambda_h)^d \\
\boldsymbol{\lambda}_h^{k+1} = \boldsymbol{\lambda}_h^k + \theta S_{2,h}^{-1}(\boldsymbol{\chi}_h - S_h\boldsymbol{\lambda}_h^k), \quad k \geq 0,
\end{array}
\right. \tag{5.3.59}
$$

applied directly to problem (5.3.54).

The convergence of the sequence $\boldsymbol{\lambda}_h^k$, with respect to the norm of $(\Lambda)^d$, follows from Theorem 4.2.2. In fact, we have already noted that both of the discrete operators $S_{1,h}$ and $S_{2,h}$ is symmetric and continuous in $(\Lambda_h)^d$, uniformly with respect to h. Moreover, S_h and $S_{2,h}$ are uniformly coercive in $(\Lambda_h)^d$, hence all the assumptions of Theorem 4.2.2 are satisfied (see Remark 4.2.4). The rate of convergence is independent of h.

A similar result holds for the Neumann–Neumann iterative scheme, basing the proof of convergence on Theorem 4.2.5.

Remark 5.3.12 The matrix associated with the Steklov–Poincaré equation is symmetric and positive definite. Therefore, this system can be efficiently solved by several methods, besides the preconditioned Richardson scheme that was outlined above. An a priori more efficient solver is the preconditioned conjugate gradient method. Several preconditioners enjoying optimal spectral properties can be proposed in order for the associated conjugate gradient iterations to converge at a rate independent of the number of grid points that are used in each subdomain: two examples are furnished by the Dirichlet–Neumann and Neumann–Neumann preconditioners.

In each iteration, computing the action of S_h on a given $\boldsymbol{\eta}_h^k$ will require solving two Stokes problems to find $\mathbf{U}_{i,h}\boldsymbol{\eta}_h^k$, $P_{i,h}\boldsymbol{\eta}_h^k$. For more details on this approach see Quarteroni (1989). □

In the case where $\Gamma_N \neq \emptyset$, the task of constructing the finite element vector function $\boldsymbol{\Psi}_h$ solution to (5.3.47) can be avoided, at the expense of solving, instead of (5.3.48), the following discrete problem:

$$\begin{cases} \text{find } \mathbf{w}_h \in (V_h)^d \subset (H^1_{\Gamma_D}(\Omega))^d,\ p_h \in Q_h\ : \\ s(\mathbf{w}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \mathcal{F}_h^*(\mathbf{v}_h) \qquad \forall\ \mathbf{v}_h \in (V_h)^d \\ b(\mathbf{w}_h, q_h) = -b(\boldsymbol{\Phi}_h, q_h) \qquad \forall\ q_h \in Q_h, \end{cases} \tag{5.3.60}$$

where

$$\mathcal{F}_h^*(\mathbf{v}_h) := (\mathbf{f}, \mathbf{v}_h)_\Omega + (\boldsymbol{\varphi}_N, \mathbf{v}_{h|\Gamma_N})_{\Gamma_N} - s(\boldsymbol{\Phi}_h, \mathbf{v}_h).$$

Also, this problem can be solved by a Dirichlet–Neumann iterative scheme, which reads as follows: given $\boldsymbol{\xi}_h^0 \in (\Lambda_h)^d$, solve for $k \geq 0$

$$\begin{cases} \text{find } (\mathbf{w}_{1,h}^{k+1}, p_{1,h}^{k+1}) \in (V_{1,h})^d \times Q_{1,h}\ : \\ s_1(\mathbf{w}_{1,h}^{k+1}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, p_{1,h}^{k+1}) = \mathcal{F}_{1,h}^*(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\mathbf{w}_{1,h}^{k+1}, q_{1,h}) = -b_1(\boldsymbol{\Phi}_{h|\Omega_1}, q_{1,h}) \quad \forall\ q_{1,h} \in Q_{1,h} \\ \mathbf{w}_{1,h}^{k+1} = \boldsymbol{\xi}_h^k \qquad \text{on } \Gamma, \end{cases} \tag{5.3.61}$$

then

$$\begin{cases} \text{find } (\mathbf{w}_{2,h}^{k+1}, p_{2,h}^{k+1}) \in (V_{2,h})^d \times Q_{2,h}\ : \\ s_2(\mathbf{w}_{2,h}^{k+1}, \mathbf{v}_{2,h}) + b_2(\mathbf{v}_{2,h}, p_{2,h}^{k+1}) = \mathcal{F}_{2,h}^*(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\ b_2(\mathbf{w}_{2,h}^{k+1}, q_{2,h}) = -b_2(\boldsymbol{\Phi}_{h|\Omega_2}, q_{2,h}) \quad \forall\ q_{2,h} \in Q_{2,h} \\ s_2(\mathbf{w}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}^{k+1}) \\ \qquad = \mathcal{F}_{2,h}^*(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + \mathcal{F}_{1,h}^*(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) \\ \qquad\quad -s_1(\mathbf{w}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}^{k+1}) \quad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d \end{cases} \tag{5.3.62}$$

and

$$\boldsymbol{\xi}_h^{k+1} = \theta \mathbf{w}_{2,h|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\xi}_h^k \qquad \text{on } \Gamma. \tag{5.3.63}$$

It can be shown that the sequence $\boldsymbol{\gamma}_h^k := \boldsymbol{\xi}_h^k - \mathbf{w}_{h|\Gamma}$ satisfies the recursive equation

$$\boldsymbol{\gamma}_h^{k+1} = \boldsymbol{\gamma}_h^k - \theta S_{2,h}^{-1} S_h \boldsymbol{\gamma}_h^k,$$

where the Steklov–Poincaré operators $S_{i,h}$ are defined as in (5.3.50), and $S_h := S_{1,h} + S_{2,h}$. As a consequence of Theorem 4.2.2, $\boldsymbol{\gamma}_h^k$ converges to $\mathbf{0}$ (equivalently, $\boldsymbol{\xi}_h^k$ converges to $\mathbf{w}_{h|\Gamma} = \mathbf{u}_{h|\Gamma} - \boldsymbol{\Phi}_{h|\Gamma}$) with respect to the norm of $(\Lambda)^d$, at a rate that is independent of h.

A similar result holds for the Neumann–Neumann iterative scheme, which reads: for $i = 1, 2$,

$$\text{(5.3.64)} \qquad \begin{cases} \text{find } (\mathbf{w}_{i,h}^{k+1}, p_{i,h}^{k+1}) \in (V_{i,h})^d \times Q_{i,h} \ : \\ s_i(\mathbf{w}_{i,h}^{k+1}, \mathbf{v}_{i,h}) + b_i(\mathbf{v}_{i,h}, p_{i,h}^{k+1}) = \mathcal{F}_{i,h}^*(\mathbf{v}_{i,h}) \quad \forall\ \mathbf{v}_{i,h} \in (V_{i,h}^0)^d \\ b_i(\mathbf{w}_{i,h}^{k+1}, q_{i,h}) = -b_i(\boldsymbol{\Phi}_{h|\Omega_i}, q_{i,h}) \quad \forall\ q_{i,h} \in Q_{i,h} \\ \mathbf{w}_{i,h}^{k+1} = \boldsymbol{\xi}_h^k \qquad \text{on } \Gamma, \end{cases}$$

then

$$\text{(5.3.65)} \qquad \begin{cases} \text{find } (\boldsymbol{\psi}_{1,h}^{k+1}, \pi_{1,h}^{k+1}) \in (V_{1,h})^d \times Q_{1,h} \ : \\ s_1(\boldsymbol{\psi}_{1,h}^{k+1}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, \pi_{1,h}^{k+1}) = 0 \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\boldsymbol{\psi}_{1,h}^{k+1}, q_{1,h}) = 0 \quad \forall\ q_{1,h} \in Q_{1,h} \\ s_1(\boldsymbol{\psi}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) + b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, \pi_{1,h}^{k+1}) \\ \qquad = -\mathcal{F}_{1,h}^*(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - \mathcal{F}_{2,h}^*(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) \\ \qquad\quad + s_1(\mathbf{w}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) + b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}^{k+1}) \\ \qquad\quad + s_2(\mathbf{w}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}^{k+1}) \quad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d \end{cases}$$

$$
\left\{\begin{array}{l}
\text{find } (\boldsymbol{\psi}_{2,h}^{k+1}, \pi_{2,h}^{k+1}) \in (V_{2,h})^d \times Q_{2,h} \ : \\
s_2(\boldsymbol{\psi}_{2,h}^{k+1}, \mathbf{v}_{2,h}) + b_2(\mathbf{v}_{2,h}, \pi_{2,h}^{k+1}) = 0 \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\
b_2(\boldsymbol{\psi}_{2,h}^{k+1}, q_{2,h}) = 0 \quad \forall\ q_{2,h} \in Q_{2,h} \\
s_2(\boldsymbol{\psi}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, \pi_{2,h}^{k+1}) \\
\qquad = \mathcal{F}_{1,h}^*(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) + \mathcal{F}_{2,h}^*(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) \\
\qquad\quad -s_1(\mathbf{w}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}^{k+1}) \\
\qquad\quad -s_2(\mathbf{w}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) - b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}^{k+1}) \quad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d,
\end{array}\right. \tag{5.3.66}
$$

setting finally

$$
\boldsymbol{\xi}_h^{k+1} = \boldsymbol{\xi}_h^k - \theta(\sigma_1 \boldsymbol{\psi}_{1,h|\Gamma}^{k+1} - \sigma_2 \boldsymbol{\psi}_{2,h|\Gamma}^{k+1}) \qquad \text{on } \Gamma. \tag{5.3.67}
$$

This scheme is equivalent to the preconditioned Richardson iteration

$$
\boldsymbol{\gamma}_h^{k+1} = \boldsymbol{\gamma}_h^k - \theta(\sigma_1 S_{1,h}^{-1} + \sigma_2 S_{2,h}^{-1}) S_h \boldsymbol{\gamma}_h^k,
$$

where $\boldsymbol{\gamma}_h^k := \boldsymbol{\xi}_h^k - \mathbf{w}_{h|\Gamma}$ and $\mathbf{w}_h$ is the solution to (5.3.60). Therefore, the proof of convergence, at a rate independent of h, follows from Theorem 4.2.5.

Let us consider now the pure Dirichlet case, namely, let $\Gamma_N = \emptyset$. The finite element variational approximation of the Stokes problem (5.3.24) reads:

$$
\left\{\begin{array}{l}
\text{find } \mathbf{W}_h \in (V_h)^d \subset (H_0^1(\Omega))^d,\ p_h \in Q_h \cap L_0^2(\Omega) \ : \\
s(\mathbf{W}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = \mathcal{F}_h(\mathbf{v}_h) \qquad \forall\ \mathbf{v}_h \in (V_h)^d \\
b(\mathbf{W}_h, q_h) = 0 \qquad \forall\ q_h \in Q_h \cap L_0^2(\Omega),
\end{array}\right. \tag{5.3.68}
$$

where

$$
\mathcal{F}_h(\mathbf{v}_h) := (\mathbf{f}, \mathbf{v}_h)_\Omega - s(\boldsymbol{\Phi}_h + \boldsymbol{\Psi}_h, \mathbf{v}_h).
$$

Note that the solution $\mathbf{W}_h$ to (5.3.68) indeed satisfies $b(\mathbf{W}_h, q_h) = 0$ for all functions $q_h \in Q_h$. In fact, let q_h be any function of Q_h; its average is

$$
(q_h)_\Omega =: \frac{1}{\text{meas } \Omega} \int_\Omega q_h.
$$

One has from (5.3.68)

$$b(\mathbf{W}_h, q_h) = b(\mathbf{W}_h, q_h - (q_h)_\Omega) + (q_h)_\Omega b(\mathbf{W}_h, 1)$$
$$= -(q_h)_\Omega \int_{\partial\Omega} \mathbf{W}_{h|\partial\Omega} \cdot \mathbf{n}^* = 0.$$

The last equality holds since $\mathbf{W}_{h|\partial\Omega} = \mathbf{0}$ when $\Gamma_N = \emptyset$.

The corresponding two-domain formulation (which is the finite dimensional counterpart of problem (5.3.25)) is given by: find $(\mathbf{W}_{1,h}, p_{1,h}) \in (V_{1,h})^d \times Q_{1,h}$, $(\mathbf{W}_{2,h}, p_{2,h}) \in (V_{2,h})^d \times Q_{2,h}$ such that

$$\text{(5.3.69)} \quad \begin{cases} s_1(\mathbf{W}_{1,h}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, p_{1,h}) = \mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\mathbf{W}_{1,h}, q_{1,h}) = 0 \quad \forall\ q_{1,h} \in Q_{1,h} \cap L_0^2(\Omega_1) \\ \mathbf{W}_{1,h} = \mathbf{W}_{2,h} \qquad \text{on } \Gamma \\ s_2(\mathbf{W}_{2,h}, \mathbf{v}_{2,h}) + b_2(\mathbf{v}_{2,h}, p_{2,h}) = \mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\ b_2(\mathbf{W}_{2,h}, q_{2,h}) = 0 \quad \forall\ q_{2,h} \in Q_{2,h} \\ s_2(\mathbf{W}_{2,h}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}) \\ \qquad = \mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + \mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) \\ \qquad\qquad -s_1(\mathbf{W}_{1,h}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}) \qquad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d \\ \displaystyle\int_{\Omega_1} p_{1,h} = -\int_{\Omega_2} p_{2,h}, \end{cases}$$

where

$$\mathcal{F}_{i,h}(\mathbf{v}_{i,h}) := (\mathbf{f}, \mathbf{v}_{i,h})_{\Omega_i} - s_i(\boldsymbol{\Phi}_{h|\Omega_i} + \boldsymbol{\Psi}_{h|\Omega_i}, \mathbf{v}_{i,h})$$

and $\boldsymbol{\mathcal{R}}_{i,h}$ denotes any extension operator from $(\Lambda_h)^d$ to $(V_{i,h})^d$, $i = 1, 2$.

By following the guidelines stated in the proof of Lemma 5.3.9, we have the following result.

Lemma 5.3.13 *The discrete Stokes problem* (5.3.48) *is equivalent to the two-domain formulation* (5.3.49)*, in the usual sense that* $\mathbf{W}_{h|\Omega_i} = \mathbf{W}_{i,h}$*,* $p_{h|\Omega_i} = p_{i,h}$ *for* $i = 1, 2$*.*

Let us denote by $(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, P_{i,h}\boldsymbol{\eta}_h) \in (V_{i,h})^d \times Q_{i,h} \cap L_0^2(\Omega)$ and $(\mathbf{U}_{i,h}^*, P_{i,h}^*) \in (V_{i,h}^0)^d \times Q_{i,h} \cap L^2(\Omega_i)$ the solutions to the problems (5.3.52), (5.3.53), respectively, where the space of test functions $Q_{i,h}$ in the continuity equation has been substituted by $Q_{i,h} \cap L_0^2(\Omega_i)$, $i = 1, 2$.

The discrete local Steklov–Poincaré operators $S_{i,h}$ and the linear functionals $\boldsymbol{\chi}_{i,h}$, $i = 1, 2$, are still defined as in (5.3.50) and (5.3.51), respectively, with $(\mathbf{U}_{i,h}\boldsymbol{\eta}_h, P_{i,h}\boldsymbol{\eta}_h)$ and $(\mathbf{U}_{i,h}^*, P_{i,h}^*)$ instead of $(\mathbf{U}_i\boldsymbol{\eta}, P_i\boldsymbol{\eta})$ and $(\mathbf{U}_i^*, P_i^*)$.

In order to derive the Steklov–Poincaré equation on Γ, it is useful to analyse the properties of the trace on Γ of the discrete solution $\mathbf{W}_h$ of the Stokes problem (5.3.68). Denoting by χ^{Ω_i} the characteristic function of Ω_i, the solution $\mathbf{W}_h$ satisfies

$$b(\mathbf{W}_h, \chi^{\Omega_i}) = 0,$$

as $\chi^{\Omega_i} \in Q_h$. Consequently,

$$\int_\Gamma \mathbf{W}_{h|\Gamma} \cdot \mathbf{n} = -b_1(\mathbf{W}_{h|\Omega_1}, 1) = -b(\mathbf{W}_h, \chi^{\Omega_1}) = 0, \tag{5.3.70}$$

because the solution $\mathbf{W}_h$ vanishes on $\partial\Omega_1 \setminus \Gamma$ (since $\Gamma_N = \emptyset$). Therefore, we introduce the space

$$\widehat{\mathbf{\Lambda}}_h := \left\{ \widehat{\boldsymbol{\eta}}_h \in (\Lambda_h)^d \mid \int_\Gamma \widehat{\boldsymbol{\eta}}_h \cdot \mathbf{n} = 0 \right\}, \tag{5.3.71}$$

which is the trace space on Γ of the discrete functions $\mathbf{v}_h \in (V_h)^d$ satisfying the continuity equation $b(\mathbf{v}_h, q_h) = 0$ for each $q_h \in Q_h$. In particular, as we have seen, $\mathbf{W}_{h|\Gamma} \in \widehat{\mathbf{\Lambda}}_h$.

Before stating the finite dimensional counterpart of Lemma 5.3.10, we note that the $(L^2(\Gamma))^d$-orthogonal complement of $\widehat{\mathbf{\Lambda}}_h$ in $(\Lambda_h)^d$ is a one-dimensional space. We denote by $\boldsymbol{\nu}_h^*$ the basis element of $(\widehat{\mathbf{\Lambda}}_h)^\perp$ satisfying

$$\frac{1}{\operatorname{meas}\Gamma} \int_\Gamma \boldsymbol{\nu}_h^* \cdot \mathbf{n} = 1.$$

We have

Lemma 5.3.14 *If we set $\widehat{\boldsymbol{\lambda}}_h := \mathbf{W}_{h|\Gamma}$, where $(\mathbf{W}_h, p_h)$ is the solution to* (5.3.68)*, then $\widehat{\boldsymbol{\lambda}}_h$ satisfies the following Steklov–Poincaré equation on Γ:*

$$\widehat{\boldsymbol{\lambda}}_h \in \widehat{\mathbf{\Lambda}}_h \ : \ \langle S_h \widehat{\boldsymbol{\lambda}}_h, \widehat{\boldsymbol{\mu}}_h \rangle = \langle \boldsymbol{\chi}_h, \widehat{\boldsymbol{\mu}}_h \rangle \qquad \forall\ \widehat{\boldsymbol{\mu}}_h \in \widehat{\mathbf{\Lambda}}_h. \tag{5.3.72}$$

Conversely, from the solution $\widehat{\boldsymbol{\lambda}}_h$ of (5.3.72) *we recover the solution of* (5.3.68) *by setting*

$$\begin{aligned} \mathbf{W}_{h|\Omega_i} &:= \mathbf{U}_{i,h}\boldsymbol{\lambda}_h + \mathbf{U}_{i,h}^* \\ p_{h|\Omega_i} &:= P_{i,h}\boldsymbol{\lambda}_h + P_{i,h}^* + \hat{p}_{i,h}, \end{aligned} \tag{5.3.73}$$

where $\hat{p}_{1,h}$ and $\hat{p}_{2,h}$ are obtained by solving the linear system:

$$\begin{cases} \hat{p}_{1,h} - \hat{p}_{2,h} = \dfrac{1}{\operatorname{meas}\Gamma} \langle S_h \widehat{\boldsymbol{\lambda}}_h - \boldsymbol{\chi}_h, \boldsymbol{\nu}_h^* \rangle \\ \hat{p}_{1,h} \operatorname{meas}\Omega_1 + \hat{p}_{2,h} \operatorname{meas}\Omega_2 = 0. \end{cases} \tag{5.3.74}$$

Proof We follow the guidelines stated in the proof of Lemma 5.3.10. The only point that deserves special attention concerns the verification that the functions given in (5.3.73) satisfy the interface relations stated in (5.3.69). Using (5.3.50) and (5.3.51) this reduces to proving that

$$\langle S_h\widehat{\boldsymbol{\lambda}}_h - \boldsymbol{\chi}_h, \boldsymbol{\mu}_h\rangle - (\hat{p}_{1,h} - \hat{p}_{2,h})\int_\Gamma \boldsymbol{\mu}_h \cdot \mathbf{n} = 0$$

for each $\boldsymbol{\mu}_h \in (\Lambda_h)^d$. If we define

$$(\mu_h)_\Gamma := \frac{1}{\text{meas }\Gamma}\int_\Gamma \boldsymbol{\mu}_h \cdot \mathbf{n},$$

then the $L^2(\Gamma)$-orthogonal projection of $\boldsymbol{\mu}_h$ on $\widehat{\boldsymbol{\Lambda}}_h$ is given by $\boldsymbol{\mu}_h - (\mu_h)_\Gamma \boldsymbol{\nu}_h^*$. Then we have

$$\begin{aligned}\langle S_h\widehat{\boldsymbol{\lambda}}_h - \boldsymbol{\chi}_h, \boldsymbol{\mu}_h\rangle &= \langle S_h\widehat{\boldsymbol{\lambda}}_h - \boldsymbol{\chi}_h, \boldsymbol{\mu}_h - (\mu_h)_\Gamma\boldsymbol{\nu}_h^*\rangle + (\mu_h)_\Gamma\langle S_h\widehat{\boldsymbol{\lambda}}_h - \boldsymbol{\chi}_h, \boldsymbol{\nu}_h^*\rangle \\ &= (\hat{p}_{1,h} - \hat{p}_{2,h})\int_\Gamma \boldsymbol{\mu}_h \cdot \mathbf{n},\end{aligned}$$

having used (5.3.72) and $(5.3.74)_1$. □

Also in this case, the Steklov–Poincaré operators S_h, $S_{1,h}$ and $S_{2,h}$ are symmetric, and continuous in $(\Lambda_h)^d$, uniformly with respect to h. Moreover, this time both operators $S_{1,h}$ and $S_{2,h}$ are coercive, uniformly with respect to h (as in the case where $\Gamma_N \neq \emptyset$, with the decomposition of Fig. 5.3.4, left).

The Dirichlet–Neumann scheme now reads: given $\widehat{\boldsymbol{\lambda}}_h^0 \in \widehat{\boldsymbol{\Lambda}}_h$, for $k \geq 0$

$$\text{(5.3.75)}\qquad \begin{cases} \text{find } (\mathbf{W}_{1,h}^{k+1}, p_{1,h}^{k+1}) \in (V_{1,h})^d \times Q_{1,h} \ : \\ s_1(\mathbf{W}_{1,h}^{k+1}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, p_{1,h}^{k+1}) = \mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\mathbf{W}_{1,h}^{k+1}, q_{1,h}) = 0 \quad \forall\ q_{1,h} \in Q_{1,h} \cap L_0^2(\Omega) \\ \mathbf{W}_{1,h}^{k+1} = \widehat{\boldsymbol{\lambda}}_h^k \qquad \text{on } \Gamma \\ \displaystyle\int_{\Omega_1} p_{1,h}^{k+1} = -\int_{\Omega_2} p_{2,h}^k, \end{cases}$$

followed by (5.3.57) and (5.3.58).

This iteration is equivalent to the following Richardson scheme

$$\text{(5.3.76)}\qquad \begin{cases} \widehat{\boldsymbol{\lambda}}_h^0 \in \widehat{\boldsymbol{\Lambda}}_h \\ \widehat{\boldsymbol{\lambda}}_h^{k+1} = \widehat{\boldsymbol{\lambda}}_h^k + \theta S_{2,h}^{-1}(\boldsymbol{\chi}_h - S_h\widehat{\boldsymbol{\lambda}}_h^k), \quad k \geq 0, \end{cases}$$

whose convergence, with respect to the norm of $(\Lambda)^d$ and at a rate independent of h, is proved in the same way as for the scheme (5.3.59).

The same result can be proved for the Neumann–Neumann iterative scheme.

Remark 5.3.15 Other approaches for the iterative solution of the Stokes problem in the framework of non-overlapping domain partitioning are proposed in Pasciak (1989), Bramble and Pasciak (1989), Fischer and Rønquist (1994), Rønquist (1996), Casarin (1996), Le Tallec and Patra (1997), Pavarino and Widlund (1997), and Pavarino (1998).

For the construction of block-diagonal or block-triangular preconditioners based on overlapping subdomains we refer to Klawonn (1996, 1998*a*, *b*), Klawonn and Pavarino (1998*a*, *b*), and Klawonn and Starke (1999). □

5.3.4 Finite dimensional approximation: the case of continuous pressure

Let us consider now the case of *continuous* discrete pressures; namely, the case in which the finite element spaces satisfy $V_h \subset H^1_{\Gamma_D}(\Omega)$, $Q_h = X_h^q \subset C^0(\overline{\Omega})$ for $q \geq 1$. Their subdomain and interface counterparts, denoted $V_{i,h}$, $V^0_{i,h}$, Λ_h, $Q_{i,h}$, are defined as in (5.3.41). We still assume that (5.3.42) is satisfied, and set

$$\begin{aligned} Q^0_{i,h} &:= \{q_{i,h} \in Q_{i,h} \,|\, q_{i,h|\Gamma} = 0\} \\ \Xi_h &:= \{q_{h|\Gamma} \,|\, q_h \in Q_h\}. \end{aligned} \tag{5.3.77}$$

Note that the space Ξ_h is also the trace space on Γ of both $Q_{i,h}$ and $Q_{i,h} \cap L^2_0(\Omega_i)$, for each $i = 1, 2$.

We assume that the inf–sup condition (5.3.43) is satisfied for the finite element spaces $(V_h)^d \subset (H^1_0(\Omega))^d$ and $Q_h \cap L^2_0(\Omega)$ (when $\Gamma_N = \emptyset$), or that the inf–sup condition (5.3.45) is satisfied for the spaces $(V_h)^d \subset (H^1_{\Gamma_D}(\Omega))^d$ and Q_h (when $\Gamma_N \neq \emptyset$), with a constant β independent of h. Similarly, in the case where $\Gamma_N = \emptyset$ we assume that (5.3.44) is satisfied for the couple of spaces $(V^0_{i,h})^d \subset (H^1_0(\Omega_i))^d$ and $Q^0_{i,h} \cap L^2_0(\Omega_i)$, as well as for the couple of spaces $(V_{i,h})^d$ and $Q_{i,h}$, with constants β_i independent of h. In the case where $\Gamma_N \neq \emptyset$ we assume that (5.3.46) is satisfied for $(V^0_{i,h})^d \subset (H^1_{\Gamma_D \cup \Gamma}(\Omega_i))^d$ and $Q^0_{i,h}$ and for $(V_{i,h})^d$ and $Q_{i,h}$, with constants β_i independent of h.

As noted before, the inf–sup conditions (5.3.44) for $(V^0_{i,h})^d$ and $Q^0_{i,h} \cap L^2_0(\Omega_i)$ are very often a consequence of (5.3.43) for $(V_h)^d$ and $Q_h \cap L^2_0(\Omega)$; similarly, the inf–sup conditions (5.3.46) for $(V^0_{i,h})^d$ and $Q^0_{i,h}$, or $(V_{i,h})^d$ and $Q_{i,h}$ are very often a consequence of (5.3.45) for $(V_h)^d$ and Q_h.

The finite element approximation of the Stokes problem is still defined as in (5.3.48) (for $\Gamma_N \neq \emptyset$) or (5.3.68) (for $\Gamma_N = \emptyset$), but now the spaces V_h and Q_h are chosen as indicated above. Denoting by $\rho_{i,h}$ any extension operator from Ξ_h to $Q_{i,h}$, in the case where $\Gamma_N \neq \emptyset$ the two-domain equivalent formulation reads: find $(\mathbf{W}_{1,h}, p_{1,h}) \in (V_{1,h})^d \times Q_{1,h}$, $(\mathbf{W}_{2,h}, p_{2,h}) \in (V_{2,h})^d \times Q_{2,h}$ such that

$$
\begin{cases}
s_1(\mathbf{W}_{1,h},\mathbf{v}_{1,h})+b_1(\mathbf{v}_{1,h},p_{1,h})=\mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h}\in (V_{1,h}^0)^d\\
b_1(\mathbf{W}_{1,h},q_{1,h})=0 \quad \forall\ q_{1,h}\in Q_{1,h}^0\\
\mathbf{W}_{1,h}=\mathbf{W}_{2,h} \qquad \text{on } \Gamma\\
p_{1,h}=p_{2,h} \qquad \text{on } \Gamma\\
s_2(\mathbf{W}_{2,h},\mathbf{v}_{2,h})+b_2(\mathbf{v}_{2,h},p_{2,h})=\mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h}\in (V_{2,h}^0)^d\\
b_2(\mathbf{W}_{2,h},q_{2,h})=0 \quad \forall\ q_{2,h}\in Q_{2,h}^0\\
s_2(\mathbf{W}_{2,h},\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h,p_{2,h})\\
\qquad =\mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+\mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)\\
\qquad\qquad -s_1(\mathbf{W}_{1,h},\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)-b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h,p_{1,h}) \qquad \forall\ \boldsymbol{\mu}_h\in(\Lambda_h)^d\\
b_2(\mathbf{W}_{2,h},\rho_{2,h}\psi_h)=-b_1(\mathbf{W}_{1,h},\rho_{1,h}\psi_h) \qquad \forall\ \psi_h\in\Xi_h.
\end{cases}
\tag{5.3.78}
$$

Similarly, in the case where $\Gamma_N=\emptyset$ the two-domain equivalent formulation reads: find $(\mathbf{W}_{1,h},p_{1,h})\in (V_{1,h})^d\times Q_{1,h}$, $(\mathbf{W}_{2,h},p_{2,h})\in (V_{2,h})^d\times Q_{2,h}$ such that

$$
\begin{cases}
s_1(\mathbf{W}_{1,h},\mathbf{v}_{1,h})+b_1(\mathbf{v}_{1,h},p_{1,h})=\mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h}\in (V_{1,h}^0)^d\\
b_1(\mathbf{W}_{1,h},q_{1,h})=0 \quad \forall\ q_{1,h}\in Q_{1,h}^0\cap L_0^2(\Omega_1)\\
\mathbf{W}_{1,h}=\mathbf{W}_{2,h} \qquad \text{on } \Gamma\\
p_{1,h}=p_{2,h} \qquad \text{on } \Gamma\\
s_2(\mathbf{W}_{2,h},\mathbf{v}_{2,h})+b_2(\mathbf{v}_{2,h},p_{2,h})=\mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h}\in (V_{2,h}^0)^d\\
b_2(\mathbf{W}_{2,h},q_{2,h})=0 \quad \forall\ q_{2,h}\in Q_{2,h}^0\\
s_2(\mathbf{W}_{2,h},\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h,p_{2,h})\\
\qquad =\mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h)+\mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)\\
\qquad\qquad -s_1(\mathbf{W}_{1,h},\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h)-b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h,p_{1,h}) \qquad \forall\ \boldsymbol{\mu}_h\in(\Lambda_h)^d\\
b_2(\mathbf{W}_{2,h},\rho_{2,h}\psi_h)=-b_1(\mathbf{W}_{1,h},\widehat{\rho}_{1,h}\psi_h) \qquad \forall\ \psi_h\in\Xi_h\\
\displaystyle\int_{\Omega_1} p_{1,h}=-\int_{\Omega_2} p_{2,h},
\end{cases}
\tag{5.3.79}
$$

where $\widehat{\rho}_{1,h}$ denotes any extension operator from Ξ_h to $Q_{1,h} \cap L^2_0(\Omega_1)$. Note that the usual interpolation operator in Ω_1, modified at one node to ensure that its mean value is zero, can be used as $\widehat{\rho}_{1,h}$.

Lemma 5.3.16 *When $\Gamma_N \neq \emptyset$, the discrete Stokes problem* (5.3.48) *is equivalent to the two-domain formulation* (5.3.78). *When $\Gamma_N = \emptyset$, the discrete Stokes problem* (5.3.68) *is equivalent to the two-domain formulation* (5.3.79).

Proof Let us confine ourselves to the case where $\Gamma_N = \emptyset$, because the proof in the case where $\Gamma_N \neq \emptyset$ is easier.

Let $(\mathbf{W}_h, p_h)$ be a solution to (5.3.68): the proof that its restrictions $\mathbf{W}_{i,h} := \mathbf{W}_{h|\Omega_i}$ and $p_{i,h} := p_{h|\Omega_i}$ satisfy (5.3.79) is easy. In fact, the local equations in Ω_i, $i = 1, 2$, are satisfied by extending by zero the local test functions $\mathbf{v}_{i,h}$ and $q_{i,h}$. The two equations on Γ are satisfied because the momentum equation in (5.3.68) holds for every $\mathbf{v}_h \in (V_h)^d$; in particular, for $\mathbf{v}_h = \boldsymbol{\mathcal{R}}\boldsymbol{\mu}_h$, where $\boldsymbol{\mu}_h \in (\Lambda_h)^d$ and

$$\boldsymbol{\mathcal{R}}\boldsymbol{\mu}_h := \begin{cases} \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h & \text{in } \Omega_1 \\ \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h & \text{in } \Omega_2 \end{cases},$$

and the continuity equation in (5.3.68) holds for every $q_h \in Q_h$; in particular, for $q_h = \rho_h\psi_h$, where $\psi_h \in \Xi_h$ and

$$\rho_h\psi_h := \begin{cases} \widehat{\rho}_{1,h}\psi_h & \text{in } \Omega_1 \\ \rho_{2,h}\psi_h & \text{in } \Omega_2 \end{cases}.$$

Conversely, if $(\mathbf{W}_{i,h}, p_{i,h})$, $i = 1, 2$, is the solution to (5.3.79), let us set

$$\mathbf{W}_h := \begin{cases} \mathbf{W}_{1,h} & \text{in } \Omega_1 \\ \mathbf{W}_{2,h} & \text{in } \Omega_2 \end{cases}, \quad p_h := \begin{cases} p_{1,h} & \text{in } \Omega_1 \\ p_{2,h} & \text{in } \Omega_2 \end{cases}.$$

Clearly, $\mathbf{W}_h \in (V_h)^d$ and $p_h \in Q_h \cap L^2_0(\Omega)$. Proceeding as in the proof of Lemma 5.3.9 one sees that the only property that is not immediate is

$$b(\mathbf{W}_h, q_h) = 0 \qquad \forall\ q_h \in Q_h \cap L^2_0(\Omega).$$

To verify this, first take $q_h \in Q_h \cap L^2_0(\Omega)$ such that $q_{h|\Omega_1} \in L^2_0(\Omega_1)$. Setting $\psi_h := q_{h|\Gamma} \in \Xi_h$, it follows that

$$\begin{aligned} b(\mathbf{W}_h, q_h) &= \sum_{i=1}^{2} b_i(\mathbf{W}_{i,h}, q_{h|\Omega_i}) \\ &= b_1(\mathbf{W}_{1,h}, q_{h|\Omega_1} - \widehat{\rho}_{1,h}\psi_h) + b_1(\mathbf{W}_{1,h}, \widehat{\rho}_{1,h}\psi_h) \\ &\quad + b_2(\mathbf{W}_{2,h}, q_{h|\Omega_2} - \rho_{2,h}\psi_h) + b_2(\mathbf{W}_{2,h}, \rho_{2,h}\psi_h) \\ &= b_1(\mathbf{W}_{1,h}, \widehat{\rho}_{1,h}\psi_h) + b_2(\mathbf{W}_{2,h}, \rho_{2,h}\psi_h) = 0. \end{aligned}$$

If now q_h is an arbitrary function in $Q_h \cap L^2_0(\Omega)$, we set

$$(q_h)_{\Omega_1} := \frac{1}{\operatorname{meas}\Omega_1}\int_{\Omega_1} q_{h|\Omega_1}$$

and obtain

$$\begin{aligned} b(\mathbf{W}_h, q_h) &= b(\mathbf{W}_h, q_h - (q_h)_{\Omega_1}) + (q_h)_{\Omega_1} b(\mathbf{W}_h, 1) \\ &= -(q_h)_{\Omega_1}\int_{\partial\Omega} \mathbf{W}_{h|\partial\Omega}\cdot\mathbf{n}^* = 0. \end{aligned}$$

The last equality holds since $\mathbf{W}_{h|\partial\Omega} = \mathbf{0}$ when $\Gamma_N = \emptyset$. □

For the two-domain formulation (5.3.78) or (5.3.79), we can formulate the Dirichlet–Neumann procedure as follows. Let us consider only the case where $\Gamma_N = \emptyset$, the other being similar. Given $\boldsymbol{\lambda}_h^0 \in (\Lambda_h)^d$, $\sigma^0 \in \Xi_h$, for each $k \geq 0$

$$\text{(5.3.80)}\quad \begin{cases} \text{find } (\mathbf{W}_{1,h}^{k+1}, p_{1,h}^{k+1}) \in (V_{1,h})^d \times Q_{1,h} \ : \\ s_1(\mathbf{W}_{1,h}^{k+1}, \mathbf{v}_{1,h}) + b_1(\mathbf{v}_{1,h}, p_{1,h}^{k+1}) = \mathcal{F}_{1,h}(\mathbf{v}_{1,h}) \quad \forall\ \mathbf{v}_{1,h} \in (V_{1,h}^0)^d \\ b_1(\mathbf{W}_{1,h}^{k+1}, q_{1,h}) = 0 \quad \forall\ q_{1,h} \in Q_{1,h}^0 \cap L_0^2(\Omega_1) \\ \mathbf{W}_{1,h}^{k+1} = \boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma \\ p_{1,h}^{k+1} = \sigma_h^k \qquad \text{on } \Gamma \\ \displaystyle\int_{\Omega_1} p_{1,h}^{k+1} = -\int_{\Omega_2} p_{2,h}^k, \end{cases}$$

then

$$\text{(5.3.81)}\quad \begin{cases} \text{find } (\mathbf{W}_{2,h}^{k+1}, p_{2,h}^{k+1}) \in (V_{2,h})^d \times Q_{2,h} \ : \\ s_2(\mathbf{W}_{2,h}^{k+1}, \mathbf{v}_{2,h}) + b_2(\mathbf{v}_{2,h}, p_{2,h}^{k+1}) = \mathcal{F}_{2,h}(\mathbf{v}_{2,h}) \quad \forall\ \mathbf{v}_{2,h} \in (V_{2,h}^0)^d \\ b_2(\mathbf{W}_{2,h}^{k+1}, q_{2,h}) = 0 \quad \forall\ q_{2,h} \in Q_{2,h}^0 \\ s_2(\mathbf{W}_{2,h}^{k+1}, \boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + b_2(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h, p_{2,h}^{k+1}) \\ \qquad = \mathcal{F}_{2,h}(\boldsymbol{\mathcal{R}}_{2,h}\boldsymbol{\mu}_h) + \mathcal{F}_{1,h}(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) \\ \qquad\quad -s_1(\mathbf{W}_{1,h}^{k+1}, \boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h) - b_1(\boldsymbol{\mathcal{R}}_{1,h}\boldsymbol{\mu}_h, p_{1,h}^{k+1}) \qquad \forall\ \boldsymbol{\mu}_h \in (\Lambda_h)^d \\ b_2(\mathbf{W}_{2,h}^{k+1}, \rho_{2,h}\psi_h) = -b_1(\mathbf{W}_{1,h}^{k+1}, \widehat{\rho}_{1,h})\psi_h \qquad \forall\ \psi_h \in \Xi_h, \end{cases}$$

and finally set

$$\text{(5.3.82)}\qquad \boldsymbol{\lambda}_h^{k+1} = \theta\mathbf{W}_{2,h|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}_h^k \qquad \text{on } \Gamma$$

$$\sigma_h^{k+1} = \theta^* p_{2,h|\Gamma}^{k+1} + (1-\theta^*)\sigma_h^k \qquad \text{on } \Gamma. \tag{5.3.83}$$

The existence and uniqueness of the solutions to (5.3.80) and (5.3.81) is ensured by the validity of the inf–sup condition (5.3.44) for the discrete spaces $(V_{1,h}^0)^d$ and $Q_{1,h}^0 \cap L_0^2(\Omega_1)$, and $(V_{2,h})^d$ and $Q_{2,h}$, respectively.

The interpretation of the Dirichlet–Neumann method (5.3.80)–(5.3.83) as a preconditioned scheme for a Steklov–Poincaré equation is an open question, as is also its convergence for suitable choices of the relaxation parameters θ and θ^*.

5.3.5 Methods based on the Uzawa pressure operator

Another approach for the Stokes problem that is amenable to the use of domain decomposition algorithms is as follows. Taking for simplicity $\boldsymbol{\varphi}_D = \mathbf{0}$ and $\Gamma_N = \emptyset$, and denoting by $\mathcal{G}$ the resolvent operator which associates with a given function $\mathbf{g}$ the solution $\mathbf{w}$ of the elliptic boundary value problem:

$$\begin{cases} -\nu\Delta\mathbf{w} = \mathbf{g} & \text{in } \Omega \\ \\ \mathbf{w} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{5.3.84}$$

the continuity equation $\operatorname{div}\mathbf{u} = 0$ in (5.3.2) provides the following equation for the pressure:

$$\mathcal{U}p = G, \tag{5.3.85}$$

where $\mathcal{U}$ is sometimes called the Uzawa (or pressure) operator and is defined by $\mathcal{U} := -\operatorname{div}\mathcal{G}\nabla$, while $G := -\operatorname{div}\mathcal{G}\mathbf{f}$.

Proposition 5.3.17 *The Uzawa operator $\mathcal{U}$ is symmetric and coercive in the space $L_0^2(\Omega)$.*

Proof The operator $-\operatorname{div}$ is the transpose of the operator ∇, and we have

$$(\mathcal{U}p, q) = (-\operatorname{div}\mathcal{G}\nabla p, q) = \langle \nabla q, \mathcal{G}\nabla p\rangle \qquad \forall\ p, q \in L^2(\Omega),$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Setting $\mathbf{w}^p := \mathcal{G}\nabla p$ and $\mathbf{w}^q := \mathcal{G}\nabla q$, we have that

$$(\mathcal{U}p, q) = \langle -\nu\Delta\mathbf{w}^q, \mathbf{w}^p\rangle = \nu \sum_{i,j=1}^{d} (D_i w_j^p, D_i w_j^q).$$

Moreover

$$(\mathcal{U}q, q) = \nu \sum_{i,j=1}^{d} ||D_i w_j^p||_{0,\Omega}^2 \geq \alpha_1 ||\mathbf{w}^q||_{1,\Omega}^2.$$

From the estimate

$$||\Delta\mathbf{w}^q||_{-1,\Omega} \leq C||\mathbf{w}^q||_{1,\Omega},$$

we finally find

$$(\mathcal{U}q, q) \geq \alpha_2 ||\nu \Delta \mathbf{w}^q||^2_{-1,\Omega} = \alpha_2 ||\nabla q||^2_{-1,\Omega} \geq \alpha_0 ||q||^2_{0,\Omega}.$$

In the last inequality we have made use of the estimate

$$||q||_{0,\Omega} \leq C_0 ||\nabla q||_{-1,\Omega} \qquad \forall\ q \in L^2_0(\Omega)$$

(see, for example, Girault and Raviart 1986, p. 20). □

The finite dimensional counterpart of the Uzawa operator is therefore a symmetric and positive definite matrix. It is a full matrix due to the presence of the 'inverse' operator $\mathcal{G}$. A convenient approach for solving the associated linear system can therefore be based on conjugate gradient iterations. At each step the computation of the residual amounts to the solution of an elliptic problem like (5.3.84). In turn, the latter can be reformulated in a multi-domain fashion as in Chapter 2, and solved iteratively as illustrated in Chapter 3.

We refer to Elman and Golub (1994), Bramble *et al.* (1997), and Maday *et al.* (1993) for other domain decomposition results for the Stokes problem based on the Uzawa operator.

5.4 The Stokes problem for compressible flows

We consider in this section a different type of Stokes problem that is related to the time-discretisation of the (linearised) Navier–Stokes equations for compressible flows.

The differential problem is given by the following set of equations:

$$\begin{cases} \alpha \mathbf{u} - \mathbf{div}\widehat{\mathbf{T}}(\mathbf{u}, \sigma) = \mathbf{f} & \text{in } \Omega \\ \alpha\sigma + \operatorname{div} \mathbf{u} = g & \text{in } \Omega \\ \mathbf{u} = \boldsymbol{\varphi}_D & \text{on } \Gamma_D \\ \nu \displaystyle\sum_{j=1}^{d} (D_l u_j + D_j u_l) n^*_j + \left(\gamma - \frac{2\nu}{d}\right) \operatorname{div} \mathbf{u}\, n^*_l & \\ \qquad -\beta\sigma n^*_l = (\boldsymbol{\varphi}_N)_l & \text{on } \Gamma_N,\ l = 1, \ldots, d, \end{cases} \tag{5.4.1}$$

where $\alpha > 0$, $\mathbf{f}$, $\boldsymbol{\varphi}_D$ and $\boldsymbol{\varphi}_N$ are given vector fields, and g is a given scalar function. The stress tensor $\widehat{\mathbf{T}}(\mathbf{w}, \pi)$ reads

$$\widehat{T}_{lj}(\mathbf{w}, \pi) := \nu(D_l w_j + D_j w_l) + \left(\gamma - \frac{2\nu}{d}\right) \operatorname{div} \mathbf{w}\, \delta_{lj} - \beta\pi\, \delta_{lj}, \tag{5.4.2}$$

where $\nu > 0$, $\gamma \geq 0$ and $\beta > 0$ are constants.

The model (5.4.1) has been introduced by Bristeau *et al.* (1987) as an intermediate stage of a fractional step algorithm to solve the Navier–Stokes equations for compressible flows. In this context, $\mathbf{u}$ denotes the fluid velocity, while σ is the logarithm of the fluid density.

A more familiar version of the momentum equation in (5.4.1) is obtained by noting that

$$-\mathbf{div}\widehat{\mathbf{T}}(\mathbf{w},\pi) = -\nu\Delta\mathbf{w} - \gamma^*\nabla\,\mathrm{div}\,\mathbf{w} + \beta\nabla\pi,$$

where $\gamma^* := \gamma + (d-2)\nu/d$.

Setting, as usual, $\mathbf{u}_i = \mathbf{u}_{|\Omega_i}$ and $\sigma_i = \sigma_{|\Omega_i}$, $i = 1, 2$, problem (5.4.1) can be split in a multi-domain fashion, and the interface conditions on Γ become

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{u}_2 \qquad \text{on } \Gamma\\
&\nu\sum_{j=1}^{d}(D_l u_{1,j} + D_j u_{1,l})n_j + \left(\gamma - \frac{2\nu}{d}\right)\mathrm{div}\,\mathbf{u}_1 n_l - \beta\sigma_1 n_l\\
&\quad = \nu\sum_{j=1}^{d}(D_l u_{2,j} + D_j u_{2,l})n_j + \left(\gamma - \frac{2\nu}{d}\right)\mathrm{div}\,\mathbf{u}_2 n_l\\
&\qquad\qquad -\beta\sigma_2 n_l \qquad \text{on } \Gamma,\ l = 1,\dots,d,
\end{aligned} \tag{5.4.3}$$

and impose continuity of the fluid velocity as well as that of normal stresses.

Remark 5.4.1 Note that system (5.4.1) can also be regarded as a mixed-type formulation for the elasticity problem (5.2.1). In fact, setting $\sigma := -\,\mathrm{div}\,\mathbf{u}$, equation $(5.2.1)_1$ becomes

$$\begin{cases} -\hat{\mu}\Delta\mathbf{u} - \hat{\lambda}\nabla\,\mathrm{div}\,\mathbf{u} + \hat{\mu}\nabla\sigma = \mathbf{f} & \text{in } \Omega\\ \sigma + \mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega, \end{cases}$$

which has a structure very similar to that of the Stokes system for compressible flows (only the term $\alpha\mathbf{u}$ is missing in the momentum equation). Therefore, all the results that we shall present in this section can be extended to the elasticity problem as well, provided that $\Gamma_D \neq \emptyset$. □

5.4.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

We introduce now the bilinear form

$$\widehat{s}(\mathbf{w},\mathbf{v}) := \int_\Omega \left[\alpha\mathbf{w}\cdot\mathbf{v} + \left(\gamma - \frac{2\nu}{d}\right)\mathrm{div}\,\mathbf{w}\ \mathrm{div}\,\mathbf{v} + \frac{\nu}{2}\sum_{l,m=1}^{d}(D_l w_m + D_m w_l)(D_l v_m + D_m v_l)\right].$$

Assuming that $\mathbf{f} \in (L^2(\Omega))^d$, $g \in L^2(\Omega)$, $\boldsymbol{\varphi}_D \in (H^{1/2}(\Gamma_D))^d$ and $\boldsymbol{\varphi}_N \in L^2(\Gamma_N))^d$, the variational formulation of (5.4.1) is given by

$$
\text{(5.4.4)}\quad
\begin{cases}
\text{find } (\mathbf{W},\sigma)\in (H^1_{\Gamma_D}(\Omega))^d\times L^2(\Omega)\ : \\
\widehat{s}(\mathbf{W},\mathbf{v})+\beta b(\mathbf{v},\sigma)=(\mathbf{f},\mathbf{v}) \\
\qquad\qquad +(\boldsymbol{\varphi}_N,\mathbf{v}_{|\Gamma_N})_{\Gamma_N}-\widehat{s}(\widehat{\boldsymbol{\varphi}}_D,\mathbf{v})\quad \forall\ \mathbf{v}\in (H^1_{\Gamma_D}(\Omega))^d \\
\alpha(\sigma,\pi)-b(\mathbf{W},\pi)=(g,\pi)+b(\widehat{\boldsymbol{\varphi}}_D,\pi)\quad \forall\ \pi\in L^2(\Omega),
\end{cases}
$$

where $\widehat{\boldsymbol{\varphi}}_D\in (H^1(\Omega))^d$ denotes any extension to Ω of the non-homogeneous Dirichlet datum $\boldsymbol{\varphi}_D$. The solution $\mathbf{u}$ of (5.4.1) is obtained by adding $\mathbf{W}$ with $\widehat{\boldsymbol{\varphi}}_D$. The existence of the solution $(\mathbf{W},\sigma)$ to (5.4.4) is a straightforward consequence of the Lax–Milgram lemma, because the bilinear form

$$
\begin{aligned}
\mathcal{A}^{(\nu)}[(\mathbf{w},\rho),(\mathbf{v},\pi)] :=\ & \widehat{s}(\mathbf{w},\mathbf{v})+\beta b(\mathbf{v},\rho) \\
& -\beta b(\mathbf{w},\pi)+\beta\alpha(\rho,\pi)
\end{aligned}
$$

is continuous and coercive in $(H^1_{\Gamma_D}(\Omega))^d\times L^2(\Omega)$. In fact, it can be proved that

$$
\begin{aligned}
\widehat{s}(\mathbf{w},\mathbf{w}) &= \int_\Omega \left[\alpha|\mathbf{w}|^2+\left(\gamma-\frac{2\nu}{d}\right)(\operatorname{div}\mathbf{w})^2\right. \\
&\qquad \left. +\frac{\nu}{2}\sum_{l,j=1}^{d}(D_l w_j+D_j w_l)^2\right] \\
&\geq K_0\|\mathbf{w}\|^2_{1,\Omega},
\end{aligned}
\tag{5.4.5}
$$

where $K_0>0$ only depends on α, ν, γ, d and Ω (see, for example, Secchi and Valli 1983).

Introducing the local bilinear forms

$$
\begin{aligned}
\widehat{s}_i(\mathbf{w}_i,\mathbf{v}_i) :=\ & \int_{\Omega_i}\left[\alpha\mathbf{w}_i\cdot\mathbf{v}_i+\left(\gamma-\frac{2\nu}{d}\right)\operatorname{div}\mathbf{w}_i\ \operatorname{div}\mathbf{v}_i\right. \\
& \left.+\frac{\nu}{2}\sum_{l,m=1}^{d}(D_l w_{i,m}+D_m w_{i,l})(D_l v_{i,m}+D_m v_{i,l})\right],
\end{aligned}
\tag{5.4.6}
$$

and assuming for simplicity that $\Gamma_N=\emptyset$ and $\boldsymbol{\varphi}_D=\mathbf{0}$, the multi-domain weak formulation of problem (5.4.1) can be written as follows: find $(\mathbf{u}_1,\sigma_1)\in (V_1)^d\times L^2(\Omega_1)$, $(\mathbf{u}_2,\sigma_2)\in (V_2)^d\times L^2(\Omega_2)$ such that

$$
\left\{\begin{array}{l}
\widehat{s}_1(\mathbf{u}_1,\mathbf{v}_1)+\beta b_1(\mathbf{v}_1,\sigma_1)=(\mathbf{f},\mathbf{v}_1)_{\Omega_1}\quad\forall\ \mathbf{v}_1\in(V_1^0)^d\\
\alpha(\sigma_1,\pi_1)_{\Omega_1}-b_1(\mathbf{u}_1,\pi_1)=(g,\pi_1)_{\Omega_1}\quad\forall\ \pi_1\in L^2(\Omega_1)\\
\mathbf{u}_1=\mathbf{u}_2\qquad\text{on }\Gamma\\
\widehat{s}_2(\mathbf{u}_2,\mathbf{v}_2)+\beta b_2(\mathbf{v}_2,\sigma_2)=(\mathbf{f},\mathbf{v}_2)_{\Omega_2}\quad\forall\ \mathbf{v}_2\in(V_2^0)^d\\
\alpha(\sigma_2,\pi_2)_{\Omega_2}-b_2(\mathbf{u}_2,\pi_2)=(g,\pi_2)_{\Omega_2}\quad\forall\ \pi_2\in L^2(\Omega_2)\\
\widehat{s}_2(\mathbf{u}_2,\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})+\beta b_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu},\sigma_2)=(\mathbf{f},\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})_{\Omega_2}+(\mathbf{f},\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})_{\Omega_1}\\
\qquad\qquad -\widehat{s}_1(\mathbf{u}_1,\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})-\beta b_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu},\sigma_1)\qquad\forall\ \boldsymbol{\mu}\in(\Lambda)^d,
\end{array}\right.
\tag{5.4.7}
$$

where, as before, $\boldsymbol{\mathcal{R}}_i$ denotes any extension operator from $(\Lambda)^d$ to $(V_i)^d$, $i=1,2$, and the rest of the notation is as in Section 1.2. The last equation is the weak form of the interface condition $(5.4.3)_2$ (the continuity of the normal stress), which can be formally recovered by integration by parts.

In the present case, the Steklov–Poincaré operators can be defined as follows. First, denote by $(\mathbf{U}_i^{(\nu)}\boldsymbol{\eta}, P_i^{(\nu)}\boldsymbol{\eta})\in(V_i)^d\times L^2(\Omega_i)$ the extension of the interface datum $\boldsymbol{\eta}$ given by the solution to

$$
\left\{\begin{array}{l}
\widehat{s}_i(\mathbf{U}_i^{(\nu)}\boldsymbol{\eta},\mathbf{v}_i)+\beta b_i(\mathbf{v}_i,P_i^{(\nu)}\boldsymbol{\eta})=0\quad\forall\ \mathbf{v}_i\in(V_i^0)^d\\
\alpha(P_i^{(\nu)}\boldsymbol{\eta},\pi_i)_{\Omega_i}-b_i(\mathbf{U}_i^{(\nu)}\boldsymbol{\eta},\pi_i)=0\quad\forall\ \pi_i\in L^2(\Omega_i)\\
\mathbf{U}_i^{(\nu)}\boldsymbol{\eta}=\boldsymbol{\eta}\qquad\text{on }\Gamma,
\end{array}\right.
\tag{5.4.8}
$$

and by $(\mathbf{U}_{i,*}^{(\nu)}, P_{i,*}^{(\nu)})\in(V_i^0)^d\times L^2(\Omega_i)$ the solution to

$$
\left\{\begin{array}{l}
\widehat{s}_i(\mathbf{U}_{i,*}^{(\nu)},\mathbf{v}_i)+\beta b_i(\mathbf{v}_i,P_{i,*}^{(\nu)})=(\mathbf{f},\mathbf{v}_i)_{\Omega_i}\quad\forall\ \mathbf{v}_i\in(V_i^0)^d\\
\alpha(P_{i,*}^{(\nu)},\pi_i)_{\Omega_i}-b_i(\mathbf{U}_{i,*}^{(\nu)},\pi_i)=(g,\pi_i)_{\Omega_i}\quad\forall\ \pi_i\in L^2(\Omega_i).
\end{array}\right.
\tag{5.4.9}
$$

For every $i=1,2$ we introduce the bilinear form

$$
\begin{aligned}
\mathcal{A}_i^{(\nu)}[(\mathbf{w}_i,\rho_i),(\mathbf{v}_i,\pi_i)] :&= \widehat{s}_i(\mathbf{w}_i,\mathbf{v}_i)+\beta b_i(\mathbf{v}_i,\rho_i)\\
&\quad -\beta b_i(\mathbf{w}_i,\pi_i)+\beta\alpha(\rho_i,\pi_i)_{\Omega_i},
\end{aligned}
\tag{5.4.10}
$$

which is continuous and coercive on the product space $(V_i^0)^d\times L^2(\Omega_i)$, and the linear functional

$$
\begin{aligned}
\langle\boldsymbol{\chi}_i,\boldsymbol{\mu}\rangle :&= (\mathbf{f},\mathbf{U}_i^{(\nu)}\boldsymbol{\mu})_{\Omega_i}+\beta(g,P_i^{(\nu)}\boldsymbol{\mu})_{\Omega_i}\\
&\quad -\mathcal{A}_i^{(\nu)}[(\mathbf{U}_{i,*}^{(\nu)},P_{i,*}^{(\nu)}),(\mathbf{U}_i^{(\nu)}\boldsymbol{\mu},P_i^{(\nu)}\boldsymbol{\mu})].
\end{aligned}
\tag{5.4.11}
$$

The local Steklov–Poincaré operators are then defined as

$$\langle S_i\boldsymbol{\eta},\boldsymbol{\mu}\rangle := \mathcal{A}_i^{(\nu)}[(\mathbf{U}_i^{(\nu)}\boldsymbol{\eta}, P_i^{(\nu)}\boldsymbol{\eta}),(\mathbf{U}_i^{(\nu)}\boldsymbol{\mu}, P_i^{(\nu)}\boldsymbol{\mu})]. \tag{5.4.12}$$

It is worthwhile noting that, although the bilinear forms $\mathcal{A}_i^{(\nu)}$ are not symmetric, the local Steklov–Poincaré operators are indeed symmetric. In fact, from $(5.4.8)_2$ we obtain

$$b_i(\mathbf{U}_i^{(\nu)}\boldsymbol{\mu}, P_i^{(\nu)}\boldsymbol{\eta}) = -\alpha(P_i^{(\nu)}\boldsymbol{\mu}, P_i^{(\nu)}\boldsymbol{\eta})_{\Omega_i},$$

therefore

$$\langle S_i\boldsymbol{\eta},\boldsymbol{\mu}\rangle = \widehat{s}_i(\mathbf{U}_i^{(\nu)}\boldsymbol{\eta}, \mathbf{U}_i^{(\nu)}\boldsymbol{\mu}) + \beta\alpha(P_i^{(\nu)}\boldsymbol{\mu}, P_i^{(\nu)}\boldsymbol{\eta})_{\Omega_i} = \langle S_i\boldsymbol{\mu},\boldsymbol{\eta}\rangle.$$

A similar computation shows that

$$\begin{aligned}\mathcal{A}_i^{(\nu)}[(\mathbf{U}_{i,*}^{(\nu)}, P_{i,*}^{(\nu)}),(\mathbf{U}_i^{(\nu)}\boldsymbol{\mu}, P_i^{(\nu)}\boldsymbol{\mu})] = \widehat{s}_i(\mathbf{U}_{i,*}^{(\nu)}, \mathbf{U}_i^{(\nu)}\boldsymbol{\mu}) + \beta b_i(\mathbf{U}_i^{(\nu)}\boldsymbol{\mu}, P_{i,*}^{(\nu)})\\ +\beta(g, P_i^{(\nu)}\boldsymbol{\mu})_{\Omega_i}.\end{aligned}$$

Moreover, by well known a priori estimates for the solution of (5.4.8), both of the local Steklov–Poincaré S_i operators is easily shown to be continuous in $(\Lambda)^d$. Finally, owing to (5.4.5) and the trace inequality (1.2.5) they are coercive in $(\Lambda)^d$.

The solution $(\mathbf{u},\sigma)$ to (5.4.4) (with $\boldsymbol{\varphi}_D = \mathbf{0}$ and $\Gamma_N = \emptyset$) satisfies

$$\mathbf{u}_{|\Omega_i} = \mathbf{U}_i^{(\nu)}\mathbf{u}_{|\Gamma} + \mathbf{U}_{i,*}^{(\nu)}, \quad \sigma_{|\Omega_i} = P_i^{(\nu)}\mathbf{u}_{|\Gamma} + P_{i,*}^{(\nu)}.$$

Setting $\boldsymbol{\lambda} := \mathbf{u}_{|\Gamma}$, and inserting the latter expressions in the last equation in (5.4.7) (the stress balance condition), we easily obtain the following Steklov–Poincaré interface equation on Γ:

$$S\boldsymbol{\lambda} = \boldsymbol{\chi}, \tag{5.4.13}$$

where we have set, as usual, $S := S_1 + S_2$ and $\boldsymbol{\chi} := \boldsymbol{\chi}_1 + \boldsymbol{\chi}_2$.

5.4.2 Substructuring iterative methods

The Dirichlet–Neumann iterative scheme for solving (5.4.7) reads: given $\boldsymbol{\lambda}^0 \in (\Lambda)^d$, for each $k \geq 0$ solve

$$\begin{cases} \text{find } (\mathbf{u}_1^{k+1}, \sigma_1^{k+1}) \in (V_1)^d \times L^2(\Omega_1) \ : \\ \widehat{s}_1(\mathbf{u}_1^{k+1}, \mathbf{v}_1) + \beta b_1(\mathbf{v}_1, \sigma_1^{k+1}) = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1} \quad \forall\ \mathbf{v}_1 \in (V_1^0)^d \\ \alpha(\sigma_1^{k+1}, \pi_1)_{\Omega_1} - b_1(\mathbf{u}_1^{k+1}, \pi_1) = (g, \pi_1)_{\Omega_1} \quad \forall\ \pi_1 \in L^2(\Omega_1) \\ \mathbf{u}_1 = \boldsymbol{\lambda}^k \qquad \text{on } \Gamma, \end{cases} \tag{5.4.14}$$

then

$$
\left\{
\begin{aligned}
&\text{find } (\mathbf{u}_2^{k+1}, \sigma_2^{k+1}) \in (V_2)^d \times L^2(\Omega_2) \ : \\
&\widehat{s}_2(\mathbf{u}_2^{k+1}, \mathbf{v}_2) + \beta b_2(\mathbf{v}_2, \sigma_2^{k+1}) = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \quad \forall\ \mathbf{v}_2 \in (V_2^0)^d \\
&\alpha(\sigma_2^{k+1}, \pi_2)_{\Omega_2} - b_2(\mathbf{u}_2^{k+1}, \pi_2) = (g, \pi_2)_{\Omega_2} \quad \forall\ \pi_2 \in L^2(\Omega_2) \\
&\widehat{s}_2(\mathbf{u}_2^{k+1}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + \beta b_2(\boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}, \sigma_2^{k+1}) \\
&\qquad = (\mathbf{f}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})_{\Omega_2} + (\mathbf{f}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})_{\Omega_1} \\
&\qquad\quad -\widehat{s}_1(\mathbf{u}_1^{k+1}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) - \beta b_1(\boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}, \sigma_1^{k+1}) \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d,
\end{aligned}
\right. \tag{5.4.15}
$$

and finally set

$$
\boldsymbol{\lambda}^{k+1} := \theta \mathbf{u}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k \qquad \text{on } \Gamma. \tag{5.4.16}
$$

This iterative scheme is equivalent to the preconditioned Richardson method

$$
\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta S_2^{-1}(\boldsymbol{\chi} - S\boldsymbol{\lambda}^k) \tag{5.4.17}
$$

for the Steklov–Poincaré equation (5.4.13), as can be shown following the guidelines as in Section 1.3. The convergence of this scheme is again a consequence of Theorem 4.2.2, because we have already noted that both Steklov–Poincaré operators are symmetric, continuous, and coercive in $(\Lambda)^d$.

A similar result can be obtained for the Neumann–Neumann iterative scheme, which can be easily defined following what was carried out in Section 1.3 for the Laplace operator. It can be equivalently formulated as an iterative procedure on the interface Γ:

$$
\boldsymbol{\lambda}^{k+1} = \boldsymbol{\lambda}^k + \theta(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})(\boldsymbol{\chi} - S\boldsymbol{\lambda}^k).
$$

All the assumptions of Theorem 4.2.5 are satisfied, therefore the convergence in $(\Lambda)^d$ of $\boldsymbol{\lambda}^k$ follows at once.

5.4.3 The finite dimensional approximation

The finite element approximation of (5.4.4) can be based on the usual finite element spaces $(V_h)^d \subset (H_0^1(\Omega))^d$ introduced in (2.1.3) for the velocity field, and the corresponding local and trace spaces $(V_{i,h})^d$, $(V_{i,h}^0)^d$ and $(\Lambda_h)^d$ (see (2.1.7) and (2.1.8)). This choice corresponds to the use of continuous piecewise-polynomial finite elements of degree $r \geq 1$ for the velocity components. For the approximation of the logarithmic density σ we consider the finite element space consisting of discontinuous elements

$$
Y_h^{r-1} := \{\pi_h \in L^2(\Omega) \mid \pi_{h|K} \in \mathbf{P}_{r-1}(K)\ \forall\ K \in \mathcal{T}_h\}, \quad r \geq 1, \tag{5.4.18}
$$

and their restrictions $Y_{i,h}^{r-1}$ to Ω_i, for $i = 1, 2$.

The discrete problem corresponding to (5.4.4) (with $\boldsymbol{\varphi}_D = \mathbf{0}$ and $\Gamma_N = \emptyset$) is given by

$$\text{(5.4.19)} \qquad \begin{cases} \text{find } (\mathbf{u}_h, \sigma_h) \in (V_h)^d \times Y_h^{r-1} \ : \\ \\ \widehat{s}(\mathbf{u}_h, \mathbf{v}_h) + \beta b(\mathbf{v}_h, \sigma_h) = (\mathbf{f}, \mathbf{v}_h) \quad \forall\ \mathbf{v}_h \in (V_h)^d \\ \\ \alpha(\sigma_h, \pi_h) - b(\mathbf{u}_h, \pi_h) = (g, \pi_h) \quad \forall\ \pi_h \in Y_h^{r-1}. \end{cases}$$

The finite element two-domain formulation corresponding to (5.4.7) can be easily defined, as well as the finite dimensional counterpart of the Steklov–Poincaré interface equation (5.4.13). For the sake of exposition, let us give a precise definition of the discrete Steklov–Poincaré operators.

First, for each $\boldsymbol{\eta}_h \in (\Lambda_h)^d$ and $i = 1, 2$, we denote by $(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, P_{i,h}^{(\nu)}\boldsymbol{\eta}_h) \in (V_{i,h})^d \times Y_{i,h}^{r-1}$ the solution to

$$\text{(5.4.20)} \qquad \begin{cases} \widehat{s}_i(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \mathbf{v}_{i,h}) + \beta b_i(\mathbf{v}_{i,h}, P_{i,h}^{(\nu)}\boldsymbol{\eta}_h) = 0 \quad \forall\ \mathbf{v}_{i,h} \in (V_{i,h}^0)^d \\ \\ \alpha(P_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \pi_{i,h})_{\Omega_i} - b_i(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \pi_{i,h}) = 0 \quad \forall\ \pi_{i,h} \in Y_{i,h}^{r-1} \\ \\ \mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h = \boldsymbol{\eta}_h \qquad \text{on } \Gamma. \end{cases}$$

The discrete Steklov–Poincaré operators are defined as

$$\text{(5.4.21)} \qquad \begin{aligned} \langle S_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h \rangle \ &: = \mathcal{A}_i^{(\nu)}[(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, P_{i,h}^{(\nu)}\boldsymbol{\eta}_h), (\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\mu}_h, P_{i,h}^{(\nu)}\boldsymbol{\mu}_h)] \\ &= \widehat{s}_i(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\mu}_h) + \beta\alpha(P_{i,h}^{(\nu)}\boldsymbol{\eta}_h, P_{i,h}^{(\nu)}\boldsymbol{\mu}_h)_{\Omega_i}. \end{aligned}$$

We have already observed that properties 1 and 2 in Remark 4.1.7 hold, therefore these operators are symmetric and coercive in $(\Lambda_h)^d$, uniformly with respect to h. On the other hand, the uniform continuity of the Steklov–Poincaré operators can be proved as follows. Taking as test functions in (5.4.20)

$$\mathbf{v}_{i,h} = \mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h - \mathbf{H}_{i,h}\boldsymbol{\eta}_h, \quad \pi_{i,h} = \beta P_{i,h}^{(\nu)}\boldsymbol{\eta}_h,$$

where $\mathbf{H}_{i,h}\boldsymbol{\eta}_h$ has been introduced in (5.2.21), and adding the two resulting equations we find that

$$\begin{aligned} \widehat{s}_i(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h) \ &+ \beta\alpha(P_{i,h}^{(\nu)}\boldsymbol{\eta}_h, P_{i,h}^{(\nu)}\boldsymbol{\eta}_h)_{\Omega_i} \\ &= \widehat{s}_i(\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h, \mathbf{H}_{i,h}\boldsymbol{\eta}_h) + \beta b_i(\mathbf{H}_{i,h}\boldsymbol{\eta}_h, P_{i,h}^{(\nu)}\boldsymbol{\eta}_h). \end{aligned}$$

Therefore

$$\text{(5.4.22)} \qquad ||\mathbf{U}_{i,h}^{(\nu)}\boldsymbol{\eta}_h||_{1,\Omega_i} + ||P_{i,h}^{(\nu)}\boldsymbol{\eta}_h||_{0,\Omega_i} \leq C||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i} \qquad \forall\ \boldsymbol{\eta}_h \in (\Lambda_h)^d,$$

where the constant $C > 0$ depends on the coefficients α and β, the continuity constants of the forms $\widehat{s_i}$ and b_i, and the coerciveness constant of the form $\widehat{s_i}$, but is independent of the grid size h.

Consequently, from (5.4.21) and Theorem 4.1.3

$$\langle S_{i,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h\rangle \leq C||\mathbf{H}_{i,h}\boldsymbol{\eta}_h||_{1,\Omega_i}||\mathbf{H}_{i,h}\boldsymbol{\mu}_h||_{1,\Omega_i} \leq C||\boldsymbol{\eta}_h||_\Lambda||\boldsymbol{\mu}_h||_\Lambda,$$

provided that the family of triangulations $\mathcal{T}_h$ of Ω is regular and the family of triangulations $\mathcal{M}_h$ of Γ is quasi-uniform.

The Dirichlet–Neumann iterative scheme (5.4.14)–(5.4.16) on the finite element subspaces turns out to be equivalent to the preconditioned Richardson method for the discrete Steklov–Poincaré equation, with $S_{2,h}$ as a preconditioner.

Concerning its convergence, we can therefore resort to Theorem 4.2.2. We have already noted that all of its assumptions are satisfied, uniformly with respect to h; thus, the Dirichlet–Neumann iterative scheme on the discrete problem (5.4.19) converges, and its rate of convergence is independent of the grid size h.

A similar results holds for the Neumann–Neumann iterative scheme, using Theorem 4.2.5 instead of Theorem 4.2.2.

5.5 The Stokes problem for inviscid compressible flows

This problem is the inviscid counterpart of the Stokes problem for compressible flows (5.4.1) considered in the previous section (that is, the one in which $\nu = \gamma = 0$). The new problem reads:

$$\begin{cases} \alpha\mathbf{u} + \beta\nabla\sigma = \mathbf{f} & \text{in } \Omega \\ \alpha\sigma + \operatorname{div}\mathbf{u} = g & \text{in } \Omega \\ \mathbf{u}\cdot\mathbf{n}^* = \varphi_D & \text{on } \Gamma_D \\ \sigma = \varphi_N & \text{on } \Gamma_N, \end{cases} \tag{5.5.1}$$

where $\alpha > 0$ and $\beta > 0$ are constants, $\mathbf{f}$ is a given vector field, and g, φ_D and φ_N are given scalar functions. Note that the system above is no longer elliptic.

Following the general principle presented in Remark 1.1.1, the interface conditions are determined by requiring that the solution u belongs to a suitable space of functions defined in Ω, and that it is a distributional solution of the given problem in the whole Ω. From (5.5.1), it is apparent that the velocity field does not need to belong to the usual energy space $(H^1(\Omega))^d$, because only the operator div acts on $\mathbf{u}$. We thus introduce the Hilbert space

$$H(\operatorname{div};\Omega) := \{\mathbf{v} \in (L^2(\Omega))^d \mid \operatorname{div}\mathbf{v} \in L^2(\Omega)\}, \tag{5.5.2}$$

and require that $\mathbf{u} \in H(\operatorname{div};\Omega)$. Let us also recall that, if $\mathbf{v} \in H(\operatorname{div};\Omega)$, then its normal component $\mathbf{v}\cdot\mathbf{n}$ is continuous across every interface.

We are now in a position to state the interface conditions for $\mathbf{u}$ and σ. Setting $\mathbf{u}_{|\Omega_i} := \mathbf{u}_i$ and $\sigma_{|\Omega_i} := \sigma_i$, $i = 1, 2$, they read

$$\begin{aligned} &\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} \qquad \text{on } \Gamma \\ &\sigma_1 = \sigma_2 \qquad\qquad\quad \text{on } \Gamma. \end{aligned} \tag{5.5.3}$$

The continuity of σ across Γ corresponds to the continuity of the normal stress. In fact, for the inviscid problem, the stress tensor $\widehat{\mathbf{T}}(\mathbf{w}, \pi)$ reduces to

$$\widehat{T}_{lj}(\mathbf{w}, \pi) := -\beta\pi\, \delta_{lj},$$

and the condition $\widehat{\mathbf{T}}(\mathbf{u}_1, \sigma_1) \cdot \mathbf{n} = \widehat{\mathbf{T}}(\mathbf{u}_2, \sigma_2) \cdot \mathbf{n}$ reads

$$-\sigma_1 \mathbf{n} = -\sigma_2 \mathbf{n} \qquad \text{on } \Gamma,$$

which is $(5.5.3)_2$.

Remark 5.5.1 System (5.5.1) can be regarded as a mixed-type formulation for the elliptic problem

$$-\varepsilon\Delta\sigma + \sigma = g \qquad \text{in } \Omega.$$

In fact, setting $\mathbf{u} := -\nabla\sigma$, we find that

$$\begin{cases} \sigma + \varepsilon \operatorname{div} \mathbf{u} = g & \text{in } \Omega \\ \\ \mathbf{u} + \nabla\sigma = \mathbf{0} & \text{in } \Omega, \end{cases} \tag{5.5.4}$$

which has the same structure as the Stokes system for inviscid compressible flows. Therefore, all the results that we shall present in this section can be extended to the elliptic problem (5.5.4). □

5.5.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

Taking for simplicity $\varphi_D = 0$ and $\Gamma_N = \emptyset$ (and referring to Quarteroni and Valli 1990 for the general case), and assuming that $\mathbf{f} \in (L^2(\Omega))^d$ and $g \in L^2(\Omega)$, the weak formulation of (5.5.1) is:

$$\begin{cases} \text{find } (\mathbf{W}, \sigma) \in H_0(\operatorname{div}; \Omega) \times L^2(\Omega) \ : \\ \\ \alpha(\mathbf{W}, \mathbf{v}) + \beta b(\mathbf{v}, \sigma) = (\mathbf{f}, \mathbf{v}) \quad \forall\ \mathbf{v} \in H_0(\operatorname{div}; \Omega) \\ \\ \alpha(\sigma, \pi) - b(\mathbf{W}, \pi) = (g, \pi) \quad \forall\ \pi \in L^2(\Omega), \end{cases} \tag{5.5.5}$$

where

$$H_0(\operatorname{div}; \Omega) := \{\mathbf{v} \in H(\operatorname{div}; \Omega) \,|\, (\mathbf{v} \cdot \mathbf{n}^*)_{|\partial\Omega} = 0\}.$$

The existence of a unique solution $(\mathbf{W}, \sigma)$ of (5.5.5) follows from the Lax–Milgram lemma, by considering an equivalent formulation for the coercive bilinear form associated with the operator $\alpha I - \beta\alpha^{-1}\nabla\,\mathrm{div}$ (see Quarteroni and Valli 1990).

The multi-domain weak formulation reads as follows. First, let us introduce the spaces

$$
\begin{aligned}
W_i &:= \{\mathbf{v}_i \in H(\mathrm{div};\Omega_i) \mid (\mathbf{v}_i \cdot \mathbf{n}^*)_{|\partial\Omega\cap\partial\Omega_i} = 0\} \\
W_i^0 &:= H_0(\mathrm{div};\Omega_i) \\
\Psi &:= \{\psi : \Gamma \to \mathbf{R} \mid \psi = (\mathbf{v}\cdot\mathbf{n})_{|\Gamma},\ \mathbf{v} \in H_0(\mathrm{div};\Omega)\}.
\end{aligned}
\tag{5.5.6}
$$

The space Ψ is included in the dual space of Λ, and coincides with it if the interface Γ has no boundary (see Girault and Raviart 1986, p. 27–9).

If we set $\mathbf{u}_i := \mathbf{u}_{|\Omega_i}$ and $\sigma_i := \sigma_{|\Omega_i}$, then problem (5.5.5) is equivalent to the following one: find $(\mathbf{u}_1, \sigma_1) \in W_1 \times L^2(\Omega_1)$, $(\mathbf{u}_2, \sigma_2) \in W_2 \times L^2(\Omega_2)$ such that

$$
\begin{cases}
\alpha(\mathbf{u}_1, \mathbf{v}_1)_{\Omega_1} + \beta b_1(\mathbf{v}_1, \sigma_1) = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1} \quad \forall\ \mathbf{v}_1 \in W_1^0 \\
\alpha(\sigma_1, \pi_1)_{\Omega_1} - b_1(\mathbf{u}_1, \pi_1) = (g, \pi_1)_{\Omega_1} \quad \forall\ \pi_1 \in L^2(\Omega_1) \\
\mathbf{u}_1 \cdot \mathbf{n} = \mathbf{u}_2 \cdot \mathbf{n} \qquad \text{on } \Gamma \\
\alpha(\mathbf{u}_2, \mathbf{v}_2)_{\Omega_2} + \beta b_2(\mathbf{v}_2, \sigma_2) = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \quad \forall\ \mathbf{v}_2 \in W_2^0 \\
\alpha(\sigma_2, \pi_2)_{\Omega_2} - b_2(\mathbf{u}_2, \pi_2) = (g, \pi_2)_{\Omega_2} \quad \forall\ \pi_2 \in L^2(\Omega_2) \\
\alpha(\mathbf{u}_2, \widehat{\boldsymbol{\mathcal{R}}}_2\psi)_{\Omega_2} + \beta b_2(\widehat{\boldsymbol{\mathcal{R}}}_2\psi, \sigma_2) = (\mathbf{f}, \widehat{\boldsymbol{\mathcal{R}}}_2\psi)_{\Omega_2} + (\mathbf{f}, \widehat{\boldsymbol{\mathcal{R}}}_1\psi)_{\Omega_1} \\
\qquad\qquad -\alpha(\mathbf{u}_1, \widehat{\boldsymbol{\mathcal{R}}}_1\psi)_{\Omega_1} - \beta b_2(\widehat{\boldsymbol{\mathcal{R}}}_1\psi, \sigma_1) \qquad \forall\ \psi \in \Psi,
\end{cases}
\tag{5.5.7}
$$

where we have denoted by $\widehat{\boldsymbol{\mathcal{R}}}_i$ any extension operator from Ψ to W_i satisfying $(\widehat{\boldsymbol{\mathcal{R}}}_i\psi \cdot \mathbf{n})_{|\Gamma} = \psi$ for each $\psi \in \Psi$.

The last equation of (5.5.7) is the weak form of the interface condition $(5.5.3)_2$ (the continuity of the normal stress that in the present case coincides with the continuity of the logarithmic density), which can be formally recovered by integration by parts.

For each $\psi \in \Psi$ we can introduce the solution $(\mathbf{U}_i^{(0)}\psi, P_i^{(0)}\psi) \in W_i \times L^2(\Omega_i)$ to

$$
\begin{cases}
\alpha(\mathbf{U}_i^{(0)}\psi, \mathbf{w}_i)_{\Omega_i} + \beta b_i(\mathbf{w}_i, P_i^{(0)}\psi) = 0 \quad \forall\ \mathbf{w}_i \in W_i^0 \\
\alpha(P_i^{(0)}\psi, \pi_i)_{\Omega_i} - b_i(\mathbf{U}_i^{(0)}\psi, \pi_i) = 0 \quad \forall\ \pi_i \in L^2(\Omega_i) \\
(\mathbf{U}_i^{(0)}\psi \cdot \mathbf{n})_{|\Gamma} = \psi \qquad \text{on } \Gamma.
\end{cases}
\tag{5.5.8}
$$

Moreover, we denote by $(\mathbf{U}_{i,*}^{(0)}, P_{i,*}^{(0)}) \in W_i^0 \times L^2(\Omega_i)$ the solution to

$$\begin{cases} \alpha(\mathbf{U}_{i,*}^{(0)}, \mathbf{w}_i)_{\Omega_i} + \beta b_i(\mathbf{w}_i, P_{i,*}^{(0)}) = (\mathbf{f}, \mathbf{w}_i)_{\Omega_i} \quad \forall\ \mathbf{w}_i \in W_i^0 \\ \alpha(P_{i,*}^{(0)}, \pi_i)_{\Omega_i} - b_i(\mathbf{U}_{i,*}^{(0)}, \pi_i) = (g, \pi_i)_{\Omega_i} \quad \forall\ \pi_i \in L^2(\Omega_i). \end{cases} \tag{5.5.9}$$

The bilinear forms associated with (5.5.8) and (5.5.9) are continuous, but they are not coercive in $W_i^0 \times L^2(\Omega_i)$. Nonetheless, we have already noted that the existence of a unique solution can be proved by resorting to an equivalent coercive problem associated with the operator $\alpha I - \beta\alpha^{-1}\nabla\,\mathrm{div}$. Moreover, one has

$$||\mathbf{U}_i^{(0)}\psi||_{H(\mathrm{div};\Omega_i)} + ||P_i^{(0)}\psi||_{0,\Omega_i} \leq C||\psi||_{\Psi}. \tag{5.5.10}$$

Denoting by $\mathcal{A}_i^{(0)}$ the bilinear forms introduced in (5.4.10) (where we have taken $\nu = \gamma = 0$), the Steklov–Poincaré operators S_i are defined in the trace space Ψ as follows:

$$\begin{aligned} \langle \phi, S_i\psi \rangle :&= \mathcal{A}_i^{(0)}[(\mathbf{U}_i^{(0)}\psi, P_i^{(0)}\psi), (\mathbf{U}_i^{(0)}\phi, P_i^{(0)}\phi)] \\ &= \alpha(\mathbf{U}_i^{(0)}\psi, \mathbf{U}_i^{(0)}\phi)_{\Omega_i} + \beta\alpha(P_i^{(0)}\psi, P_i^{(0)}\phi)_{\Omega_i}, \end{aligned} \tag{5.5.11}$$

and the linear functionals $\chi_i \in \Psi'$ are given by

$$\begin{aligned} \langle \phi, \chi_i \rangle :&= (\mathbf{f}, \mathbf{U}_i^{(0)}\phi)_{\Omega_i} + \beta(g, P_i^{(0)}\phi)_{\Omega_i} \\ &\quad -\mathcal{A}_i^{(0)}[(\mathbf{U}_{i,*}^{(0)}, P_{i,*}^{(0)}), (\mathbf{U}_i^{(0)}\phi, P_i^{(0)}\phi)]. \end{aligned} \tag{5.5.12}$$

Arguing as in the previous section, the Steklov–Poincaré equation reads

$$S\lambda = \chi \qquad \text{on } \Gamma, \tag{5.5.13}$$

where we have set as usual $S := S_1 + S_2$ and $\chi := \chi_1 + \chi_2$.

Note that, similarly to all the preceding cases, we have transformed the multidomain problem into an equivalent problem on the interface Γ which expresses the continuity of the normal stress. In the present situation, this means the continuity of the logarithmic density. Clearly, one could proceed in the opposite way, coming to an interface equation where the trace of the logarithmic density is the unknown, and the continuity on Γ of the normal trace of the velocity has to be imposed.

The Steklov–Poincaré operators S_1 and S_2 are both symmetric, and continuity in Ψ follows at once from (5.5.10). Finally, taking in (5.5.7) $\pi_i = \mathrm{div}(\mathbf{U}_i^{(0)}\psi)$ one has

$$||\,\mathrm{div}(\mathbf{U}_i^{(0)}\psi)||_{0,\Omega_i} \leq C_0||P_i^{(0)}\psi||_{0,\Omega_i}.$$

Since the trace inequality states that

$$||\psi||_{\Psi} \leq C^* ||\mathbf{U}_i^{(0)}\psi||_{H(\mathrm{div}\,;\Omega_i)}$$

(see Girault and Raviart 1986, pp. 27–8), we have that

$$\begin{aligned}\langle \psi, S_i\psi\rangle &= \alpha||\mathbf{U}_i^{(0)}\psi||^2_{0,\Omega_i} + \beta\alpha||P_i^{(0)}\psi||^2_{0,\Omega_i} \\ &\geq \alpha||\mathbf{U}_i^{(0)}\psi||^2_{0,\Omega_i} + \beta\alpha C_0^{-2}||\,\mathrm{div}(\mathbf{U}_i^{(0)}\psi)||^2_{0,\Omega_i} \geq C_1||\psi||^2_{\Psi}.\end{aligned}$$

Hence both Steklov–Poincaré operators are coercive in Ψ.

5.5.2 Substructuring iterative methods

The Dirichlet–Neumann iteration scheme for problem (5.5.7) reads: given $\lambda^0 \in \Psi$, for each $k \geq 0$ solve

$$\text{(5.5.14)}\qquad \begin{cases} \text{find } (\mathbf{u}_1^{k+1}, \sigma_1^{k+1}) \in W_1 \times L^2(\Omega_1) \ : \\ \alpha(\mathbf{u}_1^{k+1}, \mathbf{v}_1)_{\Omega_1} + \beta b_1(\mathbf{v}_1, \sigma_1^{k+1}) = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1} \quad \forall\ \mathbf{v}_1 \in W_1^0 \\ \alpha(\sigma_1^{k+1}, \pi_1)_{\Omega_1} - b_1(\mathbf{u}_1^{k+1}, \pi_1) = (g, \pi_1)_{\Omega_1} \quad \forall\ \pi_1 \in L^2(\Omega_1) \\ \mathbf{u}_1 \cdot \mathbf{n} = \lambda^k \qquad \text{on } \Gamma, \end{cases}$$

then

$$\text{(5.5.15)}\qquad \begin{cases} \text{find } (\mathbf{u}_2^{k+1}, \sigma_2^{k+1}) \in W_2 \times L^2(\Omega_2) \ : \\ \alpha(\mathbf{u}_2^{k+1}, \mathbf{v}_2)_{\Omega_2} + \beta b_2(\mathbf{v}_2, \sigma_2^{k+1}) = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \quad \forall\ \mathbf{v}_2 \in W_2^0 \\ \alpha(\sigma_2^{k+1}, \pi_2)_{\Omega_2} - b_2(\mathbf{u}_2^{k+1}, \pi_2) = (g, \pi_2)_{\Omega_2} \quad \forall\ \pi_2 \in L^2(\Omega_2) \\ \alpha(\mathbf{u}_2^{k+1}, \widehat{\boldsymbol{\mathcal{R}}}_2\psi) + \beta b_2(\widehat{\boldsymbol{\mathcal{R}}}_2\psi, \sigma_2^{k+1}) \\ \qquad = (\mathbf{f}, \widehat{\boldsymbol{\mathcal{R}}}_2\psi)_{\Omega_2} + (\mathbf{f}, \widehat{\boldsymbol{\mathcal{R}}}_1\psi)_{\Omega_1} \\ \qquad\quad -\alpha(\mathbf{u}_1^{k+1}, \widehat{\boldsymbol{\mathcal{R}}}_1\psi)_{\Omega_1} - \beta b_1(\widehat{\boldsymbol{\mathcal{R}}}_1\psi, \sigma_1^{k+1}) \qquad \forall\ \psi \in \Psi, \end{cases}$$

and finally set

$$\text{(5.5.16)}\qquad \psi^{k+1} := \theta(\mathbf{u}_2^{k+1} \cdot \mathbf{n})_{|\Gamma} + (1-\theta)\psi^k \qquad \text{on } \Gamma.$$

By proceeding as in the previous sections, the Dirichlet–Neumann iterative scheme turns out to be equivalent to a Richardson procedure for the interface equation $S\lambda = \chi$, with S_2 as a preconditioner. The convergence of these iterations can be proved by using Theorem 4.2.2, because we have already noted that all its assumptions are satisfied.

Similarly, as a consequence of Theorem 4.2.5, the Neumann–Neumann iterative method applied to the solution of (5.5.7) is convergent.

5.5.3 The finite dimensional approximation

For the approximation of problem (5.5.1) we can rely on its analogy with the mixed-type formulation (5.5.4) of the Poisson problem. For the latter, classical finite elements like the Raviart–Thomas finite elements if $d = 2$ (see Raviart and Thomas 1977), or the div-*conforming* Nédélec finite elements if $d = 3$ (see Nédélec 1980, 1986) can successfully be applied.

The latter elements were introduced in Section 4.1.2 and the associated finite element space was denoted by N_h^r. Let us denote the finite dimensional space used for approximating the velocity $\mathbf{u}$ by $W_h := N_h^r \cap H_0(\mathrm{div};\Omega)$. The approximation of σ is again based on the space Y_h^{r-1} of discontinuous finite elements introduced in (5.4.18).

Setting

$$\begin{aligned} W_{i,h} &:= \{\mathbf{v}_{h|\Omega_i} \mid \mathbf{v}_h \in W_h\}, \quad i = 1,2 \\ W_{i,h}^0 &:= W_{i,h} \cap H_0(\mathrm{div};\Omega_i), \quad i = 1,2 \\ \Psi_h &:= \{(\mathbf{v}_h \cdot \mathbf{n})_{|\Gamma} \mid \mathbf{v}_h \in W_h\}, \end{aligned}$$

for each $\psi_h \in \Psi_h$ we introduce the solution $(\mathbf{U}_{i,h}^{(0)}\psi_h, P_{i,h}^{(0)}\psi_h) \in W_{i,h} \times Y_{i,h}^{r-1}$ to

$$\begin{cases} \alpha(\mathbf{U}_{i,h}^{(0)}\psi_h, \mathbf{v}_{i,h})_{\Omega_i} + \beta b_i(\mathbf{v}_{i,h}, P_{i,h}^{(0)}\psi_h) = 0 \quad \forall\ \mathbf{v}_{i,h} \in W_{i,h}^0 \\ \alpha(P_{i,h}^{(0)}\psi_h, \pi_{i,h})_{\Omega_i} - b_i(\mathbf{U}_{i,h}^{(0)}\psi_h, \pi_{i,h}) = 0 \quad \forall\ \pi_{i,h} \in Y_{i,h}^{r-1} \\ (\mathbf{U}_{i,h}^{(0)}\psi_h \cdot \mathbf{n})_{|\Gamma} = \psi_h \qquad \text{on } \Gamma. \end{cases} \tag{5.5.17}$$

The discrete Steklov–Poincaré operators $S_{i,h}$ are defined as

$$\begin{aligned} \langle \phi_h, S_{i,h}\psi_h \rangle &:= \mathcal{A}_i^{(0)}[(\mathbf{U}_{i,h}^{(0)}\psi_h, P_{i,h}^{(0)}\psi_h), (\mathbf{U}_{i,h}^{(0)}\phi_h, P_{i,h}^{(0)}\phi_h)] \\ &= \alpha(\mathbf{U}_{i,h}^{(0)}\psi_h, \mathbf{U}_{i,h}^{(0)}\phi_h) + \beta\alpha(P_{i,h}^{(0)}\psi_h, P_{i,h}^{(0)}\phi_h)_{\Omega_i}. \end{aligned} \tag{5.5.18}$$

These operators are symmetric; proceeding as in the infinite dimensional case, it can be shown that they are coercive, uniformly with respect to h. In particular, taking $\pi_{i,h} = \mathrm{div}(\mathbf{U}_{i,h}^{(0)}\psi_h)$ in (5.5.17), it follows that

$$|| \mathrm{div}(\mathbf{U}_{i,h}^{(0)}\psi_h)||_{0,\Omega_i} \leq C_0 ||P_{i,h}^{(0)}\psi_h||_{0,\Omega_i}. \tag{5.5.19}$$

Proving the uniform continuity is a more delicate task. For each $\psi_h \in \Psi_h$ and $i = 1,2$ introduce the discrete function $\mathbf{Q}_{i,h}\psi_h \in W_{i,h}$ as in (4.1.27). Taking $\mathbf{v}_{i,h} = \mathbf{U}_{i,h}^{(0)}\psi_h - \mathbf{Q}_{i,h}\psi_h$ and $\pi_{i,h} = \beta P_{i,h}^{(0)}\psi_h$ in (5.5.17), it follows that

$$\begin{aligned} \alpha(\mathbf{U}_{i,h}^{(0)}\psi_h, \mathbf{U}_{i,h}^{(0)}\psi_h)_{\Omega_i} &+\beta\alpha(P_{i,h}^{(0)}\psi_h, P_{i,h}^{(0)}\psi_h)_{\Omega_i} \\ &= \alpha(\mathbf{U}_{i,h}^{(0)}\psi_h, \mathbf{Q}_{i,h}\psi_h)_{\Omega_i} + \beta b_i(\mathbf{Q}_{i,h}\psi_h, P_{i,h}^{(0)}\psi_h). \end{aligned}$$

Therefore,

$$\langle \phi_h, S_{i,h}\psi_h \rangle \leq C||\mathbf{Q}_{i,h}\psi_h||_{H(\mathrm{div};\Omega_i)}||\mathbf{Q}_{i,h}\phi_h||_{H(\mathrm{div};\Omega_i)}.$$

The uniform continuity of the operators $S_{i,h}$ thus follows from Theorem 4.1.9, which, under the assumptions that the family of triangulations $\mathcal{T}_h$ of Ω is regular and the family of triangulations $\mathcal{M}_h$ of Γ is quasi-uniform, shows the uniform continuity of the operators $\mathbf{Q}_{i,h}$.

The finite dimensional Steklov–Poincaré equation corresponding to (5.5.13) can be solved by means of a preconditioned Richardson procedure, with $S_{2,h}$ as a preconditioner. Theorem 4.2.2 ensures the convergence of the iteration, at a rate independent of h. The same is also true for the Dirichlet–Neumann iterative scheme corresponding to (5.5.14)–(5.5.16), because it is equivalent to the Richardson scheme with preconditioner $S_{2,h}$ for the Steklov–Poincaré equation.

As a consequence of Theorem 4.2.5, an analogous result holds also for the Neumann–Neumann iterative scheme, which is equivalent to a Richardson method for solving the Steklov–Poincaré equation with $(\sigma_1 S_{1,h}^{-1} + \sigma_2 S_{2,h}^{-1})^{-1}$ as a preconditioner.

5.6 First-order equations

Several mathematical problems in fluid dynamics or wave propagation are described by non-linear advection equations of the following type

$$\frac{\partial u}{\partial t} + \mathrm{div}[\mathbf{F}(u)] = f.$$

Applying a time discretisation based on an implicit method, followed by a linearisation procedure, we are thus led to the first-order linear advection equation

$$L_0 u := \mathrm{div}(\mathbf{b}u) + a_0 u = f \qquad \text{in } \Omega, \tag{5.6.1}$$

which is the simplest model of a hyperbolic equation. In the discretisation process described above, the coefficient a_0 is proportional to $(\Delta t)^{-1}$, the inverse of the time step.

The case of hyperbolic systems will be considered in Section 7.2.

Let us denote by $\mathbf{n}^*$ the unit outward normal vector to $\partial\Omega$, and by

$$\partial\Omega^{\mathrm{in}} := \{\mathbf{x} \in \partial\Omega \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) < 0\} \tag{5.6.2}$$

the inflow region of Ω with respect to the advective field $\mathbf{b}$. The Dirichlet boundary value problem associated with L_0 is given by

$$\begin{cases} L_0 u = f & \text{in } \Omega \\ \\ u = \varphi_D & \text{on } \partial\Omega^{\mathrm{in}}. \end{cases} \tag{5.6.3}$$

The interface conditions require that the normal flux $\mathbf{b} \cdot \mathbf{n}\, u$ is continuous across Γ; namely, denoting by u_i the restriction of u in Ω_i, $i = 1, 2$:

$$\mathbf{b}_{|\Omega_1} \cdot \mathbf{n}\, u_1 = \mathbf{b}_{|\Omega_2} \cdot \mathbf{n}\, u_2 \qquad \text{on } \Gamma. \tag{5.6.4}$$

In an equivalent way, provided that $\mathbf{b} \cdot \mathbf{n}$ is continuous across Γ, the continuity of u on the inflow region of Ω_1 as well as on the inflow region of Ω_2 has to be imposed:

$$u_1 = u_2 \qquad \text{on } \Gamma^{\text{in}} \cup \Gamma^{\text{out}}. \tag{5.6.5}$$

Here, as usual, we have denoted by $\mathbf{n}$ the unit normal vector on Γ directed from Ω_1 to Ω_2, and we have set

$$\begin{aligned} \Gamma^{\text{in}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\} \\ \Gamma^{\text{out}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\}. \end{aligned}$$

We also define

$$\partial\Omega^{\text{out}} := \{\mathbf{x} \in \partial\Omega \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) > 0\},$$

and, for $i = 1, 2$, we put

$$\partial\Omega_i^{\text{in}} := \partial\Omega^{\text{in}} \cap \partial\Omega_i, \quad \partial\Omega_i^{\text{out}} := \partial\Omega^{\text{out}} \cap \partial\Omega_i.$$

5.6.1 Weak multi-domain formulation and the Steklov–Poincaré interface equation

With the aim of deriving the weak formulation of (5.6.3), we start with some notation. First, assuming that

$$\mathbf{b} \in (L^\infty(\Omega))^d, \quad D_j\mathbf{b} \in (L^\infty(\Omega))^d, \quad j = 1, \ldots, d$$

(in particular, the vector field $\mathbf{b}$ is continuous in $\overline{\Omega}$), let us define the space

$$L_b^2(\partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}}) := \left\{ \eta : \partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}} \to \mathbf{R} \,| \int_{\partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}}} |\mathbf{b} \cdot \mathbf{n}^*|\, \eta^2 < \infty \right\}, \tag{5.6.6}$$

endowed with the norm

$$||\eta||_{b,\partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}}} := \left(\int_{\partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}}} |\mathbf{b} \cdot \mathbf{n}^*|\, \eta^2 \right)^{1/2},$$

and denote by X_b the Hilbert space

$$X_b := \{v \in L^2(\Omega) \,|\, \operatorname{div}(\mathbf{b}v) \in L^2(\Omega),\ v \in L_b^2(\partial\Omega^{\text{in}} \cup \partial\Omega^{\text{out}})\}, \tag{5.6.7}$$

endowed with the norm

$$||v||_{X_b} := \Big(||v||^2_{0,\Omega} + ||\operatorname{div}(\mathbf{b}v)||^2_{0,\Omega} + ||v||^2_{b,\partial\Omega^{\rm in}\cup\partial\Omega^{\rm out}} \Big)^{1/2}.$$

Then, assuming that $a_0 \in L^\infty(\Omega)$, for each $w, v \in X_b$ we define the bilinear form

$$\begin{aligned} \mathcal{C}(w,v) := \frac{1}{2}\int_\Omega [\operatorname{div}(\mathbf{b}w)\, v - \operatorname{div}(\mathbf{b}v)\, w] + \int_\Omega \left(\frac{1}{2}\operatorname{div}\mathbf{b} + a_0\right) w\, v \\ + \frac{1}{2}\int_{\partial\Omega^{\rm out}} \mathbf{b}\cdot\mathbf{n}^*\, w\, v - \frac{1}{2}\int_{\partial\Omega^{\rm in}} \mathbf{b}\cdot\mathbf{n}^*\, w\, v. \end{aligned} \tag{5.6.8}$$

The variational formulation of (5.6.3) reads

$$\text{find } u \in X_b \ :\ \mathcal{C}(u,v) = (f,v) + (\varphi_D, v_{|\partial\Omega^{\rm in}})_{b,\partial\Omega^{\rm in}} \qquad \forall\ v \in X_b, \tag{5.6.9}$$

where we have denoted by $(\cdot,\cdot)_{b,\partial\Omega^{\rm in}}$ the scalar product in $L^2_b(\partial\Omega^{\rm in})$, and we have assumed that $f \in L^2(\Omega)$ and $\varphi_D \in L^2_b(\partial\Omega^{\rm in})$.

It can be proved that problem (5.6.9) has a unique solution u (see, for example, F. Gastaldi *et al.* 1990), provided that the condition

$$\frac{1}{2}\operatorname{div}\mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq \mu_0 > 0 \tag{5.6.10}$$

is satisfied for almost all $\mathbf{x} \in \Omega$. Moreover, the solution satisfies $u \in L^2_b(\partial\Omega^{\rm out})$.

Introducing for $i = 1, 2$ the local spaces

$$\begin{aligned} X_{i,b} &:= \{v_i \in L^2(\Omega_i) \,|\, \operatorname{div}(\mathbf{b}v_i) \in L^2(\Omega_i),\ v_i \in L^2_b(\partial\Omega_i^{\rm in} \cup \partial\Omega_i^{\rm out}), \\ &\qquad\qquad v_i \in L^2_b(\Gamma^{\rm in}\cup\Gamma^{\rm out})\} \\ X^0_{1,b} &:= \{v_1 \in X_{1,b} \,|\, v_{1|\Gamma^{\rm in}} = 0\} \\ X^0_{2,b} &:= \{v_2 \in X_{2,b} \,|\, v_{2|\Gamma^{\rm out}} = 0\}, \end{aligned} \tag{5.6.11}$$

and the local bilinear forms defined in $X_{i,b}$ by

$$\begin{aligned} \mathcal{C}_i(w_i,v_i) := \frac{1}{2}\int_{\Omega_i} [\operatorname{div}(\mathbf{b}w_i)\, v_i - \operatorname{div}(\mathbf{b}v_i)\, w_i] \\ + \int_{\Omega_i} \left(\frac{1}{2}\operatorname{div}\mathbf{b} + a_0\right) w_i\, v_i \\ + \frac{1}{2}\int_{\partial\Omega_i^{\rm out}} \mathbf{b}\cdot\mathbf{n}^*\, w_i\, v_i - \frac{1}{2}\int_{\partial\Omega_i^{\rm in}} \mathbf{b}\cdot\mathbf{n}^*\, w_i\, v_i \\ + \frac{1}{2}\int_{\Gamma^{\rm out}} \mathbf{b}\cdot\mathbf{n}\, w_i\, v_i - \frac{1}{2}\int_{\Gamma^{\rm in}} \mathbf{b}\cdot\mathbf{n}\, w_i\, v_i, \end{aligned} \tag{5.6.12}$$

the multi-domain weak formulation of (5.6.3) reads: find $u_1 \in X_{1,b}$ and $u_2 \in X_{2,b}$ such that

$$
\begin{cases}
\mathcal{C}_1(u_1, v_1) = (f, v_1)_{\Omega_1} + (\varphi_D, v_{1|\partial\Omega_1^{\rm in}})_{b,\partial\Omega_1^{\rm in}} & \forall\ v_1 \in X_{1,b}^0 \\
\mathcal{C}_2(u_2, v_2) = (f, v_2)_{\Omega_2} + (\varphi_D, v_{2|\partial\Omega_2^{\rm in}})_{b,\partial\Omega_2^{\rm in}} & \forall\ v_2 \in X_{2,b}^0 \\
(u_{1|\Gamma^{\rm in}} - u_{2|\Gamma^{\rm in}}, \mu)_{b,\Gamma^{\rm in}} & \\
\qquad +(u_{2|\Gamma^{\rm out}} - u_{1|\Gamma^{\rm out}}, \mu)_{b,\Gamma^{\rm out}} = 0 & \forall\ \mu \in L_b^2(\Gamma^{\rm in} \cup \Gamma^{\rm out}).
\end{cases} \tag{5.6.13}
$$

Note that $L_b^2(\Gamma^{\rm in})$ is the space of traces on $\Gamma^{\rm in}$ of $X_{1,b}$, as well as $L_b^2(\Gamma^{\rm out})$ being the space of traces on $\Gamma^{\rm out}$ of $X_{2,b}$.

The equivalence of (5.6.3) and (5.6.13) is easily shown by a direct computation, using the property that the solution to (5.6.3) satisfies $u \in L_b^2(\Gamma^{\rm in} \cup \Gamma^{\rm out})$, hence $u_{|\Omega_i} \in X_{i,b}$ (see F. Gastaldi and L. Gastaldi 1994, 1995). Moreover, for each $v \in X_b$ it holds that $\mathbf{b} \cdot \mathbf{n}\, v_{|\Omega_1} = \mathbf{b} \cdot \mathbf{n}\, v_{|\Omega_2}$ on $\Gamma^{\rm in} \cup \Gamma^{\rm out}$.

To derive the equivalent Steklov–Poincaré equation on the interface, we begin to introduce the extension operators K_i, $i = 1, 2$. If $\eta \in L_b^2(\Gamma^{\rm in} \cup \Gamma^{\rm out})$, $K_1\eta$ is defined in Ω_1, and satisfies

$$
K_1\eta \in X_{1,b}\ :\ \mathcal{C}_1(K_1\eta, v_1) = (\eta, v_{1|\Gamma^{\rm in}})_{b,\Gamma^{\rm in}} \qquad \forall\ v_1 \in X_{1,b}. \tag{5.6.14}
$$

Similarly, $K_2\eta$ is defined in Ω_2 and satisfies

$$
K_2\eta \in X_{2,b}\ :\ \mathcal{C}_2(K_2\eta, v_2) = (\eta, v_{2|\Gamma^{\rm out}})_{b,\Gamma^{\rm out}} \qquad \forall\ v_2 \in X_{2,b}. \tag{5.6.15}
$$

The solution $K_1\eta$ can be shown to satisfy the a priori estimate

$$
\mu_0 ||K_1\eta||_{0,\Omega_1}^2 + \frac{1}{2}||K_1\eta||_{b,\Gamma^{\rm out}\cup\partial\Omega_1^{\rm out}}^2 \le \frac{1}{2}||\eta||_{b,\Gamma^{\rm in}}^2, \tag{5.6.16}
$$

and similarly

$$
\mu_0 ||K_2\eta||_{0,\Omega_2}^2 + \frac{1}{2}||K_2\eta||_{b,\Gamma^{\rm in}\cup\partial\Omega_2^{\rm out}}^2 \le \frac{1}{2}||\eta||_{b,\Gamma^{\rm out}}^2 \tag{5.6.17}
$$

(see, for example, F. Gastaldi *et al.* 1990).

Introduce, moreover, the solution u_i^* of

$$
u_i^* \in X_{i,b}\ :\ \mathcal{C}_i(u_i^*, v_i) = (f, v_i)_{\Omega_i} + (\varphi_D, v_{i|\partial\Omega_i^{\rm in}})_{b,\partial\Omega_i^{\rm in}} \qquad \forall\ v_i \in X_{i,b}. \tag{5.6.18}
$$

Denoting by λ the trace on $\Gamma^{\rm in} \cup \Gamma^{\rm out}$ of the solution u to (5.6.3), it is readily seen that the restriction $u_{|\Omega_i}$ satisfies

$$
u_{|\Omega_i} = K_i\lambda + u_i^*. \tag{5.6.19}
$$

Therefore, imposing that $K_1\lambda + u_1^*$ and $K_2\lambda + u_2^*$ are the solutions to (5.6.13), we find the Steklov–Poincaré equation:

$$
\begin{aligned}
(\lambda -&(K_2\lambda)_{|\Gamma^{\rm in}} - u^*_{2|\Gamma^{\rm in}}, \mu)_{b,\Gamma^{\rm in}} \\
&+(\lambda - (K_1\lambda)_{|\Gamma^{\rm out}} - u^*_{1|\Gamma^{\rm out}}, \mu)_{b,\Gamma^{\rm out}} = 0 \qquad \forall\ \mu \in L^2_b(\Gamma^{\rm in} \cup \Gamma^{\rm out}),
\end{aligned} \tag{5.6.20}
$$

which in a strong form reads

$$
\begin{cases}
\lambda - (K_2\lambda)_{|\Gamma^{\rm in}} = u^*_{2|\Gamma^{\rm in}} & \text{on } \Gamma^{\rm in} \\
\lambda - (K_1\lambda)_{|\Gamma^{\rm out}} = u^*_{1|\Gamma^{\rm out}} & \text{on } \Gamma^{\rm out}.
\end{cases}
$$

Defining in $L^2_b(\Gamma^{\rm in} \cup \Gamma^{\rm out})$ the Steklov–Poincaré operator $S = S_1 + S_2$, where

$$
\begin{aligned}
\langle S_1\eta, \mu\rangle &:= (\eta, \mu)_{b,\Gamma^{\rm in}} - ((K_1\eta)_{|\Gamma^{\rm out}}, \mu)_{b,\Gamma^{\rm out}} \\
\langle S_2\eta, \mu\rangle &:= (\eta, \mu)_{b,\Gamma^{\rm out}} - ((K_2\eta)_{|\Gamma^{\rm in}}, \mu)_{b,\Gamma^{\rm in}}
\end{aligned} \tag{5.6.21}
$$

and the function $\chi = \chi_1 + \chi_2$, where

$$
\begin{aligned}
\langle \chi_1, \mu\rangle &:= (u^*_{1|\Gamma^{\rm out}}, \mu)_{b,\Gamma^{\rm out}} \\
\langle \chi_2, \mu\rangle &:= (u^*_{2|\Gamma^{\rm in}}, \mu)_{b,\Gamma^{\rm in}},
\end{aligned} \tag{5.6.22}
$$

the Steklov–Poincaré equation can be written as

$$
\lambda \in L^2_b(\Gamma^{\rm in} \cup \Gamma^{\rm out}) \ :\ \langle S\lambda, \mu\rangle = \langle \chi, \mu\rangle \qquad \forall\ \mu \in L^2_b(\Gamma^{\rm in} \cup \Gamma^{\rm out}). \tag{5.6.23}
$$

From (5.6.16) and (5.6.17), we see that the operator S is continuous in $L^2_b(\Gamma^{\rm in} \cup \Gamma^{\rm out})$. Moreover, using (5.6.14) and (5.6.15), it can also be written as

$$
\langle S\eta, \mu\rangle = \sum_{i=1}^{2} \mathcal{C}_i(K_i\eta, K_i\mu) - ((K_1\eta)_{|\Gamma^{\rm out}}, \mu)_{b,\Gamma^{\rm out}} - ((K_2\eta)_{|\Gamma^{\rm in}}, \mu)_{b,\Gamma^{\rm in}}.
$$

The operator S_1 satisfies

$$
\begin{aligned}
\langle S_1\eta, \eta\rangle &= ||\eta||^2_{b,\Gamma^{\rm in}} - ((K_1\eta)_{|\Gamma^{\rm out}}, \eta)_{b,\Gamma^{\rm out}} \\
&= \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm in}} + \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm in}} - ((K_1\eta)_{|\Gamma^{\rm out}}, \eta)_{b,\Gamma^{\rm out}} \\
&\geq \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm in}} + \mu_0||K_1\eta||^2_{0,\Omega_1} + \frac{1}{2}||K_1\eta||^2_{b,\partial\Omega_1^{\rm out}} \\
&\quad + \frac{1}{2}||K_1\eta||^2_{b,\Gamma^{\rm out}} - ((K_1\eta)_{|\Gamma^{\rm out}}, \eta)_{b,\Gamma^{\rm out}},
\end{aligned}
$$

having used (5.6.16). We also have

$$
\frac{1}{2}||K_1\eta||^2_{b,\Gamma^{\rm out}} - ((K_1\eta)_{|\Gamma^{\rm out}}, \eta)_{b,\Gamma^{\rm out}} = \frac{1}{2}||\eta - K_1\eta||^2_{b,\Gamma^{\rm out}} - \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm out}},
$$

therefore

$$\begin{aligned}\langle S_1\eta,\eta\rangle \geq \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm in}} + \mu_0||K_1\eta||^2_{0,\Omega_1} + \frac{1}{2}||K_1\eta||^2_{b,\partial\Omega_1^{\rm out}} \\ +\frac{1}{2}||\eta - K_1\eta||^2_{b,\Gamma^{\rm out}} - \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm out}}.\end{aligned} \tag{5.6.24}$$

Similarly, we find that

$$\begin{aligned}\langle S_2\eta,\eta\rangle \geq \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm out}} + \mu_0||K_2\eta||^2_{0,\Omega_2} + \frac{1}{2}||K_2\eta||^2_{b,\partial\Omega_2^{\rm out}} \\ +\frac{1}{2}||\eta - K_2\eta||^2_{b,\Gamma^{\rm in}} - \frac{1}{2}||\eta||^2_{b,\Gamma^{\rm in}},\end{aligned} \tag{5.6.25}$$

hence

$$\begin{aligned}\langle S\eta,\eta\rangle \geq \mu_0||K_1\eta||^2_{0,\Omega_1} + \mu_0||K_2\eta||^2_{0,\Omega_2} \\ +\frac{1}{2}||K_1\eta||^2_{b,\partial\Omega_1^{\rm out}} + \frac{1}{2}||K_2\eta||^2_{b,\partial\Omega_2^{\rm out}}.\end{aligned} \tag{5.6.26}$$

Therefore, the operator S is positive, i.e.

$$\langle S\eta,\eta\rangle > 0 \quad \text{for } \eta \neq 0.$$

5.6.2 Substructuring iterative methods

The following iterative method has been proposed by F. Gastaldi and L. Gastaldi (1994, 1995): given $\lambda^0_{\rm in} \in L^2_b(\Gamma^{\rm in})$, for each $k \geq 0$ solve

$$\begin{cases} \text{find } u_1^{k+1} \in X_{1,b} \ : \\ \mathcal{C}_1(u_1^{k+1}, v_1) = (f, v_1)_{\Omega_1} + (\varphi_D, v_{1|\partial\Omega_1^{\rm in}})_{b,\partial\Omega_1^{\rm in}} \\ \qquad\qquad +(\lambda^k_{\rm in}, v_{1|\Gamma^{\rm in}})_{b,\Gamma^{\rm in}} \qquad \forall\ v_1 \in X_{1,b}, \end{cases} \tag{5.6.27}$$

then

$$\begin{cases} \text{find } u_2^{k+1} \in X_{2,b} \ : \\ \mathcal{C}_2(u_2^{k+1}, v_2) = (f, v_2)_{\Omega_2} + (\varphi_D, v_{2|\partial\Omega_2^{\rm in}})_{b,\partial\Omega_2^{\rm in}} \\ \qquad\qquad +(u^{k+1}_{1|\Gamma^{\rm out}}, v_{2|\Gamma^{\rm out}})_{b,\Gamma^{\rm out}} \qquad \forall\ v_2 \in X_{2,b}, \end{cases} \tag{5.6.28}$$

and finally set

$$\lambda^{k+1}_{\rm in} := u^{k+1}_{2|\Gamma^{\rm in}}. \tag{5.6.29}$$

In (5.6.27) we are imposing the Dirichlet condition $u_1^{k+1} = \lambda^k_{\rm in}$ on $\Gamma^{\rm in}$, while in (5.6.28) we are imposing the Dirichlet condition $u_2^{k+1} = u_1^{k+1}$ on $\Gamma^{\rm out}$.

For each $\eta \in L^2(\Gamma^{\rm in} \cup \Gamma^{\rm out})$ denote by $\eta_{\rm in}$ and $\eta_{\rm out}$ its restrictions to $\Gamma^{\rm in}$ and $\Gamma^{\rm out}$, respectively, and introduce the operators $J_1 : L^2_b(\Gamma^{\rm in}) \to L^2_b(\Gamma^{\rm out})$ and $J_2 : L^2_b(\Gamma^{\rm out}) \to L^2_b(\Gamma^{\rm in})$ as follows:

$$J_1\eta_{\rm in} := (K_1\eta_{\rm in})_{|\Gamma^{\rm out}}, \quad J_2\eta_{\rm out} := (K_2\eta_{\rm out})_{|\Gamma^{\rm in}}. \tag{5.6.30}$$

The iteration operator associated with (5.6.27)–(5.6.29) is given by J_2J_1, and in F. Gastaldi and L. Gastaldi (1994, 1995) it has been proved that it is a contraction in $L^2_b(\Gamma^{\rm in})$. Therefore, the iterative method (5.6.27)–(5.6.29) is convergent in $L^2_b(\Gamma^{\rm in})$, and the limit function $\lambda_{\rm in}$ is equal to $u_{|\Gamma^{\rm in}}$, where u is the solution to (5.6.9). Using a similar proof the map J_1J_2 is also shown to be a contraction in $L^2_b(\Gamma^{\rm out})$.

The dual space of $L^2_b(\Gamma^{\rm in}\cup\Gamma^{\rm out})$ is given by

$$L^2_{1/b}(\Gamma^{\rm in}\cup\Gamma^{\rm out}) := \Big\{\eta : \Gamma^{\rm in}\cup\Gamma^{\rm out}\to\mathbf{R}\,|\, \int_{\Gamma^{\rm in}\cup\Gamma^{\rm out}} |\mathbf{b}\cdot\mathbf{n}|^{-1}\,\eta^2 < \infty\Big\}.$$

It is easily seen that the Steklov–Poincaré operator

$$S : L^2_b(\Gamma^{\rm in}\cup\Gamma^{\rm out}) \to L^2_{1/b}(\Gamma^{\rm in}\cup\Gamma^{\rm out})$$

can be rewritten as

$$S\eta = \begin{cases} -\mathbf{b}\cdot\mathbf{n}\,\eta_{\rm in} + \mathbf{b}\cdot\mathbf{n}\,J_2\eta_{\rm out} & \text{on } \Gamma^{\rm in} \\ -\mathbf{b}\cdot\mathbf{n}\,J_1\eta_{\rm in} + \mathbf{b}\cdot\mathbf{n}\,\eta_{\rm out} & \text{on } \Gamma^{\rm out} \end{cases}.$$

Introducing the operator $Q : L^2_{1/b}(\Gamma^{\rm in}\cup\Gamma^{\rm out}) \to L^2_b(\Gamma^{\rm in}\cup\Gamma^{\rm out})$ defined as

$$Q\psi := \frac{1}{\mathbf{b}\cdot\mathbf{n}} S\left(\frac{1}{\mathbf{b}\cdot\mathbf{n}}\psi\right), \tag{5.6.31}$$

we find that

$$(I-QS)\eta = \begin{cases} J_2J_1\eta_{\rm in} & \text{on } \Gamma^{\rm in} \\ J_1J_2\eta_{\rm out} & \text{on } \Gamma^{\rm out} \end{cases}.$$

Hence, the operator $I-QS$ is a contraction in $L^2_b(\Gamma^{\rm in}\cup\Gamma^{\rm out})$. As a consequence, the operator QS is coercive in the same space; in fact, from the inequality

$$||\eta - QS\eta||^2_{b,\Gamma^{\rm in}\cup\Gamma^{\rm out}} \le K^2||\eta||^2_{b,\Gamma^{\rm in}\cup\Gamma^{\rm out}},$$

where $0<K<1$, it follows that

$$2(QS\eta,\eta)_{b,\Gamma^{\rm in}\cup\Gamma^{\rm out}} \ge (1-K^2)||\eta||^2_{b,\Gamma^{\rm in}\cup\Gamma^{\rm out}} + ||QS\eta||^2_{b,\Gamma^{\rm in}\cup\Gamma^{\rm out}}.$$

The iteration procedure (5.6.27)–(5.6.29) can be seen as a preconditioned Richardson method *without relaxation* (i.e. for $\theta=1$) applied to the operator S, with Q^{-1} as a preconditioner. Since the operator QS is coercive, other iterative procedures could be devised as well. Note that, unlike for elliptic equations, relaxation is not mandatory at the interface.

The finite element approximation of (5.6.9) and the finite dimensional iteration procedure corresponding to (5.6.27)–(5.6.29) have also been studied by F. Gastaldi and L. Gastaldi (1994, 1995). Their approximation is based on discontinuous piecewise-polynomial finite elements (see Lesaint and Raviart 1974; and Quarteroni and Valli 1994, p. 487). The rate of convergence of the iteration-by-subdomain procedure is proved to be independent of the finite element mesh size.

5.7 The time-harmonic Maxwell equations

The Maxwell equations describe the wave propagation phenomena in electromagnetism. They read:

$$\frac{\partial \boldsymbol{\mathcal{D}}}{\partial t} = \operatorname{rot} \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{J}}$$
$$\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t} = -\operatorname{rot} \boldsymbol{\mathcal{E}},$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ are the electric and magnetic field, and $\boldsymbol{\mathcal{D}}$ and $\boldsymbol{\mathcal{B}}$ are the electric and magnetic induction, respectively, and $\boldsymbol{\mathcal{J}}$ is the density of the electric current. The following constitutive relations are assumed to hold:

$$\boldsymbol{\mathcal{D}} = \varepsilon \boldsymbol{\mathcal{E}}, \quad \boldsymbol{\mathcal{B}} = \mu \boldsymbol{\mathcal{H}}$$

(where ε and μ are the dielectric permittivity and magnetic permeability, respectively), as well as Ohm's law

$$\boldsymbol{\mathcal{J}} = \sigma \boldsymbol{\mathcal{E}}$$

(where σ is the electric conductivity). The quantities ε, μ and σ are, in general, symmetric matrices, depending on the space variable $\mathbf{x}$, and are assumed to be uniformly positive definite in a conductor. In a perfect insulator σ is zero, while ε and μ are positive constants. For a complete presentation and for several theoretical mathematical results and approximation methods for the Maxwell equations see Duvaut and Lions (1976) and Bossavit (1993).

The so-called *time-harmonic* Maxwell equations are derived from the complete Maxwell system when one seeks solutions of the form

$$\boldsymbol{\mathcal{E}}(t, \mathbf{x}) = \operatorname{Re}[\mathbf{E}(\mathbf{x}) \exp(i\alpha t)], \quad \boldsymbol{\mathcal{H}}(t, \mathbf{x}) = \operatorname{Re}[\mathbf{H}(\mathbf{x}) \exp(i\alpha t)], \tag{5.7.1}$$

where i is the imaginary unit and $\alpha \neq 0$ is a parameter (the angular frequency).

The unknowns in the resulting problem are the complex vector fields $\mathbf{E}$ and $\mathbf{H}$, and these have to satisfy

$$\begin{cases} \operatorname{rot} \mathbf{H} - i\varepsilon\alpha \mathbf{E} - \sigma \mathbf{E} = \mathbf{0} & \text{in } \Omega \\ \operatorname{rot} \mathbf{E} + i\mu\alpha \mathbf{H} = \mathbf{0} & \text{in } \Omega \\ \mathbf{n} \times \mathbf{E} = \boldsymbol{\Psi} & \text{on } \partial\Omega, \end{cases} \tag{5.7.2}$$

where Ω is a bounded domain of $\mathbf{R}^3$, and the tangential vector field $\boldsymbol{\Psi}$ is a given datum defined on $\partial\Omega$.

Rewriting the system above in terms of the unknown $\mathbf{E}$, we find that

$$\begin{cases} \operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{E}) - \alpha^2\varepsilon\mathbf{E} + i\alpha\sigma\mathbf{E} = \mathbf{0} & \text{in } \Omega \\ \\ \mathbf{n}\times\mathbf{E} = \boldsymbol{\Psi} & \text{on } \partial\Omega. \end{cases} \tag{5.7.3}$$

Setting $\mathbf{u} := \mathbf{E} - \mathbf{E}_\Psi$, where $\mathbf{E}_\Psi$ is defined in Ω and satisfies $\mathbf{n}\times\mathbf{E}_\Psi = \boldsymbol{\Psi}$ on $\partial\Omega$, problem (5.7.3) can be finally rewritten as

$$\begin{cases} \operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{u}) - \alpha^2\varepsilon\mathbf{u} + i\alpha\sigma\mathbf{u} = \mathbf{F} & \text{in } \Omega \\ \\ \mathbf{n}\times\mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{5.7.4}$$

where $\mathbf{F} := -\operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{E}_\Psi) + \alpha^2\varepsilon\mathbf{E}_\Psi - i\alpha\sigma\mathbf{E}_\Psi$.

The two-domain formulation of (5.7.4) is as follows. Denoting by $\mathbf{u}_i$ the restriction of $\mathbf{u}$ to the subdomain Ω_i, $i = 1, 2$, in each subdomain $\mathbf{u}_i$ satisfies (5.7.4). In addition, the interface conditions are given by (see, for example, Alonso and Valli 1997)

$$\begin{aligned} &\mathbf{n}\times\mathbf{u}_1 = \mathbf{n}\times\mathbf{u}_2 && \text{on } \Gamma \\ &\mathbf{n}\times(\mu^{-1}\operatorname{rot}\mathbf{u}_1) = \mathbf{n}\times(\mu^{-1}\operatorname{rot}\mathbf{u}_2) && \text{on } \Gamma. \end{aligned} \tag{5.7.5}$$

5.7.1 Weak multi-domain formulation

Introducing the Hilbert spaces

$$\begin{aligned} H(\operatorname{rot};\Omega) &:= \{\mathbf{v}\in(L^2(\Omega))^3 \mid \operatorname{rot}\mathbf{v}\in(L^2(\Omega))^3\} \\ H_0(\operatorname{rot};\Omega) &:= \{\mathbf{v}\in H(\operatorname{rot};\Omega) \mid (\mathbf{n}^*\times\mathbf{v})_{|\partial\Omega} = \mathbf{0}\}, \end{aligned} \tag{5.7.6}$$

and the bilinear form

$$\begin{aligned} m(\mathbf{w},\mathbf{v}) := \int_\Omega (&\mu^{-1}\operatorname{rot}\mathbf{w}\cdot\operatorname{rot}\overline{\mathbf{v}} - \alpha^2\varepsilon\mathbf{w}\cdot\overline{\mathbf{v}} \\ &+ i\alpha\sigma\mathbf{w}\cdot\overline{\mathbf{v}}) \qquad \forall\ \mathbf{w},\mathbf{v}\in H(\operatorname{rot};\Omega), \end{aligned} \tag{5.7.7}$$

the weak formulation of (5.7.4) reads

$$\text{find } \mathbf{u}\in H_0(\operatorname{rot};\Omega) \ :\ m(\mathbf{u},\mathbf{v}) = (\mathbf{F},\mathbf{v}) \qquad \forall\ \mathbf{v}\in H_0(\operatorname{rot};\Omega). \tag{5.7.8}$$

Let us now restrict our analysis to the case of a conductor, so that ε, μ and σ are assumed to be symmetric matrices, uniformly positive definite in Ω. The case of an heterogeneous medium will be considered in Section 8.5. Under these assumptions, the bilinear form $m(\cdot,\cdot)$ is continuous and coercive in $H(\operatorname{rot};\Omega)$; therefore, the existence and uniqueness of a solution follow from the Lax–Milgram lemma. In fact, we have that

$$
\begin{aligned}
|m(\mathbf{v},\mathbf{v})|^2 &= \left(\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{v}\cdot\mathrm{rot}\,\overline{\mathbf{v}} - \alpha^2\int_\Omega \varepsilon\mathbf{v}\cdot\overline{\mathbf{v}}\right)^2 \\
&\quad + \left(\alpha\int_\Omega \sigma\mathbf{v}\cdot\overline{\mathbf{v}}\right)^2 \\
&= \left(\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{v}\cdot\mathrm{rot}\,\overline{\mathbf{v}}\right)^2 + \left(\alpha^2\int_\Omega \varepsilon\mathbf{v}\cdot\overline{\mathbf{v}}\right)^2 \\
&\quad - 2\left(\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{v}\cdot\mathrm{rot}\,\overline{\mathbf{v}}\right)\left(\alpha^2\int_\Omega \varepsilon\mathbf{v}\cdot\overline{\mathbf{v}}\right) \\
&\quad + \alpha^2\left(\int_\Omega \sigma\mathbf{v}\cdot\overline{\mathbf{v}}\right)^2.
\end{aligned}
$$

Noting that

$$
\sigma_0||\mathbf{v}||^2_{0,\Omega} \le \int_\Omega \sigma\mathbf{v}\cdot\overline{\mathbf{v}}, \quad \int_\Omega \varepsilon\mathbf{v}\cdot\overline{\mathbf{v}} \le \varepsilon_1||\mathbf{v}||^2_{0,\Omega},
$$

where σ_0 is a uniform lower bound for the minimum eigenvalue of $\sigma(\mathbf{x})$, and ε_1 is a uniform upper bound for the maximum eigenvalue of $\varepsilon(\mathbf{x})$, from the inequality

$$
2AB \le \delta A^2 + \delta^{-1}B^2, \quad \delta > 0,
$$

we find, for $\delta < 1$,

$$
\begin{aligned}
|m(\mathbf{v},\mathbf{v})|^2 &\ge (1-\delta)\left(\int_\Omega \mu^{-1}\,\mathrm{rot}\,\mathbf{v}\cdot\mathrm{rot}\,\overline{\mathbf{v}}\right)^2 \\
&\quad + \alpha^2\left[1 - \frac{\alpha^2(1-\delta)\varepsilon_1^2}{\delta\sigma_0^2}\right]\left(\int_\Omega \sigma\mathbf{v}\cdot\overline{\mathbf{v}}\right)^2.
\end{aligned}
$$

Therefore, coerciveness follows on choosing

$$
\frac{\alpha^2\varepsilon_1^2}{\alpha^2\varepsilon_1^2 + \sigma_0^2} < \delta < 1.
$$

Note that in the so-called *low-frequency* case that corresponds to omitting the term

$$
-\int_\Omega \alpha^2\varepsilon\mathbf{w}\cdot\overline{\mathbf{v}}
$$

in (5.7.7), the coerciveness result still holds with an easier proof.

To derive the equivalent two-domain formulation, for $j = 1, 2$ let us denote by $m_j(\cdot,\cdot)$ the restriction of $m(\cdot,\cdot)$ to Ω_j and introduce the following spaces:

$$
\begin{aligned}
&Z_j := \{\mathbf{v}_j \in H(\mathrm{rot};\Omega_j) \mid (\mathbf{n}^* \times \mathbf{v}_j)_{|\partial\Omega\cap\partial\Omega_j} = \mathbf{0}\} \\
&Z_j^0 := H_0(\mathrm{rot};\Omega_j) \qquad (5.7.9) \\
&\mathcal{X}_\Gamma := \{\boldsymbol{\psi} : \Gamma \to \mathbf{R}^3 \mid \boldsymbol{\psi} = (\mathbf{n}\times\mathbf{v})_{|\Gamma},\ \mathbf{v} \in H_0(\mathrm{rot};\Omega)\}.
\end{aligned}
$$

The variational formulation of the two-domain problem reads: find $(\mathbf{u}_1, \mathbf{u}_2) \in Z_1 \times Z_2$ such that

$$
\left\{\begin{array}{l}
m_1(\mathbf{u}_1,\mathbf{v}_1) = (\mathbf{F},\mathbf{v}_1)_{\Omega_1} \qquad \forall\ \mathbf{v}_1 \in Z_1^0 \\
\mathbf{n}\times\mathbf{u}_1 = \mathbf{n}\times\mathbf{u}_2 \qquad \text{on } \Gamma \\
m_2(\mathbf{u}_2,\mathbf{v}_2) = (\mathbf{F},\mathbf{v}_2)_{\Omega_2} \qquad \forall\ \mathbf{v}_2 \in Z_2^0 \\
m_2(\mathbf{u}_2,\boldsymbol{\mathcal{R}}_2^\star\boldsymbol{\psi}) = (\mathbf{F},\boldsymbol{\mathcal{R}}_2^\star\boldsymbol{\psi})_{\Omega_2} \\
\qquad\qquad +(\mathbf{F},\boldsymbol{\mathcal{R}}_1^\star\boldsymbol{\psi})_{\Omega_1} - m_1(\mathbf{u}_1,\boldsymbol{\mathcal{R}}_1^\star\boldsymbol{\psi}) \qquad \forall\ \boldsymbol{\psi}\in\mathcal{X}_\Gamma,
\end{array}\right. \tag{5.7.10}
$$

where we have denoted by $\boldsymbol{\mathcal{R}}_j^\star$ any operator from $\mathcal{X}_\Gamma$ to Z_j satisfying $(\mathbf{n}\times\boldsymbol{\mathcal{R}}_j^\star\boldsymbol{\psi})_{|\Gamma} = \boldsymbol{\psi}$ for each $\boldsymbol{\psi}\in\mathcal{X}_\Gamma$.

5.7.2 The finite dimensional Steklov–Poincaré interface equation

We start by considering the finite element approximation of problem (5.7.8) and of its two-domain formulation (5.7.10). It is based on the so-called rot-*conforming* Nédélec finite elements, introduced in Section 4.1.3 and there denoted by M_h^r. Let us denote the finite element space used for the approximation of the unknown $\mathbf{u}$ by $Z_h := M_h^r \cap H_0(\mathrm{rot};\Omega)$, and set

$$
\begin{aligned}
Z_{j,h} &:= \{\mathbf{v}_{h|\Omega_j} \mid \mathbf{v}_h \in Z_h\}, \quad j=1,2 \\
Z_{j,h}^0 &:= Z_{j,h} \cap H_0(\mathrm{rot};\Omega_j), \quad j=1,2 \\
\mathcal{X}_{\Gamma,h} &:= \{(\mathbf{n}\times\mathbf{v}_{1,h})_{|\Gamma} \,|\, \mathbf{v}_{1,h}\in Z_{1,h}\} = \{(\mathbf{n}\times\mathbf{v}_{2,h})_{|\Gamma} \,|\, \mathbf{v}_{2,h}\in Z_{2,h}\}.
\end{aligned}
$$

The finite element approximation of (5.7.8) is given by

$$
\text{find } \mathbf{u}_h \in Z_h \ : m(\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{F},\mathbf{v}_h) \qquad \forall\ \mathbf{v}_h\in Z_h. \tag{5.7.11}
$$

On the other hand, the induced approximation of the two-domain formulation (5.7.10) reads: find $(\mathbf{u}_{1,h},\mathbf{u}_{2,h}) \in Z_{1,h}\times Z_{2,h}$ such that

$$
\left\{\begin{array}{l}
m_1(\mathbf{u}_{1,h},\mathbf{v}_{1,h}) = (\mathbf{F},\mathbf{v}_{1,h})_{\Omega_1} \qquad \forall\ \mathbf{v}_{1,h}\in Z_{1,h}^0 \\
\mathbf{n}\times\mathbf{u}_{1,h} = \mathbf{n}\times\mathbf{u}_{2,h} \qquad \text{on } \Gamma \\
m_2(\mathbf{u}_{2,h},\mathbf{v}_{2,h}) = (\mathbf{F},\mathbf{v}_{2,h})_{\Omega_2} \qquad \forall\ \mathbf{v}_{2,h}\in Z_{2,h}^0 \\
m_2(\mathbf{u}_{2,h},\boldsymbol{\mathcal{R}}_{2,h}^\star\boldsymbol{\psi}_h) = (\mathbf{F},\boldsymbol{\mathcal{R}}_{2,h}^\star\boldsymbol{\psi}_h)_{\Omega_2} + (\mathbf{F},\boldsymbol{\mathcal{R}}_{1,h}^\star\boldsymbol{\psi}_h)_{\Omega_1} \\
\qquad\qquad -m_1(\mathbf{u}_{1,h},\boldsymbol{\mathcal{R}}_{1,h}^\star\boldsymbol{\psi}_h) \qquad \forall\ \boldsymbol{\psi}_h\in\mathcal{X}_{\Gamma,h},
\end{array}\right. \tag{5.7.12}
$$

where we have denoted by $\boldsymbol{\mathcal{R}}_{j,h}^\star$ any operator from $\mathcal{X}_{\Gamma,h}$ to $Z_{j,h}$ satisfying $(\mathbf{n}\times\boldsymbol{\mathcal{R}}_{j,h}^\star\boldsymbol{\psi}_h)_{|\Gamma} = \boldsymbol{\psi}_h$ for each $\boldsymbol{\psi}_h\in\mathcal{X}_{\Gamma,h}$.

Since the bilinear forms $m_j(\cdot,\cdot)$ are continuous and coercive in $H(\mathrm{rot};\Omega_j)$, $j=1,2$, we can introduce for each $\boldsymbol{\eta}_h\in\mathcal{X}_{\Gamma,h}$ the finite element solution $\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h \in Z_{j,h}$ to

$$\text{(5.7.13)}\qquad \begin{cases} m_j(\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h, \mathbf{v}_{j,h}) = 0 & \forall\ \mathbf{v}_{j,h} \in Z^0_{j,h} \\ (\mathbf{n} \times \boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h)_{|\Gamma} = \boldsymbol{\eta}_h & \text{on } \Gamma, \end{cases}$$

and the solution $\boldsymbol{\mathcal{E}}^*_{j,h} \in Z^0_{j,h}$ to

$$\text{(5.7.14)}\qquad m_j(\boldsymbol{\mathcal{E}}^*_{j,h}, \mathbf{v}_{j,h}) = (\mathbf{F}, \mathbf{v}_{j,h})_{\Omega_j} \qquad \forall\ \mathbf{v}_{j,h} \in Z^0_{j,h}.$$

The finite dimensional Steklov–Poincaré equation now reads

$$\text{(5.7.15)}\qquad \langle S_h\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h\rangle = \langle \boldsymbol{\chi}_h, \boldsymbol{\mu}_h\rangle \qquad \forall\ \boldsymbol{\mu}_h \in \mathcal{X}_{\Gamma,h},$$

where $S_h := S_{1,h} + S_{2,h}$, $\boldsymbol{\chi}_h := \boldsymbol{\chi}_{1,h} + \boldsymbol{\chi}_{2,h}$ and

$$\text{(5.7.16)}\qquad \langle S_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h\rangle := m_j(\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\mu}_h) \qquad \forall\ \boldsymbol{\eta}_h, \boldsymbol{\mu}_h \in \mathcal{X}_{\Gamma,h},$$

$$\text{(5.7.17)}\qquad \langle \boldsymbol{\chi}_{j,h}, \boldsymbol{\mu}_h\rangle := (\mathbf{F}, \boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\mu}_h)_{\Omega_j} - m_j(\boldsymbol{\mathcal{E}}^*_{j,h}, \boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\mu}_h) \qquad \forall\ \boldsymbol{\mu}_h \in \mathcal{X}_{\Gamma,h}.$$

5.7.3 Substructuring iterative methods

The Dirichlet–Neumann iterative scheme for solving the two-domain problem (5.7.12) reads: given $\boldsymbol{\lambda}^0_h \in \mathcal{X}_{\Gamma,h}$, for each $k \geq 0$ solve

$$\text{(5.7.18)}\qquad \begin{cases} \text{find } \mathbf{u}^{k+1}_{1,h} \in Z_{1,h} \ : \\ m_1(\mathbf{u}^{k+1}_{1,h}, \mathbf{v}_{1,h}) = (\mathbf{F}, \mathbf{v}_{1,h})_{\Omega_1} \qquad \forall\ \mathbf{v}_{1,h} \in Z^0_{1,h} \\ (\mathbf{n} \times \mathbf{u}^{k+1}_{1,h})_{|\Gamma} = \boldsymbol{\lambda}^k_h \qquad \text{on } \Gamma, \end{cases}$$

then

$$\text{(5.7.19)}\qquad \begin{cases} \text{find } \mathbf{u}^{k+1}_{2,h} \in Z_{2,h} \ : \\ m_2(\mathbf{u}^{k+1}_{2,h}, \mathbf{v}_{2,h}) = (\mathbf{F}, \mathbf{v}_{2,h})_{\Omega_2} \qquad \forall\ \mathbf{v}_{2,h} \in Z^0_{2,h} \\ m_2(\mathbf{u}^{k+1}_{2,h}, \boldsymbol{\mathcal{R}}^\star_{2,h}\boldsymbol{\psi}_h) = (\mathbf{F}, \boldsymbol{\mathcal{R}}^\star_{2,h}\boldsymbol{\psi}_h)_{\Omega_2} + (\mathbf{F}, \boldsymbol{\mathcal{R}}^\star_{1,h}\boldsymbol{\psi}_h)_{\Omega_1} \\ \qquad\qquad -m_1(\mathbf{u}^{k+1}_{1,h}, \boldsymbol{\mathcal{R}}^\star_{1,h}\boldsymbol{\psi}_h) \qquad \forall\ \boldsymbol{\psi}_h \in \mathcal{X}_{\Gamma,h}, \end{cases}$$

and finally set

$$\text{(5.7.20)}\qquad \boldsymbol{\lambda}^{k+1}_h := \theta(\mathbf{n} \times \mathbf{u}^{k+1}_{2,h})_{|\Gamma} + (1-\theta)\boldsymbol{\lambda}^k_h \qquad \text{on } \Gamma.$$

This iterative scheme is equivalent to the following preconditioned Richardson iterations for the Steklov–Poincaré problem (5.7.15)

$$\boldsymbol{\lambda}_h^{k+1} = \boldsymbol{\lambda}_h^k + \theta S_{2,h}^{-1}(\boldsymbol{\chi} - S_h \boldsymbol{\lambda}_h^k) \qquad \text{on } \Gamma. \tag{5.7.21}$$

Its convergence can be proved by resorting to Theorem 4.2.10 and Corollary 4.2.11, because we are dealing with *complex non-symmetric* Steklov–Poincaré operators. The complex matrices associated with the discrete Steklov–Poincaré operators are defined as

$$((\Sigma_{j,h}\boldsymbol{\eta}, \boldsymbol{\mu})) := \langle S_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\mu}_h \rangle,$$

where $\boldsymbol{\eta}$, $\boldsymbol{\mu} \in \mathbf{C}_h^M$ are the vectors corresponding to the tangential traces $\boldsymbol{\eta}_h$, $\boldsymbol{\mu}_h \in \mathcal{X}_{\Gamma,h}$, respectively, and $((\cdot,\cdot))$ denotes the scalar product in $\mathbf{C}^{M_h}$. We want to prove that (4.2.14) is satisfied, and that the constants C_0 and $\widehat{C}_0$ in (4.2.15) and (4.2.16) can be bounded independently of h.

Let us now restrict ourselves to the *low-frequency* case; that is, suppose that the frequency α is so small in comparison with the other coefficients μ^{-1} and $\alpha\sigma$ that the coefficient $\alpha^2\varepsilon$ in (5.7.4) can be omitted. In this case, the operators $S_{j,h}$, $j = 1, 2$, satisfy

$$\begin{aligned} |\langle S_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\eta}_h \rangle| &\geq c_1(||\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h||^2_{0,\Omega_j} + ||\operatorname{rot}\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h||^2_{0,\Omega_j}) \\ &\geq c_2||\boldsymbol{\eta}_h||^2_{\mathcal{X}_\Gamma} \qquad \forall\ \boldsymbol{\eta}_h \in \mathcal{X}_{\Gamma,h}, \end{aligned} \tag{5.7.22}$$

where we used the coerciveness of the forms $m_j(\cdot,\cdot)$ and the continuity of the tangential trace operator from $H(\operatorname{rot};\Omega_j)$ into $\mathcal{X}_\Gamma$ that was proved in Alonso and Valli (1996). Moreover,

$$\begin{aligned} \operatorname{Re}\langle S_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\eta}_h \rangle &= \int_{\Omega_j} \mu^{-1} \operatorname{rot}\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h \cdot \operatorname{rot}\boldsymbol{\mathcal{E}}_{j,h}\overline{\boldsymbol{\eta}_h} \geq 0 \\ \operatorname{Im}\langle S_{j,h}\boldsymbol{\eta}_h, \boldsymbol{\eta}_h \rangle &= \alpha \int_{\Omega_j} \sigma\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h \cdot \boldsymbol{\mathcal{E}}_{j,h}\overline{\boldsymbol{\eta}_h} \geq 0. \end{aligned} \tag{5.7.23}$$

Hence

$$\begin{aligned} &\operatorname{Re}\,((\Sigma_h\boldsymbol{\eta},\boldsymbol{\eta}))\operatorname{Re}((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta})) \\ &\qquad +\operatorname{Im}((\Sigma_h\boldsymbol{\eta},\boldsymbol{\eta}))\operatorname{Im}((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta})) \\ &= \operatorname{Re}((\Sigma_{1,h}\boldsymbol{\eta},\boldsymbol{\eta}))\operatorname{Re}((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta})) \\ &\qquad +\operatorname{Im}((\Sigma_{1,h}\boldsymbol{\eta},\boldsymbol{\eta}))\operatorname{Im}((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta})) + |((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta}))|^2 \\ &\geq |((\Sigma_{2,h}\boldsymbol{\eta},\boldsymbol{\eta}))|^2, \end{aligned}$$

and (4.2.14) and (4.2.15) are satisfied, with $C_0 = 1$.

The proof that (4.2.16) holds with a constant $\widehat{C}_0$ independent of h is more involved. First of all, from (5.7.22) and the continuity of the form $m_1(\cdot,\cdot)$ one has

$$\frac{|((\Sigma\boldsymbol{\eta},\boldsymbol{\eta}))|}{|((\Sigma_2\boldsymbol{\eta},\boldsymbol{\eta}))|} \leq 1 + \frac{|((\Sigma_1\boldsymbol{\eta},\boldsymbol{\eta}))|}{|((\Sigma_2\boldsymbol{\eta},\boldsymbol{\eta}))|} \leq 1 + c_3\frac{||\boldsymbol{\mathcal{E}}_{1,h}\boldsymbol{\eta}_h||^2_{H(\operatorname{rot};\Omega_1)}}{||\boldsymbol{\eta}_h||^2_{\mathcal{X}_\Gamma}}.$$

The estimate

$$||\boldsymbol{\mathcal{E}}_{1,h}\boldsymbol{\eta}_h||_{H(\mathrm{rot};\Omega_1)} \leq c_4||\boldsymbol{\eta}_h||_{\mathcal{X}_\Gamma}, \tag{5.7.24}$$

is obtained as follows. For each $j = 1, 2$, let us first introduce the discrete functions $\boldsymbol{\mathcal{F}}_{1,h}\boldsymbol{\eta}_h$ as in (4.1.39). Taking in (5.7.13) the test function $\mathbf{v}_{j,h} = \boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h - \boldsymbol{\mathcal{F}}_{j,h}\boldsymbol{\eta}_h$, $j = 1, 2$, the coerciveness and continuity of the form $m_j(\cdot,\cdot)$ yields

$$||\boldsymbol{\mathcal{E}}_{j,h}\boldsymbol{\eta}_h||_{H(\mathrm{rot};\Omega_j)} \leq c_5||\boldsymbol{\mathcal{F}}_{j,h}\boldsymbol{\eta}_h||_{H(\mathrm{rot};\Omega_j)}.$$

Finally, the inequality

$$||\boldsymbol{\mathcal{F}}_{j,h}\boldsymbol{\eta}_h||_{H(\mathrm{rot};\Omega_j)} \leq c_6||\boldsymbol{\eta}_h||_{\mathcal{X}_\Gamma},$$

is proved in Theorem 4.1.11, under the assumptions that the family of triangulations $\mathcal{T}_h$ of Ω is regular, the family of triangulations $\mathcal{M}_h$ induced by $\mathcal{T}_h$ on Γ is quasi-uniform, and Γ is a convex portion of $\partial\Omega_j$.

For the proof of the convergence of the Dirichlet–Neumann iterative scheme (5.7.18)–(5.7.20) we need that (5.7.24) is satisfied only for $\boldsymbol{\mathcal{E}}_{1,h}\boldsymbol{\eta}_h$, and not for $\boldsymbol{\mathcal{E}}_{2,h}\boldsymbol{\eta}_h$. Hence, we obtain the result under the assumption that Γ is a convex portion of $\partial\Omega_1$.

A similar procedure can be used for showing the convergence of the Neumann–Neumann iterative scheme applied to the solution of (5.7.12). As usual, it can be seen that it is equivalent to the Richardson iterative scheme for solving the Steklov–Poincaré equation (5.7.15) with $(\sigma_1 S_{1,h}^{-1} + \sigma_2 S_{2,h}^{-1})^{-1}$ as a preconditioner.

We have already verified that assumptions (4.2.20) and (4.2.21) of Theorem 4.2.13 are satisfied (see (5.7.22) and (5.7.23)). Moreover, a bound on the constants C_1 and C_2 in (4.2.22), uniform with respect to h, is a consequence of (5.7.24) for $\boldsymbol{\mathcal{E}}_{1,h}\boldsymbol{\eta}_h$ and of the analogous estimate for $\boldsymbol{\mathcal{E}}_{2,h}\boldsymbol{\eta}_h$. Hence the convergence result follows under the assumptions that the family of triangulations $\mathcal{T}_h$ of Ω is regular, the family of triangulations $\mathcal{M}_h$ induced by $\mathcal{T}_h$ on Γ is quasi-uniform, and Γ is a convex portion of both $\partial\Omega_1$ and $\partial\Omega_2$; namely, a plane surface.

For the approximation of Maxwell equations by means of the additive Schwarz method see Toselli (1999).

In Table 5.7.1, on p. 217, we summarise the interface conditions associated with the Laplace operator and with the other partial differential equations considered in this chapter.

Table 5.7.1 *The interface conditions for different partial differential operators*

	P.D.E.	Dirichlet continuity	Neumann continuity
Laplace	$-\Delta u = f$	u	$\partial u/\partial n$
Advection–diffusion	$-\sum_{l,j} D_l(a_{lj} D_j u) + \operatorname{div}(\mathbf{b}u) + a_0 u = f$	u	$\partial u/\partial n_L$
Elasticity	$-\mathbf{div}\boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f}$ $\big[\sigma_{lj} := \hat{\mu}(D_l u_j + D_j u_l) + \hat{\lambda} \operatorname{div}\mathbf{u}\, \delta_{lj}\big]$	$\mathbf{u}$	$\boldsymbol{\sigma}(\mathbf{u}) \cdot \mathbf{n}$
Oseen (Stokes: $\mathbf{u}^* = \mathbf{0}$)	$\begin{cases} -\mathbf{div}\mathbf{T}(\mathbf{u}, p) + (\mathbf{u}^* \cdot \nabla)\mathbf{u} = \mathbf{f} \\ \operatorname{div}\mathbf{u} = 0 \end{cases}$ $\big[T_{lj} := \nu(D_l u_j + D_j u_l) - p\, \delta_{lj}\big]$	$\mathbf{u}$	$\mathbf{T}(\mathbf{u}, p) \cdot \mathbf{n}$
Compressible Stokes	$\begin{cases} \alpha\mathbf{u} - \mathbf{div}\widehat{\mathbf{T}}(\mathbf{u}, \sigma) = \mathbf{f} \\ \alpha\sigma + \operatorname{div}\mathbf{u} = g \end{cases}$ $\big[\widehat{T}_{lj} := \nu(D_l u_j + D_j u_l) - \beta\, \sigma\, \delta_{lj} + (\gamma - 2\nu/d) \operatorname{div}\mathbf{u}\, \delta_{lj}\big]$	$\mathbf{u}$	$\widehat{\mathbf{T}}(\mathbf{u}, \sigma) \cdot \mathbf{n}$
Compressible inviscid Stokes	$\begin{cases} \alpha\mathbf{u} + \beta\nabla\sigma = \mathbf{f} \\ \alpha\sigma + \operatorname{div}\mathbf{u} = g \end{cases}$	$\mathbf{u} \cdot \mathbf{n}$	σ
Advection	$\operatorname{div}(\mathbf{b}u) + a_0 u = f$	$\mathbf{b} \cdot \mathbf{n}\, u$	
Maxwell time-harmonic	$\operatorname{rot}(\mu^{-1} \operatorname{rot}\mathbf{E}) - \alpha^2 \varepsilon \mathbf{E} + i\alpha\sigma\mathbf{E} = \mathbf{f}$	$\mathbf{n} \times \mathbf{E}$	$\mathbf{n} \times (\mu^{-1} \operatorname{rot}\mathbf{E})$

6

ADVECTION–DIFFUSION EQUATIONS

This chapter deals with a class of elliptic equations that are not symmetric, due to the presence of first-order terms in the differential operator. These operators describe advection–diffusion processes, and typically arise in fluid dynamics. Our emphasis is on the case in which the transport phenomena, driven by the advective terms, are dominant with respect to the diffusive ones, which are related to the principal second-order part of the operator. The case of a non-symmetric advection–diffusion equation with dominant diffusion was considered in Section 5.1.

When these equations are used as computational kernels in the simulation of Navier–Stokes problems, the ratio between convective and diffusive terms is expressed by the flow Reynolds number, which may be very large in many applications.

We investigate the solution of advection–diffusion equations in the framework of non-overlapping multi-domain partitions. Domain decomposition methods based on the Schwarz overlapping technique are described, for example, in Cai (1990, 1995), Widlund (1992), Garbey (1996), and Garbey and Kaper (1997) and the references therein.

Domain decomposition methods based on substructuring iterations are very effective when the diffusive part is dominant; whereas, if the convective part becomes more relevant, then the natural interface conditions may produce instabilities.

A first strategy for avoiding these instabilities is based on imposing interface conditions that are consistent with the hyperbolic limit of the advection–diffusion equation, taking into consideration the direction of the characteristic curves on the interface. The interface is split into subsets according to the direction of the *flow* related to the convective part, and the interface conditions are accordingly devised. Some iterative schemes have been proposed in this spirit (see Carlenzoli and Quarteroni 1995; Nataf and Rogier 1995; and Auge *et al.* 1997). This approach avoids the creation of internal layers and then makes the iterative procedure more stable. We present several methods stemming from this idea, which are based on different treatments of interfaces between adjacent subdomains.

A second strategy makes use of different interface conditions that are more germane to those of the symmetric elliptic equations. Splitting the operator into its symmetric and skew-symmetric part, the associated bilinear forms turn out to be coercive in each subdomain Ω_i, without requiring any special adjustment of the interface conditions according to the direction of the convective field on Γ. The interface conditions are invariably of the Dirichlet and Robin type, and can

be used to introduce a Steklov–Poincaré equation on Γ, equivalent to the original problem. For this approach we prove convergence of the subdomain iterations; in some cases, at a rate independent of the mesh size.

Our discussion begins with the analysis of several multi-domain formulations of advection–diffusion equations. Although equivalent at the level of the differential problem, the different sets of interface conditions behind these formulations give rise to various iterative substructuring methods.

6.1 The advection–diffusion problem and its multi-domain formulations

We consider the boundary value problem

$$\text{(6.1.1)} \qquad \begin{cases} L_\varepsilon u := -\varepsilon \Delta u + \operatorname{div}(\mathbf{b}u) + a_0 u = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 2, 3$, $\varepsilon > 0$ is a constant diffusion coefficient, $\mathbf{b} = \mathbf{b}(\mathbf{x})$ denotes the given flow velocity and $a_0 = a_0(\mathbf{x})$ an absorption term. Finally, $f = f(\mathbf{x})$ represents a given body force.

The characteristic quantity $\omega := (2\varepsilon)^{-1}|\mathbf{b}|$ (essentially the analogue of the Reynolds number for Navier–Stokes equations) will play an important role in what follows; in particular, we will be primarily concerned with the advection-dominated case, in which $\omega \gg 1$.

For the existence and uniqueness analysis of (6.1.1) it is useful to assume that $\mathbf{b}$, a_0 and f satisfy

$$\mathbf{b} \in (L^\infty(\Omega))^d, \quad \operatorname{div} \mathbf{b} \in L^\infty(\Omega)$$
$$a_0 \in L^\infty(\Omega), \quad f \in L^2(\Omega)$$

and

$$\text{(6.1.2)} \qquad \frac{1}{2} \operatorname{div} \mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq 0 \quad \text{for almost every } \mathbf{x} \in \Omega.$$

Under this restriction, it is known (see, for example, Quarteroni and Valli 1994, Chapter 6) that there exists a unique solution to the following weak form of (6.1.1)

$$\text{(6.1.3)} \qquad u \in H_0^1(\Omega) \ : \ a^0(u, v) = (f, v) \quad \forall\, v \in H_0^1(\Omega),$$

where we have denoted by

$$\text{(6.1.4)} \qquad a^0(w, v) := \int_\Omega [\varepsilon \nabla w \cdot \nabla v + \operatorname{div}(\mathbf{b}w)\, v + a_0\, w\, v] \quad w, v \in H^1(\Omega),$$

the bilinear form associated with the operator L_ε. This bilinear form is continuous and coercive in $H_0^1(\Omega)$ owing to (6.1.2).

As in Chapter 1, let us consider a partition of Ω into two non-overlapping open subdomains Ω_i, $i = 1, 2$, and denote by $\Gamma := \overline{\Omega_1} \cap \overline{\Omega_2}$ the common boundary between Ω_1 and Ω_2, and by $\mathbf{n}^i$ the unit outward normal vector on Γ, setting $\mathbf{n} := \mathbf{n}^1$. Under these assumptions, problem (6.1.1) can be reformulated as follows: for $i = 1, 2$ find $u_i = u_{|\Omega_i}$ such that

$$\begin{cases} L_\varepsilon u_i = f & \text{in } \Omega_i, \quad i = 1, 2 \\ u_i = 0 & \text{on } \partial\Omega \cap \partial\Omega_i, \quad i = 1, 2 \\ u_1 = u_2 & \text{on } \Gamma \\ \varepsilon \dfrac{\partial u_1}{\partial n} = \varepsilon \dfrac{\partial u_2}{\partial n} & \text{on } \Gamma. \end{cases} \tag{6.1.5}$$

This is called the *Dirichlet–Neumann* (DN) formulation.

We now set, for $i = 1, 2$,

$$V_i := \{v_i \in H^1(\Omega_i) \,|\, v_{i|\partial\Omega\cap\partial\Omega_i} = 0\}$$

and for each $w_i, v_i \in V_i$

$$a_i^0(w_i, v_i) := \int_{\Omega_i} [\varepsilon \nabla w_i \cdot \nabla v_i + \operatorname{div}(\mathbf{b} w_i)\, v_i + a_0\, w_i\, v_i]. \tag{6.1.6}$$

Proceeding as in Lemma 1.2.1 it is readily seen that (6.1.4) is equivalent to the following *multi-domain* problem: find $u_1 \in V_1$, $u_2 \in V_2$ such that

$$\begin{cases} a_1^0(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\, v_1 \in H_0^1(\Omega_1) \\ u_1 = u_2 \qquad \text{on } \Gamma \\ a_2^0(u_2, v_2) = (f, v_2)_{\Omega_2} \quad \forall\, v_2 \in H_0^1(\Omega_2) \\ \displaystyle\sum_{i=1}^{2} a_i^0(u_i, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\, \mu \in \Lambda, \end{cases} \tag{6.1.7}$$

where Λ is the space of traces on Γ of the functions of $H_0^1(\Omega)$ (see (1.2.4)), and $\mathcal{R}_i\mu$ denotes any possible extension of μ to Ω_i (for example, $\mathcal{R}_i\mu = H_i\mu$, its harmonic extension in Ω_i, see (1.2.14)).

We assume that Γ is a Lipschitz interface and we distinguish three subsets

$$\begin{aligned} \Gamma^0 &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0\} \\ \Gamma^{\text{in}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\} \\ \Gamma^{\text{out}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\}, \end{aligned} \tag{6.1.8}$$

FIG. 6.1.1. Partition of the domain Ω.

which are identified through the local direction of the flow field $\mathbf{b}(\mathbf{x})$ at the subdomain interface (see Fig. 6.1.1).

If we assume that Γ^0 has zero surface measure, then we may replace (6.1.5) by the following equivalent *Robin–Neumann* formulation (RN):

$$
\text{(6.1.9)} \qquad \begin{cases} L_\varepsilon u_i = f & \text{in } \Omega_i, \quad i = 1,2 \\ u_i = 0 & \text{on } \partial\Omega \cap \partial\Omega_i, \quad i = 1,2 \\ \varepsilon \dfrac{\partial u_1}{\partial n} - \mathbf{b}\cdot\mathbf{n} u_1 = \varepsilon \dfrac{\partial u_2}{\partial n} - \mathbf{b}\cdot\mathbf{n} u_2 & \text{on } \Gamma \\ \varepsilon \dfrac{\partial u_1}{\partial n} = \varepsilon \dfrac{\partial u_2}{\partial n} & \text{on } \Gamma. \end{cases}
$$

The new interface conditions derive from a different weak formulation of (6.1.1), which is alternative to (6.1.3), (6.1.4). The difference stems from the fact that this time we also integrate the convective term by parts, and obtain:

$$
\text{(6.1.10)} \qquad \text{find } u \in H_0^1(\Omega) \ : \ a^R(u,v) = (f,v) \quad \forall\ v \in H_0^1(\Omega),
$$

where

$$
a^R(w,v) := \int_\Omega [\varepsilon \nabla w \cdot \nabla v - w\,\mathbf{b}\cdot\nabla v + a_0\, w\, v].
$$

The variational formulation of (6.1.9) takes a form similar to (6.1.7) and reads: find $u_1 \in V_1$, $u_2 \in V_2$ such that

$$
\begin{cases}
a_1^0(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\, v_1 \in H_0^1(\Omega_1) \\
\displaystyle\sum_{i=1}^{2} a_i^R(u_i, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\, \mu \in \Lambda \\
a_2^0(u_2, v_2) = (f, v_2)_{\Omega_2} \quad \forall\, v_2 \in H_0^1(\Omega_2) \\
\displaystyle\sum_{i=1}^{2} a_i^0(u_i, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\, \mu \in \Lambda,
\end{cases}
\tag{6.1.11}
$$

where

$$
a_i^R(w_i, v_i) := \int_{\Omega_i} [\varepsilon \nabla w_i \cdot \nabla v_i - w_i\, \mathbf{b} \cdot \nabla v_i + a_0\, w_i\, v_i], \quad i = 1, 2.
\tag{6.1.12}
$$

Note that

$$
a_i^R(w_i, v_i) = a_i^0(w_i, v_i) - \int_\Gamma \mathbf{b} \cdot \mathbf{n}^i\, w_i\, v_i, \quad w_i, v_i \in V_i,
$$

hence the two bilinear forms $a_i^0(\cdot,\cdot)$ and $a_i^R(\cdot,\cdot)$ are coincident on the space $H_0^1(\Omega_i)$. For this reason a_1^0 could be replaced by a_1^R in the first equation of (6.1.11), and a_2^0 by a_2^R in the third equation of (6.1.11).

Finally, for any type of Γ^0 (that is, even if it has positive surface measure), a third set of interface conditions can be used. The associated multi-domain problem, alternative to either (6.1.5) and (6.1.9), is called the *β-Robin–Neumann* ($\mathrm{R}_\beta\mathrm{N}$) formulation, and reads

$$
\begin{cases}
L_\varepsilon u_i = f & \text{in } \Omega_i, \quad i = 1, 2 \\
u_i = 0 & \text{on } \partial\Omega \cap \partial\Omega_i, \quad i = 1, 2 \\
\varepsilon \dfrac{\partial u_1}{\partial n} - \beta u_1 = \varepsilon \dfrac{\partial u_2}{\partial n} - \beta u_2 & \text{on } \Gamma \\
\varepsilon \dfrac{\partial u_1}{\partial n} = \varepsilon \dfrac{\partial u_2}{\partial n} & \text{on } \Gamma,
\end{cases}
\tag{6.1.13}
$$

where $\beta \in L^\infty(\Gamma)$ is a given function, almost everywhere different from 0. In particular, note that (6.1.9) is obtained for $\beta = \mathbf{b} \cdot \mathbf{n}$. Its variational formulation is given by: find $u_1 \in V_1$, $u_2 \in V_2$ such that

$$
\begin{cases}
a_1^0(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\, v_1 \in H_0^1(\Omega_1) \\
\displaystyle\sum_{i=1}^{2} a_i^{\beta}(u_i, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\, \mu \in \Lambda \\
a_2^0(u_2, v_2) = (f, v_2)_{\Omega_2} \quad \forall\, v_2 \in H_0^1(\Omega_2) \\
\displaystyle\sum_{i=1}^{2} a_i^{0}(u_i, \mathcal{R}_i\mu) = \sum_{i=1}^{2} (f, \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\, \mu \in \Lambda,
\end{cases} \tag{6.1.14}
$$

where

$$
a_i^{\beta}(w_i, v_i) := a_i^{R}(w_i, v_i) + \int_{\Gamma} (\mathbf{b}\cdot\mathbf{n} - \beta)\mathbf{n}\cdot\mathbf{n}^i\, w_i\, v_i. \tag{6.1.15}
$$

In conclusion, we can consider either one of the multi-domain forms that are given by equations (6.1.5), (6.1.9) and (6.1.13). Any such choice can be rigorously justified and the equivalence between them can be proved (see F. Gastaldi *et al.* 1996). However, although all sets of conditions DN, RN and R$_\beta$N are equivalent to one another, they will generate different iterative substructuring procedures between Ω_1 and Ω_2. The reason for choosing a specific set of equations (out of the three possibilities) depends primarily on the data for the problem under consideration.

Two further iterative procedures will be introduced in Section 6.4, based on multi-domain formulations that are different from (6.1.5), (6.1.9) and (6.1.13).

6.2 Iterative substructuring methods for one-dimensional problems

The previous multi-domain formulations suggest the application of suitable iterative methods that reduce problem (6.1.1) to a sequence of mixed boundary value problems on each subdomain.

Before introducing these methods in the general case, let us consider the simple (albeit meaningful) one-dimensional problem

$$
\begin{cases}
L_\varepsilon u := -\varepsilon u_{xx} + b u_x + a_0 u = f \quad \text{in } \Omega = (0,1) \\
u(0) = u(1) = 0,
\end{cases} \tag{6.2.1}
$$

where $\varepsilon > 0$, $b \neq 0$ and $a_0 \geq 0$ are constant coefficients. In F. Gastaldi *et al.* (1996) the following iteration-by-subdomain method to solve (6.2.1) has been analysed: given λ^0, solve for $k \geq 0$

$$\begin{cases} L_\varepsilon u_1^{k+1} = f \qquad \text{in } \Omega_1 = (0,c) \\ u_1^{k+1}(0) = 0 \\ \varepsilon u_{1x}^{k+1}(c) - \left(\frac{1}{2}b + A\right) u_1^{k+1}(c) = \lambda^k, \end{cases} \tag{6.2.2}$$

then

$$\begin{cases} L_\varepsilon u_2^{k+1} = f \qquad \text{in } \Omega_2 = (c,1) \\ u_2^{k+1}(1) = 0 \\ \varepsilon u_{2x}^{k+1}(c) - \left(\frac{1}{2}b + B\right) u_2^{k+1}(c) = \varepsilon u_{1x}^{k+1}(c) - \left(\frac{1}{2}b + B\right) u_1^{k+1}(c), \end{cases} \tag{6.2.3}$$

and finally set

$$\lambda^{k+1} = \varepsilon u_{2x}^{k+1}(c) - \left(\frac{1}{2}b + A\right) u_2^{k+1}(c), \tag{6.2.4}$$

where $0 < c < 1$, and A and B are real parameters, with $A \neq B$. In fact, the cases $A = \pm\infty$ and $B \in \mathbf{R}$, or $A \in \mathbf{R}$ and $B = \pm\infty$ can also be considered. The first choice corresponds to a Dirichlet boundary condition at $x = c$ in (6.2.2), the second one to a Dirichlet boundary condition at $x = c$ in (6.2.3).

A straightforward computation shows that this method converges provided that

$$|\rho_\varepsilon(A,B)| < 1, \tag{6.2.5}$$

where

$$\rho_\varepsilon(A,B) := \frac{\tau\coth(\tau c) - B/\varepsilon}{\tau\coth(\tau c) - A/\varepsilon}\,\frac{\tau\coth[\tau(1-c)] + A/\varepsilon}{\tau\coth[\tau(1-c)] + B/\varepsilon},$$

and

$$\tau := \frac{\sqrt{b^2 + 4\varepsilon a_0}}{2\varepsilon}$$

(see F. Gastaldi *et al.* 1996, in which $\alpha = \frac{1}{2}\,b + A$ and $\beta = \frac{1}{2}\,b + B$).

Introducing a relaxation parameter $\theta \neq 0$, we can consider a more general iterative scheme, still based on (6.2.2), (6.2.3), where now (6.2.4) is replaced by

$$\lambda^{k+1} = \theta\left[\varepsilon u_{2x}^{k+1}(c) - \left(\frac{1}{2}\,b + A\right) u_2^{k+1}(c)\right] + (1-\theta)\lambda^k. \tag{6.2.6}$$

In this case we have convergence if

$$|1 - \theta[1 - \rho_\varepsilon(A,B)]| < 1. \tag{6.2.7}$$

Noticing that $\rho_\varepsilon(A,B) \neq 1$ for $A \neq B$, this yields the following limitation on θ:

$$\begin{cases} 0 < \theta < \dfrac{2}{1-\rho_\varepsilon(A,B)} & \text{for } \rho_\varepsilon(A,B) < 1 \\ \dfrac{2}{1-\rho_\varepsilon(A,B)} < \theta < 0 & \text{for } \rho_\varepsilon(A,B) > 1 \end{cases} . \tag{6.2.8}$$

However, since the focus here is on advection-dominated problems, we are looking for methods that converge when θ is chosen independently of ε as $\varepsilon \to 0^+$. A direct calculation shows that

$$\rho_0(A,B) := \lim_{\varepsilon \to 0^+} \rho_\varepsilon(A,B) = \frac{|b|/2 - B}{|b|/2 + B} \frac{|b|/2 + A}{|b|/2 - A}, \tag{6.2.9}$$

hence, the choices $A = |b|/2$ and $B = -|b|/2$ (which would be forbidden in the limiting case $\varepsilon = 0$) are expected to lead to an inefficient scheme even for $\varepsilon \neq 0$ but small. These choices correspond to imposing the value of the normal derivative u_x at the inflow region, or the value of the conormal derivative $\varepsilon u_x - bu$ at the outflow region.

From this analysis we can easily conclude that the sign of b needs to be taken into account when one chooses the type of interface conditions to be used. The methods we shall present in Section 6.3 will be tailored in a way consistent with the hyperbolic limit of the convection–diffusion equation; namely, the Dirichlet (or Robin) boundary condition is imposed on the inflow boundary, while the Neumann condition is enforced on the outflow boundary.

Let us also point out that, when the asymptotic reduction factor $\rho_0(A,B)$ belongs to the interval $(-1,1)$, the relaxation parameter can be chosen in the whole interval $(0,1]$, leading to efficient iterative schemes. From (6.2.9) it is seen at once that the choice $A \leq 0$ and $B \geq 0$ (with $A \neq B$) implies that $-1 < \rho_0(A,B) < 1$.

The condition $A \leq 0$ and $B \geq 0$ is closely related to the coerciveness of the bilinear forms associated with the boundary value problems (6.2.2) and (6.2.3). In fact, these forms are given respectively by

$$\begin{aligned} a_1^\flat(w_1, v_1) &:= \int_0^c (\varepsilon w_{1,x}\, v_{1,x} + a_0\, w_1\, v_1) \\ &\quad + \frac{1}{2} \int_0^c b(v_1\, w_{1,x} - w_1\, v_{1,x}) - A\, w_1(c)\, v_1(c) \\ a_2^\flat(w_2, v_2) &:= \int_c^1 (\varepsilon w_{2,x}\, v_{2,x} + a_0\, w_2\, v_2) \\ &\quad + \frac{1}{2} \int_c^1 b(v_2\, w_{2,x} - w_2\, v_{2,x}) + B\, w_2(c)\, v_2(c); \end{aligned}$$

hence, they are coercive in $H^1(0,c)$ and $H^1(c,1)$, respectively, for $a_0 > 0$ and for every choice of the parameter ε if and only if $A \leq 0$ and $B \geq 0$. In other words, coerciveness implies that $-1 < \rho_0(A,B) < 1$, and consequently the iterative

method converges for every relaxation parameter in the interval $(0, 1]$.

This analysis for the one-dimensional case leads us to propose, also for the general case (6.1.1), iterative methods based on bilinear forms that are coercive in $H^1(\Omega_1)$ and $H^1(\Omega_2)$, respectively: this will be done in Section 6.4.

6.3 Adaptive iterative substructuring methods: ADN, ARN and AR$_\beta$N

Now we return to the multi-dimensional problem (6.1.1). As mentioned in the previous section, the difficulties arising when the parameter ε is small can be given a heuristic explanation in the limiting situation $\varepsilon \to 0^+$. When providing each subproblem in Ω_1 and Ω_2 with a boundary condition at the interface, one should account for the local direction of the characteristic curves in order to be consistent with the hyperbolic limit. Hence, the Neumann interface condition has to be enforced at the *outflow* boundary of a subdomain, whereas the Dirichlet condition has to be enforced at the *inflow* boundary.

Let the boundary $\partial\Omega$ be partitioned as $\partial\Omega = \partial\Omega^{\rm in} \cup \partial\Omega^{\rm out} \cup \partial\Omega_0$, where

$$\partial\Omega^{\rm in} := \{\mathbf{x} \in \partial\Omega \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) < 0\} \tag{6.3.1}$$

$$\partial\Omega^{0} := \{\mathbf{x} \in \partial\Omega \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) = 0\} \tag{6.3.2}$$

and

$$\partial\Omega^{\rm out} := \{\mathbf{x} \in \partial\Omega \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) > 0\}, \tag{6.3.3}$$

where $\mathbf{n}^*$ is the unit outward normal vector to $\partial\Omega$. Then we set, for $i = 1, 2$,

$$\partial\Omega_i^{\rm in} := \partial\Omega^{\rm in} \cap \partial\Omega_i. \tag{6.3.4}$$

If we want to consider the hyperbolic limit case of (6.1.1) (taking formally $\varepsilon = 0$), following what we have done in Section 5.6.2, we should operate as follows: for given values λ^0 and μ^0, we look for a sequence $\{u_1^{k+1}, u_2^{k+1}\}$ with $k \geq 0$ such that:

$$\begin{cases} L_0 u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1^{\rm in} \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma^{\rm in} \end{cases} \tag{6.3.5}$$

$$\begin{cases} L_0 u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2^{\rm in} \\ u_2^{k+1} = \mu^k & \text{on } \Gamma^{\rm out}, \end{cases} \tag{6.3.6}$$

with

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma^{\text{in}} \tag{6.3.7}$$

$$\mu^{k+1} = \theta u_{1|\Gamma}^{k+1} + (1-\theta)\mu^k \qquad \text{on } \Gamma^{\text{out}}. \tag{6.3.8}$$

To be consistent with the hyperbolic limit (6.3.5)–(6.3.8), we will therefore split the interface conditions for the advection–diffusion problem (6.1.1) according to the local flow direction.

The idea is to define a sequence $\{u_1^k, u_2^k\}$ where u_i^k satisfies $L_\varepsilon u_i^k = f$ in Ω_i, along with suitable boundary conditions at the subdomain boundary Γ that depend on the local direction of the flow field $\mathbf{b}(\mathbf{x})$. These conditions at the interface are selected from the set DN, RN or R$_\beta$N, so that one of them is associated with u_1^k, the other with u_2^k. The corresponding algorithms will be called *adaptive*, because the role played by the boundary condition on Γ varies according to the flow conditions there. Consequently, we will have Adaptive Dirichlet–Neumann (ADN), Adaptive Robin–Neumann (ARN), or Adaptive β-Robin–Neumann (AR$_\beta$N) algorithms depending upon the choice of interface conditions. These methods have been proposed by Carlenzoli and Quarteroni (1995), motivated by the observation that, for the one-dimensional problem (6.2.1) with $a_0 = 0$, they guarantee a convergence rate of order $\exp(-b/\varepsilon)$. They have been further developed by Ciccoli (1995), Trotta (1996) and F. Gastaldi *et al.* (1996). In order to keep our discussion as simple as possible, our approach will be based on the differential (instead of variational) formulations. The variational formulation of ADN and ARN methods for the finite element approximation is presented in Section 6.5.

Let us start by presenting the *Adaptive Dirichlet–Neumann algorithm* (ADN): given u_i^0 in Ω_i, $i = 1, 2$, solve for each $k \geq 0$

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma^{\text{in}} \cup \Gamma^0 \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} = \varepsilon \dfrac{\partial u_2^k}{\partial n} & \text{on } \Gamma^{\text{out}} \end{cases} \tag{6.3.9}$$

and

$$
\begin{cases}
L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\
u_2^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\
u_2^{k+1} = \mu^{k+1} & \text{on } \Gamma^{\text{out}} \\
\varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma^{\text{in}} \cup \Gamma^0,
\end{cases}
\tag{6.3.10}
$$

with

$$
\lambda^k := \theta' u^k_{2|\Gamma^{\text{in}} \cup \Gamma^0} + (1-\theta') u^k_{1|\Gamma^{\text{in}} \cup \Gamma^0} \qquad \text{on } \Gamma^{\text{in}} \cup \Gamma^0 \tag{6.3.11}
$$

and

$$
\mu^{k+1} := \theta'' u^{k+1}_{1|\Gamma^{\text{out}}} + (1-\theta'') u^k_{2|\Gamma^{\text{out}}} \qquad \text{on } \Gamma^{\text{out}}, \tag{6.3.12}
$$

θ' and θ'' being two positive parameters that are used to allow possible *under-relaxation* (if needed) to ensure convergence. Typically, a single parameter θ suffices (two parameters allow more flexibility in achieving an optimal convergence), and sometimes $\theta = 1$ (no relaxation at all) is a possible choice.

Should the sequence $\{u_1^k, u_2^k\}$ converge in a suitable sense, its limit $\{u_1, u_2\}$ would be the desired solution, because it would satisfy (at least formally) the differential problem (6.1.5). The proof of the convergence of the ADN iteration has been given in F. Gastaldi *et al.* (1996), for a problem set in the square $(0,1)^2$, with a constant advective field $\mathbf{b} = (b, 0)$ and a constant absorption term a_0.

On the set Γ^0 one can either impose a Dirichlet or Neumann condition. Clearly, once the choice is made, on the complementary domain, say Ω_2, one has to impose the condition that has not been enforced in Ω_1 (see Fig. 6.3.1 for a geometric representation of the role of the interface conditions).

The formulation (6.3.9)–(6.3.12) is *sequential*, because it yields the solution of problem (6.3.10) in Ω_2 only after having solved problem (6.3.9) in Ω_1. Although this sequence guarantees the quickest convergence, it might not be the best option in view of parallelism, especially when many subdomains are used. With this aim, an obvious modification of the previous algorithm consists of solving again problem (6.3.9) in Ω_1, and, *simultaneously*, the following modified problem in Ω_2:

FIG. 6.3.1. The role of Dirichlet and Neumann conditions at the subdomain interface in the ADN method.

$$\begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\ u_2^{k+1} = \eta^k & \text{on } \Gamma^{\text{out}} \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = \varepsilon \dfrac{\partial u_1^k}{\partial n} & \text{on } \Gamma^{\text{in}} \cup \Gamma^0, \end{cases} \tag{6.3.13}$$

with

$$\eta^k = \theta'' u_{1|\Gamma^{\text{out}}}^k + (1-\theta'') u_{2|\Gamma^{\text{out}}}^k \qquad \text{on } \Gamma^{\text{out}}. \tag{6.3.14}$$

This approach generalises straightforwardly to the case in which Ω is partitioned into many subdomains.

Iterative algorithms based on other interface conditions can be defined similarly. Let us state here that based on Robin–Neumann matching conditions (6.1.9). The *Adaptive Robin–Neumann algorithm* (ARN) reads: assuming that Γ^0 has a zero surface measure, given u_i^0 in Ω_i, $i = 1, 2$, solve for each $k \geq 0$

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ \psi(u_1^{k+1}) = \lambda^k & \text{on } \Gamma^{\text{in}} \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} = \varepsilon \dfrac{\partial u_2^k}{\partial n} & \text{on } \Gamma^{\text{out}} \end{cases} \tag{6.3.15}$$

and

$$\text{(6.3.16)}\qquad \begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\ \psi(u_2^{k+1}) = \mu^{k+1} & \text{on } \Gamma^{\text{out}} \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} & \text{on } \Gamma^{\text{in}}, \end{cases}$$

with

$$\text{(6.3.17)}\qquad \psi(v) := \varepsilon \frac{\partial v}{\partial n} - \mathbf{b}\cdot\mathbf{n}\, v$$

$$\text{(6.3.18)}\qquad \lambda^k := \theta' \psi(u_2^k)_{|\Gamma^{\text{in}}} + (1-\theta')\psi(u_1^k)_{|\Gamma^{\text{in}}} \qquad \text{on } \Gamma^{\text{in}}$$

and

$$\text{(6.3.19)}\qquad \mu^{k+1} := \theta'' \psi(u_1^{k+1})_{|\Gamma^{\text{out}}} + (1-\theta'')\psi(u_2^k)_{|\Gamma^{\text{out}}} \qquad \text{on } \Gamma^{\text{out}}.$$

Again, we are enforcing a Neumann condition on the outflow part of the interface, whereas a Robin (rather than Dirichlet) condition is imposed at the inflow. The convergence of the ARN scheme will be proved in Section 6.4 as a particular case of a family of iteration-by-subdomain methods (see Theorem 6.4.2 and Remark 6.4.3).

The adaptive *β-Robin–Neumann algorithm* (AR$_\beta$N) can be obtained from (6.3.15) and (6.3.16) by simply replacing the flux in (6.3.17) by the modified flux:

$$\text{(6.3.20)}\qquad \psi(v) := \varepsilon \frac{\partial v}{\partial n} - \beta v.$$

In this case, the assumption on Γ^0 can be omitted, and, for instance, on Γ^0 we may impose the same condition as on Γ^{in}. To ensure the solvability of (6.3.15) and (6.3.16) in the AR$_\beta$N scheme we have to assume that

$$\beta \le \frac{1}{2}\mathbf{b}\cdot\mathbf{n} \ \text{ on } \Gamma^{\text{in}} \cup \Gamma^0, \quad \beta \ge \frac{1}{2}\mathbf{b}\cdot\mathbf{n} \ \text{ on } \Gamma^{\text{out}}.$$

As noted for the ADN algorithm, a 'parallel' version can be easily generated by simply replacing $(6.3.16)_3$, $(6.3.16)_4$ and (6.3.19) respectively by

$$\text{(6.3.21)}\qquad \psi(u_2^{k+1}) = \mu^k \quad \text{on } \Gamma^{\text{out}}$$

$$\varepsilon \frac{\partial u_2^{k+1}}{\partial n} = \varepsilon \frac{\partial u_1^k}{\partial n} \quad \text{on } \Gamma^{\text{in}} \tag{6.3.22}$$

$$\mu^k := \theta'' \psi(u_1^k)_{|\Gamma^{\text{out}}} + (1-\theta'')\psi(u_2^k)_{|\Gamma^{\text{out}}} \qquad \text{on } \Gamma^{\text{out}}. \tag{6.3.23}$$

6.3.1 The damped form of the iterative algorithms: d-ADN, d-ARN and d-AR$_\beta$N

When $\omega \gg 1$, namely, advection is strongly dominating the diffusion, the domain decomposition algorithms previously introduced can be modified as follows on the outflow part of the interface boundary. The Neumann condition expressing the continuity of normal derivatives is replaced by a *homogeneous* Neumann condition. The corresponding algorithms will be referred to as the *damped* forms of their predecessors, and indicated by the acronyms: d-ADN, d-ARN and d-AR$_\beta$N.

The d-ADN *algorithm* consists therefore in solving the following problems:

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma^{\text{in}} \cup \Gamma^0 \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} = 0 & \text{on } \Gamma^{\text{out}} \end{cases} \tag{6.3.24}$$

and

$$\begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\ u_2^{k+1} = \mu^{k+1} & \text{on } \Gamma^{\text{out}} \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = 0 & \text{on } \Gamma^{\text{in}} \cup \Gamma^0. \end{cases} \tag{6.3.25}$$

A similar modification can be carried out on both of the ARN and AR$_\beta$N methods, obtaining the following d-ARN *algorithm* (if ψ is as in (6.3.17)) and d-AR$_\beta$N *algorithm* (if ψ is as in (6.3.20)):

$$\text{(6.3.26)}\qquad \begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ \psi(u_1^{k+1}) = \lambda^k & \text{on } \Gamma^{\text{in}} \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} = 0 & \text{on } \Gamma^{\text{out}} \end{cases}$$

and

$$\text{(6.3.27)}\qquad \begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\ \psi(u_2^{k+1}) = \mu^{k+1} & \text{on } \Gamma^{\text{out}} \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = 0 & \text{on } \Gamma^{\text{in}}. \end{cases}$$

The damped algorithms weaken the coupling between u_1^{k+1} and u_2^{k+1} at the interface Γ, thus allowing a faster convergence of the corresponding sequence. In the particular case when $\mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Gamma$, we have that $\Gamma = \Gamma^{\text{out}}$, and the d-ADN algorithm (without relaxation) becomes

$$\text{(6.3.28)}\qquad \begin{cases} L_\varepsilon u_1 = f & \text{in } \Omega_1 \\ u_1 = 0 & \text{on } \partial\Omega \cap \partial\Omega_1 \\ \varepsilon \dfrac{\partial u_1}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

and

$$\text{(6.3.29)}\qquad \begin{cases} L_\varepsilon u_2 = f & \text{in } \Omega_2 \\ u_2 = 0 & \text{on } \partial\Omega \cap \partial\Omega_2 \\ u_2 = u_1 & \text{on } \Gamma. \end{cases}$$

Clearly, in this case u_1 is independent of u_2, and u_2 can be obtained in a single step after solving (6.3.28).

For the same case, the d-ARN is obtained from (6.3.28) and (6.3.29) provided the third equation of (6.3.29) is replaced by the Robin condition:

$$\text{(6.3.30)}\qquad \varepsilon \frac{\partial u_2}{\partial n} - \mathbf{b} \cdot \mathbf{n}\, u_2 = \varepsilon \frac{\partial u_1}{\partial n} - \mathbf{b} \cdot \mathbf{n}\, u_1 (= -\mathbf{b} \cdot \mathbf{n}\, u_1) \qquad \text{on } \Gamma.$$

Note that, in general, the limit function of the damped iterations differs from

the solution of the original problem, because we have modified the continuity equation for the normal derivative. However, the difference between the two functions can be estimated in suitable norms and turns out to be small for small values of ε (see Ciccoli 1995 and F. Gastaldi *et al.* 1996). Moreover, unlike the original undamped solution, the limiting damped solution for the ARN approach fails to be continuous at subdomain interfaces. However, the jump discontinuity is small in the advection-dominated case $\omega \gg 1$ (see Ciccoli 1995 and Trotta 1996).

6.4 Coercive iterative substructuring methods: γ-DR and γ-RR

Other iterative methods can be proposed that do not necessarily pay attention to the local direction of the advective field $\mathbf{b}$ on Γ. Instead, what is required is that the bilinear form associated with each individual boundary value problem set in Ω_1 or Ω_2 for the subdomain iterations is *coercive* in $H^1(\Omega_1)$ or $H^1(\Omega_2)$, respectively, under the sole assumption that $\frac{1}{2}\operatorname{div}\mathbf{b} + a_0 \geq \mu_0 > 0$ in Ω, without requiring the boundary value to vanish on any part of $\partial\Omega_i$, $i = 1, 2$. In this respect, it is worthwhile to note that the bilinear forms $a_1^0(\cdot,\cdot)$ and $a_2^0(\cdot,\cdot)$ introduced in (6.1.6) are coercive in $V_1 \cap H^1_{\Gamma_{\text{in}}}(\Omega_1)$ and $V_2 \cap H^1_{\Gamma_{\text{out}}}(\Omega_2)$, respectively, and the bilinear forms $a_1^R(\cdot,\cdot)$ and $a_2^R(\cdot,\cdot)$ introduced in (6.1.12) are coercive in $V_1 \cap H^1_{\Gamma_{\text{out}}}(\Omega_1)$ and $V_2 \cap H^1_{\Gamma_{\text{in}}}(\Omega_2)$, respectively, but not in $H^1(\Omega_1)$ or $H^1(\Omega_2)$.

With respect to the adaptive iterative substructuring methods introduced in the preceding section, the drawback is that we will now use somewhat more complicated bilinear forms (see (6.4.4) below) compared with (6.1.6) or (6.1.12), and a further parameter γ has to be considered besides the relaxation parameter θ. On the other hand, no attention has to be paid to the flow direction, and the methods can be easily extended to systems of equations.

6.4.1 The γ-DR iterative method

The following scheme has been proposed by Alonso *et al.* (1998), and was given the name of γ-Dirichlet–Robin (γ-DR) method: let λ^0 be given, for each $k \geq 0$ solve

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma \end{cases} \tag{6.4.1}$$

$$
\begin{cases}
L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\
u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\
\varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} - \left(\dfrac{1}{2}\,\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_2^{k+1} & \\
\qquad = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2}\,\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} & \text{on } \Gamma,
\end{cases}
\tag{6.4.2}
$$

and set

$$
\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma,
\tag{6.4.3}
$$

where $\theta \neq 0$ is a relaxation parameter.

In (6.4.2) $\gamma = \gamma(\mathbf{x})$ is a given non-negative function belonging to $L^\infty(\Gamma)$, which, in general, influences the rate of convergence of the method. If applied to the one-dimensional problem (6.2.1), the γ-DR method corresponds to the scheme (6.2.2)–(6.2.4) with the choice $A = -\infty$ and $B = \gamma$.

Let us introduce the local bilinear forms

$$
\begin{aligned}
a_i^\flat(w_i, v_i) : = & \int_{\Omega_i} \left[\varepsilon \nabla w_i \cdot \nabla v_i + \left(\frac{1}{2}\ \mathrm{div}\,\mathbf{b} + a_0\right) w_i\, v_i\right] \\
& + \frac{1}{2} \int_{\Omega_i} (v_i\, \mathbf{b}\cdot\nabla w_i - w_i\, \mathbf{b}\cdot\nabla v_i).
\end{aligned}
\tag{6.4.4}
$$

Note that, owing to (6.1.2), there exist constants $\beta_i^\flat$ and $\alpha_i^\flat$, $i = 1, 2$, such that

$$
a_i^\flat(w_i, v_i) \leq \beta_i^\flat\, ||w_i||_{1,\Omega_i}\, ||v_i||_{1,\Omega_i} \qquad \forall\ w_i, v_i \in H^1(\Omega_i)
\tag{6.4.5}
$$

and

$$
a_i^\flat(v_i, v_i) \geq \alpha_i^\flat\, ||v_i||_{1,\Omega_i}^2 \qquad \forall\ v_i \in H^1(\Omega_i).
\tag{6.4.6}
$$

Define V_i, V_i^0, $i = 1, 2$, and Λ as in (1.2.4). The variational formulation of the γ-DR iterative scheme (6.4.1)–(6.4.3) reads:

$$
\begin{cases}
\text{find } u_1^{k+1} \in V_1\ : & \\
a_1^\flat(u_1^{k+1}, v_1) = (f, v_1)_{\Omega_1} \qquad \forall\ v_1 \in V_1^0 & \\
u_{1|\Gamma}^{k+1} = \lambda^k &
\end{cases}
\tag{6.4.7}
$$

$$
\begin{cases}
\text{find } u_2^{k+1} \in V_2 \ : \\
a_2^\flat(u_2^{k+1}, v_2) = (f, v_2)_{\Omega_2} \qquad \forall\, v_2 \in V_2^0 \\
a_2^\flat(u_2^{k+1}, \mathcal{R}_2\mu) + \displaystyle\int_\Gamma \gamma u_{2|\Gamma}^{k+1}\mu = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} \\
\qquad\qquad -a_1^\flat(u_1^{k+1}, \mathcal{R}_1\mu) + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1}\mu \qquad \forall\, \mu \in \Lambda,
\end{cases} \tag{6.4.8}
$$

setting finally

$$
\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \qquad \text{on } \Gamma, \tag{6.4.9}
$$

where $\mathcal{R}_i$ denotes any extension operator from Λ to V_i.

Problem (6.4.8) can be rewritten in the equivalent form

$$
\begin{cases}
\text{find } u_2^{k+1} \in V_2 \ : \\
a_2^\flat(u_2^{k+1}, v_2) + \displaystyle\int_\Gamma \gamma u_{2|\Gamma}^{k+1} v_{2|\Gamma} = (f, v_2)_{\Omega_2} + (f, \mathcal{R}_1 v_{2|\Gamma})_{\Omega_1} \\
\qquad\qquad -a_1^\flat(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2,
\end{cases} \tag{6.4.10}
$$

which is coercive in V_2, for any $\gamma \geq 0$. Hence, the iterative scheme (6.4.7)–(6.4.9) is correctly defined.

These iterations are different from the ADN scheme, because the Dirichlet boundary condition is imposed on the whole interface Γ, no matter whether it is an inflow or an outflow boundary. However, in the particular situation when the flow has the same direction on the whole set of Γ, say $\mathbf{b}\cdot\mathbf{n} < 0$ on Γ, by choosing $\gamma = -\frac{1}{2}\,\mathbf{b}\cdot\mathbf{n}$ we recover the ADN scheme.

We also propose a modified algorithm, which is slightly more complicated to implement, but enjoys better convergence properties. It is obtained by replacing the integral on Γ by the scalar product in the trace space Λ, say $((\eta, \mu))_\Lambda$ (for a similar choice, see also Dryja 1982 and P.-L. Lions 1990). Thus, we solve, instead of (6.4.10), the following problem:

$$
\begin{cases}
\text{find } u_2^{k+1} \in V_2 \ : \\
a_2^\flat(u_2^{k+1}, v_2) + \gamma((u_{2|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda = (f, v_2)_{\Omega_2} + (f, \mathcal{R}_1 v_{2|\Gamma})_{\Omega_1} \\
\qquad\qquad -a_1^\flat(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \gamma((u_{1|\Gamma}^{k+1}, v_{2|\Gamma}))_\Lambda \qquad \forall\, v_2 \in V_2,
\end{cases} \tag{6.4.11}
$$

for a constant $\gamma \geq 0$.

6.4.2 Convergence of the γ-DR iterative method

For proving the convergence of the scheme (6.4.7), (6.4.11), (6.4.9) we need some preliminary results. First of all, for $i = 1, 2$ and for each $\eta \in \Lambda$, let us introduce as in (5.1.10) the solution $E_i^\flat \eta \in V_i$ of the Dirichlet boundary value problem:

$$\begin{cases} a_i^\flat(E_i^\flat \eta, v_i) = 0 \qquad \forall\ v_i \in V_i^0 \\ \\ (E_i^\flat \eta)_{|\Gamma} = \eta. \end{cases} \tag{6.4.12}$$

By well known a priori estimates for elliptic problems, the extension operator $E_i^\flat : \Lambda \to V_i$ is continuous; that is, there exists $k_i > 0$ such that

$$||E_i^\flat \eta||_{1,\Omega_i} \leq k_i ||\eta||_\Lambda \quad \forall\ \eta \in \Lambda. \tag{6.4.13}$$

Moreover, the trace inequality (1.2.5) yields

$$||\eta||_\Lambda \leq C_i^* ||E_i^\flat \eta||_{1,\Omega_i} \quad \forall\ \eta \in \Lambda. \tag{6.4.14}$$

For each $\eta, \mu \in \Lambda$ and $i = 1, 2$ let us define now the Steklov–Poincaré operators $S_i : \Lambda \to \Lambda'$ as in (5.1.9):

$$\langle S_i \eta, \mu \rangle := a_i^\flat(E_i^\flat \eta, E_i^\flat \mu).$$

Moreover, let us set

$$\begin{aligned} \langle S_1^{(\gamma)} \eta, \mu \rangle &:= a_1^\flat(E_1^\flat \eta, E_1^\flat \mu) - \gamma((\eta, \mu))_\Lambda \\ \langle S_2^{(\gamma)} \eta, \mu \rangle &:= a_2^\flat(E_2^\flat \eta, E_2^\flat \mu) + \gamma((\eta, \mu))_\Lambda, \end{aligned} \tag{6.4.15}$$

and define

$$S = S_1 + S_2 = S_1^{(\gamma)} + S_2^{(\gamma)}.$$

Both operators $S_i^{(\gamma)}$, $i = 1, 2$, are continuous, because from (6.4.5) and (6.4.13) we have

$$\langle S_i^{(\gamma)} \eta, \mu \rangle \leq (\beta_i^\flat k_i^2 + \gamma) ||\eta||_\Lambda ||\mu||_\Lambda. \tag{6.4.16}$$

Moreover, $S_2^{(\gamma)}$ is coercive for each $\gamma \geq 0$, because from (6.4.6) and (6.4.14) we have

$$\langle S_2^{(\gamma)} \eta, \eta \rangle \geq [\alpha_2^\flat (C_2^*)^{-2} + \gamma] ||\eta||_\Lambda^2. \tag{6.4.17}$$

Proceeding as in Section 1.3 for the Poisson problem, the iterations (6.4.7), (6.4.11), (6.4.9) are equivalent to a preconditioned Richardson method for the Steklov–Poincaré operator S, with $S_2^{(\gamma)}$ as a preconditioner. We are now in a position to prove the following convergence result.

Theorem 6.4.1 *There exists $\gamma^* \geq 0$ such that for each $\gamma \geq \gamma^*$ and for each $\lambda^0 \in \Lambda$ the iterative scheme* (6.4.7), (6.4.11), (6.4.9) *is convergent in Λ, provided that the relaxation parameter θ is chosen in a suitable interval $(0, \theta_\gamma)$.*

Proof Owing to (6.4.16) and (6.4.17), assumptions (a) and (b) of Theorem 4.2.2 are satisfied provided we set

$$\alpha_2 := \alpha_2^\flat (C_2^*)^{-2} + \gamma, \quad \beta_i := \beta_i^\flat k_i^2 + \gamma, \quad i = 1, 2.$$

We shall prove that there exists $\gamma^* \geq 0$ such that for each $\gamma \geq \gamma^*$ assumption (c) of Theorem 4.2.2 is also satisfied; namely

$$\langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle + \langle S\eta, \eta \rangle \geq \kappa^* ||\eta||_\Lambda^2 \qquad \forall\, \eta \in \Lambda. \tag{6.4.18}$$

We have

$$\begin{aligned} \langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle + \langle S\eta, \eta \rangle &= 2\langle S\eta, \eta \rangle + \langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle - \langle S\eta, \eta \rangle \\ &\geq 2\langle S\eta, \eta \rangle - |\langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle - \langle S\eta, \eta \rangle|. \end{aligned}$$

Setting $\mu = (S_2^{(\gamma)})^{-1} S\eta$, we obtain

$$\begin{aligned} |\langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle - \langle S\eta, \eta \rangle| &= |a_2^\flat (E_2^\flat \eta, E_2^\flat \mu) - a_2^\flat (E_2^\flat \mu, E_2^\flat \eta)| \\ &= \left| \int_{\Omega_2} \mathbf{b} \cdot (E_2^\flat \mu \nabla E_2^\flat \eta - E_2^\flat \eta \nabla E_2^\flat \mu) \right| \\ &\leq 2 ||\mathbf{b}||_{L^\infty(\Omega_2)} ||E_2^\flat \eta||_{1,\Omega_2} ||E_2^\flat \mu||_{1,\Omega_2} \\ &\leq 2 ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 ||\eta||_\Lambda ||(S_2^{(\gamma)})^{-1} S\eta||_\Lambda. \end{aligned}$$

From the definition of S and (6.4.5), (6.4.6), (6.4.13) and (6.4.14) it follows that

$$\begin{aligned} &\langle S\eta, \mu \rangle \leq (\beta_1^\flat k_1^2 + \beta_2^\flat k_2^2)\, ||\eta||_\Lambda^2 ||\mu||_\Lambda^2 \\ &\langle S\eta, \eta \rangle \geq [\alpha_1^\flat (C_1^*)^{-2} + \alpha_2^\flat (C_2^*)^{-2})\, ||\eta||_\Lambda^2. \end{aligned}$$

Therefore, setting $\beta := \beta_1^\flat k_1^2 + \beta_2^\flat k_2^2$ and $\alpha := \alpha_1^\flat (C_1^*)^{-2} + \alpha_2^\flat (C_2^*)^{-2}$, we have

$$\begin{aligned} \langle S_2^{(\gamma)} \eta, (S_2^{(\gamma)})^{-1} S\eta \rangle + \langle S\eta, \eta \rangle &\geq 2\alpha ||\eta||_\Lambda^2 - 2||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha_2} ||\eta||_\Lambda^2 \\ &= 2 \left(\alpha - ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha_2} \right) ||\eta||_\Lambda^2. \end{aligned}$$

Condition (6.4.18) is satisfied provided that

$$\kappa^* := 2 \left(\alpha - ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha_2} \right) > 0,$$

i.e.

$$\alpha_2 > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \frac{\beta}{\alpha}.$$

Recalling the definition of α_2 it is therefore sufficient to take $\gamma \geq \gamma^*$, where

$$\begin{cases} \gamma^* = 0 & \text{if } \alpha_2^\flat (C_2^*)^{-2} > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \dfrac{\beta}{\alpha} \\ \gamma^* > ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \dfrac{\beta}{\alpha} - \alpha_2^\flat (C_2^*)^{-2} & \text{if } \alpha_2^\flat (C_2^*)^{-2} \leq ||\mathbf{b}||_{L^\infty(\Omega_2)} k_2^2 \dfrac{\beta}{\alpha} \end{cases}.$$

Since all the assumptions of Theorem 4.2.2 are satisfied, the conclusion follows. □

It is worthwhile to note that the rate of convergence of the iterative scheme (6.4.7), (6.4.11), (6.4.9) depends only on the parameters $\beta_i^\flat$, $\alpha_i^\flat$ in (6.4.5) and (6.4.6), k_i, C_i^* in (6.4.13) and (6.4.14), $i = 1, 2$.

6.4.3 The γ-DR iterative method for systems of advection–diffusion equations

The γ-DR iterative method can be easily extended to the case of advection–diffusion systems of the form

$$\begin{cases} -\varepsilon \Delta \mathbf{u} + \sum_{j=1}^{d} D_j (B^{(j)} \mathbf{u}) + A_0 \mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{6.4.19}$$

where $B^{(j)}$, $j = 1, \ldots, d$, and A_0 are $q \times q$ *symmetric* matrices.

In particular, a splitting procedure à la Chorin–Temam applied to the linearised non-stationary Navier–Stokes equations for incompressible flows (see Section 7.3.1) can yield a system of this type.

We assume that the coefficients of $B^{(j)}$ and A_0 belong to $L^\infty(\Omega)$, and that the coefficients of $\sum_j D_j B^{(j)}$ belong to $L^\infty(\Omega)$. Moreover, we require that the matrix

$$M(\mathbf{x}) := \frac{1}{2} \sum_{j=1}^{d} D_j B^{(j)}(\mathbf{x}) + A_0(\mathbf{x}) \tag{6.4.20}$$

is positive semi-definite for almost all $\mathbf{x} \in \Omega$, which corresponds to the coerciveness assumption (6.1.2).

We can introduce the associated bilinear form:

$$\begin{aligned} a^\flat(\mathbf{w}, \mathbf{v}) := & \int_\Omega \left[\varepsilon \nabla \mathbf{w} \cdot \nabla \mathbf{v} + (M\, \mathbf{w}) \cdot \mathbf{v} \right] \\ & + \frac{1}{2} \int_\Omega \sum_{j=1}^{d} [(B^{(j)} \mathbf{v}) \cdot D_j \mathbf{w} - (B^{(j)} D_j \mathbf{v}) \cdot \mathbf{w}], \end{aligned} \tag{6.4.21}$$

which can be used to rewrite the Dirichlet boundary value problem (6.4.19) in the variational form

$$\mathbf{u} \in (H_0^1(\Omega))^d \ : \ a^\flat(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\ \mathbf{v} \in (H_0^1(\Omega))^d. \tag{6.4.22}$$

Let γ be a $q \times q$ positive semi-definite matrix with constant coefficients. Similarly to (6.4.7), (6.4.11), (6.4.9), the γ-DR scheme for problem (6.4.22) reads:

$$\begin{cases} \text{find } \ \mathbf{u}_1^{k+1} \in (V_1)^q \ : \\ a_1^\flat(\mathbf{u}_1^{k+1}, \mathbf{v}_1) = (\mathbf{f}, \mathbf{v}_1)_{\Omega_1} \qquad \forall\ \mathbf{v}_1 \in (H_0^1(\Omega_1))^q \\ \mathbf{u}_{1|\Gamma}^{k+1} = \boldsymbol{\lambda}^k \end{cases} \tag{6.4.23}$$

$$\begin{cases} \text{find } \ \mathbf{u}_2^{k+1} \in (V_2)^q \ : \\ a_2^\flat(\mathbf{u}_2^{k+1}, \mathbf{v}_2) = (\mathbf{f}, \mathbf{v}_2)_{\Omega_2} \qquad \forall\ \mathbf{v}_2 \in (H_0^1(\Omega_2))^q \\ a_2^\flat(\mathbf{u}_2^{k+1}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu}) + ((\gamma \mathbf{u}_{2|\Gamma}^{k+1}, \boldsymbol{\mu}))_\Lambda = (\mathbf{f}, \boldsymbol{\mathcal{R}}_2\boldsymbol{\mu})_{\Omega_2} + (\mathbf{f}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu})_{\Omega_1} \\ \qquad\qquad -a_1^\flat(\mathbf{u}_1^{k+1}, \boldsymbol{\mathcal{R}}_1\boldsymbol{\mu}) + ((\gamma \mathbf{u}_{1|\Gamma}^{k+1}, \boldsymbol{\mu}))_\Lambda \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^q, \end{cases} \tag{6.4.24}$$

and finally

$$\boldsymbol{\lambda}^{k+1} := \theta \mathbf{u}_{2|\Gamma}^{k+1} + (1-\theta)\boldsymbol{\lambda}^k \qquad \text{on } \Gamma, \tag{6.4.25}$$

with obvious meaning of notation.

The convergence of this iterative scheme can be shown as in Section 6.4.2. More precisely, the γ-DR scheme is proved to converge provided that the matrix γ satisfies

$$(\gamma\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq \gamma^* \boldsymbol{\xi} \cdot \boldsymbol{\xi} \qquad \forall\ \boldsymbol{\xi} \in \mathbf{R}^q,$$

for a suitable $\gamma^* \geq 0$, depending on the data of the problem.

Substituting in (6.4.24) the scalar product in $(\Lambda)^q$ with the $(L^2(\Gamma))^q$-scalar product, one can also consider another γ-DR scheme for problem (6.4.22), corresponding to (6.4.7), (6.4.8), (6.4.9).

6.4.4 The γ-RR iterative method

In this section we present and analyse another iteration-by-subdomain procedure, proposed by Alonso *et al.* (1998), and called γ-*Robin/Robin* (γ-RR). It extends to the non-symmetric case the Robin method described for symmetric elliptic operators in Section 1.3. The scheme reads as follows: given λ^0 in $L^2(\Gamma)$, for each $k \geq 0$ solve

$$\begin{cases} L_\varepsilon u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = 0 & \text{on } \partial\Omega_1 \cap \partial\Omega \\ \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2}\,\mathbf{b}\cdot\mathbf{n} - \gamma\right) u_1^{k+1} = \lambda^k & \text{on } \Gamma \end{cases} \tag{6.4.26}$$

and

$$\begin{cases} L_\varepsilon u_2^{k+1} = f & \text{in } \Omega_2 \\ u_2^{k+1} = 0 & \text{on } \partial\Omega_2 \cap \partial\Omega \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} - \left(\dfrac{1}{2}\,\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_2^{k+1} & \\ \qquad = \varepsilon \dfrac{\partial u_1^{k+1}}{\partial n} - \left(\dfrac{1}{2}\,\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} & \text{on } \Gamma, \end{cases} \tag{6.4.27}$$

where

$$\lambda^{k+1} := \varepsilon \frac{\partial u_2^{k+1}}{\partial n} - \left(\frac{1}{2}\,\mathbf{b}\cdot\mathbf{n} - \gamma\right) u_2^{k+1} \qquad \text{on } \Gamma, \tag{6.4.28}$$

$\gamma = \gamma(\mathbf{x})$ being a given function in $L^\infty(\Gamma)$ satisfying $\gamma(\mathbf{x}) \geq \hat{\gamma} > 0$ for almost all $\mathbf{x} \in \Gamma$. In the one-dimensional case, this corresponds to the scheme (6.2.2)–(6.2.4) with the choice $A = -\gamma$ and $B = \gamma$.

Noting that

$$\begin{aligned} \lambda^{k+1} &= \varepsilon \frac{\partial u_1^{k+1}}{\partial n} - \left(\frac{1}{2}\,\mathbf{b}\cdot\mathbf{n} + \gamma\right) u_1^{k+1} + 2\gamma u_2^{k+1} \\ &= \lambda^k + 2\gamma(u_2^{k+1} - u_1^{k+1}), \end{aligned} \tag{6.4.29}$$

we can rewrite the scheme above in the variational form

$$\begin{cases} \text{find } u_1^{k+1} \in V_1 \ : \\ a_1^\flat(u_1^{k+1}, v_1) + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{1|\Gamma} \\ \qquad\qquad = \displaystyle\int_{\Omega_1} f v_1 + \int_\Gamma \lambda^k v_{1|\Gamma} \qquad \forall\ v_1 \in V_1, \end{cases} \tag{6.4.30}$$

then

$$
\left\{
\begin{array}{l}
\text{find } u_2^{k+1} \in V_2 \ : \\
a_2^\flat(u_2^{k+1}, v_2) + \displaystyle\int_\Gamma \gamma u_{2|\Gamma}^{k+1} v_{2|\Gamma} \\
\qquad = \displaystyle\int_{\Omega_2} f v_2 + \int_{\Omega_1} f \mathcal{R}_1 v_{2|\Gamma} - a_1^\flat(u_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) \\
\qquad + \displaystyle\int_\Gamma \gamma u_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2,
\end{array}
\right. \tag{6.4.31}
$$

and finally

$$
\lambda^{k+1} = \lambda^k + 2\gamma(u_{2|\Gamma}^{k+1} - u_{1|\Gamma}^{k+1}) \qquad \text{on } \Gamma, \tag{6.4.32}
$$

where $a_i^\flat$ have been introduced in (6.4.4). Since $\lambda^0 \in L^2(\Gamma)$ and $\gamma \in L^\infty(\Gamma)$, then $\lambda^{k+1} \in L^2(\Gamma)$ for each $k \geq 0$.

Let us emphasise that the bilinear forms

$$
a_i^\flat(w_i, v_i) + \int_\Gamma \gamma w_i v_i, \quad i = 1, 2,
$$

which are used in (6.4.30) and (6.4.31), are coercive in V_i, for every $\gamma \geq 0$.

6.4.5 Convergence of the γ-RR iterative method

We obtain the following convergence theorem, which is proved in Alonso *et al.* (1998), and is inspired by the results of P.-L. Lions (1990) and Nataf and Rogier (1995).

Theorem 6.4.2 *Assume either that Ω is a Lipschitz polygonal domain or that $\partial\Omega$ is regular enough, say $\partial\Omega \in C^2$. Suppose, moreover, that $\mathbf{b}_{|\Gamma} \in (L^\infty(\Gamma))^d$. For each $\lambda^0 \in L^2(\Gamma)$ and for each $i = 1, 2$, the sequences u_i^k converge in $H^1(\Omega_i)$ to the restriction $u_{|\Omega_i}$ of the solution u of* (6.1.1).

Proof The exact solution u of (6.1.1) satisfies

$$
\begin{aligned}
a_1^\flat(u_{|\Omega_1}, v_1) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma} &= \int_{\Omega_1} f v_1 + \int_{\Omega_2} f \mathcal{R}_2 v_{1|\Gamma} \\
&\quad - a_2^\flat(u_{|\Omega_2}, \mathcal{R}_2 v_{1|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma} \qquad \forall\, v_1 \in V_1,
\end{aligned}
$$

and

$$
\begin{aligned}
a_2^\flat(u_{|\Omega_2}, v_2) + \int_\Gamma \gamma u_{|\Gamma} v_{2|\Gamma} &= \int_{\Omega_2} f v_2 + \int_{\Omega_1} f \mathcal{R}_1 v_{2|\Gamma} \\
&\quad - a_1^\flat(u_{|\Omega_1}, \mathcal{R}_1 v_{2|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{2|\Gamma} \qquad \forall\, v_2 \in V_2.
\end{aligned}
$$

From well known regularity results for elliptic equations (see, for example, J.-L. Lions and Magenes 1972 and Dauge 1988), the solution u belongs to $H^{3/2+\delta}(\Omega)$ for a suitable $\delta > 0$, and consequently $\frac{\partial u}{\partial n} \in L^2(\Gamma)$. Therefore we can also write

$$\int_{\Omega_2} f\mathcal{R}_2 v_{1|\Gamma} - a_2^\flat(u_{|\Omega_2}, \mathcal{R}_2 v_{1|\Gamma}) + \int_\Gamma \gamma u_{|\Gamma} v_{1|\Gamma} = \int_\Gamma \left(\varepsilon \frac{\partial u}{\partial n} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}\, u + \gamma\, u\right)_{|\Gamma} v_{1|\Gamma}.$$

Setting now, for each $k \geq 0$, $e_i^k := u_i^k - u_{|\Omega_i}$ and

$$\omega^k := \lambda^k - \left(\varepsilon \frac{\partial u}{\partial n} - \frac{1}{2}\mathbf{b}\cdot\mathbf{n}u + \gamma u\right)_{|\Gamma},$$

the error equations can be written as

$$a_1^\flat(e_1^{k+1}, v_1) + \int_\Gamma \gamma e_{1|\Gamma}^{k+1} v_{1|\Gamma} = \int_\Gamma \omega^k v_{1|\Gamma} \qquad \forall\ v_1 \in V_1, \tag{6.4.33}$$

and

$$\begin{aligned} &a_2^\flat(e_2^{k+1}, v_2) + \int_\Gamma \gamma e_{2|\Gamma}^{k+1} v_{2|\Gamma} \\ &\quad = -a_1^\flat(e_1^{k+1}, \mathcal{R}_1 v_{2|\Gamma}) + \int_\Gamma \gamma e_{1|\Gamma}^{k+1} v_{2|\Gamma} \qquad \forall\ v_2 \in V_2, \end{aligned} \tag{6.4.34}$$

where

$$\omega^{k+1} = \omega^k + 2\gamma(e_{2|\Gamma}^{k+1} - e_{1|\Gamma}^{k+1}). \tag{6.4.35}$$

Taking $v_1 = e_1^{k+1}$ in (6.4.33) and $v_2 = e_2^{k+1}$ in (6.4.34), we have that

$$a_1^\flat(e_1^{k+1}, e_1^{k+1}) = \int_\Gamma (\omega^k - \gamma e_{1|\Gamma}^{k+1}) e_{1|\Gamma}^{k+1} \tag{6.4.36}$$

and

$$a_2^\flat(e_2^{k+1}, e_2^{k+1}) = -a_1^\flat(e_1^{k+1}, \mathcal{R}_1 e_{2|\Gamma}^{k+1}) + \int_\Gamma \gamma(e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) e_{2|\Gamma}^{k+1}. \tag{6.4.37}$$

Choosing $v_1 = \mathcal{R}_1 e_{2|\Gamma}^{k+1}$ in (6.4.33), we also obtain

$$a_1^\flat(e_1^{k+1}, \mathcal{R}_1 e_{2|\Gamma}^{k+1}) = \int_\Gamma (\omega^k - \gamma e_{1|\Gamma}^{k+1}) e_{2|\Gamma}^{k+1}, \tag{6.4.38}$$

and using this equality in (6.4.37) we have

$$a_2^\flat(e_2^{k+1}, e_2^{k+1}) = \int_\Gamma (2\gamma e_{1|\Gamma}^{k+1} - \gamma e_{2|\Gamma}^{k+1} - \omega^k) e_{2|\Gamma}^{k+1}. \tag{6.4.39}$$

Adding (6.4.36) and (6.4.39) we find that

$$
\begin{aligned}
a_1^\flat(e_1^{k+1}, e_1^{k+1}) + & a_2^\flat(e_2^{k+1}, e_2^{k+1}) \\
&= \int_\Gamma \frac{1}{\gamma}[\gamma\omega^k(e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) - \gamma^2(e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1})^2] \\
&= \int_\Gamma -\frac{1}{4\gamma}[2\gamma(e_{1|\Gamma}^{k+1} - e_{2|\Gamma}^{k+1}) - \omega^k]^2 + \int_\Gamma \frac{1}{4\gamma}(\omega^k)^2.
\end{aligned}
$$

Recalling (6.4.35), we finally obtain

$$
a_1^\flat(e_1^{k+1}, e_1^{k+1}) + a_2^\flat(e_2^{k+1}, e_2^{k+1}) + \int_\Gamma \frac{1}{4\gamma}(\omega^{k+1})^2 = \int_\Gamma \frac{1}{4\gamma}(\omega^k)^2. \tag{6.4.40}
$$

Summing up from $k = 0$ to $k = M - 1$, it follows that

$$
\begin{aligned}
\sum_{k=1}^{M}[a_1^\flat(e_1^k, e_1^k) + & a_2^\flat(e_2^k, e_2^k)] + \int_\Gamma \frac{1}{4\gamma}(\omega^M)^2 \\
&= \int_\Gamma \frac{1}{4\gamma}(\omega^0)^2,
\end{aligned} \tag{6.4.41}
$$

hence the series

$$
\sum_{k=1}^{\infty}[a_1^\flat(e_1^k, e_1^k) + a_2^\flat(e_2^k, e_2^k)]
$$

is convergent, and, as a consequence of the coerciveness of $a_i^\flat(\cdot,\cdot)$ in V_i, e_i^k converge to 0 in $H^1(\Omega_i)$, $i = 1, 2$. □

Remark 6.4.3 It is worthwhile to note that the γ-RR method generalises several other methods proposed recently. For instance, we have

$$
\gamma = \begin{cases}
\frac{1}{2}|\mathbf{b}\cdot\mathbf{n}| & \text{ARN method (without relaxation)} \\
\frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4a_0\varepsilon} & \text{Nataf and Rogier (1995)} \\
\frac{1}{2}\sqrt{|\mathbf{b}\cdot\mathbf{n}|^2 + 4\kappa\varepsilon}, \quad \kappa > 0 & \text{Auge } \textit{et al.} \text{ (1996).}
\end{cases}
$$

Therefore, Theorem 6.4.2 yields, in particular, the convergence of the unrelaxed ARN method introduced in Section 6.3. Conversely, this is not the case for the AR$_\beta$N method. □

Remark 6.4.4 The γ-RR iterative method can also be easily extended to advection–diffusion systems (see Alonso *et al.* 1998). □

6.5 The finite element realisation of the iterative algorithms

Adopting the notation of Chapter 2, the Galerkin finite element approximation of the boundary value problem (6.1.1) reads

$$\text{find } u_h \in V_h \ : \ a^0(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h. \tag{6.5.1}$$

By analogy with what was carried out in Section 6.1 for the differential problem, (6.5.1) is clearly equivalent to the following multi-domain formulation: find $u_{1,h} \in V_{1,h}$, $u_{2,h} \in V_{2,h}$ such that

$$\begin{cases} a_1^0(u_{1,h}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \quad \forall\, v_{1,h} \in V_{1,h}^0 \\ u_{1,h} = u_{2,h} \quad \text{on } \Gamma \\ a_2^0(u_{2,h}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \quad \forall\, v_{2,h} \in V_{2,h}^0 \\ a_2^0(u_{2,h}, \mathcal{R}_{2,h}\mu_h) = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\ \qquad\qquad -a_1^0(u_{1,h}, \mathcal{R}_{1,h}\mu_h) \quad \forall\, \mu_h \in \Lambda_h. \end{cases} \tag{6.5.2}$$

As usual, $\mathcal{R}_{i,h}$ denotes any extension operator from Λ_h to $V_{i,h}$, $i = 1, 2$. In practice, one often chooses the interpolant $\pi_{i,h}\mu_h$; namely, the element of the finite element space $V_{i,h}$ that assumes the value of μ_h on Γ and vanishes at all the other finite element nodes in $\overline{\Omega_i}$.

Problem (6.5.2) can be regarded as the finite element counterpart of (6.1.7). It provides the basis for the iterative algorithm ADN for finite elements: given $u_{1,h}^0 \in V_{1,h}$ and $u_{2,h}^0 \in V_{2,h}$, for all $k \geq 0$

$$\begin{cases} \text{find } u_{1,h}^{k+1} \in V_{1,h} \ : \\ a_1^0(u_{1,h}^{k+1}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \qquad \forall\, v_{1,h} \in V_{1,h}^0 \\ u_{1,h}^{k+1} = \theta' u_{2,h}^k + (1-\theta') u_{1,h}^k \qquad \text{on } \Gamma^{\text{in}} \cup \Gamma^0 \\ a_1^0(u_{1,h}^{k+1}, \mathcal{R}_{1,h}^{\text{out}}\mu_h^{\text{out}}) = (f, \mathcal{R}_{1,h}^{\text{out}}\mu_h^{\text{out}})_{\Omega_1} + (f, \mathcal{R}_{2,h}^{\text{out}}\mu_h^{\text{out}})_{\Omega_2} \\ \qquad\qquad -a_2^0(u_{2,h}^k, \mathcal{R}_{2,h}^{\text{out}}\mu_h^{\text{out}}) \qquad \forall\, \mu_h^{\text{out}} \in \Lambda_h^{\text{out}} \end{cases} \tag{6.5.3}$$

and

$$\begin{cases} \text{find } u_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2^0(u_{2,h}^{k+1}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \qquad \forall\, v_{2,h} \in V_{2,h}^0 \\ u_{2,h}^{k+1} = \theta'' u_{1,h}^{k+1} + (1-\theta'') u_{2,h}^k \qquad \text{on } \Gamma^{\text{out}} \\ a_2^0(u_{2,h}^{k+1}, \mathcal{R}_{2,h}^{\text{in},0}\mu_h^{\text{in},0}) = (f, \mathcal{R}_{2,h}^{\text{in},0}\mu_h^{\text{in},0})_{\Omega_2} + (f, \mathcal{R}_{1,h}^{\text{in},0}\mu_h^{\text{in},0})_{\Omega_1} \\ \qquad\qquad -a_1^0(u_{1,h}^{k+1}, \mathcal{R}_{1,h}^{\text{in},0}\mu_h^{\text{in},0}) \quad \forall\, \mu_h^{\text{in},0} \in \Lambda_h^{\text{in},0}, \end{cases} \tag{6.5.4}$$

where

$$\Lambda_h^{\rm out} := \{v_{h|\Gamma^{\rm out}} \,|\, v_h \in V_h\}$$
$$\Lambda_h^{\rm in,0} := \{v_{h|\Gamma^{\rm in}\cup\Gamma^0} \,|\, v_h \in V_h\},$$

and $\mathcal{R}_{i,h}^{\rm out}$ denotes any extension operator from $\Lambda_h^{\rm out}$ to $V_{i,h}$, $i = 1, 2$, vanishing at the nodes internal to $\Gamma^{\rm in} \cup \Gamma^0$. An analogous definition holds for $\mathcal{R}_{i,h}^{\rm in,0}$.

In a similar manner, assuming that the surface measure of Γ^0 is zero, the ARN algorithm is defined as (6.5.3), (6.5.4) provided the Dirichlet conditions on $\Gamma^{\rm in}$ for $u_{1,h}^{k+1}$ and on $\Gamma^{\rm out}$ for $u_{2,h}^{k+1}$ are replaced by the Robin conditions:

$$a_1^R(u_{1,h}^{k+1}, \mathcal{R}_{1,h}^{\rm in}\mu_h^{\rm in}) = \theta'[(f, \mathcal{R}_{1,h}^{\rm in}\mu_h^{\rm in})_{\Omega_1} + (f, \mathcal{R}_{2,h}^{\rm in}\mu_h^{\rm in})_{\Omega_2} - a_2^R(u_{2,h}^k, \mathcal{R}_{2,h}^{\rm in}\mu_h^{\rm in})]$$
$$+(1-\theta')a_1^R(u_{1,h}^k, \mathcal{R}_{1,h}^{\rm in}\mu_h^{\rm in}) \qquad \forall\ \mu_h^{\rm in} \in \Lambda_h^{\rm in}$$

and

$$a_2^R(u_{2,h}^{k+1}, \mathcal{R}_{2,h}^{\rm out}\mu_h^{\rm out}) = \theta''[(f, \mathcal{R}_{2,h}^{\rm out}\mu_h^{\rm out})_{\Omega_2} + (f, \mathcal{R}_{1,h}^{\rm out}\mu_h^{\rm out})_{\Omega_1}$$
$$-a_1^R(u_{1,h}^{k+1}, \mathcal{R}_{1,h}^{\rm out}\mu_h^{\rm out})]$$
$$+(1-\theta'')a_2^R(u_{2,h}^k, \mathcal{R}_{2,h}^{\rm out}\mu_h^{\rm out}) \qquad \forall\ \mu_h^{\rm out} \in \Lambda_h^{\rm out},$$

respectively.

The damped forms of the algorithms ADN and ARN are obtained by omitting in (6.5.3) and (6.5.4) the contribution from the complementary domain in the equation at the outflow (i.e. on $\Gamma^{\rm out}$ for the domain Ω_1, and on $\Gamma^{\rm in} \cup \Gamma^0$ for the domain Ω_2). The last equations in (6.5.3) and (6.5.4) therefore become

$$a_1^0(u_{1,h}^{k+1}, \mathcal{R}_{1,h}^{\rm out}\mu_h^{\rm out}) = (f, \mathcal{R}_{1,h}^{\rm out}\mu_h^{\rm out})_{\Omega_1} \qquad \forall\ \mu_h^{\rm out} \in \Lambda_h^{\rm out}$$

and

$$a_2^0(u_{2,h}^{k+1}, \mathcal{R}_{2,h}^{\rm in,0}\mu_h^{\rm in,0}) = (f, \mathcal{R}_{2,h}^{\rm in,0}\mu_h^{\rm in,0})_{\Omega_2} \qquad \forall\ \mu_h^{\rm in,0} \in \Lambda_h^{\rm in,0},$$

respectively, which are the weak form of the homogeneous Neumann condition.

The finite element formulation of the γ-DR scheme reads: given $\lambda_h^0 \in \Lambda_h$, for each $k \geq 0$ solve

$$\text{(6.5.5)} \qquad \begin{cases} \text{find } u_{1,h}^{k+1} \in V_{1,h} \ : \\ a_1^\flat(u_{1,h}^{k+1}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \qquad \forall\ v_{1,h} \in V_{1,h}^0 \\ u_{1,h|\Gamma}^{k+1} = \lambda_h^k \end{cases}$$

$$
(6.5.6)\quad \begin{cases} \text{find } u_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2^\flat(u_{2,h}^{k+1}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \qquad \forall\ v_{2,h} \in V_{2,h}^0 \\ a_2^\flat(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) + \displaystyle\int_\Gamma \gamma u_{2,h|\Gamma}^{k+1}\mu_h = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\ \qquad\qquad -a_1^\flat(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) + \displaystyle\int_\Gamma \gamma u_{1,h|\Gamma}^{k+1}\mu_h \quad \forall\ \mu_h \in \Lambda_h, \end{cases}
$$

setting finally

$$
(6.5.7)\qquad \lambda_h^{k+1} := \theta u_{2,h|\Gamma}^{k+1} + (1-\theta)\lambda_h^k \qquad \text{on } \Gamma.
$$

The second step (6.5.6) can be substituted by

$$
(6.5.8)\quad \begin{cases} \text{find } u_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2^\flat(u_{2,h}^{k+1}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \qquad \forall\ v_{2,h} \in V_{2,h}^0 \\ a_2^\flat(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) + \gamma((u_{2,h|\Gamma}^{k+1}, \mu_h))_\Lambda = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\ \qquad\qquad -a_1^\flat(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) + \gamma((u_{1,h|\Gamma}^{k+1}, \mu_h))_\Lambda \quad \forall\ \mu_h \in \Lambda_h, \end{cases}
$$

for a constant $\gamma \geq 0$.

As we have already noted, the iterative scheme (6.5.5), (6.5.8), (6.5.7) converges for $\gamma \geq \gamma^*$ (see Theorem 6.4.1), and the rate of convergence only depends on the parameters $\beta_i^\flat$, $\alpha_i^\flat$ in (6.4.5) and (6.4.6), k_i, C_i^* in (6.4.13) and (6.4.14), $i = 1, 2$. In the finite dimensional context, all these constants, possibly except k_i, are independent of the total number of degrees of freedom. Therefore, the rate of convergence is independent of h, provided that we can find a uniform bound for k_i. In other words, it is necessary to prove the uniform extension result

$$
(6.5.9)\qquad ||E_{i,h}^\flat \eta_h||_{1,\Omega_i} \leq k_i ||\eta_h||_\Lambda \quad \forall\ \eta_h \in \Lambda_h,
$$

where for each $\eta \in \Lambda$, $E_{i,h}^\flat \eta$ is the finite dimensional counterpart of the extension $E_i^\flat \eta$ introduced in (6.4.12). This result has been proved in Section 5.1, assuming that the family of triangulations $\mathcal{T}_h$ of Ω is regular, and the induced family of triangulations $\mathcal{M}_h$ of Γ is quasi-uniform.

The convergence of the iterative scheme (6.5.5), (6.5.6), (6.5.7) can be proved by a similar argument. In fact, for discrete functions all the norms are equivalent; hence, there exists a constant $\kappa_h > 0$ such that

$$
\kappa_h ||\eta_h||_\Lambda^2 \leq ||\eta_h||_{0,\Gamma}^2 \quad \forall\ \eta_h \in \Lambda_h.
$$

By using this estimate, we only have to substitute the constant $\alpha_2 := \alpha_2^\flat (C_2^*)^{-2} + \gamma$ in (6.4.17) with

$$\alpha_{2,h} := \alpha_2^\flat (C_2^*)^{-2} + \gamma \kappa_h,$$

and convergence is ensured provided that $\inf_\Omega \gamma \geq \gamma_h^* := \gamma^*/\kappa_h$.

Therefore, in this case we are not in a position to prove that the rate of convergence is independent of h. However, the numerical results obtained in Alonso *et al.* (1998) suggest that this is in fact the case.

Remark 6.5.1 Though the convergence result in Theorem 6.4.1 holds only for γ sufficiently large, numerical evidence shows that the γ-DR iterative scheme converges for any $\gamma \geq 0$; in particular, for $\gamma = 0$ (see Alonso *et al.* 1998). Moreover, an efficient choice of the relaxation parameter θ is achieved by taking it into the neighbourhood of $1/2$. □

The finite element γ-RR scheme reads: given $\lambda^0 \in L^2(\Gamma)$, for each $k \geq 0$ solve

$$\begin{cases} \text{find } u_{1,h}^{k+1} \in V_{1,h} \ : \\ a_1^\flat(u_{1,h}^{k+1}, v_{1,h}) + \displaystyle\int_\Gamma \gamma u_{1,h|\Gamma}^{k+1} v_{1,h|\Gamma} \\ \qquad = \displaystyle\int_{\Omega_1} f v_{1,h} + \int_\Gamma \lambda^k v_{1,h|\Gamma} \qquad \forall\ v_{1,h} \in V_{1,h}, \end{cases} \tag{6.5.10}$$

then

$$\begin{cases} \text{find } u_{2,h}^{k+1} \in V_{2,h} \ : \\ a_2^\flat(u_{2,h}^{k+1}, v_{2,h}) + \displaystyle\int_\Gamma \gamma u_{2,h|\Gamma}^{k+1} v_{2,h|\Gamma} \\ \qquad = \displaystyle\int_{\Omega_2} f v_{2,h} + \int_{\Omega_1} f \mathcal{R}_{1,h} v_{2,h|\Gamma} - a_1^\flat(u_{1,h}^{k+1}, \mathcal{R}_{1,h} v_{2,h|\Gamma}) \\ \qquad\quad + \displaystyle\int_\Gamma \gamma u_{1,h|\Gamma}^{k+1} v_{2,h|\Gamma} \qquad \forall\ v_{2,h} \in V_{2,h}, \end{cases} \tag{6.5.11}$$

and finally

$$\lambda^{k+1} = \lambda^k + 2\gamma(u_{2,h|\Gamma}^{k+1} - u_{1,h|\Gamma}^{k+1}) \qquad \text{on } \Gamma. \tag{6.5.12}$$

The convergence result in Theorem 6.4.2 holds true for this scheme. In fact, for each $\mu_h \in \Lambda_h$ we can write

$$\begin{aligned} \int_{\Omega_2} f \mathcal{R}_{2,h} \mu_h - a_2^\flat(u_{h|\Omega_2}, \mathcal{R}_{2,h}\mu_h) + \int_\Gamma \gamma u_{h|\Gamma} \mu_h \\ = \int_\Gamma g_h \mu_h \end{aligned}$$

for a suitable $g_h \in \Lambda_h$, and the proof of convergence follows as in the infinite dimensional case.

Though we have no information about the rate of convergence, which, in principle, can depend on h, the numerical results in Alonso *et al.* (1998) show that this is not the case, and the number of subdomain iterations is independent of the mesh size, for suitable choices of the parameter γ (possibly depending on h).

Remark 6.5.2 A possible strategy for choosing the parameter γ is reported in Alonso *et al.* (1998). □

At this point, we would like to point out a potential advantage of using the domain decomposition iterative approach. In the advection-dominated case ($\omega \gg 1$), the standard finite element approximation (6.5.1) of the advection–diffusion problem might be unstable, unless suitably stabilised by an upwinding mechanism, such as those yielding the SUPG or GALS methods (see, for example, Johnson 1987; Franca *et al.* 1992; and Quarteroni and Valli 1994, Chapter 8). The stabilisation procedure, which should be used in the whole domain Ω when using a standard single-domain finite element approach, can be limited solely to those subdomains that contain boundary or internal layers. In all other subdomains the less expensive, better conditioned Galerkin finite element method can be used. Since the stabilisation terms do not affect the subdomain boundaries, the interface treatment in the previous iterative procedures will not change. What should change is the bilinear form $a_i^0(\cdot,\cdot)$ (or $a_i^R(\cdot,\cdot)$, $a_i^\flat(\cdot,\cdot)$), due to the inclusion of the stabilising terms. For a more detailed discussion on this issue see Trotta (1996).

There is another, less conventional, way to combine finite element and iterative domain decomposition algorithms. This time a single-domain, stabilised finite element approximation of the whole advection–diffusion problem (6.1.3) is carried out first, yielding an algebraic system $A\mathbf{u} = \mathbf{f}$ whose matrix is sparse, positive definite, with a large condition number. Typical methods for solving this kind of system require GMRES or BiCGSTAB iterations and a suitable preconditioner. The latter can be sought in the class of algebraic preconditioners (ILU or polynomials preconditioners), which, however, are unsuitable for parallel implementations. Otherwise, we could use few steps of any of the iterative domain decomposition algorithms presented in this chapter as a *preconditioner* of the residual at each step of the outer GMRES or BiCGSTAB iteration. Obviously, by this second approach we will find in the limit the original finite element solution expressed by the vector $\mathbf{u}$ of its grid values. The reader may find a more detailed discussion of these issues concerning parallel preconditioners for non-symmetric accelerators, for instance, in Quarteroni and Valli (1994) and in the references quoted therein.

7

TIME-DEPENDENT PROBLEMS

In this chapter we illustrate how the domain decomposition techniques that have been presented so far can be applied in the process of solving time-dependent problems. As usual we assume that Ω is a bounded domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz boundary. The initial-boundary value problem to be considered will have the abstract form

$$\begin{cases} \dfrac{\partial u}{\partial t} + L(u) = f & \text{in } Q_T := \Omega \times (0, T) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{7.1}$$

with suitable boundary conditions for each $t > 0$ on the whole boundary $\partial\Omega$ or on a subset of it, depending upon the nature of the differential operator L.

In (7.1) f and u_0 are given functions, while the operator L can be either linear or non-linear. In Section 7.1 we will address the parabolic case (when L is a second-order linear elliptic operator), while hyperbolic equations (of both first-order and second-order) are discussed in Section 7.2.

After reformulating (7.1) in a variational way, we will introduce its semi-discrete continuous-in-time approximation using the Galerkin method for the spatial discretisation. For the time discretisation, different kinds of approaches (based on either finite differences, or finite elements, or fractional steps) can be adopted, yielding either explicit or implicit schemes. In the latter case, at every time level we are left with a fully coupled boundary value problem that can be faced by the domain decomposition iterative techniques previously introduced. However, the convergence behaviour of these iterations will benefit from the special structure of the associated spatial operator, which depends on the time-step parameter Δt: indeed, in the limit as $\Delta t \to 0$, the resulting operator approaches the identity.

An intermediate situation between the fully explicit and fully implicit is one in which the interface variables are the first to be updated at the new time level. They provide, therefore, the new boundary values that make the individual subdomain problems independent of each other.

In Section 7.3 we will consider the case of non-linear operators $L(u)$, with the purpose of illustrating how domain decomposition concepts can be conveniently adapted to the time-advancing schemes that are most commonly used for equations of this kind. In particular, we address the Navier–Stokes equations for both incompressible and compressible flow problems, and the Euler equations of gas dynamics.

7.1 Parabolic problems

Consider the second-order linear symmetric differential operator

$$Lw := -\sum_{l,j=1}^{d} D_l(a_{lj} D_j w) + a_0 w, \tag{7.1.1}$$

where $a_{lj} = a_{jl}$ for each $l, j = 1, \ldots, d$. Let us recall that L is *elliptic* if there exists a constant $\alpha_0 > 0$ such that

$$\sum_{l,j=1}^{d} a_{lj}(\mathbf{x})\xi_l \xi_j \geq \alpha_0 |\boldsymbol{\xi}|^2, \tag{7.1.2}$$

for each $\boldsymbol{\xi} \in \mathbf{R}^d$ and for almost every $\mathbf{x} \in \Omega$.

When L is elliptic, the operator

$$\frac{\partial}{\partial t} + L \tag{7.1.3}$$

is parabolic. A classical example is provided by the heat equation $\frac{\partial u}{\partial t} - \Delta u = f$, in which case $L = -\Delta$.

For simplicity, we only treat the case of a homogeneous Dirichlet boundary condition on $\partial\Omega$.

We are therefore dealing with the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f & \text{in } Q_T := \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{7.1.4}$$

where $f = f(\mathbf{x}, t)$ and $u_0 = u_0(\mathbf{x})$ are given data.

As in the time-independent case considered until now, a weak formulation can be provided. Let $V = H_0^1(\Omega)$ and denote by $L^2(0,T;V)$ the space

$$L^2(0,T;V) := \Big\{ v : (0,T) \to V \,|\, v \text{ is measurable in } (0,T) \text{ and } \int_0^T ||v(t)||_1^2 \, dt < \infty \Big\}.$$

Similarly, the space $C^0([0,T]; L^2(\Omega))$ is defined as

$$C^0([0,T]; L^2(\Omega)) := \left\{ v : [0,T] \to L^2(\Omega) \,|\, v \text{ is continuous in } [0,T] \right\}.$$

The weak formulation of (7.1.4) reads as follows: given $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, find $u \in L^2(0,T;V) \cap C^0([0,T]; L^2(\Omega))$ such that

$$\begin{cases} \dfrac{d}{dt}(u(t),v) + a^*(u(t),v) = (f(t),v) \quad \forall\ v \in V \\ u(0) = u_0, \end{cases} \tag{7.1.5}$$

where $(\cdot,\cdot)$ denotes the scalar product in $L^2(\Omega)$, the bilinear form $a^*(\cdot,\cdot)$ has been introduced in (1.4.3), and the above equation has to be intended in the sense of distributions in $(0,T)$.

The existence and uniqueness of the solution to (7.1.5) is proved, for example, in J.-L. Lions and Magenes (1972), under the assumption that the bilinear form $a^*(\cdot,\cdot)$ is continuous and *weakly coercive* in V; that is, there exist two constants $\alpha > 0$ and $\lambda \geq 0$ such that

$$a^*(v,v) + \lambda ||v||_0^2 \geq \alpha ||v||_1^2 \qquad \forall\ v \in V. \tag{7.1.6}$$

Very often, this inequality is satisfied with $\lambda = 0$; namely, the bilinear form $a^*(\cdot,\cdot)$ is coercive. For example, this is the case for the heat equation. In particular, continuity and weak coerciveness are satisfied if the coefficients a_{lj} and a_0 belong to $L^\infty(\Omega)$; in this case, (7.1.6) holds for $\lambda > ||a_0||_{L^\infty(\Omega)}$.

Let us note, moreover, that, if we introduce the change of variable $u_\lambda(\mathbf{x},t) := e^{-\lambda t}u(\mathbf{x},t)$, where u is the solution of (7.1.4), the new unknown u_λ satisfies

$$\frac{\partial u_\lambda}{\partial t} + Lu_\lambda + \lambda u_\lambda = e^{-\lambda t} f \quad \text{in } Q_T.$$

If (7.1.6) holds for $a^*(w,v)$, the bilinear form $a^*_\lambda(w,v) := a^*(w,v) + \lambda(w,v)$ associated with this last problem is coercive; that is, it satisfies (7.1.6) with $\lambda = 0$. Therefore, if we replace f with $e^{-\lambda t}f$ and L with $L + \lambda I$, I being the identity operator, without loss of generality we can assume that the bilinear form associated with the initial-boundary value problem (7.1.4) satisfies (7.1.6) with $\lambda = 0$. This will always be assumed in the rest of this chapter.

7.1.1 Multi-domain formulation and space discretisation

We adopt the same notation as in Section 1.2. Upon setting $u_i := u_{|\Omega_i}$ for $i = 1,2$ and all $t > 0$, problem (7.1.5) admits the following equivalent two-domain formulation: find $u_i \in L^2(0,T;V_i) \cap C^0([0,T];L^2(\Omega_i))$, $i = 1,2$, such that

$$
\left\{
\begin{array}{l}
\dfrac{d}{dt}(u_1, v_1)_{\Omega_1} + a_1^*(u_1, v_1) = (f, v_1)_{\Omega_1} \quad \forall\ v_1 \in V_1^0 \\
\\
u_1 = u_2 \quad \text{on } \Gamma \\
\\
\dfrac{d}{dt}(u_2, v_2)_{\Omega_2} + a_2^*(u_2, v_2) = (f, v_2)_{\Omega_2} \quad \forall\ v_2 \in V_2^0 \\
\dfrac{d}{dt}(u_2, \mathcal{R}_2\mu)_{\Omega_2} + a_2^*(u_2, \mathcal{R}_2\mu) = (f, \mathcal{R}_2\mu)_{\Omega_2} + (f, \mathcal{R}_1\mu)_{\Omega_1} \\
\qquad\qquad -\dfrac{d}{dt}(u_1, \mathcal{R}_1\mu)_{\Omega_1} - a_1^*(u_1, \mathcal{R}_1\mu) \quad \forall\ \mu \in \Lambda,
\end{array}
\right. \tag{7.1.7}
$$

where $a_i^*(\cdot,\cdot)$ are defined in (1.4.7), $\mathcal{R}_i$ denotes any possible extension operator from Λ to V_i, and the equations have to be satisfied in the distributional sense on $(0,T)$. Clearly, the initial condition $u_i(\mathbf{x},0) = u_{0|\Omega_i}(\mathbf{x})$ has also to be imposed.

The proof that (7.1.7) is equivalent to the single-domain problem (7.1.5) follows from that for the steady case (see Lemma 1.2.1), and is left to the reader.

We note that the interface condition $(7.1.7)_4$ can be equivalently reformulated in the more 'physical' way (see (1.4.5)):

$$
\frac{\partial u_1}{\partial n_L} = \frac{\partial u_2}{\partial n_L} \quad \text{on } \Gamma, \quad \text{for each } t > 0, \tag{7.1.8}
$$

this equality being satisfied in a weak sense.

Now a semi-discretisation (continuous-in-time) can be defined via a Galerkin approximation of the space V by finite element subspaces V_h; for instance, those introduced in (2.1.3). Assuming that Ω is a polygonal domain with a Lipschitz boundary, the semi-discrete approximate problem reads as follows: given that $f \in L^2(Q_T)$ and $u_{0,h} \in V_h$ being a suitable approximation of the initial datum u_0, find $u_h(t) \in V_h$ such that, for each $t \in (0,T)$,

$$
\left\{
\begin{array}{l}
\dfrac{d}{dt}(u_h(t), v_h) + a^*(u_h(t), v_h) \\
\qquad\qquad = (f(t), v_h) \qquad \forall\ v_h \in V_h \\
\\
u_h(0) = u_{0,h}.
\end{array}
\right. \tag{7.1.9}
$$

Problem (7.1.9) is a system of ordinary differential equations. Writing

$$
u_h(t) = \sum_j \xi_j(t)\varphi_j,
$$

where $\{\varphi_j\}$, $j = 1, \ldots, N_h$, is a basis of V_h, and $u_{0,h} = \sum_j \xi_{0,j}\varphi_j$, problem (7.1.9) can be written as

$$\begin{cases} M\dfrac{d}{dt}\boldsymbol{\xi}(t) + A\boldsymbol{\xi}(t) = \mathbf{f}(t) \\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \end{cases} \tag{7.1.10}$$

where

$$M_{ij} := (\varphi_i, \varphi_j), \quad A_{ij} := a^*(\varphi_j, \varphi_i),$$
$$f_i(t) := (f(t), \varphi_i), \quad i, j = 1, \ldots, N_h.$$

Since the mass matrix M is positive definite, there exists a unique solution $\boldsymbol{\xi}(t)$ of (7.1.10).

Proceeding as for the continuous problem, this semi-discrete problem can be formulated in the following equivalent way, upon setting $u_{i,h} := u_{h|\Omega_i}$, $i = 1, 2$ for each $t > 0$:

$$\begin{cases} \dfrac{d}{dt}(u_{1,h}, v_{1,h})_{\Omega_1} + a_1^*(u_{1,h}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \quad \forall\ v_{1,h} \in V_{1,h}^0 \\ u_{1,h} = u_{2,h} \quad \text{on } \Gamma \\ \dfrac{d}{dt}(u_{2,h}, v_{2,h})_{\Omega_2} + a_2^*(u_{2,h}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \quad \forall\ v_{2,h} \in V_{2,h}^0 \\ \dfrac{d}{dt}(u_{2,h}, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + a_2^*(u_{2,h}, \mathcal{R}_{2,h}\mu_h) \\ \qquad = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\ \qquad\quad - \dfrac{d}{dt}(u_{1,h}, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} - a_1^*(u_{1,h}, \mathcal{R}_{1,h}\mu_h) \quad \forall\ \mu_h \in \Lambda_h, \end{cases} \tag{7.1.11}$$

where $\mathcal{R}_{i,h}$ denotes any possible extension operator from Λ_h to $V_{i,h}$. The initial condition $u_{i,h}(\mathbf{x}, 0) = u_{0,h|\Omega_i}(\mathbf{x})$ has also to be imposed.

The continuous-in-time problem (7.1.11) is at the heart of the *waveform-relaxation* domain decomposition algorithms. Denoting with $Q_{i,T} := \Omega_i \times (0, T)$, $i = 1, 2$, the cylinder identified by the ith subdomain Ω_i, one could devise the following iterative algorithm: given $\lambda_h^k \in \Lambda_h$, find for $k \geq 0$ the solution $u_h^{k+1}(t) \in V_{1,h}$ of

$$\begin{cases} \dfrac{d}{dt}(u_{1,h}^{k+1}, v_{1,h})_{\Omega_1} + a_1^*(u_{1,h}^{k+1}, v_{1,h}) = (f, v_{1,h})_{\Omega_1} \quad \forall\ v_{1,h} \in V_{1,h}^0 \\ u_{1,h}^{k+1} = \lambda_h^k \quad \text{on } \Gamma \\ u_{1,h|t=0}^{k+1} = u_{0,h} \quad \text{in } \Omega_1, \end{cases} \tag{7.1.12}$$

then

$$
\begin{cases}
\dfrac{d}{dt}(u_{2,h}^{k+1}, v_{2,h})_{\Omega_2} + a_2^*(u_{2,h}^{k+1}, v_{2,h}) = (f, v_{2,h})_{\Omega_2} \quad \forall\, v_{2,h} \in V_{2,h}^0 \\
\dfrac{d}{dt}(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + a_2^*(u_{2,h}^{k+1}, \mathcal{R}_{2,h}\mu_h) \\
\qquad = (f, \mathcal{R}_{2,h}\mu_h)_{\Omega_2} + (f, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} \\
\qquad\quad -\dfrac{d}{dt}(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h)_{\Omega_1} - a_1^*(u_{1,h}^{k+1}, \mathcal{R}_{1,h}\mu_h) \quad \forall\, \mu_h \in \Lambda_h \\
u_{2,h|t=0}^{k+1} = u_{0,h} \quad \text{in } \Omega_2,
\end{cases}
\tag{7.1.13}
$$

finally setting

$$
\lambda_h^{k+1} := \theta u_{2,h|\Gamma}^{k+1} + (1-\theta)\lambda_h^k \qquad \text{on } \Gamma. \tag{7.1.14}
$$

At each step one has to solve one parabolic problem in the cylinder $Q_{1,T}$ with a Dirichlet boundary condition on Γ, and one problem in the cylinder $Q_{2,T}$ with a Neumann boundary condition on Γ. Note that k is an iteration level, and not a time level. For every k, continuous-in-time problems in $Q_{1,T}$ and $Q_{2,T}$ need to be fully discretised in time by either implicit or explicit methods.

A dual approach to that previously introduced in (7.1.9) is a continuous-in-space time discretisation of the original parabolic problem. Using for the sake of simplicity the first-order backward Euler scheme, denoting by $\Delta t > 0$ the time-step, $t^n := n\Delta t$ the nth time level, and using the superscript n to refer to the nth time level, we obtain the following problem for each $n \geq 0$:

$$
\begin{cases}
L_{\Delta t} u^{n+1} := (I + \Delta t\, L)\, u^{n+1} = u^n + \Delta t\, f^{n+1} & \text{in } \Omega \\
u^{n+1} = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{7.1.15}
$$

where $u^0 = u_0$. For each n we have a time-independent elliptic boundary value problem, whose weak formulation (for either a single- or multi-domain framework) can be easily devised.

The new operator $L_{\Delta t}$ is better conditioned than L owing to the presence of the identity operator I and the factor Δt. This feature enhances the rate of convergence of any substructuring iterative method that is used for solving (7.1.15) in a multi-domain framework. This desirable property is obviously reflected by the space discretisation of (7.1.15), as discussed in the following section.

7.1.2 Implicit time discretisation and subdomain iterations

The actual solution of the parabolic initial-boundary value problem is accomplished by a full space–time discretisation. Still for the ease of exposition, let us consider the backward Euler method for time discretisation. We can start from (7.1.15) and approximate the space operator $L_{\Delta t}$ by means of the finite element subspace V_h. This yields the fully discrete problem: given $u_h^0 := u_{0,h}$, for each $n \geq 0$ find $u_h^{n+1} \in V_h$ such that

$$(7.1.16)\quad \left(u_h^{n+1}, v_h\right) + \Delta t\, a^*(u_h^{n+1}, v_h) = (u_h^n, v_h) + \Delta t\, (f^{n+1}, v_h) \qquad \forall\ v_h \in V_h.$$

Obviously the same problem would have been obtained if we had started from (7.1.9), and applied the backward Euler method. At each step (7.1.16) yields a system whose matrix is $L^{\mathrm{FE}}_{h,\Delta t} := M + \Delta t\, A$, M and A being the mass and stiffness matrices, respectively.

Using finite differences rather than finite elements for the space discretisation of (7.1.15) would give the following fully discrete problem:

$$(7.1.17)\qquad L^{\mathrm{FD}}_{h,\Delta t}\mathbf{u}^{n+1} := (I + \Delta t\, L_h)\, \mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t\, \mathbf{f}^{n+1}$$

at all nodes internal to Ω, where L_h is the finite difference discretisation of L and $\mathbf{u}^n$ denotes the vector of grid values at the time level t^n.

The eigenvalues η_j^{FE} of $L^{\mathrm{FE}}_{h,\Delta t}$ satisfy

$$(7.1.18)\qquad C_1 h^d(1 + \Delta t) \le \eta_j^{\mathrm{FE}} \le C_2 h^d(1 + \Delta t\, h^{-2})$$

(see, for example, Quarteroni and Valli 1994, Section 6.3.2). Similarly, the eigenvalues η_j^{FD} of $L^{\mathrm{FD}}_{h,\Delta t}$ satisfy

$$(7.1.19)\qquad C_1^*(1 + \Delta t) \le \eta_j^{\mathrm{FD}} \le C_2^*(1 + \Delta t\, h^{-2}).$$

Therefore, the condition number $\kappa_{h,\Delta t}$ of both $L^{\mathrm{FE}}_{h,\Delta t}$ and $L^{\mathrm{FD}}_{h,\Delta t}$ is of order $O(1 + \Delta t\, h^{-2})$. If Δt is small, e.g. $\Delta t = O(h)$ or $\Delta t = O(h^2)$, then $\kappa_{h,\Delta t}$ is much smaller than the condition number of L_h.

This more favourable condition number has interesting consequences for domain decomposition algorithms that are used to solve (7.1.16). Precisely, as pointed out by several authors (see, for example, Chan and Mathew 1994*a* and the references therein), in these cases a coarse grid problem may not be required for global communication of information. Equivalently, the singular perturbation character of the operator $L_{h,\Delta t}$ is used to suppress the global communication within each time-step.

For the case of overlapping domain decomposition, observing that the entries in the lth row of the discrete Green function $G_{h,\Delta t} := (I + \Delta t L_h)^{-1}$ decay rapidly as the distance between the nodes $\{\mathbf{x}_s\}$ increases; precisely $|(G_{h,\Delta t})_{ls}| \le \epsilon$ when $|\mathbf{x}_l - \mathbf{x}_s| \ge C\sqrt{\Delta t}\log(\epsilon^{-1})$ (see Kuznetsov 1988, 1991; and Meurant 1991), it is possible to estimate the number of Schwarz iterations that are needed to guarantee that the convergence error is less than ϵ, a tolerance that equals, for instance, the local truncation error (see Kuznetsov 1991).

In particular, one finds that if the minimum size of overlap among subdomains is $O(\sqrt{\Delta t}\log(\epsilon^{-1}))$, then just one subdomain iteration of the Schwarz method is enough to guarantee that the error is $O(\epsilon)$. If $\epsilon = O(h^2)$ and $\Delta t = O(h)$, then the extended subdomains must have a minimum overlap of the size $O(\sqrt{h}|\log h|)$ (see Kuznetsov 1988).

If the Schwarz method is used as preconditioner P_{as} (see (3.5.8)) for problem (7.1.17), then Cai (1991) has proved that $\kappa(P_{\mathrm{as}}^{-1}L_{h,\Delta t})$ is bounded by a constant

independent of h, H and Δt, provided that $\Delta t \leq CH^2$. For larger Δt, the same estimate is obtained for $\kappa(P_{\rm cas}^{-1} L_{h,\Delta t})$ (where $P_{\rm cas}$ has been defined in (3.6.11)). Coarse grid correction is therefore needed to maintain a constant rate of convergence when Δt is large. Similar results hold for the multiplicative Schwarz preconditioner.

Consider now the case of the decomposition of Ω into M subdomains Ω_i without overlapping. We assume that there is a black–white ordering, with black subregions holding a Neumann interface condition and white subregions holding a Dirichlet condition. Let Ω_1 and Ω_2 denote the union of white and black subregions, respectively, and Γ denote the union of all subdomain interfaces. Following Section 2.3, the algebraic system associated with (7.1.16) can be represented as follows:

$$\hat{A}\mathbf{u}^{n+1} = \begin{pmatrix} \hat{A}_{11} & 0 & \hat{A}_{1\Gamma} \\ 0 & \hat{A}_{22} & \hat{A}_{2\Gamma} \\ \hat{A}_{\Gamma 1} & \hat{A}_{\Gamma 2} & \hat{A}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1^{n+1} \\ \mathbf{u}_2^{n+1} \\ \mathbf{u}_\Gamma^{n+1} \end{pmatrix} = \begin{pmatrix} \mathbf{g}_1^{n+1} \\ \mathbf{g}_2^{n+1} \\ \mathbf{g}_\Gamma^{n+1} \end{pmatrix}, \tag{7.1.20}$$

where $\hat{A} := M + \Delta t\, A = L_{h,\Delta t}^{FE}$, $A_{\Gamma i} = A_{i\Gamma}^T$, $i = 1, 2$, and $\mathbf{g}^{n+1} := M\mathbf{u}^n + \Delta t\, \mathbf{f}^{n+1}$. The matrix $\hat{A}_{11}$ represents the coupling between pairs of degrees of freedom associated with the set Ω_1, $\hat{A}_{1\Gamma}$ the coupling between pairs associated with Ω_1 and the interface Γ, and the other entries of $\hat{A}$ are defined similarly.

Let $\hat{\Sigma}_h$ denote the Schur complement of $\hat{A}$ with respect to $\hat{A}_{\Gamma\Gamma}$. Then (see (2.3.9), (2.3.10))

$$\hat{\Sigma}_h = \hat{\Sigma}_{1,h} + \hat{\Sigma}_{2,h},$$

with

$$\hat{\Sigma}_{i,h} := \hat{A}_{\Gamma\Gamma}^{(i)} - \hat{A}_{\Gamma i} \hat{A}_{ii}^{-1} \hat{A}_{i\Gamma}.$$

The matrices $\hat{A}_{\Gamma\Gamma}^{(i)}$ are such that $\hat{A}_{\Gamma\Gamma} = \hat{A}_{\Gamma\Gamma}^{(1)} + \hat{A}_{\Gamma\Gamma}^{(2)}$; for the stiffness matrix A, they have been introduced in (2.3.3).

In the two-dimensional finite element case (7.1.16), Dryja (1991) has proved that

$$\kappa(\hat{\Sigma}_{2,h}^{-1} \hat{\Sigma}_h) \leq C \left(1 + \log \frac{H}{h}\right)^2, \tag{7.1.21}$$

where $C > 0$ is a constant independent of h, H and Δt. The right-hand side of (7.1.21) governs, therefore, the rate of convergence of the Dirichlet–Neumann subdomain iterations (see Sections 2.3 and 3.2).

As pointed out in Section 2.3.1, knowledge of the preconditioner $\hat{\Sigma}_{2,h}$ for $\hat{\Sigma}_h$ allows the setting up of a preconditioner $\hat{Q}_h$ for the whole matrix $\hat{A}$. Following Section 2.3.1, the construction of $\hat{Q}_h$ is carried out as follows. Since $\hat{A} = \hat{L}\hat{U}$, where

$$\hat{L} := \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ \hat{A}_{\Gamma 1}\hat{A}_{11}^{-1} & \hat{A}_{\Gamma 2}\hat{A}_{22}^{-1} & I_\Gamma \end{pmatrix}$$

$$\hat{U} := \begin{pmatrix} \hat{A}_{11} & 0 & \hat{A}_{1\Gamma} \\ 0 & \hat{A}_{22} & \hat{A}_{2\Gamma} \\ 0 & 0 & \hat{\Sigma}_{1,h} + \hat{\Sigma}_{2,h} \end{pmatrix},$$

we can define

$$\hat{Q}_h := \hat{L}\widetilde{U}, \tag{7.1.22}$$

with

$$\widetilde{U} := \begin{pmatrix} \hat{A}_{11} & 0 & \hat{A}_{1\Gamma} \\ 0 & \hat{A}_{22} & \hat{A}_{2\Gamma} \\ 0 & 0 & \hat{\Sigma}_{2,h} \end{pmatrix}. \tag{7.1.23}$$

The preconditioned stiffness matrix $\hat{Q}_h^{-1}\hat{A}$ has the same eigenvalues as the matrix $\hat{\Sigma}_{2,h}^{-1}\hat{\Sigma}_h$, plus the eigenvalue 1 (see Section 2.3.1), therefore

$$\kappa(\hat{Q}_h^{-1}\hat{A}) \leq C\left(1 + \log\frac{H}{h}\right)^2.$$

Using the preconditioner $\hat{Q}_h$ in a conjugate gradient method ensures that a solution with accuracy ϵ can be obtained after an order of $(\log\frac{1}{\epsilon})(1 + \log\frac{H}{h})$ iterations. For solution algorithms to be used at each iteration see Dryja (1991).

Remark 7.1.1 Substructuring methods for space–time discontinuous finite elements are reported in Lube *et al.* (1998). A Neumann–Neumann iterative method is described in Dryja (1991). For other implicit methods see Israeli *et al.* (1993). □

Alternative approaches to fully implicit methods stem from the idea of computing first the interface values $u_{h|\Gamma}^{n+1}$ on the interface Γ, using an explicit scheme or even an implicit scheme in a small neighbourhood of Γ. Then these boundary values are used at the new time level to provide Dirichlet data for independent subproblems to be solved in each subdomain. We term these methods *predictor–corrector* domain decomposition methods.

Perhaps the most natural predictor–corrector domain decomposition method is easily illustrated on the two-domain formulation (7.1.11). For $i = 1, 2$ denote by $\varphi_s^{\Omega_i}$, $s = 1, \ldots, N_i$, the basis functions supported in Ω_i, and by φ_l^Γ, $l = 1, \ldots, N_\Gamma$, the basis functions associated with the nodes on Γ. Given u_h^n defined in Ω, for $i = 1, 2$ seek the solutions

$$\tilde{u}_{i,h}^{n+1} = \sum_{l=1}^{N_\Gamma} \tilde{u}_{i,l}^{n+1}\varphi_l^\Gamma + \sum_{s=1}^{N_i} u_{i,s}^n \varphi_s^{\Omega_i} \tag{7.1.24}$$

of

$$\sum_{i=1}^{2}\left(\frac{\tilde{u}_{i,h}^{n+1}-u_{i,h}^{n}}{\Delta t},\pi_{i,h}\mu_h\right)_{\Omega_i}+a_i^*(\tilde{u}_{i,h}^{n+1},\pi_{i,h}\mu_h)=\sum_{i=1}^{2}(f^{n+1},\pi_{i,h}\mu_h)_{\Omega_i}\quad\forall\ \mu_h\in\Lambda_h, \tag{7.1.25}$$

together with the matching condition

$$\tilde{u}_{1,h|\Gamma}^{n+1}-\tilde{u}_{2,h|\Gamma}^{n+1}=0, \tag{7.1.26}$$

where $\pi_{i,h}\mu_h$ denotes the interpolant of μ_h in Ω_i, $i=1,2$.

These equations provide a reduced system whose solution yields the set of new variables $u_{h|\Gamma}^{n+1}$ $(=\tilde{u}_{2,h|\Gamma}^{n+1}=\tilde{u}_{1,h|\Gamma}^{n+1})$ on Γ. After this predictor step, we can advance $(7.1.11)_1$ and $(7.1.11)_3$ and use $u_{h|\Gamma}^{n+1}$ as the Dirichlet datum for the two subproblems.

The splitting introduced in (7.1.24) permits us to decouple the computation of the solution within the subdomains. However, the solution generated by this scheme does not coincide with that of the original finite element problem (7.1.16).

For the implementation of a similar predictor–corrector domain decomposition method in the context of finite differences, see Kuznetsov (1988).

On the basis of this idea, instead of (7.1.25) Dawson and Du (1991) and Dawson *et al.* (1991) use a similar equation in which the test functions are defined on a larger stencil than that of μ_h. This is an explicit scheme on a coarser mesh. This enlarged support guarantees a better coupling and, consequently, a stability condition on the time-step less strict than that of the fully explicit scheme.

Another predictor–corrector domain decomposition procedure, proposed by Blum *et al.* (1992), is concerned with overlapping partitions. Setting $\Gamma_i:=\partial\Omega_i\setminus\partial\Omega$, the predictor step consists now of generating preliminarily the values $\hat{u}_{h|\Gamma_i}^{n+1}$, $i=1,2$, at the new time level t^{n+1}, by either a linear extrapolation

$$\hat{u}_{h|\Gamma_i}^{n+1}:=2u_{h|\Gamma_i}^{n}-u_{h|\Gamma_i}^{n-1}$$

or by a quadratic extrapolation

$$\hat{u}_{h|\Gamma_i}^{n+1}:=3u_{h|\Gamma_i}^{n}-3u_{h|\Gamma_i}^{n-1}+u_{h|\Gamma_i}^{n-2}$$

from previous time levels. Then a second-order Crank–Nicolson scheme is used to discretise equations (7.1.11) in each subdomain, using the previously computed values as Dirichlet data. This step is completely parallel. Finally, at a third step, a global single-valued function $u_h^{n+1}\in V_h$ is constructed from the patchwise solutions $u_{i,h}^{n+1}$ through a suitable averaging process between the two solutions computed in the overlapping region.

This method is stable under a step-size condition of the form

$$\Delta t \leq \frac{1}{\alpha} K_\beta h^2,$$

where $K_\beta \simeq K_0 \beta^2 |\log \beta|^{-2}$ is independent of the number of subdomais, $K_0 \simeq 2$, βh is the size of the overlapping zones and α is the coerciveness constant of the bilinear form $a^*(\cdot,\cdot)$.

7.2 Hyperbolic problems

Hyperbolic equations provide the mathematical model for the description of wave propagation processes in many different areas of application; notably, gas dynamics, geophysics, electromagnetism, and the dynamics of elastic structures.

A mathematical feature common with hyperbolic problems is that information propagates along characteristic curves.

Partitioning the spatial domain into subdomains is especially well suited for dealing with heterogeneous media, because the latter yield sudden changes of the wave speeds when crossing the interfaces of different materials.

We illustrate the basic mathematical concepts behind domain decomposition, first for an advection equation, then also for problems arising from propagation of acoustic and elastic waves. For each problem we describe the associated initial-boundary value problem, as well as its multi-domain formulation. Then we show how domain decomposition algorithms can be derived after time discretisation.

The first initial-boundary value problem we consider is a simple scalar conservation law (see also Section 5.6)

$$\begin{cases} \dfrac{\partial u}{\partial t} + \operatorname{div}(\mathbf{b}u) = f & \text{in } Q_T := \Omega \times (0,T) \\ u = \varphi_D & \text{on } \partial\Omega^{\text{in}} \times (0,T) \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases} \tag{7.2.1}$$

where $T > 0$ is an upper time level, f, φ_D, u_0 and $\mathbf{b}$ are given functions, and

$$\partial\Omega^{\text{in}} := \{\mathbf{x} \in \partial\Omega \mid \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) < 0\}$$

is the inflow boundary.

Assuming that $\mathbf{b}$ is smooth enough, and defining the characteristic curves $\boldsymbol{\xi}(t)$ in the $(\mathbf{x},t)$-space as the solution to

$$\begin{cases} \boldsymbol{\xi}'(t) = \mathbf{b}(\boldsymbol{\xi}(t),t) & \text{for } 0 < t < T \\ \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \end{cases}$$

it follows that

$$\frac{d}{dt}[u(\boldsymbol{\xi}(t),t)] = f(\boldsymbol{\xi}(t),t) - (\operatorname{div}\mathbf{b})(\boldsymbol{\xi}(t),t)\, u(\boldsymbol{\xi}(t),t).$$

In the case in which $f = 0$ and $\mathbf{b}$ is divergence free, the solution u is constant along the characteristics and it is simply given by the wave travelling with local speed $\mathbf{b}$.

The unknown u can represent different physical variables, such as, for example, the concentration of a certain quantity.

We now consider the case of a second-order hyperbolic equation that models acoustic waves. In a bounded inhomogeneous medium Ω the acoustic dilatation is represented by the solution $u(\mathbf{x}, t)$ (the pressure field) to the governing equation

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho c^2}\frac{\partial u}{\partial t}\right) - \operatorname{div}\left(\frac{1}{\rho}\nabla u\right) = f \qquad \text{in } Q_T, \tag{7.2.2}$$

supplemented by the initial conditions

$$u_{|t=0} = u_0, \quad \frac{\partial u}{\partial t}_{|t=0} = v_0 \qquad \text{in } \Omega, \tag{7.2.3}$$

and by suitable boundary conditions on the whole boundary of Ω, which, for simplicity of exposition, we will assume to be of a Dirichlet type, i.e.

$$u = \varphi_D \qquad \text{on } \partial\Omega \times (0, T). \tag{7.2.4}$$

In (7.2.2) $\rho(\mathbf{x})$ is the density, $c(\mathbf{x})$ the wave velocity, and $f(\mathbf{x}, t)$ the source forcing term. Finally, $u_0(\mathbf{x})$, $v_0(\mathbf{x})$ and $\varphi_D(\mathbf{x}, t)$ are given data.

In practical geological applications, the Dirichlet condition (7.2.4) is often replaced on certain portions of $\partial\Omega$ by a Neumann condition that holds at the free surface, or by more complex conditions (the absorbing boundary conditions) involving a combination between time and space derivatives to be enforced at fictitious (non-physical) boundaries (see, for example, Engquist and Majda 1977; Collino 1993).

The third example concerns elastic waves propagating in an elastic structure with constant thickness in the regime of small strains and small displacements. Denoting by $\mathbf{w} = (w_1, \ldots, w_d)$ the displacement field, by

$$\boldsymbol{\varepsilon}(\mathbf{w}) = (\varepsilon_{lj}(\mathbf{w})) = \left(\frac{1}{2}(D_l w_j + D_j w_l)\right), \quad l, j = 1, \ldots, d, \tag{7.2.5}$$

the strain field and by $\boldsymbol{\sigma}(\mathbf{w}) = (\sigma_{lj}(\mathbf{w}))$ the stress field produced in the body by the application of known distributions of external actions, the equilibrium equations read:

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{f} \qquad \text{in } Q_T \tag{7.2.6}$$

where

$$(\mathbf{div}\,\boldsymbol{\sigma}(\mathbf{w}))_l := \sum_{j=1}^{d} \frac{\partial \sigma_{lj}(\mathbf{w})}{\partial x_j}, \quad l = 1, \ldots, d.$$

The constitutive law provides the stress components as

$$\sigma_{lj}(\mathbf{w}) = 2\hat{\mu}\varepsilon_{lj}(\mathbf{w}) + \hat{\lambda}\,\mathrm{div}\,\mathbf{w}\,\delta_{lj}, \quad l, j = 1, \ldots, d, \tag{7.2.7}$$

where $\hat{\mu} > 0$ and $\hat{\lambda} \geq 0$ are the Lamé's constants (see Section 5.2).

The differential equations (7.2.6) need to be supplemented by initial conditions, say

$$\mathbf{u}_{|t=0} = \mathbf{u}_0 \ \text{ and } \ \frac{\partial \mathbf{u}}{\partial t}_{|t=0} = \mathbf{v}_0 \qquad \text{in } \Omega, \tag{7.2.8}$$

and by boundary conditions that for simplicity we will assume to be of a Dirichlet type

$$\mathbf{u} = \boldsymbol{\varphi}_D \qquad \text{on } \partial\Omega \times (0, T). \tag{7.2.9}$$

One of the most common solution strategies leads to the so-called *displacement formulation* of the stress analysis problem (7.2.6), taking into account (7.2.5) and the constitutive equations (7.2.7). The resulting system reads

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} + \mathbf{L}\mathbf{u} = \mathbf{f} \qquad \text{in } Q_T \tag{7.2.10}$$

with

$$\mathbf{L}\mathbf{w} := -\hat{\mu}\Delta\mathbf{w} - (\hat{\mu} + \hat{\lambda})\nabla\,\mathrm{div}\,\mathbf{w}. \tag{7.2.11}$$

7.2.1 Multi-domain formulation

For the sake of exposition we still assume that Ω is partitioned into two subdomains. Having defined, as in Section 5.6, that $\partial\Omega_i^{\mathrm{in}} := \partial\Omega^{\mathrm{in}} \cap \partial\Omega_i$, a multi-domain formulation of problem (7.2.1) reads as follows: the restriction u_i of u to Ω_i satisfies

$$\begin{cases} \dfrac{\partial u_i}{\partial t} + \mathrm{div}(\mathbf{b}u_i) = f & \text{in } \Omega_i \times (0, T) \\ u_i = \varphi_D & \text{on } \partial\Omega_i^{\mathrm{in}} \times (0, T) \\ u_{i|t=0} = u_0 & \text{in } \Omega_i \end{cases} \tag{7.2.12}$$

for $i = 1, 2$, as well as the following matching condition at the interface Γ:

$$(\mathbf{b} \cdot \mathbf{n})u_1 = (\mathbf{b} \cdot \mathbf{n})u_2 \qquad \text{on } \Gamma \times (0, T), \tag{7.2.13}$$

as for the steady equation (see (5.6.4)). This condition implies that $u_1 = u_2$ at all points of Γ except those where the flow field $\mathbf{b}$ is tangential to the interface.

Referring for notation to Section 5.6, the weak multi-domain problem corresponding to (7.2.12), (7.2.13) reads: find $u_i \in C^0([0,T];X_{i,b})$ such that

$$
\begin{cases}
\dfrac{d}{dt}(u_i(t),v_i)_{\Omega_i} + \mathcal{C}_i(u_i(t),v_i) = (f(t),v_i)_{\Omega_i} \\
\qquad\qquad +(\varphi_D(t), v_{i|\partial\Omega_i^{\rm in}})_{b,\partial\Omega_i^{\rm in}} \qquad \forall\ v_i \in X_{i,b}^0 \\
(u_{1|\Gamma^{\rm in}}(t) - u_{2|\Gamma^{\rm in}}(t),\mu)_{b,\Gamma^{\rm in}} + (u_{2|\Gamma^{\rm out}}(t) - u_{1|\Gamma^{\rm out}}(t),\mu)_{b,\Gamma^{\rm out}} = 0 \\
\qquad\qquad \forall\ \mu \in L_b^2(\Gamma^{\rm in}\cup\Gamma^{\rm out}),\quad \forall\ t\in[0,T] \\
u_i(0) = u_{0|\Omega_i},
\end{cases}
\tag{7.2.14}
$$

where equations (7.2.14) have to be understood in the sense of distributions in $(0,T)$. The interface equation in $(7.2.14)_2$ is equivalent to the flux-continuity condition (7.2.13), this latter being satisfied in a weak sense.

The multi-domain version of the acoustics problem (7.2.2)–(7.2.4) is formulated as follows:

$$
\begin{cases}
\dfrac{\partial}{\partial t}\left(\dfrac{1}{\rho c^2}\dfrac{\partial u_i}{\partial t}\right) - \operatorname{div}\left(\dfrac{1}{\rho}\nabla u_i\right) = f & \text{in } \Omega_i\times(0,T) \\
u_{i|t=0} = u_0,\quad \dfrac{\partial u_i}{\partial t}_{|t=0} = v_0 & \text{in } \Omega_i \\
u_i = \varphi_D & \text{on } (\partial\Omega_i\cap\partial\Omega)\times(0,T)
\end{cases}
\tag{7.2.15}
$$

for $i=1,2$, supplemented by the following transmission conditions at subdomain interfaces, which are inherited from the (elliptic) structure of the spatial operator $-\operatorname{div}(\frac{1}{\rho}\nabla)$:

$$
\begin{aligned}
& u_1 = u_2 && \text{on } \Gamma\times(0,T) \\
& \frac{1}{\rho_{|\Omega_1}}\frac{\partial u_1}{\partial n} = \frac{1}{\rho_{|\Omega_2}}\frac{\partial u_2}{\partial n} && \text{on } \Gamma\times(0,T).
\end{aligned}
\tag{7.2.16}
$$

This time, besides the continuity of the solution (see $(7.2.16)_1$), we have required continuity of its normal flux as well (see $(7.2.16)_2$). Note the analogy with the interface conditions $(1.1.1)_3$, $(1.1.1)_4$ for elliptic operators.

Assuming for simplicity that $\varphi_D = 0$, the weak multi-domain formulation of (7.2.15), (7.2.16) reads: find $u_i \in C^0([0,T];V_i)\cap C^1([0,T];L^2(\Omega_i))$ such that

$$
\begin{cases}
\dfrac{d^2}{dt^2}\left(\dfrac{1}{\rho c^2}u_i(t), v_i\right)_{\Omega_i} + \left(\dfrac{1}{\rho}\nabla u_i(t), \nabla v_i\right)_{\Omega_i} \\
\qquad\qquad = (f(t), v_i)_{\Omega_i} \qquad \forall\ v_i \in V_i^0 \\
u_1(t) = u_2(t) \qquad \text{on } \Gamma, \quad \forall\ t \in [0,T] \\
\displaystyle\sum_{i=1}^{2}\left\{\frac{d^2}{dt^2}\left(\frac{1}{\rho c^2}u_i(t), \mathcal{R}_i\mu\right)_{\Omega_i} + \left(\frac{1}{\rho}\nabla u_i(t), \nabla\mathcal{R}_i\mu\right)_{\Omega_i}\right\} \\
\qquad\qquad = \displaystyle\sum_{i=1}^{2}(f(t), \mathcal{R}_i\mu)_{\Omega_i} \qquad \forall\ \mu \in \Lambda \\
u_i(0) = u_{0|\Omega_i}, \quad \dfrac{\partial u_i}{\partial t}(0) = v_{0|\Omega_i},
\end{cases}
\tag{7.2.17}
$$

where V_i, V_i^0 and Λ have been defined in (1.2.4), and $\mathcal{R}_i$ indicates any possible extension operator from Λ to V_i. Equations $(7.2.17)_1$, $(7.2.17)_3$ have to be understood in the sense of distributions in $(0,T)$. In particular, $(7.2.17)_3$ is equivalent to the normal flux condition $(7.2.16)_2$.

Finally, let us turn to the linear elasticity problem. Denoting by $\mathbf{u}_i$ the restriction of the horizontal displacements $\mathbf{u}$ to Ω_i, $i = 1, 2$, it can be written in the following equivalent multi-domain form:

$$
\begin{cases}
\dfrac{\partial^2 \mathbf{u}_i}{\partial t^2} - \mathbf{L}\mathbf{u}_i = \mathbf{f} & \text{in } \Omega_i \times (0,T) \\
\mathbf{u}_{i|t=0} = \mathbf{u}_0, \quad \dfrac{\partial \mathbf{u}_i}{\partial t}_{|t=0} = \mathbf{v}_0 & \text{in } \Omega_i \\
\mathbf{u}_i = \boldsymbol{\varphi}_D & \text{on } (\partial\Omega_i \cap \partial\Omega) \times (0,T)
\end{cases}
\tag{7.2.18}
$$

for $i = 1, 2$, with the interface transmission conditions (see (5.2.2))

$$
\begin{aligned}
&\mathbf{u}_1 = \mathbf{u}_2 && \text{on } \Gamma \times (0,T) \\
&\sum_{j=1}^{d}\sigma_{lj}(\mathbf{u}_1)n_j = \sum_{j=1}^{d}\sigma_{lj}(\mathbf{u}_2)n_j && \text{on } \Gamma \times (0,T), \quad l = 1, \ldots, d,
\end{aligned}
\tag{7.2.19}
$$

which enforce the continuity of displacements as well as that of normal stresses at the interface of the subdomains.

In order to restate (7.2.18) in a weak form, we introduce the following definition (for $i = 1, 2$):

$$
e_i(\mathbf{w}_i, \mathbf{v}_i) := 2\hat{\mu}\sum_{l,j=1}^{d}\int_{\Omega_i}\varepsilon_{lj}(\mathbf{w}_i)\,\varepsilon_{lj}(\mathbf{v}_i) + \hat{\lambda}\int_{\Omega_i}\operatorname{div}\mathbf{w}_i\operatorname{div}\mathbf{v}_i.
\tag{7.2.20}
$$

Note that $e_i(\cdot,\cdot)$ is the bilinear form associated with the operator $\mathbf{L}$ (see (7.2.11)) restricted to Ω_i.

If we assume, for simplicity, that $\boldsymbol{\varphi}_D = \mathbf{0}$ on $\partial\Omega$ in (7.2.9) (i.e. we consider the homogeneous Dirichlet boundary condition), then problem (7.2.18), (7.2.19) can be reformulated as follows: find $\mathbf{u}_i \in (C^0([0,T];V_i) \cap C^1([0,T];L^2(\Omega_i)))^d$ such that

$$
\text{(7.2.21)} \qquad \begin{cases} \dfrac{d^2}{dt^2}(\mathbf{u}_i(t),\mathbf{v}_i)_{\Omega_i} + e_i(\mathbf{u}_i(t),\mathbf{v}_i) = (\mathbf{f}(t),\mathbf{v}_i)_{\Omega_i} \qquad \forall\ \mathbf{v}_i \in (V_i^0)^d \\ \mathbf{u}_1(t) = \mathbf{u}_2(t) \qquad \text{on } \Gamma, \quad \forall\ t \in [0,T] \\ \displaystyle\sum_{i=1}^{2} \left\{ \frac{d^2}{dt^2}(\mathbf{u}_i(t),\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} + e_i(\mathbf{u}_i(t),\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}) \right\} \\ \qquad\qquad = \displaystyle\sum_{i=1}^{2} (\mathbf{f}(t),\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d \\ \mathbf{u}_i(0) = \mathbf{u}_{0|\Omega_i}, \quad \dfrac{\partial \mathbf{u}_i}{\partial t}(0) = \mathbf{v}_{0|\Omega_i}, \end{cases}
$$

where $\boldsymbol{\mathcal{R}}_i$ indicates any possible extension operator from $(\Lambda)^d$ to $(V_i)^d$, the spaces V_i, V_i^0 and Λ are defined in (1.2.4), and equations $(7.2.21)_1$, $(7.2.21)_3$ have to be understood in the sense of distributions in $(0,T)$.

For each $i = 1, 2$, equation $(7.2.21)_1$ is equivalent to $(7.2.18)_1$, while $(7.2.21)_3$ is the counterpart of the stress continuity equations $(7.2.19)_2$.

Similarly to what we have done in Section 7.1 for parabolic problems, a spatial discretisation of the problems above can be realised, for example, by Galerkin or finite difference methods. The continuous-in-time, semi-discrete version of the multi-domain formulations are straightforward to deduce.

7.2.2 Implicit time discretisation and subdomain iterations

We are interested in the case of implicit time discretisation of the semi-discrete problems that we introduced at the end of the previous section, and their solution in the framework of subdomain iterations.

As before, it is customary to use finite difference schemes for the temporal derivative; however, owing to the hyperbolic nature of the original problem, discontinuous finite elements with least-square stabilisation offer an interesting alternative, see, for example, Shakib and Hughes (1991), and also Lube *et al.* (1998).

As usual, let us introduce a time-step $\Delta t > 0$ and the time levels $t^n = n\Delta t$ with $n = 0, \ldots, \mathcal{N}$ and $t^{\mathcal{N}} = T$, and consider first the advection problem (7.2.12). With the aim of simplifying our notation, we do not use the superscript n to identify the current time level, and simply denote by u_i the function u_i^{n+1}.

Advancing from t^n to t^{n+1} $(n \geq 0)$, (7.2.12) becomes

$$\begin{cases} a_0 u_i + \operatorname{div}(\mathbf{b} u_i) = F & \text{in } \Omega_i, \quad i = 1,2 \\ u_i = \varphi_D & \text{on } \partial\Omega^{\text{in}} \cap \partial\Omega_i, \quad i = 1,2 \\ (\mathbf{b}\cdot\mathbf{n}) u_1 = (\mathbf{b}\cdot\mathbf{n}) u_2 & \text{on } \Gamma, \end{cases} \tag{7.2.22}$$

where $a_0 > 0$ is proportional to the inverse of Δt, and F depends on f and u_i at previous time levels.

The coupled problem (7.2.22) can be solved iteratively as illustrated in Section 5.6.2 by alternating a boundary value problem in Ω_1 with one in Ω_2. Perhaps the simplest way is to construct two sequences of functions $\{u_1^k\}$ and $\{u_2^k\}$, $k \geq 0$, that satisfy:

$$\begin{cases} a_0 u_i^{k+1} + \operatorname{div}(\mathbf{b}_i u_i^{k+1}) = F & \text{in } \Omega_i \\ u_i^{k+1} = \varphi_D & \text{on } \partial\Omega^{\text{in}} \cap \partial\Omega_i \\ u_i^{k+1} = u_{[i]}^k & \text{on } \Gamma_i^{\text{in}}, \end{cases} \tag{7.2.23}$$

for $i = 1, 2$, where

$$[i] := \begin{cases} 2 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \end{cases},$$

and Γ_i^{in} is the portion of Γ on which $\mathbf{b}$ is pointing into Ω_i.

For each k we have, therefore, two independent inflow–outflow boundary value problems to be solved, one in Ω_1, the other in Ω_2.

As for the parabolic case, the smaller Δt, the larger a_0, and therefore the convergence rate is higher.

Had Ω been decomposed into $M > 2$ subdomains, the conclusion would be the same; namely, M (rather than two) inflow–outflow independent subproblems have to be solved at each iteration step.

Remark 7.2.1 An additive Schwarz method for the Galerkin approximation of the advection equation (7.2.1) is analysed in Wu *et al.* (1998), where convergence is proved under the time-step limitation $\Delta t = O(h)$. □

Let us turn now to the acoustic wave problem (7.2.15), (7.2.16). The second-order time derivative can be discretised by implicit finite differences, e.g. the two-step, second-order backward differences (Gear 1971) or the family of one-step Newmark schemes (e.g. Raviart and Thomas 1983) that include either first- or second-order methods. In all cases, after advancing from t^n to t^{n+1}, if we keep denoting by u_i the updated function $u_i(t^{n+1})$, then we are left with the new problem:

$$
\begin{cases}
a_0 u_i - \operatorname{div}\left(\dfrac{1}{\rho}\nabla u_i\right) = F & \text{in } \Omega_i, \quad i = 1,2 \\
u_i = \varphi_D & \text{on } \partial\Omega_i \cap \partial\Omega, \quad i = 1,2 \\
u_1 = u_2 & \text{on } \Gamma \\
\dfrac{1}{\rho_{|\Omega_1}}\dfrac{\partial u_1}{\partial n} = \dfrac{1}{\rho_{|\Omega_2}}\dfrac{\partial u_2}{\partial n} & \text{on } \Gamma,
\end{cases} \tag{7.2.24}
$$

where $a_0 > 0$ depends on the finite difference scheme that is used for the discretisation of the time derivative, and is proportional to $(\Delta t)^{-2}$.

At each time level this two-domain formulation is clearly equivalent to a second-order elliptic boundary value problem in Ω, which can be therefore solved by any one of the domain decomposition algorithms introduced in Sections 1.3 and 1.4. If conforming finite elements are used for the space discretisation, then the matrix associated with (7.2.24) is $L^{FE}_{h,\Delta t} = M + a_0^{-1}A$, where M and A are the mass and stiffness matrix, respectively. Its eigenvalues are bounded as in (7.1.18), with Δt replaced by a_0^{-1}. Thus, the spectral condition number of $L^{FE}_{h,\Delta t}$ is bounded by $C(1 + (\Delta t)^2 h^{-2})$, and no coarse grid is needed, as already noted for parabolic problems.

An alternative point of view consists of reformulating the original initial-boundary value problem (7.2.2)–(7.2.4) as a *first-order hyperbolic system* with constant coefficients.

Before analysing the specific system obtained by (7.2.24), let us consider the following one-dimensional problem

$$
\frac{\partial \mathbf{U}}{\partial t} + A\frac{\partial \mathbf{U}}{\partial x} = \mathbf{F} \qquad \text{in } \Omega \times (0,T), \tag{7.2.25}
$$

where Ω is an interval, $\mathbf{U} = (U_1, \ldots, U_p)$ and A is a $p \times p$ matrix with constant coefficients. This system is said to be *hyperbolic* if A is diagonalisable with real eigenvalues, so that one can consider the decomposition

$$
A = L^{-1}\Lambda L, \tag{7.2.26}
$$

where $\Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_p)$ is the diagonal matrix of eigenvalues, and L is the matrix whose rows are given by the left eigenvectors of A, i.e.

$$
\mathbf{l}^r A = \lambda_r \mathbf{l}^r, \quad r = 1, \ldots, p.
$$

Introducing the *characteristic variables* $\tilde{\mathbf{U}} := L\mathbf{U}$ in (7.2.25) gives

$$
\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \Lambda\frac{\partial \tilde{\mathbf{U}}}{\partial x} = L\mathbf{F}. \tag{7.2.27}
$$

This decouples into p independent scalar advection equations

$$\frac{\partial \tilde{U}_r}{\partial t} + \lambda_r \frac{\partial \tilde{U}_r}{\partial x} = (L\mathbf{F})_r, \quad r = 1, \ldots, p. \tag{7.2.28}$$

The curves $X(t) = x_0 + \lambda_r t$ of the (x,t)-plane satisfying $X'(t) = \lambda_r$ are the r-characteristics. Any characteristic variable $\tilde{U}_r$ is constant along each corresponding r-characteristic, hence the value of $\tilde{U}_r$ at any point (x,t) can be recovered from the data of the problem tracing back the characteristic line passing through (x,t).

Let us return to problem (7.2.24). Introducing the new set of unknowns $\mathbf{U} := (D_t u, D_1 u, \ldots, D_d u)$ (with $D_l := \frac{\partial}{\partial x_l}$, $l = 1, \ldots, d$, $D_t := \frac{\partial}{\partial t}$), setting $\mathbf{F} := (f, 0, \ldots, 0)$ and assuming for the sake of simplicity $\rho = c = 1$, we obtain

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{l=1}^{d} A^{(l)} D_l \mathbf{U} = \mathbf{F} \qquad \text{in } Q_T, \tag{7.2.29}$$

where $A^{(l)}$, $l = 1, \ldots, d$, are $(d+1)\times(d+1)$ symmetric matrices with the following entries: $a^{(l)}_{l+1,1} = a^{(l)}_{1,l+1} = -1$, $a^{(l)}_{rs} = 0$ otherwise. Initial and boundary conditions are derived accordingly.

The multi-domain version of (7.2.29) can be easily obtained after generalising (7.2.12), (7.2.13), which were derived for the scalar advection equation. Let us present this procedure for a general hyperbolic system of the form (7.2.29), where $A^{(l)}$, $l = 1, \ldots, d$, are $p \times p$ matrices (possibly with variable coefficients) such that for each direction $\boldsymbol{\xi} \in \mathbf{R}^d$ the matrix $\sum_l \xi_l A^{(l)}$ is diagonalisable with real eigenvalues (this is surely the case if each matrix $A^{(l)}$ is symmetric).

For any point $\mathbf{x} \in \Gamma$ define the characteristic matrix $C = C(\mathbf{n}) = \sum_l n_l A^{(l)}$, where, as usual, $\mathbf{n}$ is the unit normal vector on Γ, directed from Ω_1 to Ω_2. By the hyperbolicity assumption, C can be diagonalised as $\Lambda = LCL^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_r)$ with $\lambda_r \in \mathbf{R}$, $r = 1, \ldots, p$, and L is the matrix whose rows are the left eigenvectors of C.

With the usual notational convention, the restrictions $\mathbf{U}_i$ of $\mathbf{U}$ to Ω_i, $i = 1, 2$, satisfy

$$\frac{\partial \mathbf{U}_i}{\partial t} + \sum_{l=1}^{d} A^{(l)} D_l \mathbf{U}_i = \mathbf{F} \qquad \text{in } \Omega_i \times (0,T), \quad i = 1, 2, \tag{7.2.30}$$

$$C(\mathbf{n})\,\mathbf{U}_1 = C(\mathbf{n})\,\mathbf{U}_2 \qquad \text{on } \Gamma \times (0,T). \tag{7.2.31}$$

Since we can rewrite (7.2.29) in the conservative form

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{l=1}^{d} D_l(A^{(l)}\mathbf{U}) = \mathbf{F} - \sum_{l=1}^{d} (D_l A^{(l)})\mathbf{U} \qquad \text{in } Q_T,$$

the interface condition (7.2.31) is indeed a matching condition for the normal flux.

When we iterate between the subdomains, this condition ought to be split into incoming and outgoing characteristics. For this reason, for $i = 1, 2$ we introduce the variables $\tilde{\mathbf{U}}_i := L\mathbf{U}_i$ (which coincide with the characteristic variables if C is a constant matrix), and distinguish between positive and negative eigenvalues. Assume, for example, that $\lambda_r > 0$ for $r \leq q$ and $\lambda_r < 0$ if $r > q$ for a suitable $0 \leq q \leq p$ (in particular, C is non-singular). Then (7.2.31) can be written equivalently as

$$\begin{aligned} &\tilde{U}_{1,r} = \tilde{U}_{2,r} \qquad \text{for } r > q \\ &\tilde{U}_{2,r} = \tilde{U}_{1,r} \qquad \text{for } r \leq q, \end{aligned} \tag{7.2.32}$$

where $\tilde{U}_{i,r}$ denotes the rth component of $\tilde{\mathbf{U}}_i$, for $i = 1, 2$ and $r = 1, \ldots, p$. If some eigenvalue of C is equal to zero, then (7.2.31) is equivalent to imposing the matching of all the components $\tilde{U}_r$ of $L\mathbf{U}$ for which the corresponding eigenvalue λ_r is different from zero.

If (7.2.30) is advanced in time from t^n to t^{n+1} by an implicit finite difference scheme (e.g. the backward Euler method), the resulting boundary value problem at the time level t^{n+1} can be solved by the following subdomain iteration method ($k \geq 0$ is the subdomain iteration counter, while we have omitted the superscript $n + 1$ that indicates the time level):

$$\begin{cases} a_0 \mathbf{U}_1^{k+1} + \displaystyle\sum_{l=1}^{d} A^{(l)} D_l \mathbf{U}_1^{k+1} = \mathbf{G}_1 & \text{in } \Omega_1 \\ \tilde{U}_{1,r}^{k+1} = \tilde{U}_{2,r}^{k} & \text{on } \Gamma\ , \quad r > q \end{cases} \tag{7.2.33}$$

$$\begin{cases} a_0 \mathbf{U}_2^{k+1} + \displaystyle\sum_{l=1}^{d} A^{(l)} D_l \mathbf{U}_2^{k+1} = \mathbf{G}_2 & \text{in } \Omega_2 \\ \tilde{U}_{2,r}^{k+1} = \tilde{U}_{1,r}^{k} & \text{on } \Gamma, \quad r \leq q, \end{cases} \tag{7.2.34}$$

where $a_0 := 1/\Delta t$ and $\mathbf{G}_i := \mathbf{F}_i(t^n) + a_0 \mathbf{U}_i^n$, $i = 1, 2$, $\mathbf{U}_i^n$ being the approximation of $\mathbf{U}_i(t^n)$ obtained at the previous time-step, as the limit of the iteration by subdomain described above.

Note that for both problems (7.2.33) and (7.2.34) we are providing the values of the *incoming* 'characteristic' variables on Γ. These conditions, together with the boundary conditions prescribed on $\partial\Omega$, provide the correct number of boundary conditions to solve both subdomain problems (7.2.33) and (7.2.34).

The convergence of the sequence $\{\mathbf{U}_i^k\}$ to $\mathbf{U}_i$ as $k \to \infty$, for $i = 1, 2$, has been proved in the one-dimensional case for a constant matrix A, by analysing the behaviour of the corresponding characteristic variables $\tilde{\mathbf{U}}_i^k = L\mathbf{U}_i^k$ (see Quarteroni 1990; see also Bjørhus 1995). For a different way of enforcing interface conditions, based on the penalty residual in the framework of finite difference approximations, see Carpenter *et al.* (1997).

Turning now to the elastic wave problem addressed above, it is clear that all methods discussed for the problem of acoustic waves apply to this as well. The change of notation is obvious and the conclusions are quite similar (see, for example, Faccioli *et al.* 1996, 1997).

For this latter case, let us also show how an *explicit* time-advancing scheme can be introduced and implemented in the multi-domain framework. We assume that problem (7.2.21) is discretised in space by a Galerkin finite element method. Formally speaking, this can be easily accomplished if V_i, V_i^0 and Λ are replaced by suitable finite element subspaces, say $V_{i,h}$, $V_{i,h}^0$, and Λ_h. The easiest way to generate these spaces is to start from a master space V_h, which is a classical finite element subspace of V, and then proceed as in (2.1.7) and (2.1.8). Then $\mathbf{u}_i(t)$, $\mathbf{v}_i$ and $\boldsymbol{\mu}$ are respectively replaced by $\mathbf{u}_{i,h}(t)$, $\mathbf{v}_{i,h}$ and $\boldsymbol{\mu}_h$, while $\boldsymbol{\mathcal{R}}_i\boldsymbol{\mu}$ should be replaced by $\boldsymbol{\mathcal{R}}_{i,h}\boldsymbol{\mu}_h \in \mathbf{V}_{i,h}$, a finite element extension of $\boldsymbol{\mu}_h$. With the aim of simplifying the notation we will drop the subscript h everywhere and therefore we will refer to (7.2.21) as the finite element problem.

Concerning the discretisation of the temporal derivative, a classical instance of an explicit scheme is provided by the second-order, *leapfrog* method, which transforms at each time level t^n the problem (7.2.21) into the multi-domain boundary value problem

$$(\delta(\mathbf{u}_i^n), \mathbf{v}_i)_{\Omega_i} + e_i(\mathbf{u}_i^n, \mathbf{v}_i) = (\mathbf{f}(t^n), \mathbf{v}_i)_{\Omega_i} \qquad \forall\ \mathbf{v}_i \in (V_i^0)^d, \quad i = 1, 2 \tag{7.2.35}$$

$$\mathbf{u}_1^{n+1} = \mathbf{u}_2^{n+1} \qquad \text{on } \Gamma \tag{7.2.36}$$

$$\sum_{i=1}^{2}[(\delta(\mathbf{u}_i^n), \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} + e_i(\mathbf{u}_i^n, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})] = \sum_{i=1}^{2}(\mathbf{f}(t^n), \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d, \tag{7.2.37}$$

having set

$$\delta(\mathbf{u}_i^n) := \frac{\mathbf{u}_i^{n+1} - 2\mathbf{u}_i^n + \mathbf{u}_i^{n-1}}{(\Delta t)^2}.$$

An alternative approach consists of coupling (7.2.35), (7.2.36) with the new interface equation

$$\begin{aligned}\sum_{i=1}^{2}[\delta_b(\mathbf{u}_i^{n+1}), \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} + e_i(\mathbf{u}_i^{n+1}, \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})] \\ = \sum_{i=1}^{2}(\mathbf{f}(t^{n+1}), \boldsymbol{\mathcal{R}}_i\boldsymbol{\mu})_{\Omega_i} \qquad \forall\ \boldsymbol{\mu} \in (\Lambda)^d,\end{aligned} \tag{7.2.38}$$

where

$$\delta_b(\mathbf{u}_i^{n+1}) := \frac{2\mathbf{u}_i^{n+1} - 5\mathbf{u}_i^n + 4\mathbf{u}_i^{n-1} - \mathbf{u}_i^{n-2}}{(\Delta t)^2}$$

now denotes the second-order backward implicit discretisation of $\frac{d^2\mathbf{u}}{dt^2}$ at $t = t^{n+1}$.

Since the values $\mathbf{u}_i^{n+1}$ are available at all internal finite element nodes after applying (7.2.35), the new 'implicit' equations (7.2.38) yield for each finite element node $\mathbf{x}_l$ on Γ a simple algebraic equation that provides the common value $\mathbf{u}_1^{n+1}(\mathbf{x}_l) = \mathbf{u}_2^{n+1}(\mathbf{x}_l)$.

Therefore, (7.2.35), (7.2.36) and (7.2.38) provide a '*semi-explicit*' method that has roughly the same computational complexity as the fully explicit method (7.2.35)–(7.2.37).

A dual approach is inspired by the predictor–corrector method illustrated in Section 7.1.2. It consists of first advancing the interface equations (7.2.37) in which $\mathbf{u}_i^{n+1}$ is replaced by $\tilde{\mathbf{u}}_i^{n+1}$ (the latter vector is defined in analogy with (7.1.24)). Together with (7.2.36), this yields the updated values of $\mathbf{u}^{n+1}$ on Γ, which are then used as boundary data for the implicit equations

$$(\delta_b(\mathbf{u}_i^{n+1}), \mathbf{v}_i)_{\Omega_i} + e_i(\mathbf{u}_i^{n+1}, \mathbf{v}_i) = (\mathbf{f}(t^{n+1}), \mathbf{v}_i)_{\Omega_i} \qquad \forall\ \mathbf{v}_i \in (V_i^0)^d,$$

for $i = 1, 2$.

Remark 7.2.2 The domain decomposition approach for problems of wave propagation is also well-suited to dealing with material inhomogeneity. Many illustrative examples for the cases of both acoustic and elastic waves are provided in Seriani and Priolo (1994) and Faccioli *et al.* (1997). □

7.3 Non-linear time-dependent problems

The time discretisation in (7.1) can be based on implicit, explicit or semi-implicit schemes. Any implicit method yields at each time level a non-linear boundary value problem resembling (7.1.15) of the following form:

$$L_{\Delta t}(u^{n+1}) := u^{n+1} + \Delta t\, L(u^{n+1}) = g^{n+1} \qquad \text{in } \Omega, \tag{7.3.1}$$

with $u^0 := u_0$, and the boundary condition inherited from the original initial-boundary value problem.

Domain decomposition methods can be called into play for problem (7.3.1) according to two different paradygms.

1. Problem (7.3.1) can first be linearised by a Newton method

$$\left(I + \Delta t\, L'(u^{n+1,k})\right)(u^{n+1,k+1} - u^{n+1,k}) = g^{n+1} - L_{\Delta t}(u^{n+1,k})$$

   for $k \geq 0$, with $u^{n+1,0} := u^n$, then at each step k the associated linear boundary value problem can be solved by a Krylov iteration process using a domain decomposition preconditioner of either the Schwarz or substructuring type.
2. Problem (7.3.1) is reformulated as a multi-domain problem with the correct interface conditions, then suitable iterative substructuring methods are applied to generate a sequence of problems in the subdomains that are still non-linear and need, therefore, to be linearised at the subdomain level.

Other approaches can exploit the specific form of the non-linear problem (7.1) in order to linearise it by semi-implicit time discretisation or fractional-step methods. Hence, one or more linear boundary value problems are derived at every time level, and suitable domain decomposition algorithms can be applied directly to these subproblems.

In what follows we briefly address three different types of non-linear equations that are commonly used to model fluid flows: the Navier–Stokes equations for viscous incompressible flows; the Navier–Stokes equations for viscous compressible flows; and the Euler equations for inviscid compressible flows.

7.3.1 Navier–Stokes equations for incompressible flows

The Navier–Stokes equations for a viscous incompressible fluid confined in a region Ω of $\mathbf{R}^d$, $d = 2, 3$, read (see Landau and Lifshitz 1959; Ladyzhenskaya 1969; and Temam 1984):

$$\text{(7.3.2)} \qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla \pi = \mathbf{f} - (\mathbf{u} \cdot \nabla)\mathbf{u} & \text{in } Q_T := \Omega \times (0, T) \\ \operatorname{div} \mathbf{u} = 0 & \text{in } Q_T \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \times (0, T) \\ \mathbf{u}_{|t=0} = \mathbf{u}_0 & \text{in } \Omega, \end{cases}$$

where $\nu > 0$ is the viscosity coefficient, $\mathbf{f}(x, t)$ and $\mathbf{u}_0(x)$ are given data, while $\mathbf{u}$ denotes the velocity field and π the pressure.

These equations can be advanced in time by a fully implicit method (for example, the first-order backward Euler scheme), then formulated in a multi-domain context with interface conditions that coincide with those for the Stokes problem (see Section 5.3). Then we proceed as indicated in 1.

However, most often the complexity is reduced by suitable time-advancing schemes with the aim of decoupling the convective term from the other terms, or the computation of the pressure from that of the velocity field.

An example of the first kind is provided by the Crank–Nicolson/Adams–Bashforth scheme, which approximates (7.3.2) by

$$\text{(7.3.3)} \qquad \begin{cases} \dfrac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \dfrac{1}{2}\nu\Delta(\mathbf{u}^{n+1} + \mathbf{u}^n) + \dfrac{1}{2}\nabla(\pi^{n+1} + \pi^n) \\ \qquad = \dfrac{1}{2}(\mathbf{f}^{n+1} + \mathbf{f}^n) - \dfrac{3}{2}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^n + \dfrac{1}{2}(\mathbf{u}^{n-1} \cdot \nabla)\mathbf{u}^{n-1} & \text{in } \Omega \\ \operatorname{div} \mathbf{u}^{n+1} = 0 & \text{in } \Omega \\ \mathbf{u}^{n+1} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

In this way we find at each time level a Stokes problem that can be solved using the algorithms analysed in Section 5.3.

A different way consists of using a projection method à la Chorin–Temam (see Quarteroni and Valli 1994, Section 13.5; Prohl 1997). In this case, the momentum equations are advanced first to generate a preliminary velocity field $\mathbf{u}^{n+1/2}$ that is not divergence free:

$$\begin{cases} \dfrac{1}{\Delta t}(\mathbf{u}^{n+1/2} - \mathbf{u}^n) - \nu\Delta\mathbf{u}^{n+1/2} + \nabla\pi_*^n \\ \qquad +(\mathbf{u}^n \cdot \nabla)\mathbf{u}_*^n + \frac{1}{2}(\operatorname{div}\mathbf{u}^n)\mathbf{u}_*^n = \mathbf{f}^{n+1/2} \quad \text{in } \Omega \\ \mathbf{u}^{n+1/2} = \mathbf{0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{7.3.4}$$

This scheme for the momentum equation corresponds to a skew-symmetric treatment of the convective term $(\mathbf{u} \cdot \nabla)\mathbf{u}$.

At this stage, the pressure term is either disregarded ($\pi_*^n = 0$) or treated explicitly ($\pi_*^n = \pi^n$), while the convective term is treated explicitly ($\mathbf{u}_*^n = \mathbf{u}^n$) or semi-implicitly ($\mathbf{u}_*^n = \mathbf{u}^{n+1/2}$). In the former case, we obtain a second-order symmetric elliptic equation for each velocity component, which can be treated as shown in Sections 1.3 and 1.4, while in the latter we obtain a linear advection–diffusion equation, which can be solved as discussed in Chapter 6.

Then one computes the new pressure π^{n+1} and the new velocity $\mathbf{u}^{n+1}$ as follows:

$$\begin{cases} \dfrac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^{n+1/2}) + \nabla(\pi^{n+1} - \pi_*^n) = 0 \quad \text{in } \Omega \\ \operatorname{div}\mathbf{u}^{n+1} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in } \Omega \\ \mathbf{u}^{n+1} \cdot \mathbf{n} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega. \end{cases} \tag{7.3.5}$$

The latter is called the projection step, because it amounts to projecting the intermediate velocity field $\mathbf{u}^{n+1/2}$ onto the space of divergence-free functions. This step can be reformulated as a boundary value problem for the pressure π^{n+1}:

$$\begin{cases} \Delta\pi^{n+1} = \Delta\pi_*^n + \dfrac{1}{\Delta t}\operatorname{div}\mathbf{u}^{n+1/2} \quad \text{in } \Omega \\ \dfrac{\partial\pi^{n+1}}{\partial n} = \dfrac{\partial\pi_*^n}{\partial n} \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{7.3.6}$$

together with the final updating for the velocity field

$$\mathbf{u}^{n+1} = \mathbf{u}^{n+1/2} - \Delta t\nabla(\pi^{n+1} - \pi_*^n). \tag{7.3.7}$$

Thus we have a Poisson equation for the pressure (with a Neumann boundary condition), which can be treated as indicated in Chapter 1 (see in particular Section 1.4.1).

7.3.2 Navier–Stokes equations for compressible flows

In the case of a compressible fluid, the Navier–Stokes equations, which express the conservation of mass, momentum and energy, can be written as (see, for example, Landau and Lifshitz 1959)

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{div}\,\mathbf{F}(\mathbf{W}) = \mathbf{div}\,\mathbf{G}(\mathbf{W}) \qquad \text{in } \Omega \times (0,T), \tag{7.3.8}$$

where $\Omega \subset \mathbf{R}^d$, $d = 2, 3$. The array $\mathbf{W}$ contains the conserved variables, $\mathbf{W} = (\rho, \rho\mathbf{u}, \rho E)$, ρ being the density, $\mathbf{u}$ the velocity vector, and E the total energy per unit mass (E is the sum of the internal thermodynamic energy per unit mass e and the kinetic energy per unit mass $|\mathbf{u}|^2/2$). The convective and diffusive terms $\mathbf{F}(\mathbf{W})$ and $\mathbf{G}(\mathbf{W})$ are the $(d+2) \times d$ matrices defined as

$$\mathbf{F}(\mathbf{W}) := \begin{pmatrix} \rho\mathbf{u} \\ \rho\mathbf{u} \otimes \mathbf{u} + p\mathbf{I} \\ (\rho E + p)\mathbf{u} \end{pmatrix}, \quad \mathbf{G}(\mathbf{W}) := \begin{pmatrix} 0 \\ \boldsymbol{\tau} \\ \boldsymbol{\tau} \cdot \mathbf{u} - \mathbf{q} \end{pmatrix}.$$

Here, p is the pressure, $\mathbf{u} \otimes \mathbf{u}$ denotes the tensor whose components are $u_l u_j$, $\mathbf{I}$ is the unit tensor δ_{lj}, $\mathbf{q}$ is the heat flux, and finally $(\boldsymbol{\tau} \cdot \mathbf{u})_l := \sum_j \tau_{lj} u_j$, where $\boldsymbol{\tau}$ is the viscous stress tensor, the components of which are defined as

$$\tau_{lj} := \mu(D_l u_j + D_j u_l) + \left(\zeta - \frac{2\mu}{d}\right) \operatorname{div} \mathbf{u}\, \delta_{lj}, \tag{7.3.9}$$

with $\mu > 0$ and $\zeta \geq 0$ being the shear and bulk viscosity coefficients, respectively. The heat flux $\mathbf{q}$ is related to the absolute temperature θ by the standard Fourier law

$$\mathbf{q} = -\kappa \nabla \theta, \tag{7.3.10}$$

where $\kappa > 0$ is the heat conductivity coefficient.

In (7.3.8), the divergence of $\mathbf{F}(\mathbf{W})$ is the $(d+2)$-vector

$$\mathbf{div}\,\mathbf{F}(\mathbf{W}) = (\operatorname{div}(\rho\mathbf{u}), \mathbf{div}\,(\rho\mathbf{u} \otimes \mathbf{u} + p\mathbf{I}), \operatorname{div}[(\rho E + p)\mathbf{u}])$$

(and similarly for $\mathbf{div}\,\mathbf{G}(\mathbf{W})$), and the divergence of a tensor $\mathbf{T}$ is the vector with components

$$(\mathbf{div}\,\mathbf{T})_l := \sum_j D_j T_{lj}.$$

The entropy per unit mass s can be introduced via the second law of thermodynamics

$$\theta\, ds = de - \frac{p}{\rho^2} d\rho. \tag{7.3.11}$$

For ideal polytropic gases, the pressure p and the internal energy e are related to the other thermodynamic quantities ρ and θ through the equations of state

$$p = R\rho\theta, \quad e = c_V\theta,$$

where $R > 0$ is the difference between the specific heat at constant pressure $c_P > 0$ and the specific heat at constant volume $c_V > 0$. From these relations it follows that

$$p = (\gamma - 1)\rho e,$$

with $\gamma > 1$ being the ratio of specific heats. Moreover, from (7.3.11) we find that

$$p = k\rho^\gamma \exp(s/c_V)$$

for a suitable constant $k > 0$.

Equations (7.3.8) provide an incomplete parabolic system (the continuity equation for the density ρ is hyperbolic). The hyperbolic Euler equations for inviscid and non-conductive flows can be obtained by dropping all the diffusive terms; that is, by taking $\mu = \zeta = \kappa = 0$, or, equivalently, $\boldsymbol{\tau} = \mathbf{0}$ and $\mathbf{q} = \mathbf{0}$.

Since, for compressible flows, explicit as well as implicit methods are used, our analysis of the multi-domain formulation will be carried out without explicitly indicating the discretisation of the time derivative.

Let the domain Ω be partitioned into Ω_1 and Ω_2, as in Section 1.1. Denoting by $\mathbf{W}_i$ the restriction of $\mathbf{W}$ on Ω_i, $i = 1, 2$, equations (7.3.8) can be reformulated in the following split form

$$\text{(7.3.12)} \quad \begin{cases} \dfrac{\partial \mathbf{W}_i}{\partial t} + \mathbf{div}\,\mathbf{F}(\mathbf{W}_i) = \mathbf{div}\,\mathbf{G}(\mathbf{W}_i) & \text{in } \Omega_i \times (0,T) \\ \mathbf{u}_1 = \mathbf{u}_2\ ,\ E_1 = E_2 & \text{on } \Gamma \times (0,T) \\ [\mathbf{F}(\mathbf{W}_1) - \mathbf{G}(\mathbf{W}_1)] \cdot \mathbf{n} = [\mathbf{F}(\mathbf{W}_2) - \mathbf{G}(\mathbf{W}_2)] \cdot \mathbf{n} & \text{on } \Gamma \times (0,T), \end{cases}$$

for $i = 1, 2$, where $\mathbf{F}(\mathbf{W}) \cdot \mathbf{n}$ denotes the $(d+2)$-vector

$$(\rho\mathbf{u} \cdot \mathbf{n}, \rho\mathbf{u}(\mathbf{u} \cdot \mathbf{n}) + p\mathbf{n}, (\rho E + p)\mathbf{u} \cdot \mathbf{n})$$

(and similarly for $\mathbf{G}(\mathbf{W}) \cdot \mathbf{n}$), and $\mathbf{n}$ is the unit normal vector on Γ, directed from Ω_1 to Ω_2.

In other words, the subdomain restrictions $\mathbf{W}_1$ and $\mathbf{W}_2$ satisfy the Navier–Stokes equations separately in Ω_1 and Ω_2, together with suitable interface conditions. Clearly, equations (7.3.12) inherit the same boundary and initial conditions prescribed for $\mathbf{W}$ on $\partial\Omega$ and at $t = 0$, respectively.

The interface conditions $(7.3.12)_2$ are a consequence of the following fact: the unknowns $\mathbf{u}$ and θ appear in (7.3.8) through their second-order derivatives, hence they must be continuous across Γ. Since $E = e + |\mathbf{u}|^2/2 = c_V\theta + |\mathbf{u}|^2/2$, continuity holds also for the total energy E.

The interface condition $(7.3.12)_3$ states the continuity of the normal fluxes, and can be rewritten as

$$
\begin{aligned}
&\rho_1 \mathbf{u}_1 \cdot \mathbf{n} = \rho_2 \mathbf{u}_2 \cdot \mathbf{n} \\
&\rho_1 u_{1,l} \mathbf{u}_1 \cdot \mathbf{n} + p_1 n_l - \sum_{j=1}^{d} \tau_{1,lj}\, n_j \\
&\qquad = \rho_2 u_{2,l} \mathbf{u}_2 \cdot \mathbf{n} + p_2 n_l - \sum_{j=1}^{d} \tau_{2,lj}\, n_j, \quad l = 1, \ldots, d \\
&(\rho_1 E_1 + p_1) \mathbf{u}_1 \cdot \mathbf{n} - \sum_{j,l=1}^{d} \tau_{1,lj} u_{1,j} n_l - \kappa \nabla \theta_1 \cdot \mathbf{n} \\
&\qquad = (\rho_2 E_2 + p_2) \mathbf{u}_2 \cdot \mathbf{n} - \sum_{j,l=1}^{d} \tau_{2,lj} u_{2,j} n_l - \kappa \nabla \theta_2 \cdot \mathbf{n}.
\end{aligned}
\tag{7.3.13}
$$

In particular, since $\mathbf{u}$ is continuous across Γ, from the first relation it follows that

$$
\rho_1 = \rho_2 \qquad \text{at all points on } \Gamma \times (0,T) \text{ where } \mathbf{u} \cdot \mathbf{n} \neq 0. \tag{7.3.14}
$$

Finally, the flux matching property $(7.3.12)_3$ is a natural consequence of the fact that the variable $\mathbf{W}$ is a distributional solution to (7.3.8) in Ω.

Remark 7.3.1 Within the frame of iterative substructuring methods, equations $(7.3.12)_2$ can provide the Dirichlet conditions for $\mathbf{u}$ and θ on Γ for one subdomain, while $(7.3.13)_2$ and $(7.3.13)_3$ can yield the Neumann conditions on Γ for the other subdomain. Concerning the interface condition $(7.3.13)_1$ (or, equivalently, (7.3.14)), it must be split into a Dirichlet condition for ρ_1 on $\Gamma \cap \{\mathbf{u} \cdot \mathbf{n} < 0\}$, and a Dirichlet condition for ρ_2 on $\Gamma \cap \{\mathbf{u} \cdot \mathbf{n} > 0\}$. □

7.3.3 Euler equations for compressible flows

As already noted, the Euler equations are obtained from the Navier–Stokes equations by taking the viscosity coefficients $\mu = \zeta = 0$ and the heat conductivity coefficient $\kappa = 0$; that is, disregarding all the diffusive terms $\mathbf{G}(\mathbf{W})$. The equations read

$$
\frac{\partial \mathbf{W}}{\partial t} + \mathbf{div}\, \mathbf{F}(\mathbf{W}) = \mathbf{0} \qquad \text{in } \Omega \times (0,T). \tag{7.3.15}
$$

Still denoting by $\mathbf{W}_i$ the restriction of $\mathbf{W}$ to Ω_i, we have the equivalent formulation

$$
\begin{cases}
\dfrac{\partial \mathbf{W}_i}{\partial t} + \mathbf{div}\, \mathbf{F}(\mathbf{W}_i) = \mathbf{0} & \text{in } \Omega_i \times (0,T)\ , \quad i = 1,2 \\
\mathbf{F}(\mathbf{W}_1) \cdot \mathbf{n} = \mathbf{F}(\mathbf{W}_2) \cdot \mathbf{n} & \text{on } \Gamma \times (0,T).
\end{cases}
\tag{7.3.16}
$$

This time the only matching conditions on Γ are those prescribing the continuity of the normal inviscid flux, which can be rewritten as

$$
\begin{aligned}
&\rho_1 \mathbf{u}_1 \cdot \mathbf{n} = \rho_2 \mathbf{u}_2 \cdot \mathbf{n} \\
&\rho_1 u_{1,l} \mathbf{u}_1 \cdot \mathbf{n} + p_1 n_l = \rho_2 u_{2,l} \mathbf{u}_2 \cdot \mathbf{n} + p_2 n_l, \quad l = 1, \ldots, d \\
&(\rho_1 E_1 + p_1) \mathbf{u}_1 \cdot \mathbf{n} = (\rho_2 E_2 + p_2) \mathbf{u}_2 \cdot \mathbf{n}.
\end{aligned} \tag{7.3.17}
$$

As we have noted already, the first condition is equivalent to (7.3.14). This latter condition is in agreement with the physical properties of compressible fluid flows, which admit two types of discontinuities: shock waves or contact discontinuities. In particular, on contact discontinuities the normal velocity is zero, the pressure is continuous, but density, as well as tangential velocity and temperature, may have a non-zero jump.

If the interface Γ is not kept fixed but moves along in time, then denoting by $\sigma(t)$ its velocity at time t along the normal direction $\mathbf{n} = \mathbf{n}(t)$, the matching condition $(7.3.16)_2$ has to be replaced by

$$
[\mathbf{W}_1(t) - \mathbf{W}_2(t)]\sigma(t) = [\mathbf{F}(\mathbf{W}_1(t)) - \mathbf{F}(\mathbf{W}_2(t))] \cdot \mathbf{n}(t) \qquad \text{on } \Gamma(t). \tag{7.3.18}
$$

If $\Gamma(t)$ intercepts (or coincides with) a shock front $\delta(t)$, then (7.3.18) can be easily recognised as the Rankine–Hugoniot jump condition across $\delta(t)$ (see, for example, Hirsch 1990, p. 136).

A characteristic analysis of Euler equations enlightens the role that the interface conditions could play in the framework of a substructuring iterative method. To simplify our analysis, first we consider one-dimensional flows, in which case (7.3.15) becomes

$$
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{7.3.19}
$$

Ω being an interval. This time, the unknowns quantities are $\mathbf{W} = (\rho, \rho u, \rho E)$, while the inviscid flux can be written as a three-dimensional vector

$$
\mathbf{F}(\mathbf{W}) = \left(\rho u, \rho u^2 + p, (\rho E + p)u\right).
$$

Let us recall some known facts about the characteristic form of the one-dimensional Euler equations. Using (7.3.11) to express the derivatives of e in terms of s and ρ, the quasi-linear form of (7.3.19) in terms of the vector of primitive variables $\mathbf{U} = (\rho, u, s)$ is given by

$$
\frac{\partial \mathbf{U}}{\partial t} + A \frac{\partial \mathbf{U}}{\partial x} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{7.3.20}
$$

where

$$
A := \begin{pmatrix} u & \rho & 0 \\ c^2/\rho & u & p_s/\rho \\ 0 & 0 & u \end{pmatrix} \tag{7.3.21}
$$

and we have set $c := \sqrt{\frac{\partial p}{\partial \rho}}$ (the speed of sound) and $p_s := \frac{\partial p}{\partial s}$.

The eigenvalues of A are

$$\lambda_1 = u + c, \quad \lambda_2 = u - c, \quad \lambda_3 = u, \tag{7.3.22}$$

and the matrix L of left eigenvectors is

$$L := \begin{pmatrix} c/\rho & 1 & p_s/(\rho c) \\ -c/\rho & 1 & -p_s/(\rho c) \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.3.23}$$

For subsonic flows in which the left end of the interval Ω is the upstream boundary, i.e. $0 < u < c$, the two eigenvalues λ_1 and λ_3 are positive, while λ_2 is negative. If the flow is supersonic ($u > c > 0$), all the eigenvalues are positive.

In principle, system (7.3.20) can be diagonalised by using the eigenvectors of A, leading to a fully decoupled problem. A suitable set of variables, the *characteristic variables* $\tilde{\mathbf{U}}$, is introduced by means of the following differential form (see, for example, Hirsch 1990, p. 162)

$$d\tilde{\mathbf{U}} := L d\mathbf{U} = \left(\frac{c}{\rho} d\rho + du + \frac{p_s}{\rho c} ds, -\frac{c}{\rho} d\rho + du - \frac{p_s}{\rho c} ds, ds \right). \tag{7.3.24}$$

The introduction of the above relations based on variations, and not upon the quantities themselves, is motivated by the fact that the coefficients of the system; that is, the elements of A, are not constant and depend on the solution itself.

Since $LAL^{-1} = \Lambda = \text{diag}\,(\lambda_1, \lambda_2, \lambda_3)$, owing to (7.3.24) equations (7.3.20) become

$$\frac{\partial \tilde{\mathbf{U}}}{\partial t} + \Lambda \frac{\partial \tilde{\mathbf{U}}}{\partial x} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{7.3.25}$$

which splits into three scalar independent equations

$$\frac{\partial \tilde{U}_r}{\partial t} + \lambda_r \frac{\partial \tilde{U}_r}{\partial x} = 0 \qquad \text{in } \Omega \times (0, T), \quad r = 1, 2, 3. \tag{7.3.26}$$

It follows that, for each $r = 1, 2, 3$, the component $\tilde{U}_r$ is constant along the *characteristic curve*

$$\{(x_r(t), t) \,|\, x_r'(t) = \lambda_r\}.$$

Note the formal analogy between system (7.3.26) and the characteristic system (7.2.28). Assuming that we know $\tilde{\mathbf{U}}$, we can then proceed as for problem (7.2.28), and enforce the continuity of the incoming characteristic variables on each subdomain at every iteration.

This procedure can be illustrated on the isentropic Euler equations, which read

$$\frac{\partial \mathbf{V}}{\partial t} + B \frac{\partial \mathbf{V}}{\partial x} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{7.3.27}$$

where $\mathbf{V} = (\rho, u)$ and

$$B := \begin{pmatrix} u & \rho \\ c^2/\rho & u \end{pmatrix}. \tag{7.3.28}$$

Here, the entropy s is assumed to be constant, so that the pressure p depends only on the density ρ; precisely, $p = K\rho^\gamma$ for a suitable constant $K > 0$.

The eigenvalues of B are

$$\lambda_1 = u + c, \quad \lambda_2 = u - c,$$

where $c = \sqrt{K\gamma\rho^{\gamma-1}}$, and the matrix L is given by

$$L := \begin{pmatrix} c/\rho & 1 \\ -c/\rho & 1 \end{pmatrix}.$$

The characteristic variables $\tilde{\mathbf{V}}$ are defined as

$$d\tilde{\mathbf{V}} := Ld\mathbf{V} = \left(\frac{c}{\rho}d\rho + du, -\frac{c}{\rho}d\rho + du\right); \tag{7.3.29}$$

hence, by direct integration one finds that

$$\begin{aligned} \tilde{V}_1 &= u + \int \frac{c}{\rho}d\rho = u + \int \sqrt{K\gamma}\rho^{\gamma/2-3/2}d\rho \\ &= u + \frac{2}{\gamma-1}c + \text{const}, \end{aligned}$$

and similarly for $\tilde{V}_2$. In conclusion we have

$$\tilde{\mathbf{V}} = (R_+, R_-), \quad R_\pm := u \pm \frac{2}{\gamma-1}c. \tag{7.3.30}$$

The Riemann invariant R_+ is constant on $C_+ = \{(x(t), t) \,|\, x'(t) = u+c\}$, whereas R_- is constant on $C_- = \{(x(t), t) \,|\, x'(t) = u - c\}$.

In this case, assuming for instance that the flow is subsonic ($0 < u < c$), the iterative method would read as follows:

$$\begin{cases} \dfrac{\partial \mathbf{V}_1^{k+1}}{\partial t} + B\dfrac{\partial \mathbf{V}_1^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_1 \times (0,T) \\ R_{-,1}^{k+1} = R_{-,2}^{k} & \text{at } x_\Gamma \times (0,T) \end{cases} \tag{7.3.31}$$

$$\begin{cases} \dfrac{\partial \mathbf{V}_2^{k+1}}{\partial t} + B\dfrac{\partial \mathbf{V}_2^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\ R_{+,2}^{k+1} = R_{+,1}^{k} & \text{at } x_\Gamma \times (0,T). \end{cases} \tag{7.3.32}$$

Here, x_Γ is the interface point separating the subintervals Ω_1 and Ω_2, and we have assumed that Ω_1 is on the left side of Ω_2. Therefore, C_- is incoming in Ω_1 at x_Γ, and C_+ is incoming in Ω_2 at x_Γ, and we are imposing a boundary condition at x_Γ for any characteristic variable associated with an incoming characteristic curve. Following this procedure, for a supersonic flow both interface conditions have to be attributed to the domain Ω_2.

Furthermore, there are the initial condition and two additional boundary conditions, one assigned at the left end of the interval Ω_1 (where the characteristic C_+ is incoming) and the other at the right end of the interval Ω_2 (where the characteristic C_- is incoming). It is easily seen that, at convergence of the iterative scheme, the continuity of R_+ and R_- at x_Γ implies that $(7.3.17)_1$ and $(7.3.17)_2$ are satisfied.

Unfortunately, for the non-isentropic system (7.3.20) the functions $R_\pm$ are no longer constant along the characteristic curves $C_\pm$, and in general the characteristic variables $\tilde{\mathbf{U}}$ are not explicitly known (see, for example, Hirsch 1990, pp. 155–6), except for the entropy s, which is constant along the characteristic curve $C_0 = \{(x(t), t) \,|\, x'(t) = u\}$.

To compensate for the lack of knowledge of the characteristic variables, we consider the following iteration procedure. The Euler system is solved in Ω_1 and Ω_2, with given initial data at $t = 0$ and boundary data on $\partial\Omega$ (these latter conditions have to be determined by means of a suitable eigenvalue analysis). Concerning the interface conditions, we can proceed as for the isentropic Euler equations, and, assuming that each eigenvalue of A is different from zero, enforce the continuity of the quantities $L\mathbf{U}$ and iterate accordingly, even though $L\mathbf{U}$ are not characteristic variables. For the Euler system, since

$$\begin{aligned} L\mathbf{U} &= \left(u + c + \frac{p_s}{\rho c}s, u - c - \frac{p_s}{\rho c}s, s\right) \\ &=: (Q_+, Q_-, Q_0), \end{aligned}$$

when all the eigenvalues are non-zero we enforce the continuity of the three functions Q_+, Q_- and Q_0, that are associated with the eigenvalues $u + c$, $u - c$ and u, respectively.

For instance, in the subsonic case $0 < u < c$, the interface conditions in the iterative process become

$$Q_{-,1}^{k+1}(x_\Gamma, t) = Q_{-,2}^{k}(x_\Gamma, t),$$

because the characteristic curve C_-, associated with the eigenvalue $u - c$, is incoming in Ω_1, and

$$\begin{cases} Q_{+,2}^{k+1}(x_\Gamma, t) = Q_{+,1}^{k}(x_\Gamma, t) \\ \\ Q_{0,2}^{k+1}(x_\Gamma, t) = Q_{0,1}^{k}(x_\Gamma, t), \end{cases}$$

because the characteristic curves C_+ and C_0, associated with the eigenvalues $u+c$ and u, are incoming in Ω_2. For each subdomain we are therefore prescribing a

boundary condition at the interface for any variable associated with an incoming characteristic line.

Another possibility is to enforce the continuity of the Riemann 'invariants' R_+, R_- and $R_0 = s$. In both cases, it is straightforward to verify that the continuity of $L\mathbf{U} = (Q_+, Q_-, Q_0)$ or that of (R_+, R_-, R_0) at x_Γ implies that (7.3.17) is satisfied.

When considering discretisation, at the interface point x_Γ one has to enforce three additional conditions (besides the other three related to Q_+, Q_- and Q_0, or R_+, R_- and R_0), in order to recover all the six interface variables. This can be accomplished through the equations

$$\begin{aligned} &\left[\mathbf{l}^r \cdot \left(\frac{\partial \mathbf{U}_1}{\partial t} + \lambda_r \frac{\partial \mathbf{U}_1}{\partial x}\right)\right](x_\Gamma, t) = 0 \qquad \text{for } r = 1, 3 \\ &\left[\mathbf{l}^2 \cdot \left(\frac{\partial \mathbf{U}_2}{\partial t} + \lambda_2 \frac{\partial \mathbf{U}_2}{\partial x}\right)\right](x_\Gamma, t) = 0, \end{aligned} \tag{7.3.33}$$

where

$$\mathbf{l}^1 := \left(\frac{c}{\rho}, 1, \frac{p_s}{\rho c}\right), \quad \mathbf{l}^2 := \left(-\frac{c}{\rho}, 1, -\frac{p_s}{\rho c}\right), \quad \mathbf{l}^3 := (0, 0, 1)$$

are the left eigenvectors of A.

In each subdomain, equations (7.3.33) are obtained from the Euler system (7.3.20) by taking the scalar product with $\mathbf{l}^r$, $r = 1, 2, 3$, at the interface point, but only for those values of r whose corresponding eigenvalue λ_r identifies a characteristic line that is outgoing at x_Γ. These equations are sometimes called the *compatibility* equations.

Let us return now to the three-dimensional Euler equations, which, with respect to the primitive variables $\mathbf{U} = (\rho, u_1, u_2, u_3, s)$, can be written as

$$\frac{\partial \mathbf{U}}{\partial t} + \sum_{l=1}^{3} A^{(l)} D_l \mathbf{U} = \mathbf{0} \qquad \text{in } \Omega \times (0, T), \tag{7.3.34}$$

where

$$A^{(1)} := \begin{pmatrix} u_1 & \rho & 0 & 0 & 0 \\ c^2/\rho & u_1 & 0 & 0 & p_s/\rho \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 \\ 0 & 0 & 0 & 0 & u_1 \end{pmatrix}, \quad A^{(2)} := \begin{pmatrix} u_2 & 0 & \rho & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 \\ c^2/\rho & 0 & u_2 & 0 & p_s/\rho \\ 0 & 0 & 0 & u_2 & 0 \\ 0 & 0 & 0 & 0 & u_2 \end{pmatrix}$$

$$A^{(3)} := \begin{pmatrix} u_3 & 0 & 0 & \rho & 0 \\ 0 & u_3 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 \\ c^2/\rho & 0 & 0 & u_3 & p_s/\rho \\ 0 & 0 & 0 & 0 & u_3 \end{pmatrix}.$$

As usual, we have set $c := \sqrt{\frac{\partial p}{\partial \rho}}$ and $p_s := \frac{\partial p}{\partial s}$.

For any point $\mathbf{x} \in \Gamma$ and any time $t \in (0,T)$, denote by $C = C(\mathbf{n})$ the characteristic matrix $C = \sum_l n_l A^{(l)}$. The eigenvalues of C are given by

$$\lambda_1 = \mathbf{u}\cdot\mathbf{n} + c, \quad \lambda_2 = \mathbf{u}\cdot\mathbf{n} - c, \quad \lambda_{3,4,5} = \mathbf{u}\cdot\mathbf{n}. \tag{7.3.35}$$

Finally, denote by L the matrix of the left eigenvectors of C, which is given by

$$L := \begin{pmatrix} \frac{c}{\rho} & n_1 & n_2 & n_3 & \frac{p_s}{\rho c} \\ -\frac{c}{\rho} & n_1 & n_2 & n_3 & -\frac{p_s}{\rho c} \\ 0 & \tau_1^{(1)} & \tau_2^{(1)} & \tau_3^{(1)} & 1 \\ 0 & \tau_1^{(2)} & \tau_2^{(2)} & \tau_3^{(2)} & 1 \\ 0 & -\tau_1^{(1)} - \tau_1^{(2)} & -\tau_2^{(1)} - \tau_2^{(2)} & -\tau_3^{(1)} - \tau_3^{(2)} & 1 \end{pmatrix}, \tag{7.3.36}$$

where $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$ are two unit orthogonal vectors, spanning the plane orthogonal to $\mathbf{n}$.

Assuming that at time t the interface Γ is not characteristic at point $\mathbf{x}$; namely, that no eigenvalue is zero at $\mathbf{x}$, the matching conditions are naturally extrapolated from the one-dimensional case, and yield

$$\sum_{q=1}^{5} L_{rq} U_{1,q} = \sum_{q=1}^{5} L_{rq} U_{2,q} \qquad \text{at } \mathbf{x} \in \Gamma, \quad r = 1, \ldots, 5. \tag{7.3.37}$$

Again, we are enforcing the continuity of the 'characteristic' variables $L\mathbf{U}$. It is easily verified that, as a consequence of (7.3.37), the interface conditions (7.3.17) are satisfied.

The iteration-by-subdomain algorithm used for solving the multi-domain problem alternates the solution of the Euler equations (7.3.34) in Ω_1 and in Ω_2, with the Dirichlet boundary condition (7.3.37) imposed at a point $\mathbf{x}$ on Γ for all indices r corresponding to incoming characteristic lines. For instance, if we assume that at time t the interface point $\mathbf{x}$ is an outflow point for Ω_1 and that the flow is subsonic (namely, $0 < \mathbf{u}\cdot\mathbf{n} < c$, with $\mathbf{n}$ directed from Ω_1 to Ω_2), we have to impose at the $(k+1)$th iteration

$$\sum_{q=1}^{5} L_{2q} U_{1,q}^{k+1} = \sum_{q=1}^{5} L_{2q} U_{2,q}^{k} \qquad \text{at } \mathbf{x} \in \Gamma,$$

and

$$\sum_{q=1}^{5} L_{rq} U_{2,q}^{k+1} = \sum_{q=1}^{5} L_{rq} U_{1,q}^{k} \qquad \text{at } \mathbf{x} \in \Gamma, \quad r = 1, 3, 4, 5.$$

When a numerical discretisation is being applied, other equations, i.e. the compatibility equations, have to be imposed at $\mathbf{x}$. Proceeding as in (7.3.33), in

each subdomain Ω_i, $i = 1, 2$, they are obtained by taking the scalar product of (7.3.34) (stated for $\mathbf{U}_i^{k+1}$ instead of $\mathbf{U}$) with the rth left eigenvector $\mathbf{l}^r$, $r = 1, \ldots, 5$, but only for those values of r for which the eigenvalue λ_r is associated with a characteristic line that is directed outward from Ω_i at $\mathbf{x}$.

8

HETEROGENEOUS DOMAIN DECOMPOSITION METHODS

In this chapter we address the case of heterogeneous domain decomposition, arising whenever, in the approximation of certain physical phenomena, two *different kinds* of (initial-) boundary value problems hold within two disjoined subregions of the computational domain.

This is a generalisation of the classical, homogeneous domain decomposition approach, in which the *same kind* of problem occurs in each subdomain.

The abstract mathematical setting that we have in mind can be described as follows. We denote by L_1 and L_2 two different operators and consider the problem: find $u_i : \Omega_i \to \mathbf{R}$, $i = 1, 2$, such that

$$L_1 u_1 = f \ \text{ in } \Omega_1, \quad L_2 u_2 = f \ \text{ in } \Omega_2, \tag{8.1}$$

together with suitable boundary conditions for u_1 on $\partial\Omega_1 \setminus \Gamma$ and u_2 on $\partial\Omega_2 \setminus \Gamma$. (As usual, Ω_1 and Ω_2 denote two subdomains that provide a non-overlapping partition of the computational domain Ω, and Γ is their interface.) Moreover, the unknown functions u_1 and u_2 should satisfy proper matching conditions on Γ, which we can formulate in an abstract form as

$$\Phi(u_1) = \Phi(u_2) \quad \text{on } \Gamma' \tag{8.2}$$

$$\Psi(u_1) = \Psi(u_2) \quad \text{on } \Gamma'', \tag{8.3}$$

where the functions Φ and Ψ, as well as $\Gamma' \subset \Gamma$ and $\Gamma'' \subset \Gamma$, will depend upon the nature of the problem at hand.

In some problems we shall consider some terms in the modelling equations are dimensionally irrelevant on a certain portion (say, Ω_1) of the computational domain Ω. For such cases, if L_2 denotes the operator associated with the complete model, L_1 will denote the operator obtained from L_2 after omitting the 'irrelevant' terms. In such a way, the split problem (8.1)–(8.3) can be regarded as a reduced version of the global problem

$$L_2 w = f \quad \text{in } \Omega. \tag{8.4}$$

From a mathematical viewpoint, the crucial issue is the set-up of the matching equations on Γ' and Γ''. The derivation of such interface conditions ought to be carried out so that we model as closely as possible to the physics of the underlying problem. Most often, the leading criterion is that the solution to the

coupled problem (8.1)–(8.3) is also the limit of a sequence of solutions to global variational problems set in the whole domain Ω, following a singular perturbation approach.

We will illustrate how these general principles can be adopted to analyse several kinds of heterogeneous models, remarkably: advection–diffusion equations, the coupling of viscous and inviscid model equations for compressible and incompressible flows, and the coupling of Maxwell equations in heterogeneous media (conductor and insulator).

Some physical motivations that justify the adoption of heterogeneous decomposition models are given hereafter.

A family of problems that are heterogeneous 'in nature' is supplied by the interaction between fluids and solids. For instance, this is the case for off-shore mechanics, undersea pipelines, underwater acoustic phenomena in shallow waters (see Lie 1992), or the mutual action of blood flow and compliant vessel walls (see Perktold and Rappitsch 1995; Taylor *et al.* 1998; and Quarteroni *et al.* 1998). Typically, these problems are modelled by the coupling between the Navier–Stokes equations (or other flow equations) and the system of (linear or non-linear) structural analysis (e.g. Navier equations, the system of elasticity).

Another example is provided by the coexistence of different flow regimes. An instance arises from the simultaneous simulation of surface and subsurface hydraulics, where shallow water equations are coupled to equations describing flow in porous media. At the opposite extreme, for the simulation of the re-entry of a vehicle from the upper atmosphere, the molecular and continuous flow regimes are modelled by coupling the Boltzmann kinetic equations and the Navier–Stokes (respectively, Euler) equations for viscous (respectively, inviscid) compressible flows (see Bourgat *et al.* 1992, 1994).

The propagation of electromagnetic waves in heterogeneous media can be modelled through the Maxwell equations, whose conductivity coefficient degenerates upon a subregion of the computational domain (see Alonso and Valli 1997, and Section 8.5).

This kind of approach is nowadays known as multi-physics or multi-field analysis.

Another class is provided by those problems that, although 'homogeneous' in nature, can be solved in an heterogeneous fashion after reducing the given problem to a simplified one in a subregion of Ω. We encounter situations of this type in fluid dynamics whenever convection-dominated viscous flows yield internal and/or boundary layers. In aerodynamic simulations, ignoring the viscous effects far from sharp layers leads to the coupling of Navier–Stokes and Euler equations or Euler and full potential equations for irrotational isentropic flows. We address these models in Section 8.3.

An analogous coupling between Navier–Stokes equations for incompressible flows and their linearised form (Oseen or Stokes equations) can be suitably adopted to enforce far field boundary conditions more efficiently, see Section 8.2.2.

Similarly, ignoring the thermal conductivity coefficient in a thermodynamic problem, or else the viscous diffusivity in convection–diffusion–reaction equations, leads to the coupling of hyperbolic and parabolic equations.

The potential interest behind the heterogeneous approach is manifold. Using two rather than one model problem allows higher flexibility in devising the numerical method that fits better the nature of the physical phenomenon within each subregion. Most often, far from sharp layers, the expected solution is smooth and exhibits slow variations; hence, a very inexpensive numerical approximation of the reduced problem suffices to produce accurate results. Besides, the set-up of sound numerical algorithms allows the solution to the coupled problem to be obtained through a sequence of independent solves upon each subdomain (see, for a general presentation, Quarteroni *et al.* 1992).

8.1 Heterogeneous models for advection–diffusion equations

The first example that we consider concerns an advection–diffusion problem with dominating advection. As described in Chapter 6, this boundary value problem reads

$$\begin{cases} L_\varepsilon u := -\varepsilon \Delta u + \operatorname{div}(\mathbf{b}u) + a_0 u = f & \text{in } \Omega \\ u = \varphi_D & \text{on } \Gamma_D \\ \varepsilon \dfrac{\partial u}{\partial n^*} = \varphi_N & \text{on } \Gamma_N, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$; the subsets Γ_D and Γ_N provide a partition of $\partial\Omega$, and $\mathbf{n}^*$ is the unit outward normal vector on $\partial\Omega$.

Here f, φ_D and φ_N are prescribed functions, $\varepsilon > 0$ is a parameter, $a_0 \in L^\infty(\Omega)$, $\mathbf{b} \in (L^\infty(\Omega))^d$ with $D_j \mathbf{b} \in (L^\infty(\Omega))^d$ for every $j = 1, \ldots, d$, and satisfy

$$\frac{1}{2} \operatorname{div} \mathbf{b}(\mathbf{x}) + a_0(\mathbf{x}) \geq \mu_0 > 0 \qquad \text{for almost every } \mathbf{x} \in \Omega,$$

$$\mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) \geq 0 \qquad \text{for almost every } \mathbf{x} \in \Gamma_N.$$

These conditions ensure solvability of the problem, because the latter can be associated with a continuous and coercive bilinear form.

We are interested in the case in which the convective field is dimensionally dominating over the diffusion, i.e. $\varepsilon \ll ||\mathbf{b}||_{L^\infty(\Omega)}$, and, more specifically, the diffusion processes are relevant only in a subregion, say Ω_2, of the domain Ω. For example, in a flow problem, Ω_2 could be the region near a physical boundary, while the complementary part, Ω_1, is far from it. We are thus led to considering a reduced problem, which is

FIG. 8.1.1. The computational domain, Ω, and the convective field at the subdomains' interface.

$$
\begin{cases}
L_1 u_1 := \operatorname{div}(\mathbf{b}u_1) + a_0 u_1 = f & \text{in } \Omega_1 \\
u_1 = \varphi_{1,D} & \text{on } \Gamma_{1,D} \\
u_2 = \varphi_{2,D} & \text{on } \Gamma_{2,D} \\
\varepsilon \dfrac{\partial u_2}{\partial n^*} = \varphi_{2,N} & \text{on } \Gamma_{2,N} \\
L_{2,\varepsilon} u_2 := -\varepsilon \Delta u_2 + \operatorname{div}(\mathbf{b}u_2) + a_0 u_2 = f & \text{in } \Omega_2,
\end{cases} \tag{8.1.1}
$$

together with suitable interface conditions on Γ to be determined in a correct way. Here, we have denoted by $\Gamma_{i,D} := \partial\Omega_i \cap \Gamma_D$, $\Gamma_{i,N} := \partial\Omega_i \cap \Gamma_N$, $\varphi_{i,D} := \varphi_{D|\Gamma_{i,D}}$, $i = 1, 2$, $\varphi_{2,N} := \varphi_{N|\Gamma_{2,N}}$, and we suppose that

$$
\mathbf{b}(\mathbf{x}) \cdot \mathbf{n}^*(\mathbf{x}) < 0 \qquad \text{for almost every } \mathbf{x} \in \Gamma_{1,D}.
$$

Having defined

$$
\begin{aligned}
\Gamma^0 &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0\} \\
\Gamma^{\text{in}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\} \\
\Gamma^{\text{out}} &:= \{\mathbf{x} \in \Gamma \,|\, \mathbf{b}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) > 0\},
\end{aligned} \tag{8.1.2}
$$

where $\mathbf{n}$ is the unit normal vector on Γ directed towards Ω_2, it turns out that $\Gamma_{1,D} \cup \Gamma^{\text{in}}$ is the inflow boundary for Ω_1 relative to the convection field $\mathbf{b}$. An illustrative description is provided in Fig. 8.1.1, where arrows indicate the local direction of the vector $\mathbf{b}$.

The operator L_1 is formally the limit of $L_{\varepsilon|\Omega_1}$ as $\varepsilon \to 0+$, while $L_{2,\varepsilon}$ coincides with $L_{\varepsilon|\Omega_2}$. Therefore, the conditions at the interface can be found by the following procedure. We add an elliptic perturbation $-\delta\Delta$ (δ being a small positive

coefficient) to the hyperbolic operator L_1 and impose the associated interface conditions, which, in the current case, express the continuity of the perturbed solutions $u_{1,\delta}$ and $u_{2,\delta}$ as well as the continuity of the fluxes

$$\delta \frac{\partial u_{1,\delta}}{\partial n} - \mathbf{b}\cdot\mathbf{n}\, u_{1,\delta} = \varepsilon \frac{\partial u_{2,\delta}}{\partial n} - \mathbf{b}\cdot\mathbf{n}\, u_{2,\delta}$$

across Γ. It has been proved (see F. Gastaldi *et al.* 1990) that the limit solution, as $\delta \to 0^+$, satisfies (8.1.1) as well as the interface relationships

$$\begin{aligned} & u_1 = u_2 && \text{on } \Gamma^{\text{in}} \\ & -\mathbf{b}\cdot\mathbf{n}\, u_1 = \varepsilon \frac{\partial u_2}{\partial n} - \mathbf{b}\cdot\mathbf{n}\, u_2 && \text{on } \Gamma. \end{aligned} \tag{8.1.3}$$

The flux is therefore continuous across the whole interface, whereas the solution is continuous only across Γ^{in}. Equations (8.1.3) will therefore be added to (8.1.1) to complete the coupled advection–diffusion boundary value problem.

The coupled formulation (8.1.1), (8.1.3) allows the independent solution of a sequence of hyperbolic problems in Ω_1 and elliptic problems in Ω_2, in the framework of iterative processes between subdomains. The different possible treatments of the interface relations is what distinguishes one iterative method from another. In this respect, a very natural approach is defined as follows. Let us summarise the boundary conditions on $\partial\Omega_2 \setminus \Gamma$ in the form $B_2 u_2 = \varphi_2$, with obvious meaning of notation and let λ^0 be a given initial function on Γ. For $k \geq 0$ we define u_1^{k+1} and u_2^{k+1} to be the solutions to the problems

$$\begin{cases} L_1 u_1^{k+1} = f & \text{in } \Omega_1 \\ u_1^{k+1} = \lambda^k & \text{on } \Gamma^{\text{in}} \\ u_1^{k+1} = \varphi_{1,D} & \text{on } \Gamma_{1,D} \end{cases} \tag{8.1.4}$$

$$\begin{cases} L_{2,\varepsilon} u_2^{k+1} = f & \text{in } \Omega_2 \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} - \mathbf{b}\cdot\mathbf{n}\, u_2^{k+1} = -\mathbf{b}\cdot\mathbf{n}\, u_1^{k+1} & \text{on } \Gamma^{\text{out}} \cup \Gamma^0 \\ \varepsilon \dfrac{\partial u_2^{k+1}}{\partial n} = 0 & \text{on } \Gamma^{\text{in}} \\ B_2 u_2^{k+1} = \varphi_2 & \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases} \tag{8.1.5}$$

where

$$\lambda^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1-\theta)\lambda^k \tag{8.1.6}$$

and $\theta > 0$ is an acceleration parameter.

The convergence properties of this method are analysed in F. Gastaldi *et al.* (1990), while several numerical results can be found in Frati *et al.* (1993). Other research papers connected with this approach are those by F. Gastaldi and Quarteroni (1989), Scroggs (1990), Chan and Mathew (1994*b*) and Aguilar and Lisbona (1994).

8.1.1 The Steklov–Poincaré reformulation

Also in this heterogeneous situation, the transmission conditions (8.1.3) can be enforced through an equation on Γ (that can be solved autonomously from (8.1.1)), which is written in terms of the Steklov–Poincaré operator and reads

$$S\lambda = \chi \qquad \text{on } \Gamma. \tag{8.1.7}$$

The right-hand side χ depends on the data of the problem f, $\varphi_{1,D}$, $\varphi_{2,D}$, $\varphi_{2,N}$, while λ is a function on Γ that a posteriori will allow both u_1 in Ω_1 and u_2 in Ω_2 to be obtained via the solution of two distinct subproblems. Note that u_1 and u_2 are not equal on the whole Γ, but only on $\Gamma^{\rm in}$; hence, one could also make different choices for the interface unknown λ; for instance, $\lambda = u_{2|\Gamma^{\rm in}} = u_{1|\Gamma^{\rm in}}$.

In the case we are considering, let λ denote the unknown value of $u_{2|\Gamma}$. Once (8.1.7) has been solved one can find u_1 through the solution to $(8.1.1)_1$, $(8.1.1)_2$, with the additional condition $u_1 = \lambda_{|\Gamma^{\rm in}}$ on $\Gamma^{\rm in}$. Similarly, u_2 can be determined by solving $(8.1.1)_3$, $(8.1.1)_4$ and $(8.1.1)_5$, with the supplementary condition $u_2 = \lambda$ on Γ.

To make precise the definition of the Steklov–Poincaré S and the right-hand side χ, we start by introducing the extension operator K_1. If η is a function defined on Γ, $K_1\eta$ is defined in Ω_1, and satisfies

$$\begin{cases} L_1(K_1\eta) = 0 & \text{in } \Omega_1 \\ K_1\eta = \eta_{|\Gamma^{\rm in}} & \text{on } \Gamma^{\rm in} \\ K_1\eta = 0 & \text{on } \Gamma_{1,D}. \end{cases} \tag{8.1.8}$$

On the other hand, the function $E_2^{\flat}$ is defined in Ω_2 as the solution to

$$\begin{cases} L_{2,\varepsilon}(E_2^{\flat}\eta) = 0 & \text{in } \Omega_2 \\ E_2^{\flat}\eta = \eta & \text{on } \Gamma \\ B_2(E_2^{\flat}\eta) = 0 & \text{on } \partial\Omega_2 \setminus \Gamma \end{cases} \tag{8.1.9}$$

(see also (6.4.12)). For the sake of brevity we shall not introduce the rigorous variational setting for the above problems, but rather we stick with their differential form. The variational formulation can be easily recovered by following the guidelines of Sections 5.6 and 6.4.

The solution $K_1\eta$ can be shown to satisfy the a priori estimate

$$\mu_0||K_1\eta||^2_{0,\Omega_1} + \frac{1}{2}\int_{\Gamma^{\rm out}\cup\Gamma_{1,N}} |\mathbf{b}\cdot\mathbf{n}|(K_1\eta)^2 \leq \frac{1}{2}\int_{\Gamma^{\rm in}} |\mathbf{b}\cdot\mathbf{n}|\eta^2 \tag{8.1.10}$$

(see, for example, F. Gastaldi *et al.* 1990).

Define, moreover, the functions $u_i^* : \Omega_i \to \mathbf{R}$, $i = 1, 2$, that are solutions to the problems

$$\begin{cases} L_1 u_1^* = f & \text{in } \Omega_1 \\ u_1^* = 0 & \text{on } \Gamma^{\rm in} \\ u_1^* = \varphi_{1,D} & \text{on } \Gamma_{1,D} \end{cases} \tag{8.1.11}$$

$$\begin{cases} L_{2,\varepsilon} u_2^* = f & \text{in } \Omega_2 \\ u_2^* = 0 & \text{on } \Gamma \\ B_2 u_2^* = \varphi_2 & \text{on } \partial\Omega_2 \setminus \Gamma. \end{cases} \tag{8.1.12}$$

We are now in a position to define the function χ and the operator S appearing in (8.1.7). It is clear that the solution (u_1, u_2) to the coupled problem (8.1.1), (8.1.3) is given by

$$u_1 = K_1\lambda + u_1^*, \quad u_2 = E_2^\flat\lambda + u_2^*.$$

Therefore, the function λ we are looking for on Γ is precisely the solution to (8.1.7) with χ and S defined as

$$\chi := \mathbf{b}\cdot\mathbf{n}\, u_1^* + \varepsilon\frac{\partial u_2^*}{\partial n} \qquad \text{on } \Gamma \tag{8.1.13}$$

$$S\eta := -\mathbf{b}\cdot\mathbf{n}\, K_1\eta + \mathbf{b}\cdot\mathbf{n}\,\eta - \varepsilon\frac{\partial E_2^\flat\eta}{\partial n} \qquad \text{on } \Gamma, \tag{8.1.14}$$

which can also be written as

$$S\eta = \begin{cases} -\mathbf{b}\cdot\mathbf{n}\, K_1\eta + \mathbf{b}\cdot\mathbf{n}\,\eta - \varepsilon\dfrac{\partial E_2^\flat\eta}{\partial n} & \text{on } \Gamma^{\rm out}\cup\Gamma^0 \\ -\varepsilon\dfrac{\partial E_2^\flat\eta}{\partial n} & \text{on } \Gamma^{\rm in}. \end{cases}$$

If we define the two operators S_1 and S_2 through

$$S_1\eta := \begin{cases} -\mathbf{b}\cdot\mathbf{n}\, K_1\eta & \text{on } \Gamma^{\rm out}\cup\Gamma^0 \\ 0 & \text{on } \Gamma^{\rm in} \end{cases} \tag{8.1.15}$$

$$S_2\eta := \begin{cases} \mathbf{b}\cdot\mathbf{n}\,\eta - \varepsilon\dfrac{\partial E_2^\flat\eta}{\partial n} & \text{on } \Gamma^{\text{out}}\cup\Gamma^0 \\ -\varepsilon\dfrac{\partial E_2^\flat\eta}{\partial n} & \text{on } \Gamma^{\text{in}}, \end{cases} \tag{8.1.16}$$

we obviously have

$$S\eta = S_1\eta + S_2\eta. \tag{8.1.17}$$

The operator S turns out to be continuous and coercive in Λ, where, as usual, we have defined

$$\Lambda := \{\eta \in H^{1/2}(\Gamma)\,|\,\eta = v_{|\Gamma} \text{ for a suitable } v \in V\},$$

having set $V = H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega)\,|\,v_{|\Gamma_D} = 0\}$. In fact, denoting by $\langle\cdot,\cdot\rangle$ the duality pairing between Λ and its dual, we have

$$\langle S_2\eta,\mu\rangle = \int_{\Gamma^{\text{out}}} \mathbf{b}\cdot\mathbf{n}\,\eta\,\mu - \varepsilon\left\langle\frac{\partial E_2^\flat\eta}{\partial n},\mu\right\rangle,$$

and

$$\begin{aligned} -\varepsilon\left\langle\frac{\partial E_2^\flat\eta}{\partial n},\mu\right\rangle &= \int_{\Omega_2}\operatorname{div}[\varepsilon(\nabla E_2^\flat\eta)E_2^\flat\mu] \\ &= \varepsilon\int_{\Omega_2}\nabla E_2^\flat\eta\cdot\nabla E_2^\flat\mu + \int_{\Omega_2}(\varepsilon\Delta E_2^\flat\eta)\,E_2^\flat\mu \\ &= \varepsilon\int_{\Omega_2}\nabla E_2^\flat\eta\cdot\nabla E_2^\flat\mu + \int_{\Omega_2}[\operatorname{div}(\mathbf{b}E_2^\flat\eta) + a_0E_2^\flat\eta]\,E_2^\flat\mu. \end{aligned}$$

Moreover

$$\begin{aligned} \int_{\Omega_2}[\operatorname{div}(\mathbf{b}E_2^\flat\eta)]\,E_2^\flat\mu = &-\int_{\Omega_2}E_2^\flat\eta\,\mathbf{b}\cdot\nabla E_2^\flat\mu \\ &-\int_{\Gamma^{\text{in}}\cup\Gamma^{\text{out}}}\mathbf{b}\cdot\mathbf{n}\,\eta\,\mu + \int_{\Gamma_{2,N}}\mathbf{b}\cdot\mathbf{n}^*E_2^\flat\eta\,E_2^\flat\mu. \end{aligned}$$

In conclusion, we have obtained

$$\begin{aligned} \langle S_2\eta,\mu\rangle = &\varepsilon\int_{\Omega_2}\nabla E_2^\flat\eta\cdot\nabla E_2^\flat\mu + \int_{\Omega_2}a_0E_2^\flat\eta\,E_2^\flat\mu - \int_{\Omega_2}E_2^\flat\eta\,\mathbf{b}\cdot\nabla E_2^\flat\mu \\ &-\int_{\Gamma^{\text{in}}}\mathbf{b}\cdot\mathbf{n}\,\eta\,\mu + \int_{\Gamma_{2,N}}\mathbf{b}\cdot\mathbf{n}^*E_2^\flat\eta\,E_2^\flat\mu. \end{aligned} \tag{8.1.18}$$

On the other hand,

$$\langle S_1\eta,\mu\rangle = -\int_{\Gamma^{\text{out}}}\mathbf{b}\cdot\mathbf{n}\,(K_1\eta)\,\mu. \tag{8.1.19}$$

Therefore, continuity of S_2 and S_1 in Λ is a straightforward consequence of the a priori estimates (6.4.13) and (8.1.10).

The coerciveness is proved as follows: at first

$$\begin{aligned}-\int_{\Omega_2} E_2^\flat\eta\,\mathbf{b}\cdot\nabla E_2^\flat\eta &= -\frac{1}{2}\int_{\Omega_2}\mathbf{b}\cdot\nabla(E_2^\flat\eta)^2\\ &= \frac{1}{2}\int_{\Omega_2}(\operatorname{div}\mathbf{b})\,(E_2^\flat\eta)^2+\frac{1}{2}\int_{\Gamma^{\mathrm{in}}\cup\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2\\ &\quad -\frac{1}{2}\int_{\Gamma_{2,N}}\mathbf{b}\cdot\mathbf{n}^*\,(E_2^\flat\eta)^2,\end{aligned}$$

hence

$$\begin{aligned}\langle S_2\eta,\eta\rangle &= \varepsilon\int_{\Omega_2}|\nabla E_2^\flat\eta|^2+\int_{\Omega_2}\left(\frac{1}{2}\operatorname{div}\mathbf{b}+a_0\right)(E_2^\flat\eta)^2\\ &\quad +\frac{1}{2}\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2-\frac{1}{2}\int_{\Gamma^{\mathrm{in}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2\\ &\quad +\frac{1}{2}\int_{\Gamma_{2,N}}\mathbf{b}\cdot\mathbf{n}^*\,(E_2^\flat\eta)^2.\end{aligned}\tag{8.1.20}$$

On the other hand,

$$\begin{aligned}\langle S_1\eta,\eta\rangle &= -\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,(K_1\eta)\,\eta\\ &= -\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,K_1\eta\,(\eta-K_1\eta)-\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,(K_1\eta)^2\\ &= -\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,K_1\eta\,(\eta-K_1\eta)-\int_{\Omega_1}\operatorname{div}[\mathbf{b}\,(K_1\eta)^2]\\ &\quad +\int_{\Gamma^{\mathrm{in}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2+\int_{\Gamma_{1,N}}\mathbf{b}\cdot\mathbf{n}^*\,(K_1\eta)^2,\end{aligned}$$

and

$$\begin{aligned}-\int_{\Omega_1}\operatorname{div}[\mathbf{b}\,(K_1\eta)^2] &= -\int_{\Omega_1}\operatorname{div}(\mathbf{b}\,K_1\eta)\,K_1\eta-\frac{1}{2}\int_{\Omega_1}\mathbf{b}\cdot\nabla(K_1\eta)^2\\ &= \int_{\Omega_1}\left(\frac{1}{2}\operatorname{div}\mathbf{b}+a_0\right)(K_1\eta)^2-\frac{1}{2}\int_{\Gamma^{\mathrm{in}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2\\ &\quad -\frac{1}{2}\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,(K_1\eta)^2-\frac{1}{2}\int_{\Gamma_{1,N}}\mathbf{b}\cdot\mathbf{n}^*\,(K_1\eta)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle S_1\eta,\eta\rangle &= \int_{\Omega_1}\left(\frac{1}{2}\operatorname{div}\mathbf{b}+a_0\right)(K_1\eta)^2+\frac{1}{2}\int_{\Gamma^{\mathrm{in}}}\mathbf{b}\cdot\mathbf{n}\,\eta^2\\ &\quad -\frac{1}{2}\int_{\Gamma^{\mathrm{out}}}\mathbf{b}\cdot\mathbf{n}\,K_1\eta(2\eta-K_1\eta)+\frac{1}{2}\int_{\Gamma_{1,N}}\mathbf{b}\cdot\mathbf{n}^*\,(K_1\eta)^2.\end{aligned}\tag{8.1.21}$$

Adding (8.1.20) and (8.1.21) we finally have

$$\begin{aligned}\langle S\eta,\eta\rangle = \varepsilon\int_{\Omega_2}|\nabla E_2^\flat\eta|^2 + \int_{\Omega_2}\left(\frac{1}{2}\operatorname{div}\mathbf{b}+a_0\right)(E_2^\flat\eta)^2 \\ + \int_{\Omega_1}\left(\frac{1}{2}\operatorname{div}\mathbf{b}+a_0\right)(K_1\eta)^2 + \frac{1}{2}\int_{\Gamma^{\rm out}}\mathbf{b}\cdot\mathbf{n}\,(\eta-K_1\eta)^2 \\ +\frac{1}{2}\int_{\Gamma_{2,N}}\mathbf{b}\cdot\mathbf{n}^*\,(E_2^\flat\eta)^2 + \frac{1}{2}\int_{\Gamma_{1,N}}\mathbf{b}\cdot\mathbf{n}^*\,(K_1\eta)^2.\end{aligned} \tag{8.1.22}$$

Now the coerciveness of S and S_2 follows from (8.1.22) and (8.1.20), respectively, using (6.4.14).

By analogy with the homogeneous elliptic case considered in Chapter 1 (see, in particular, Section 1.3), it is not difficult to see that (8.1.4)–(8.1.6) is equivalent to the following iterative method for the Steklov–Poincaré equation (8.1.7):

$$S_2(\lambda^{k+1}-\lambda^k) = \theta(\chi - S\lambda^k) \qquad \text{on } \Gamma, \quad k\geq 0. \tag{8.1.23}$$

This is a preconditioned Richardson method for the interface problem (8.1.7), with S_2 as a preconditioner.

A convergence result on the whole Γ for the iterates λ^k defined in (8.1.23) could be obtained as in Section 5.1.2 by means of Theorem 4.2.2. But this would require the convective field $\mathbf{b}$ to be small in comparison with ε, which is not the case we are interested in.

However, proving convergence of λ^k to λ on the whole Γ is not strictly necessary for proving that the iterates u_1^{k+1} and u_2^{k+1} of (8.1.4)–(8.1.6) converge to u_1 and u_2. As a matter of fact, if one succeeds in showing that $\lambda^k_{|\Gamma^{\rm in}}$ converges to a value $\lambda^{\rm in}$ on $\Gamma^{\rm in}$, then a further step (8.1.4), (8.1.5) (taking $\lambda^{\rm in}$ instead of λ^k) will provide the desired solution u_1 and u_2, and therefore the solution $\lambda = u_{2|\Gamma}$ to (8.1.7).

To this end, let us restrict (8.1.7) to a problem set solely on $\Gamma^{\rm in}$. This is easily accomplished by eliminating from (8.1.7) the restriction of λ to $\Gamma^{\rm out}\cup\Gamma^0$; say, $\lambda^{out,0}$. This yields a *reduced* Steklov–Poincaré problem of the form

$$S^{\rm in}\lambda^{\rm in} = \chi^{\rm in} \qquad \text{on } \Gamma^{\rm in}. \tag{8.1.24}$$

A characterisation of $S^{\rm in}$ in terms of S_2 is as follows. Let us refer for simplicity to S_1, S_2 and λ with a 'matrix' notation of the type

$$S_1 = \begin{pmatrix} 0 & S_{12}\\ 0 & 0\end{pmatrix}, \quad S_2 = \begin{pmatrix} S_{21} & S_{22}\\ S_{23} & S_{24}\end{pmatrix}, \quad \lambda = \begin{pmatrix}\lambda^{\rm out,0}\\ \lambda^{\rm in}\end{pmatrix}.$$

Then

$$S^{\rm in} = (S_{24} - S_{23}S_{21}^{-1}S_{22}) - S_{23}S_{21}^{-1}S_{12} =: T_*^{\rm in} + T_{**}^{\rm in}. \tag{8.1.25}$$

The iterative method (8.1.4)–(8.1.6), for the variables $\lambda^{in,k} := \lambda^k_{|\Gamma^{\rm in}}$, can be shown to be equivalent to the following iterations:

$$T^{\rm in}_*(\lambda^{{\rm in},k+1} - \lambda^{{\rm in},k}) = \theta(\chi^{\rm in} - S^{\rm in}\lambda^{{\rm in},k}) \qquad \text{on } \Gamma^{\rm in}, \quad k \geq 0, \tag{8.1.26}$$

which is nothing but a Richardson method for the reduced Steklov–Poincaré problem (8.1.24) where $T^{\rm in}_*$ acts as a preconditioner. At the algebraic level, the matrix associated with the operator $T^{\rm in}_*$ is the Schur complement of the matrix associated with S_2, taken with respect to the variables $\lambda^{\rm in}$ (i.e. after elimination of $\lambda^{{\rm out},0}$).

In F. Gastaldi *et al.* (1990) it is proved that the iteration operator $I - \theta(T^{\rm in}_*)^{-1}S^{\rm in}$ is a contraction in the Hilbert space

$$L^2_b(\Gamma^{\rm in}) := \left\{ v : \Gamma^{\rm in} \to \mathbf{R} \,|\, \int_{\Gamma^{\rm in}} |\mathbf{b}\cdot\mathbf{n}|\, v^2 < \infty \right\},$$

when θ ranges in a suitable neighbourhood of $\theta = 1$. This property yields in particular that the iterative procedure (8.1.26) converges if we take $\theta = 1$. Furthermore, it ensures that $(T^{\rm in}_*)^{-1}S^{\rm in}$ is a coercive operator in $L^2_b(\Gamma^{\rm in})$ and therefore its eigenvalues ζ satisfy $\mathrm{Re}\,\zeta \geq \rho > 0$ for a suitable ρ.

The same properties are enjoyed by the finite dimensional counterparts of the above operators. This enables one to use other iterative methods to solve (8.1.24), such as, for example, non-symmetric conjugate gradient methods or the GMRESt method. In all cases, since the range of the eigenvalues of $(T^{\rm in}_*)^{-1}S^{\rm in}$ is bounded uniformly with respect to the number N of the numerical degrees of freedom, convergence is achieved at a rate independent of N (i.e. $T^{\rm in}_*$ is an optimal preconditioner for $S^{\rm in}$). In the numerical tests the convergence rate looks indeed independent of all critical parameters involved in the equation, including ε and $||\mathbf{b}||_{L^\infty(\Omega)}$.

8.1.2 The coupling for non-linear convection–diffusion equations

Let us consider the viscous Burgers equation, which reads

$$\frac{\partial w}{\partial t} + \mathrm{div}\,\mathbf{f}(w) = \mathrm{div}\,\mathbf{g}(w) \qquad \text{in } \Omega \times (0,T), \tag{8.1.27}$$

where $\mathbf{f}(\cdot)$ and $\mathbf{g}(\cdot)$ are two (a priori non-linear) vector functions, playing the role of inviscid and viscous fluxes, respectively, whilst $w = w(\mathbf{x},t)$ is the scalar unknown function. Without loss of generality we can assume that $\mathbf{g}(w) = \nu\nabla w$, where $\nu > 0$ is a given viscosity. In such a case (8.1.27) is a parabolic advection–diffusion Burgers equation.

Referring to the domain splitting introduced before, the heterogeneous problem associated with (8.1.27) is the one in which the right-hand side is neglected in the subregion Ω_1 of the given computational domain. Denoting by w_i the unknown solution in the subdomain Ω_i, $i = 1, 2$, the reduced coupled problem reads, for each $0 < t < T$:

$$\frac{\partial w_1}{\partial t} + \operatorname{div} \mathbf{f}(w_1) = 0 \qquad \text{in } \Omega_1 \times (0,T) \tag{8.1.28}$$

$$\frac{\partial w_2}{\partial t} + \operatorname{div} \mathbf{f}(w_2) = \nu \Delta w_2 \qquad \text{in } \Omega_2 \times (0,T), \tag{8.1.29}$$

with the interface conditions

$$\begin{aligned} &\nu \nabla w_2 \cdot \mathbf{n} - \mathbf{f}(w_2) \cdot \mathbf{n} = -\mathbf{f}(w_1) \cdot \mathbf{n} && \text{on } \Gamma \times (0,T) \\ &w_1 = w_2 && \text{on } \Gamma_T^{\text{in}}. \end{aligned} \tag{8.1.30}$$

The continuity of normal fluxes on Γ is enforced in equation $(8.1.30)_1$. More precisely, the normal component of the inviscid flux associated with w_1 must be equal to the normal component of the whole flux (inviscid plus viscous) associated with w_2. In equation $(8.1.30)_2$ we have made use of the following notation:

$$\Gamma_T^{\text{in}} := \left\{ (\mathbf{x},t) \in \Gamma \times (0,T) \mid \frac{\partial \mathbf{f}(w_1)}{\partial w}(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x}) < 0 \right\}. \tag{8.1.31}$$

The intersection of Γ_T^{in} with the time level t is the inflow interface; that is, the subset of Γ from which the flow at time t is entering into Ω_1.

In conclusion, the interface conditions for the coupled hyperbolic–parabolic problem (8.1.28), (8.1.29) impose the continuity of the fluxes on the whole Γ, as well as of the solution itself, but limited to the portion of Γ through which the flow is entering into the 'inviscid' domain Ω_1. These conditions generalise to the non-linear scalar case those introduced in (8.1.3) for the linear advection–diffusion equation.

The coupled problem (8.1.28)–(8.1.30) can still be solved by a splitting method that alternates the solution of the second-order boundary value problem in Ω_2 with the Neumann boundary condition induced from $(8.1.30)_1$ on $\Gamma \times (0,T)$, and a first-order problem in Ω_1 with a Dirichlet condition obtained from $(8.1.30)_2$ on the inflow boundary Γ_T^{in}.

Following the discussion in Chapter 7, if the time derivative is discretised by an explicit method, then at each time step the two boundary value problems are independent of one another. Otherwise, they are coupled through the interface conditions and an iterative procedure between subdomains can be devised by splitting the interface conditions as outlined above. Note that the inflow boundary Γ_T^{in} depends in turn on the solution and is therefore unknown; it might, however, be determined on the basis of the latest information available on w_1, such as w_1 at the previous time level, or, even better, w_1 at the previous subdomain iteration in the case of implicit methods.

8.2 Heterogeneous models for incompressible flows

Incompressible flow problems in exterior domains are encountered, for example, when investigating flow past bodies or obstacles, or flow through ducts or pipes

with artificial boundaries representing their inlets or outlets. In this process, it is sometimes convenient to resort to a mechanism of coupling between the full Navier–Stokes system (7.3.2) and a reduced model. The latter can be obtained by applying different kinds of strategies; for example, omitting the viscous stresses from the momentum equations $(7.3.2)_1$, or linearising the convective terms, or even assuming an irrotational flow regime in a subregion of the computational domain.

We analyse some of these models in what follows, after having considered a heterogeneous model for the linear Stokes equations.

8.2.1 The coupling for the linearised Stokes problem

Let us refer to an idealised geometrical situation as depicted in Fig. 8.2.1.

FIG. 8.2.1. The geometrical configuration.

Here, Ω is a bounded domain of $\mathbf{R}^d$, $d = 2, 3$, external to a body whose boundary is Γ_b, and is partitioned into a subdomain Ω_2 next to the body and a far field subdomain Ω_1; the interface between Ω_1 and Ω_2 is denoted by Γ, and we set $\Gamma_\infty := \partial\Omega \setminus \Gamma_b$. Finally, let $\mathbf{n}$ indicate the unit normal vector on Γ directed from Ω_1 to Ω_2, and $\mathbf{n}^*$ the unit outward normal vector on $\partial\Omega$.

The problem we are considering reads:

$$
\text{(8.2.1)} \qquad \begin{cases} \alpha\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_b \\ B\mathbf{u} = \boldsymbol{\varphi}_\infty & \text{on } \Gamma_\infty, \end{cases}
$$

where we have denoted by B the boundary operator on Γ_∞. This problem may arise in the process of solving the full Navier–Stokes equations, when the discretisation of the time derivative is performed by means of a scheme that is explicit in the non-linear convective term. The unknowns $\mathbf{u}$ and p represent velocity and

pressure field, respectively.

The boundary conditions on Γ_∞ have to be prescribed in a suitable way for assuring well-posedness. In this respect, on a portion $\Gamma_\infty^{\text{in}}$ of Γ_∞ an onset flow $\mathbf{u} = \mathbf{u}_\infty^{\text{in}}$ is given. However, assigning conditions on the outflow section $\Gamma_\infty^{\text{out}}$ may not be simple. It is also clear that all interesting flow features occur in the vicinity of the body due to the role of viscosity in this area.

For these reasons, Schenk and Hebeker (1993) have proposed the replacement of (8.2.1) with the following coupled problem, where the viscosity ν is set to 0 in Ω_1:

$$
\begin{cases}
\alpha \mathbf{u}_1 + \nabla p_1 = \mathbf{f} & \text{in } \Omega_1 \\
\operatorname{div} \mathbf{u}_1 = 0 & \text{in } \Omega_1 \\
\mathbf{u}_1 = \mathbf{u}_\infty^{\text{in}} & \text{on } \Gamma_\infty^{\text{in}} \\
p_1 = 0 & \text{on } \Gamma_\infty^{\text{out}} \\
\mathbf{u}_2 = \mathbf{0} & \text{on } \Gamma_b \\
\operatorname{div} \mathbf{u}_2 = 0 & \text{in } \Omega_2 \\
\alpha \mathbf{u}_2 - \nu \Delta \mathbf{u}_2 + \nabla p_2 = \mathbf{f} & \text{in } \Omega_2 \ .
\end{cases}
\tag{8.2.2}
$$

This problem is incomplete, because the matching conditions that have to be fulfilled on Γ are missing.

In Schenk and Hebeker (1993) these conditions are recovered through a singular perturbation analysis similar to that carried out for the advection–diffusion problem in Section 8.1. The limiting coupled problem is proved to be: find $p_1 \in H^1(\Omega_1)$, $p_2 \in L^2(\Omega_2)$ and $\mathbf{u}_2 \in H^1(\Omega_2)$ such that

$$
\begin{cases}
\Delta p_1 = \operatorname{div} \mathbf{f} & \text{in } \Omega_1 \\
\dfrac{\partial p_1}{\partial n^*} = (\mathbf{f} - \alpha \mathbf{u}_\infty^{\text{in}}) \cdot \mathbf{n}^* & \text{on } \Gamma_\infty^{\text{in}} \\
p_1 = 0 & \text{on } \Gamma_\infty^{\text{out}} \\
\dfrac{\partial p_1}{\partial n} = (\mathbf{f} - \alpha \mathbf{u}_2) \cdot \mathbf{n} & \text{on } \Gamma \\
\nu \displaystyle\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j - p_2 n_l = -p_1 n_l & \text{on } \Gamma, \quad l = 1, \ldots, d \\
\mathbf{u}_2 = \mathbf{0} & \text{on } \Gamma_b \\
\operatorname{div} \mathbf{u}_2 = 0 & \text{in } \Omega_2 \\
\alpha \mathbf{u}_2 - \nu \Delta \mathbf{u}_2 + \nabla p_2 = \mathbf{f} & \text{in } \Omega_2.
\end{cases} \tag{8.2.3}
$$

This coupled problem can now be solved in the framework of an iterative process as follows. Assume that $\widehat{\boldsymbol{\lambda}}^0$ is given, satisfying $\int_\Gamma \widehat{\boldsymbol{\lambda}}^0 \cdot \mathbf{n} = 0$, and for any $k \geq 0$ solve

$$
\begin{cases}
\Delta p_1^{k+1} = \operatorname{div} \mathbf{f} & \text{in } \Omega_1 \\
\dfrac{\partial p_1^{k+1}}{\partial n^*} = (\mathbf{f} - \alpha \mathbf{u}_\infty^{\text{in}}) \cdot \mathbf{n}^* & \text{on } \Gamma_\infty^{\text{in}} \\
p_1^{k+1} = 0 & \text{on } \Gamma_\infty^{\text{out}} \\
\dfrac{\partial p_1^{k+1}}{\partial n} = (\mathbf{f} - \alpha \widehat{\boldsymbol{\lambda}}^k) \cdot \mathbf{n} & \text{on } \Gamma,
\end{cases} \tag{8.2.4}
$$

then

$$
\begin{cases}
\alpha \mathbf{u}_2^{k+1} - \nu \Delta \mathbf{u}_2^{k+1} + \nabla p_2^{k+1} = \mathbf{f} & \text{in } \Omega_2 \\
\operatorname{div} \mathbf{u}_2^{k+1} = 0 & \text{in } \Omega_2 \\
\mathbf{u}_2^{k+1} = \mathbf{0} & \text{on } \Gamma_b \\
\nu \displaystyle\sum_j (D_l u_{2,j}^{k+1} + D_j u_{2,l}^{k+1}) n_j - p_2^{k+1} n_l & \\
\qquad = -p_1^{k+1} n_l & \text{on } \Gamma, \quad l = 1, \ldots, d,
\end{cases} \tag{8.2.5}
$$

and finally set

$$
\widehat{\boldsymbol{\lambda}}^{k+1} := \theta \mathbf{u}_{2|\Gamma}^{k+1} + (1 - \theta) \widehat{\boldsymbol{\lambda}}^k \qquad \text{on } \Gamma, \tag{8.2.6}
$$

where $\theta > 0$ is a relaxation parameter.

Since $\operatorname{div} \mathbf{u}_2^{k+1} = 0$ in Ω_2, the trace $\mathbf{u}_{2|\Gamma}^{k+1}$ satisfies

$$\int_\Gamma \mathbf{u}_{2|\Gamma}^{k+1} \cdot \mathbf{n} = 0;$$

hence, $\int_\Gamma \widehat{\boldsymbol{\lambda}}^k \cdot \mathbf{n} = 0$ for each $k \geq 0$. The same holds for $\mathbf{u}_{2|\Gamma}$.

The analysis of the coupled problem and the proof of convergence of the above iterative process are reported in Schenk and Hebeker (1993). They can be performed also by writing the problem in terms of the associated Steklov–Poincaré operators, and then proving convergence by applying the abstract Theorem 4.2.2. A similar approach for the coupling of the compressible Stokes equations will be presented in Section 8.4.

8.2.2 The coupling for the Navier–Stokes equations in exterior domains

We consider now the case of the Navier–Stokes equations in an unbounded exterior domain $\Omega = \mathbf{R}^d \setminus \Omega_b$, where Ω_b is the region of $\mathbf{R}^d$, $d = 2, 3$, occupied by a body (or an obstacle), and Γ_b denotes the boundary of Ω_b.

The problem reads: find $\mathbf{u} : \Omega \to \mathbf{R}^d$ and $p : \Omega \to \mathbf{R}$ satisfying

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma_b \\ \lim\limits_{|\mathbf{x}| \to \infty} \mathbf{u} = \mathbf{u}_\infty. \end{cases} \tag{8.2.7}$$

Typically, the unbounded domain Ω is replaced by a smaller bounded computational region, say Ω_2, with an artificial boundary Γ (see Fig. 8.2.2).

At this stage, we should prescribe suitable far field boundary conditions on Γ to feed the 'interior' Navier–Stokes problem in Ω_2; however, this choice can be quite critical.

A way to circumvent this difficulty is to describe the flow process in the far region Ω_1 beyond Γ by a reduced (linear) model of the Navier–Stokes equations. Clearly, the resulting coupled problem consisting of the Navier–Stokes system in Ω_2 and the reduced linear problem in Ω_1 has to be provided with matching conditions on the interface Γ.

Remark 8.2.1 A slightly different situation is that in which the physical domain Ω is actually bounded; however, the flow field has to be entirely computed only in a subregion of Ω. This is, for instance, the case illustrated in Fig. 8.2.3, where Ω may represent a long channel or a blood vessel with a bifurcation that very likely induces relevant modification of the local flow pattern.

FIG. 8.2.2. External flow problem.

FIG. 8.2.3. Flow in a diverging region.

In this case, the 'far field' Ω_1 (where the reduced problem is solved) obviates the need to prescribe boundary conditions on Γ, which could be difficult to devise with sufficient accuracy.

On the other hand, the treatment of defective boundary conditions at the far field inflow $\Gamma_\infty^{\text{in}}$ and outflow $\Gamma_\infty^{\text{out}}$ is less critical for the reduced model in Ω_1 than it would be for the complete Navier–Stokes system in Ω_2.

Although our discussion will be focused on the situation depicted in Fig. 8.2.2, adapting the results to the case of Fig. 8.2.3 is straightforward. □

Suitable models for describing the flow in the far field Ω_1 are: the incompressible Euler (or linearised Euler) equations, the inviscid full potential equations, the Stokes equations, and the Oseen equations.

The last three cases are treated in Feistauer and Schwab (1996, 1998). For the coupling of the Navier–Stokes equations in Ω_2 with the Stokes equations in Ω_1 they propose the following problem, which has to be used if the velocity field

$\mathbf{u}$ is assumed to be rather small on the interface Γ: for $i = 1, 2$ find $\mathbf{u}_i : \Omega_i \to \mathbf{R}^d$, $p_i : \Omega_i \to \mathbf{R}$ such that

$$
\begin{cases}
-\nu\Delta\mathbf{u}_2 + (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 + \nabla p_2 = \mathbf{f} & \text{in } \Omega_2 \\
\operatorname{div} \mathbf{u}_2 = 0 & \text{in } \Omega_2 \\
\mathbf{u}_2 = \mathbf{0} & \text{on } \Gamma_b \\
\mathbf{u}_2 = \mathbf{u}_1 & \text{on } \Gamma \\
\displaystyle\sum_j T_{lj}(\mathbf{u}_1, p_1) n_j = \sum_j T_{lj}(\mathbf{u}_2, p_2) n_j & \\
\qquad\qquad -\frac{1}{2}|\mathbf{u}_2|^2 n_l & \text{on } \Gamma, \quad l = 1, \ldots, d \\
\lim_{|\mathbf{x}|\to\infty} \mathbf{u}_1 = \mathbf{u}_\infty & \\
\operatorname{div} \mathbf{u}_1 = 0 & \text{in } \Omega_1 \\
-\nu\Delta\mathbf{u}_1 + \nabla p_1 = \mathbf{f} & \text{in } \Omega_1,
\end{cases}
\tag{8.2.8}
$$

where $\mathbf{u}_\infty$ is a prescribed constant vector and $T_{lj}(\mathbf{w}, \omega)$ are the components of the stress tensor

$$T_{lj}(\mathbf{w}, \omega) := \nu(D_l w_j + D_j w_l) - \omega\delta_{lj}.$$

The second interface condition derives from having written the non-linear convective term as

$$(\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 = \frac{1}{2}\nabla\left(|\mathbf{u}_2|^2\right) + (\operatorname{rot} \mathbf{u}_2) \times \mathbf{u}_2.$$

A different way of formulating it is

$$
\begin{aligned}
&\sum_j (D_l u_{1,j} + D_j u_{1,l}) n_j - p_1 n_l \\
&\quad = \sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j - \left(p_2 + \frac{1}{2}|\mathbf{u}_2|^2\right) n_l \qquad \text{on } \Gamma,
\end{aligned}
\tag{8.2.9}
$$

for $l = 1, \ldots, d$, which is perhaps more immediately understandable since the kinematic static pressure p_1 for the Stokes problem plays the same role as the dynamic pressure $p_2 + \frac{1}{2}|\mathbf{u}_2|^2$ for the Navier–Stokes problem.

Note that this choice of the flux interface condition is not guided by the strategy proposed in Remark 1.1.1.

An iterative process can be devised solving at each new step $k+1$ the exterior Stokes problem in Ω_1 with a Dirichlet condition

$$\mathbf{u}_1^{k+1} = \mathbf{u}_2^k \qquad \text{on } \Gamma, \tag{8.2.10}$$

and the interior Navier–Stokes problem in Ω_2 with the interface condition

$$
\begin{aligned}
-\frac{1}{2}|\mathbf{u}_2^{k+1}|^2 n_l + \sum_j T_{lj}(\mathbf{u}_2^{k+1}, p_2^{k+1}) n_j & \\
= \sum_j T_{lj}(\mathbf{u}_1^{k+1}, p_1^{k+1}) n_j \qquad & \text{on } \Gamma, \quad l = 1, \ldots, d,
\end{aligned}
\tag{8.2.11}
$$

with a possible relaxation procedure to update the velocity $\mathbf{u}_2^{k+1}$ on Γ.

The role of the interface conditions can also be reversed: in this case, the Stokes problem in Ω_1 is solved with the Neumann condition

$$
\sum_j T_{lj}(\mathbf{u}_1^{k+1}, p_1^{k+1}) n_j = -\frac{1}{2}|\mathbf{u}_2^k|^2 n_l + \sum_j T_{lj}(\mathbf{u}_2^k, p_2^k) n_j \quad \text{on } \Gamma, \quad l = 1, \ldots, d,
$$

then the Navier–Stokes problem in Ω_2 is solved with the Dirichlet condition

$$
\mathbf{u}_2^{k+1} = \mathbf{u}_1^{k+1} \qquad \text{on } \Gamma.
$$

The proof of the convergence of the preceding iterative schemes is still an open problem.

Another heterogeneous model is obtained by coupling Navier–Stokes equations in Ω_2 and Oseen equations (see Section 5.3) in Ω_1. The model is suitable if the velocity field $\mathbf{u}$ is supposed to be close to the far field $\mathbf{u}_\infty$ on the interface Γ. The coupled problem can still be formulated as in (8.2.8) upon replacing $(8.2.8)_8$ with the Oseen momentum equation

$$
-\nu \Delta \mathbf{u}_1 + (\mathbf{u}_\infty \cdot \nabla)\mathbf{u}_1 + \nabla p_1 = \mathbf{f} \qquad \text{in } \Omega_1,
$$

and the interface equation $(8.2.8)_5$ by

$$
\begin{aligned}
-\frac{1}{2}(\mathbf{u}_\infty \cdot \mathbf{n}) u_{1,l} + \sum_j T_{lj}(\mathbf{u}_1, p_1) n_j & \\
= -\frac{1}{2}(\mathbf{u}_2 \cdot \mathbf{n}) u_{2,l} + \sum_j T_{lj}(\mathbf{u}_2, p_2) n_j \qquad & \text{on } \Gamma, \quad l = 1, \ldots, d,
\end{aligned}
$$

which comes from having written the convective term in the skew-symmetric form

$$
(\mathbf{u}_\infty \cdot \nabla) u_{1,l} = \frac{1}{2} \sum_{j=1}^{d} D_j(u_{\infty,j} u_{1,l}) + \frac{1}{2}(\mathbf{u}_\infty \cdot \nabla) u_{1,l}
$$

for $l = 1, \ldots, d$ (see also the treatment of the convective term in the case of linear advection–diffusion equations in (5.1.3) and Section 6.4). Again, we note that the choice of this interface condition does not fit the guidelines presented in Remark 1.1.1.

The iterative procedure for the solution at each step of Navier–Stokes equations in Ω_2 and Oseen equations in Ω_1 can be defined similarly to what was carried out for problem (8.2.8).

A further heterogeneous model advocated in Feistauer and Schwab (1996) can be adopted under the assumption that the flow is inviscid and irrotational in the exterior domain Ω_1. Assuming that Ω_1 is simply connected, we can therefore postulate the existence of a velocity potential ϕ_1 such that $\nabla\phi_1 = \mathbf{u}_1$ in Ω_1. In view of the continuity equation we obtain

$$\Delta\phi_1 = 0 \qquad \text{in } \Omega_1. \tag{8.2.12}$$

Correspondingly, assuming that the external force field is irrotational in Ω_1, so that $\mathbf{f}_{|\Omega_1} = \nabla\psi_1$, the pressure p_1 is determined from ϕ_1 with the aid of the Bernoulli equation

$$p_1 + \frac{1}{2}|\nabla\phi_1|^2 - \psi_1 = C_\infty, \tag{8.2.13}$$

where the constant C_∞ is given by

$$C_\infty := p_\infty + \frac{1}{2}|\mathbf{u}_\infty|^2 - \psi_{1,\infty},$$

p_∞, $\mathbf{u}_\infty$ and $\psi_{1,\infty}$ being the constant pressure, velocity and force potential of the homogeneous flow at infinity (for $|\mathbf{x}| \to \infty$).

A first interface condition expresses the continuity across Γ of the normal component of the velocity fields $\mathbf{u}_1$ and $\mathbf{u}_2$.

Besides, since the dynamic pressure plays the same role for both the Euler and Navier–Stokes equations, the following equality holds on Γ:

$$\begin{aligned}\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j - \left(p_2 + \frac{1}{2}|\mathbf{u}_2|^2\right) n_l \\ = -\left(p_1 + \frac{1}{2}|\nabla\phi_1|^2\right) n_l = -(C_\infty + \psi_1) n_l \quad \text{on } \Gamma, \quad l = 1, \ldots, d,\end{aligned} \tag{8.2.14}$$

where the last equality follows from the Bernoulli equation.

The resulting problem reads:

$$\begin{cases} -\nu\Delta\mathbf{u}_2 + (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2 + \nabla p_2 = \mathbf{f} & \text{in } \Omega_2 \\ \operatorname{div}\mathbf{u}_2 = 0 & \text{in } \Omega_2 \\ \mathbf{u}_2 = \mathbf{0} & \text{on } \Gamma_b \\ \displaystyle\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j - \left(p_2 + \frac{1}{2}|\mathbf{u}_2|^2\right) n_l \\ \qquad = -(C_\infty + \psi_1) n_l & \text{on } \Gamma, \quad l = 1, \ldots, d \end{cases} \tag{8.2.15}$$

$$\text{(8.2.16)} \qquad \begin{cases} \Delta\phi_1 = 0 & \text{in } \Omega_1 \\ \dfrac{\partial\phi_1}{\partial n} = \mathbf{u}_2 \cdot \mathbf{n} & \text{on } \Gamma \\ \lim\limits_{|\mathbf{x}|\to\infty} (\phi_1 - \phi_{1,\infty}) = 0, \end{cases}$$

where the velocity potential $\phi_{1,\infty}$ is defined as $\phi_{1,\infty}(\mathbf{x}) := \mathbf{x} \cdot \mathbf{u}_\infty$, $\mathbf{x} \in \Omega_1$.

Although well posed from the mathematical point of view, the physical significance of the heterogeneous problem (8.2.15), (8.2.16) is questionable. Actually, in this form, the Navier–Stokes problem (8.2.15) is independent of the exterior problem and can be solved first. Then the associated velocity $\mathbf{u}_2$ is used to provide the Neumann data for the exterior potential problem (8.2.16). The Navier–Stokes problem in the near field domain is influenced by the external problem solely through the potential ψ_1 and the constant C_∞, which accounts for the far field quantities p_∞, $\mathbf{u}_\infty$ and $\psi_{1,\infty}$ (the homogeneous flow at infinity).

The coupling of the Navier–Stokes and the Euler equations for incompressible flows has also been considered by C. Xu (1996), using the spectral approximation method.

8.3 Heterogeneous models for compressible flows

Compressible flows are modelled through the Navier–Stokes equations, that express the conservation of mass, momentum, and energy of a fluid. They provide a complete description of all flow phenomena, but need empirical laws to relate the viscous coefficients and the heat conductivity coefficient to the thermodynamic flow variables. Different types of formulations can be used for these equations, depending upon the set of variables that are chosen to represent the flow field. Here, we consider the conservative form already presented in Section 7.3.2 (and use the same notation introduced therein):

$$\text{(8.3.1)} \qquad \frac{\partial \mathbf{W}}{\partial t} + \mathbf{div}\, \mathbf{F}(\mathbf{W}) = \mathbf{div}\, \mathbf{G}(\mathbf{W}) \qquad \text{in } \Omega \times (0, T).$$

Omitting the diffusive term $\mathbf{G}(\mathbf{W})$, we obtain the Euler equations

$$\text{(8.3.2)} \qquad \frac{\partial \mathbf{W}}{\partial t} + \mathbf{div}\, \mathbf{F}(\mathbf{W}) = \mathbf{0} \qquad \text{in } \Omega \times (0, T),$$

which describe the motion of inviscid and non-conductive compressible fluid flows.

The Euler system (8.3.2) shares with the Navier–Stokes system the same number of equations $d + 2$. However, the viscous stress and the heat-conduction terms are no longer considered; neither are those equations that might supplement (8.3.1) when the Navier–Stokes equations are Reynolds averaged through suitable turbulence models (e.g. Lesieur 1997).

FIG. 8.3.1. The three-subdomain decomposition around an airfoil.

When the flow field is potential and isentropic, the Euler equations can be further simplified by resorting to the so-called *full potential* equation, which is a single, scalar equation. In this case, the conservation of mass reads

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \nabla \varphi) = 0 \qquad \text{in } \Omega \times (0,T), \tag{8.3.3}$$

where φ is the velocity potential, i.e. $\mathbf{u} = \nabla\varphi$, and another equation for the potential φ has to be determined by means of a suitable conservation principle (see Hirsch 1990, and Section 8.3.2).

Since equation (8.3.3) has no mechanism to generate entropy variations across discontinuities, it allows shock discontinuities that do not satisfy the Rankine–Hugoniot relations. Indeed, both mass and energy are conserved, whereas the momentum $\rho\mathbf{u}$ is not.

For the simulation of the flow field around an airfoil, a heterogeneous model based on the simultaneous use of three different sets of equations could be envisaged: the Navier–Stokes equations are used in the boundary layer and in the downstream wake, the Euler equations in the surrounding region where the shock may develop, and the full potential equation in the far field where the flow is irrotational (see Fig. 8.3.1). This is a very sophisticated approach that requires the treatment of interaction between couples of submodels such as Navier–Stokes with Euler, and Euler with full potential. We give a short account of both these heterogeneous models in the following pages.

For ease of notation, in this section we will refer to the simple geometrical situation in which the spatial computational domain Ω is split into two disjoint subdomains Ω_1 and Ω_2, wherein the different models are used. As usual, we will denote by $\mathbf{n}$ the unit outward normal vector to Ω_1 on Γ, the latter being the interface between Ω_1 and Ω_2.

8.3.1 The coupling between the Navier–Stokes and Euler equations

This approach is motivated by the fact that experimental evidence has shown that at high Reynolds numbers (that is, when the viscosity coefficients are small, or, equivalently, the convection is dominant), apart from a thin region near the

body surface, the velocity of a fluid flow around a streamlined body is of the same order of magnitude as the free-stream velocity and the inertial forces are dominant over the viscous stresses. On the other hand, near the body surface the viscous stresses cannot be ignored even for very small viscosities, due to the presence of boundary layers. Consequently, the flow-field can be split into two regions: the near-body domain Ω_2, that includes the boundary layer, and the outer region Ω_1 (see Fig. 8.2.1).

In Ω_1 we can omit the viscous stresses from the governing equations. The resulting problem is a heterogeneous model that entails the coupling between the Navier–Stokes equations in Ω_2 and the Euler equations in Ω_1; that is,

$$\frac{\partial \mathbf{W}_1}{\partial t} + \mathbf{div}\,\mathbf{F}(\mathbf{W}_1) = \mathbf{0} \qquad \text{in } \Omega_1 \times (0,T) \tag{8.3.4}$$

$$\frac{\partial \mathbf{W}_2}{\partial t} + \mathbf{div}\,\mathbf{F}(\mathbf{W}_2) = \mathbf{div}\,\mathbf{G}(\mathbf{W}_2) \qquad \text{in } \Omega_2 \times (0,T). \tag{8.3.5}$$

Finding the matching at interface Γ is a more difficult mathematical problem. Let us start by considering the following one-dimensional parabolic model problem: find $\mathbf{W}$ satisfying

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} = \frac{\partial \mathbf{G}(\mathbf{W})}{\partial x} \qquad \text{in } \Omega \times (0,T), \tag{8.3.6}$$

where $\mathbf{W} = (W_1, W_2, W_3)$, $\Omega = (\alpha, \beta)$ is a one-dimensional interval that is supposed to be partitioned into the subdomains $\Omega_1 = (\alpha, \gamma)$, $\Omega_2 = (\gamma, \beta)$ for a suitable γ. We assume that $\mathbf{F}(\mathbf{W})$ is a hyperbolic flux, namely, its Jacobian $A(\mathbf{W}) := \partial \mathbf{F}/\partial \mathbf{W}$ is a matrix with real eigenvalues, and that the right-hand side is a second-order elliptic term. Therefore, $\mathbf{F}(\cdot)$ and $\mathbf{G}(\cdot)$ play the roles of convective and diffusive fluxes, respectively.

The coupled hyperbolic–parabolic problem associated with equation (8.3.6) is, for all $0 < t < T$,

$$\frac{\partial \mathbf{W}_1}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W}_1)}{\partial x} = \mathbf{0} \qquad \text{for } \alpha < x < \gamma \tag{8.3.7}$$

$$\frac{\partial \mathbf{W}_2}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W}_2)}{\partial x} = \frac{\partial \mathbf{G}(\mathbf{W}_2)}{\partial x} \qquad \text{for } \gamma < x < \beta. \tag{8.3.8}$$

Denoting by L and Λ, respectively, the matrices of the left eigenvectors and associated eigenvalues of $A(\mathbf{W})$ (Λ is therefore a diagonal matrix), the interface matching conditions to be satisfied at the point $x = \gamma$ are

$$\sum_{r=1}^{3} L_{qr} W_{1,r} = \sum_{r=1}^{3} L_{qr} W_{2,r} \qquad \text{at } x = \gamma, \quad t \in (0,T) \tag{8.3.9}$$

for all indices q associated with the negative eigenvalues (see Section 7.3.3), and

$$\mathbf{G}(\mathbf{W}_2) - \mathbf{F}(\mathbf{W}_2) = -\mathbf{F}(\mathbf{W}_1) \qquad \text{at } x = \gamma, \quad t \in (0, T). \tag{8.3.10}$$

Condition (8.3.10) expresses the continuity of fluxes at $x = \gamma$: the inviscid flux associated with $\mathbf{W}_1$ is equal to the complete (inviscid plus viscous) flux associated with $\mathbf{W}_2$. This is precisely the one-dimensional vector analogue of the multi-dimensional scalar condition expressed by equation $(8.1.30)_1$, which was established for the multi-dimensional scalar equation. On the other hand, (8.3.9) states the continuity of the 'characteristic' variables $(L\mathbf{W})_q$ for all indices q that are associated with a characteristic line entering into the hyperbolic subdomain Ω_1. Again, this property can be regarded as the one-dimensional vector analogue of the condition expressed by equation $(8.1.30)_2$. This analysis was carried out in F. Gastaldi and Quarteroni (1989) for the case of linear systems with constant coefficients.

We can now turn back to the non-linear, multi-dimensional case in which the Euler equations in Ω_1 are coupled with the Navier–Stokes equations in Ω_2, see (8.3.4), (8.3.5). As an immediate generalisation of the case considered in the previous subsections, following $(8.1.30)_1$ and (8.3.10) we can guess that the normal fluxes are continuous across interface Γ at all time t, i.e.

$$[\mathbf{G}(\mathbf{W}_2) - \mathbf{F}(\mathbf{W}_2)] \cdot \mathbf{n} = -\mathbf{F}(\mathbf{W}_1) \cdot \mathbf{n} \qquad \text{on } \Gamma \times (0, T). \tag{8.3.11}$$

An a priori less obvious task is how to generalise the matching conditions given by $(8.1.30)_2$ and (8.3.9) for the problem at hand. One possibility is to proceed as in Section 7.3.3, first rewriting equation (8.3.4) in non-conservative form, like (7.3.34). Recalling that the hyperbolic flux $\mathbf{F}(\mathbf{W})$ is a $(d+2) \times d$ matrix whose entries will be denoted by $F_{rl}(\mathbf{W})$, we can write (8.3.4) as

$$\frac{\partial \mathbf{W}_1}{\partial t} + \sum_{l=1}^{d} A^{(l)}(\mathbf{W}_1) D_l \mathbf{W}_1 = \mathbf{0} \qquad \text{in } \Omega_1 \times (0, T), \tag{8.3.12}$$

where the $(d+2) \times (d+2)$ matrices $A^{(l)}$ are defined as $A^{(l)}_{rs} := \partial F_{rl} / \partial W_s$.

For each time level t and each point $\mathbf{x} \in \Gamma$, denoting by $C = C(\mathbf{n})$ the characteristic matrix $C = \sum_l n_l A^{(l)}$, and by L the matrix whose rows are the left eigenvectors of C, the matching conditions at time t are

$$\sum_{r=1}^{5} L_{qr} W_{1,r} = \sum_{r=1}^{5} L_{qr} W_{2,r} \qquad \text{at } \mathbf{x} \in \Gamma, \tag{8.3.13}$$

for all indices q associated with negative eigenvalues of C.

The iteration-by-subdomain algorithm used for solving (8.3.7), (8.3.8) alternates the solution of the second-order problem (8.3.8) with the Neumann boundary condition on $\Gamma \times (0, T)$ given by (8.3.11), and the solution of the first-order

FIG. 8.3.2. Converging–diverging nozzle.

problem (8.3.12) with the Dirichlet boundary condition (8.3.13), which, at each time level t and at each point $\mathbf{x}$ of Γ, has to be imposed for all indices q that correspond to a negative eigenvalue of C.

Numerical evidence about the effectiveness of this coupling procedure is given in Quarteroni and Stolcis (1995).

8.3.2 The coupling between the Euler equations and the full potential equation

As previously pointed out, the Euler equations provide an acceptable mathematical description of compressible flows whenever the viscous stresses and the heat conducting terms can be ignored. This is the case for flows around an obstacle (e.g. an airfoil) far from the obstacle (and from the boundary layer induced in its vicinity), or flows in ducts or pipes (e.g. the converging–diverging nozzle of Fig. 8.3.2).

If there are regions of irrotational and isentropic flow, the full potential equation (8.3.3) can be used instead of the Euler equations, yielding two scalar equations rather than the $d+1$ equations of the isentropic Euler system.

Before facing the issue of the heterogeneous coupling between the Euler and full potential equations, it is worthwhile to address the full potential equation alone in order to discuss the interface conditions to be used in the event of adopting a homogeneous coupling involving solely the full potential.

Assume, therefore, that the full potential equation is considered in the whole domain Ω; we aim to consider it separately in Ω_1 and Ω_2 and couple the corresponding solutions through suitable matching conditions on interface Γ between Ω_1 and Ω_2.

Since the flow is potential and isentropic, the conservation of momentum now reads

$$\frac{\partial \nabla\varphi}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{c^2}{\rho}\nabla\rho = \mathbf{0} \qquad \text{in } \Omega\times(0,T), \tag{8.3.14}$$

where, as usual, $c := \sqrt{p'(\rho)}$ is the sound speed. On the other hand,

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 + \operatorname{rot}\mathbf{u}\times\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2$$

and

$$\frac{c^2}{\rho}\nabla\rho = \nabla\int\frac{c^2}{\rho}d\rho.$$

Since the gas is ideal and polytropic, namely, $p = R\rho\theta$ and $e = c_V\theta$, where $R = c_P - c_V$ is the difference between the specific heats, for an isentropic flow we also have $p = K\rho^\gamma$, where $K > 0$ is a suitable constant and γ is the ratio of specific heats. Hence

$$\begin{aligned}\int\frac{c^2}{\rho}d\rho &= \int K\gamma\rho^{\gamma-2}d\rho = K\frac{\gamma}{\gamma-1}\rho^{\gamma-1} + \text{const}\\ &= \frac{\gamma}{\gamma-1}\frac{p}{\rho} + \text{const} = \frac{\gamma}{\gamma-1}R\theta + \text{const}\\ &= c_P\theta + \text{const}.\end{aligned}$$

Introducing the enthalpy per unit mass, $h = e + \frac{p}{\rho}$, one has

$$h = c_V\theta + R\theta = c_P\theta.$$

Thus, the equation of conservation of momentum finally reads

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + h = H_0 \qquad \text{in } \Omega\times(0,T), \tag{8.3.15}$$

where H_0 is the (constant) stagnation total enthalpy.

In the case of isentropic flows, the second law of thermodynamics (7.3.11), expressed in terms of the enthalpy h, reads

$$dh - \frac{c^2}{\rho}d\rho = 0,$$

because the left-hand side equals θds. Hence, differentiating in t equation (8.3.15) and using (8.3.3) we find

$$\begin{aligned}\frac{\partial^2\varphi}{\partial t^2} &+ \frac{\partial h}{\partial t} + \frac{\partial\nabla\varphi}{\partial t}\cdot\nabla\varphi\\ &= \frac{\partial^2\varphi}{\partial t^2} + \frac{c^2}{\rho}\frac{\partial\rho}{\partial t} + \frac{\partial\nabla\varphi}{\partial t}\cdot\nabla\varphi\\ &= \frac{\partial^2\varphi}{\partial t^2} - \frac{c^2}{\rho}\operatorname{div}(\rho\nabla\varphi) + \frac{\partial\nabla\varphi}{\partial t}\cdot\nabla\varphi = 0.\end{aligned}$$

By a direct computation, differentiating equation (8.3.14) with respect to $\mathbf{x}$ one also has

$$
\begin{aligned}
-\frac{c^2}{\rho}\,\mathrm{div}(\rho\nabla\varphi) &= -c^2\Delta\varphi - \frac{c^2}{\rho}\nabla\varphi\cdot\nabla\rho \\
&= -c^2\Delta\varphi - \nabla\varphi\cdot\nabla h \\
&= -c^2\Delta\varphi + \nabla\varphi\cdot\left[\frac{\partial\nabla\varphi}{\partial t} + \frac{1}{2}\nabla|\nabla\varphi|^2\right].
\end{aligned}
$$

We have thus found the single equation for the potential φ:

$$
\frac{\partial^2\varphi}{\partial t^2} - c^2\Delta\varphi + \nabla\varphi\cdot[(\nabla\varphi\cdot\nabla)\nabla\varphi] + 2\nabla\varphi\cdot\frac{\partial\nabla\varphi}{\partial t} = 0 \qquad \text{in } \Omega\times(0,T). \tag{8.3.16}
$$

Upon introducing the vector unknown

$$
\mathbf{Z} := (D_t\varphi, D_1\varphi, \ldots, D_d\varphi),
$$

we also obtain the following quasi-linear vector form of the full potential equation

$$
\frac{\partial\mathbf{Z}}{\partial t} + \sum_{l=1}^{d} A^{(l)} D_l Z = \mathbf{0} \qquad \text{in } \Omega\times(0,T), \tag{8.3.17}
$$

where, for $d = 3$, the Jacobian matrices $A^{(l)}$ are given by

$$
A^{(1)} = \begin{pmatrix} 2u_1 & u_1^2 - c^2 & u_1u_2 & u_1u_3 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, A^{(2)} = \begin{pmatrix} 2u_2 & u_1u_2 & u_2^2 - c^2 & u_2u_3 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
A^{(3)} = \begin{pmatrix} 2u_3 & u_1u_3 & u_2u_3 & u_3^2 - c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
$$

If $d = 2$, they reduce to $A^{(1)}$ and $A^{(2)}$ upon omitting the last row and column.

The hyperbolic form (8.3.17) suggests the coupling mechanism at the interface. As a matter of fact, following the guidelines of Section 7.2.2 we deduce that the whole vector $C(\mathbf{n})\mathbf{Z}$ is continuous across the interface, where $C(\mathbf{n}) := \sum_{l=1}^{d} n_l A^{(l)}$ is the characteristic matrix.

To analyse what consequences can be drawn about the continuity of the unknown variables $\mathbf{Z}$, let us investigate the spectral properties of $C(\mathbf{n})$. If $d = 3$, its eigenvalues are

$$
\lambda_1 = \mathbf{u}\cdot\mathbf{n} + c, \quad \lambda_2 = \mathbf{u}\cdot\mathbf{n} - c, \quad \lambda_3 = \lambda_4 = 0.
$$

The first two of them coincide with those of the Euler system (see (7.3.35)), whereas the third and fourth are zero. The reason is that, in contrast with the Euler case, the two characteristic curves along which vorticity and entropy

propagate are missing (remember that vorticity is zero and entropy is constant for irrotational and isentropic flows).

In the one-dimensional case, the full potential equation (8.3.16) becomes

$$\frac{\partial^2\varphi}{\partial t^2} - (c^2 - u^2)\frac{\partial^2\varphi}{\partial x^2} + 2u\frac{\partial^2\varphi}{\partial t\partial x} = 0, \tag{8.3.18}$$

where $u = \frac{\partial\varphi}{\partial x}$ is the flow velocity. The eigenvalues of the characteristic matrix

$$A = \begin{pmatrix} 2u & u^2 - c^2 \\ -1 & 0 \end{pmatrix}$$

are $\lambda_1 = u + c$ and $\lambda_2 = u - c$, and the matrix of left eigenvectors is given by

$$L = \begin{pmatrix} -\dfrac{1}{c} & 1 - \dfrac{u}{c} \\ \dfrac{1}{c} & 1 + \dfrac{u}{c} \end{pmatrix}.$$

The characteristic variables are defined through the relations $d\tilde{Z} = L dZ$, i.e.

$$d\tilde{Z}_1 = -\frac{1}{c}dZ_1 + \left(1 - \frac{u}{c}\right)dZ_2, \quad d\tilde{Z}_2 = \frac{1}{c}dZ_1 + \left(1 + \frac{u}{c}\right)dZ_2, \tag{8.3.19}$$

where $Z_1 = \frac{\partial\varphi}{\partial t}$ and $Z_2 = \frac{\partial\varphi}{\partial x} = u$.

For isentropic flows the sound speed can be written as

$$\begin{aligned} c^2 &= p'(\rho) = \gamma\frac{p}{\rho} = \gamma R\theta \\ &= (\gamma - 1)c_P\theta = (\gamma - 1)h = (\gamma - 1)\left(H_0 - Z_1 - \frac{Z_2^2}{2}\right), \end{aligned} \tag{8.3.20}$$

having used (8.3.15). Then we find that

$$dZ_1 = -\frac{2}{\gamma - 1}c\,dc - Z_2\,dZ_2 = -\frac{2}{\gamma - 1}c\,dc - u\,du,$$

and therefore

$$\begin{aligned} d\tilde{Z}_1 &= \frac{2}{\gamma - 1}dc + \frac{u}{c}du + \left(1 - \frac{u}{c}\right)du = \frac{2}{\gamma - 1}dc + du \\ d\tilde{Z}_2 &= -\frac{2}{\gamma - 1}dc - \frac{u}{c}du + \left(1 + \frac{u}{c}\right)du = -\frac{2}{\gamma - 1}dc + du. \end{aligned}$$

Integrating these relations we obtain

$$\tilde{Z}_1 = u + \frac{2}{\gamma - 1}c, \quad \tilde{Z}_2 = u - \frac{2}{\gamma - 1}c, \tag{8.3.21}$$

which coincides with the first two Riemann invariants of the one-dimensional isentropic Euler system (see (7.3.30)).

To simplify the notation, let us denote for velocity u and sound speed c the functional dependence on the potential φ as follows:

$$u = u(\varphi) := \frac{\partial \varphi}{\partial x}$$
$$c = c(\varphi) := \sqrt{(\gamma - 1)\left[H_0 - \frac{\partial \varphi}{\partial t} - \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2\right]}.$$

Repeating the analysis of Section 7.3.3, we obtain the following two-domain formulation for the one-dimensional full potential equation:

$$\text{(8.3.22)} \qquad \begin{cases} \dfrac{\partial^2 \varphi_1}{\partial t^2} - (c(\varphi_1)^2 - u(\varphi_1)^2)\dfrac{\partial^2 \varphi_1}{\partial x^2} & \\ \qquad\qquad +2u(\varphi_1)\dfrac{\partial^2 \varphi_1}{\partial t \partial x} = 0 & \text{in } \Omega_1 \times (0,T) \\ \dfrac{\partial^2 \varphi_2}{\partial t^2} - (c(\varphi_2)^2 - u(\varphi_2)^2)\dfrac{\partial^2 \varphi_2}{\partial x^2} & \\ \qquad\qquad +2u(\varphi_2)\dfrac{\partial^2 \varphi_2}{\partial t \partial x} = 0 & \text{in } \Omega_2 \times (0,T) \\ u(\varphi_1) - \dfrac{2}{\gamma - 1}c(\varphi_1) = u(\varphi_2) - \dfrac{2}{\gamma - 1}c(\varphi_2) & \text{at } x_\Gamma \times (0,T) \\ u(\varphi_1) + \dfrac{2}{\gamma - 1}c(\varphi_1) = u(\varphi_2) + \dfrac{2}{\gamma - 1}c(\varphi_2) & \text{at } x_\Gamma \times (0,T), \end{cases}$$

where x_Γ is the interface point separating the subintervals Ω_1 and Ω_2. As usual, this system has to be supplemented by the initial condition and the additional boundary conditions for each characteristic variable whose associated characteristic line is incoming.

An iterative procedure like (7.3.31), (7.3.32) can be defined in order to prescribing the values of the incoming Riemann invariants for the interval at hand. Assuming that Ω_1 is on the left side of Ω_2, that u is positive at x_Γ and that the flow is subsonic, at the kth step we prescribe

$$u(\varphi_1^{k+1}) - \frac{2}{\gamma - 1}c(\varphi_1^{k+1}) = u(\varphi_2^k) - \frac{2}{\gamma - 1}c(\varphi_2^k) \qquad \text{at } x_\Gamma \times (0,T)$$
$$u(\varphi_2^{k+1}) + \frac{2}{\gamma - 1}c(\varphi_2^{k+1}) = u(\varphi_1^k) + \frac{2}{\gamma - 1}c(\varphi_1^k) \qquad \text{at } x_\Gamma \times (0,T),$$

whereas for supersonic flows we assign both values to the downstream domain Ω_2; that is,

$$u(\varphi_2^{k+1}) - \frac{2}{\gamma - 1}c(\varphi_2^{k+1}) = u(\varphi_1^k) - \frac{2}{\gamma - 1}c(\varphi_1^k) \qquad \text{at } x_\Gamma \times (0,T)$$
$$u(\varphi_2^{k+1}) + \frac{2}{\gamma - 1}c(\varphi_2^{k+1}) = u(\varphi_1^k) + \frac{2}{\gamma - 1}c(\varphi_1^k) \qquad \text{at } x_\Gamma \times (0,T).$$

Observe that the continuity of $\tilde{Z}_1$ and $\tilde{Z}_2$ at the interface ensures that of u and c, hence that of $\frac{\partial \varphi}{\partial x}$ (the potential flux) and $\frac{\partial \varphi}{\partial t}$ (through (8.3.20)), or else, equivalently, that of $\frac{\partial \varphi}{\partial x}$ and φ. This may encourage the use of a multi-domain formulation directly on the multi-dimensional scalar equation (8.3.16); in that case, the subdomain iterations should aim at enforcing the continuity of both quantities in the framework of a Dirichlet–Neumann strategy.

The above characteristic analysis suggests how to couple the full potential equation with the Euler equations in the one-dimensional isentropic case. In fact, the characteristic lines are the same (in both cases associated with the eigenvalues $u + c$ and $u - c$), as well as the Riemann invariants $u \pm \frac{2}{\gamma - 1} c$. The heterogeneous coupling is appropriate, for instance, to simulating the flow field for the converging–diverging nozzle of Fig. 8.3.2: in that case, the full potential equation is used in the two extreme sections of the duct Ω_1^{in} and Ω_1^{out}, and the Euler equations in the inner domain Ω_2. Note also that the Euler equations in the converging–diverging nozzle can indeed be reduced to a one-dimensional system of similar structure, by means of a suitable transformation in terms of the cross-section $S(x)$ (see, for example, Hirsch 1990, pp. 157–8).

We can proceed as follows. Let $\Omega_1 = (\alpha, x_\gamma)$ be the domain where we consider the full potential equation, and $\Omega_2 = (x_\gamma, \beta)$ the domain where the Euler equations are enforced. Consider first the subsonic case $0 < u < c$. Then the iterative scheme reads (for notation see (7.3.27), (7.3.28)):

$$\text{(8.3.23)} \qquad \begin{cases} \dfrac{\partial^2 \varphi_1^{k+1}}{\partial t^2} - [c(\varphi_1^{k+1})^2 - u(\varphi_1^{k+1})^2] \dfrac{\partial^2 \varphi_1^{k+1}}{\partial x^2} \\ \qquad\qquad +2u(\varphi_1^{k+1}) \dfrac{\partial^2 \varphi_1^{k+1}}{\partial t \partial x} = 0 & \text{in } \Omega_1 \times (0,T) \\ \dfrac{\partial \varphi_1^{k+1}}{\partial x} = u_2^k & \text{at } x_\Gamma \times (0,T) \end{cases}$$

$$\text{(8.3.24)} \qquad \begin{cases} \dfrac{\partial \mathbf{V}_2^{k+1}}{\partial t} + B \dfrac{\partial \mathbf{V}_2^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\ u_2^{k+1} + \dfrac{2}{\gamma - 1} c_2^{k+1} = u(\varphi_1^k) + \dfrac{2}{\gamma - 1} c(\varphi_1^k) & \text{at } x_\Gamma \times (0,T), \end{cases}$$

together with initial conditions at $t = 0$ and boundary conditions at α and β. The interface condition in Ω_1 can be substituted by

$$\frac{\partial \varphi_1^{k+1}}{\partial t} = H_0 - \frac{1}{2}(u_2^k)^2 - \frac{1}{\gamma - 1}(c_2^k)^2 \qquad \text{at } x_\Gamma \times (0,T),$$

or else by the Dirichlet condition for φ_1^{k+1} obtained by integration in time. The same iterations are used if the direction of the flow is reversed ($-c < u < 0$).

When the flow is supersonic, we have to distinguish between the inflow and the outflow case. Let us first assume that $u > c > 0$. The iterative procedure is defined as follows:

$$\begin{aligned}\frac{\partial^2\varphi_1^{k+1}}{\partial t^2} - [c(\varphi_1^{k+1})^2 - u(\varphi_1^{k+1})^2]\frac{\partial^2\varphi_1^{k+1}}{\partial x^2} \\ +2u(\varphi_1^{k+1})\frac{\partial^2\varphi_1^{k+1}}{\partial t\partial x} = 0 \qquad \text{in } \Omega_1\times(0,T)\end{aligned} \tag{8.3.25}$$

$$\begin{cases}\dfrac{\partial\mathbf{V}_2^{k+1}}{\partial t} + B\dfrac{\partial\mathbf{V}_2^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_2\times(0,T)\\ u_2^{k+1} + \dfrac{2}{\gamma-1}c_2^{k+1} = u(\varphi_1^k) + \dfrac{2}{\gamma-1}c(\varphi_1^k) & \text{at } x_\Gamma\times(0,T)\\ u_2^{k+1} - \dfrac{2}{\gamma-1}c_2^{k+1} = u(\varphi_1^k) - \dfrac{2}{\gamma-1}c(\varphi_1^k) & \text{at } x_\Gamma\times(0,T).\end{cases} \tag{8.3.26}$$

Instead, when $u < -c < 0$ we consider the iterative scheme

$$\begin{cases}\dfrac{\partial^2\varphi_1^{k+1}}{\partial t^2} - [c(\varphi_1^{k+1})^2 - u(\varphi_1^{k+1})^2]\dfrac{\partial^2\varphi_1^{k+1}}{\partial x^2} \\ \qquad +2u(\varphi_1^{k+1})\dfrac{\partial^2\varphi_1^{k+1}}{\partial t\partial x} = 0 & \text{in } \Omega_1\times(0,T)\\ \dfrac{\partial\varphi_1^{k+1}}{\partial x} = u_2^k & \text{at } x_\Gamma\times(0,T)\\ \dfrac{\partial\varphi_1^{k+1}}{\partial t} = H_0 - \dfrac{1}{2}(u_2^k)^2 - \dfrac{1}{\gamma-1}(c_2^k)^2 & \text{at } x_\Gamma\times(0,T)\end{cases} \tag{8.3.27}$$

$$\frac{\partial\mathbf{V}_2^{k+1}}{\partial t} + B\frac{\partial\mathbf{V}_2^{k+1}}{\partial x} = \mathbf{0} \qquad \text{in } \Omega_2\times(0,T). \tag{8.3.28}$$

The above procedures can also be extended to the case of a flow that is not isentropic in Ω_2. As we have already noted in Section 7.3.3, in this case the Riemann 'invariants' $u \pm \frac{2}{\gamma-1}c$ are no longer constant along the corresponding characteristic curves. However, we can decide to impose their continuity at the interface. Furthermore, the continuity of the Riemann invariant s, the entropy, that is constant along the characteristic curve $C_0 = \{(x(t),t)\,|\,x'(t) = u\}$, has to be imposed when the curve C_0 is incoming in Ω_2.

Let us denote by $\hat{s}_1$ the constant entropy in Ω_1, and consider first the subsonic case $0 < u < c$. The first step of the iterative scheme remains the same as the isentropic case (see (8.3.23)), while the second step reads (for notation see (7.3.20), (7.3.21)):

$$
\text{(8.3.29)}\qquad \begin{cases} \dfrac{\partial \mathbf{U}_2^{k+1}}{\partial t} + A\dfrac{\partial \mathbf{U}_2^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\ u_2^{k+1} + \dfrac{2}{\gamma-1}c_2^{k+1} = u(\varphi_1^k) + \dfrac{2}{\gamma-1}c(\varphi_1^k) & \text{at } x_\Gamma \times (0,T) \\ s_2^{k+1} = \hat{s}_1 & \text{at } x_\Gamma \times (0,T), \end{cases}
$$

together with initial conditions at $t = 0$ and boundary conditions at β.

If the direction of the flow is reversed ($-c < u < 0$), the eigenvalue u is negative and consequently the characteristic line associated with the entropy s is no longer incoming in Ω_2; the Dirichlet condition for s_2^{k+1} at the interface is therefore omitted in (8.3.29). On the other hand, u is not an eigenvalue for the full potential equation, and only one interface condition for the potential φ is needed; for instance, the Neumann boundary condition used in (8.3.23).

When the flow is supersonic and $0 < c < u$, the iteration in Ω_1 is still given by (8.3.25), while the step related to Ω_2 is defined as follows:

$$
\text{(8.3.30)}\qquad \begin{cases} \dfrac{\partial \mathbf{U}_2^{k+1}}{\partial t} + A\dfrac{\partial \mathbf{U}_2^{k+1}}{\partial x} = \mathbf{0} & \text{in } \Omega_2 \times (0,T) \\ u_2^{k+1} + \dfrac{2}{\gamma-1}c_2^{k+1} = u(\varphi_1^k) + \dfrac{2}{\gamma-1}c(\varphi_1^k) & \text{at } x_\Gamma \times (0,T) \\ u_2^{k+1} - \dfrac{2}{\gamma-1}c_2^{k+1} = u(\varphi_1^k) - \dfrac{2}{\gamma-1}c(\varphi_1^k) & \text{at } x_\Gamma \times (0,T) \\ s_2^{k+1} = \hat{s}_1 & \text{at } x_\Gamma \times (0,T). \end{cases}
$$

Instead, when $u < -c < 0$ no characteristic curve is incoming in Ω_2, hence all the three interface conditions in (8.3.30) must be omitted. Since u is not an eigenvalue of the full potential equation, we are led to consider the iterative scheme given by (8.3.27) and

$$
\text{(8.3.31)}\qquad \frac{\partial \mathbf{U}_2^{k+1}}{\partial t} + A\frac{\partial \mathbf{U}_2^{k+1}}{\partial x} = \mathbf{0} \qquad \text{in } \Omega_2 \times (0,T).
$$

The well-posedness of all boundary value problems we have considered for the full potential equation can be analysed by the classical energy method or the Kreiss' normal mode analysis as in Oliger and Sundström (1978).

In the two- or three-dimensional case the eigenvalues of the characteristic matrix C_{E} of the isentropic Euler equations and those of the characteristic matrix C_{FP} of the full potential equation are different, and so are the corresponding 'characteristic' variables defined by means of the matrix of left eigenvectors of C.

Indeed, we have already seen that the eigenvalues for the Euler equations are $\mathbf{u}\cdot\mathbf{n} + c$, $\mathbf{u}\cdot\mathbf{n} - c$ and $\mathbf{u}\cdot\mathbf{n}$ (with a multiplicity equal to $d-1$), whereas for the full potential equation the characteristic matrix C_{FP} is given by (for $d = 3$)

$$C_{\mathrm{FP}} = \begin{pmatrix} 2\mathbf{u}\cdot\mathbf{n} & u_1\mathbf{u}\cdot\mathbf{n} - c^2 n_1 & u_2\mathbf{u}\cdot\mathbf{n} - c^2 n_2 & u_3\mathbf{u}\cdot\mathbf{n} - c^2 n_3 \\ -n_1 & 0 & 0 & 0 \\ -n_2 & 0 & 0 & 0 \\ -n_3 & 0 & 0 & 0 \end{pmatrix},$$

with eigenvalues $\mathbf{u}\cdot\mathbf{n} + c$, $\mathbf{u}\cdot\mathbf{n} - c$ and 0 (with a multiplicity equal to $d-1$).

Besides, a straightforward computation shows that the matrix of left eigenvectors for the Euler equations is

$$L_{\mathrm{E}} := \begin{pmatrix} c/\rho & n_1 & n_2 & n_3 \\ -c/\rho & n_1 & n_2 & n_3 \\ 0 & \tau_1^{(1)} & \tau_2^{(1)} & \tau_3^{(1)} \\ 0 & \tau_1^{(2)} & \tau_2^{(2)} & \tau_3^{(2)} \end{pmatrix},$$

(see (7.3.36)), while that associated with the full potential equation is given by

$$L_{\mathrm{FP}} := \begin{pmatrix} -1/c & -\dfrac{u_1\mathbf{u}\cdot\mathbf{n} - c^2 n_1}{c(\mathbf{u}\cdot\mathbf{n} + c)} & -\dfrac{u_2\mathbf{u}\cdot\mathbf{n} - c^2 n_2}{c(\mathbf{u}\cdot\mathbf{n} + c)} & -\dfrac{u_3\mathbf{u}\cdot\mathbf{n} - c^2 n_3}{c(\mathbf{u}\cdot\mathbf{n} + c)} \\ 1/c & \dfrac{u_1\mathbf{u}\cdot\mathbf{n} - c^2 n_1}{c(\mathbf{u}\cdot\mathbf{n} - c)} & \dfrac{u_2\mathbf{u}\cdot\mathbf{n} - c^2 n_2}{c(\mathbf{u}\cdot\mathbf{n} - c)} & \dfrac{u_3\mathbf{u}\cdot\mathbf{n} - c^2 n_3}{c(\mathbf{u}\cdot\mathbf{n} - c)} \\ 0 & \tau_1^{(1)} & \tau_2^{(1)} & \tau_3^{(1)} \\ 0 & \tau_1^{(2)} & \tau_2^{(2)} & \tau_3^{(2)} \end{pmatrix}.$$

Hence, the extension of the above algorithms to these multi-dimensional cases is not straightforward and needs some further investigation.

8.4 The coupling for the compressible Stokes equations

The next example we shall consider is the so-called compressible Stokes problem, a stage of the linearisation process of the solution to the Navier–Stokes equations for compressible viscous flows through a fractional step approach (see Bristeau *et al.* 1987). We have presented this problem in Section 5.4, as well as its inviscid counterpart related to the Euler equations in Section 5.5.

We are now interested in a problem in which the viscous terms do not give significant contributions in a subregion, say Ω_1, of the whole domain Ω (Ω_1 could be, for instance, the far field region in a flow problem). Therefore, we propose to omit them from the equations, and consider an inviscid problem in Ω_1, to be coupled with the viscous problem in Ω_2.

The viscous–inviscid coupled problem reads as follows: find $\mathbf{u}_i$, σ_i, $i = 1, 2$, such that

$$
\begin{cases}
\alpha\mathbf{u}_1 + \beta\nabla\sigma_1 = \mathbf{f} & \text{in } \Omega_1 \\
\alpha\sigma_1 + \operatorname{div}\mathbf{u}_1 = g & \text{in } \Omega_1 \\
\mathbf{u}_1\cdot\mathbf{n}^* = \varphi_{1,D} & \text{on } \Gamma_{1,D} \\
\sigma_1 = \varphi_{1,N} & \text{on } \Gamma_{1,N} \\
\nu\displaystyle\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j^* + (\gamma - 2\nu/d)\operatorname{div}\mathbf{u}_2 n_l^* & \\
\qquad -\beta\sigma_2 n_l^* = (\boldsymbol{\varphi}_{2,N})_l, \quad l = 1,\dots,d, & \text{on } \Gamma_{2,N} \\
\mathbf{u}_2 = \boldsymbol{\varphi}_{2,D} & \text{on } \Gamma_{2,D} \\
\alpha\sigma_2 + \operatorname{div}\mathbf{u}_2 = g & \text{in } \Omega_2 \\
\alpha\mathbf{u}_2 - \nu\Delta\mathbf{u}_2 - \gamma^*\nabla\operatorname{div}\mathbf{u}_2 + \beta\nabla\sigma_2 = \mathbf{f} & \text{in } \Omega_2,
\end{cases}
\tag{8.4.1}
$$

together with suitable interface conditions on Γ.

Here, $\alpha > 0$, $\nu > 0$, $\gamma > 0$, $\gamma^* := \gamma + (d-2)\nu/d$ and $\beta > 0$ are constants, $\mathbf{f}$, $\boldsymbol{\varphi}_{2,D}$ and $\boldsymbol{\varphi}_{2,N}$ are given vector fields, and g, $\varphi_{1,D}$ and $\varphi_{1,N}$ are given scalar functions.

As in the coupling for the advection–diffusion equations described in Section 8.1, the interface conditions can be found by elliptic perturbation; that is, by adding $-\delta\Delta\mathbf{u}_1$ to the left-hand side of $(8.4.1)_1$, where δ is a small positive parameter. Passing to the limit as $\delta \to 0^+$ in the following interface conditions of the perturbed problem:

$$
\begin{aligned}
&\mathbf{u}_1^\delta = \mathbf{u}_2 \qquad \text{on } \Gamma \\
&\delta\sum_j D_j u_{1,l}^\delta n_j - \beta\sigma_1^\delta n_l = \nu\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j \\
&\qquad +(\gamma - 2\nu/d)\operatorname{div}\mathbf{u}_2 n_l - \beta\sigma_2 n_l \qquad \text{on } \Gamma, \quad l = 1,\dots,d,
\end{aligned}
$$

we find for the limit solution (see Quarteroni *et al.* 1991)

$$
\begin{aligned}
&\mathbf{u}_1\cdot\mathbf{n} = \mathbf{u}_2\cdot\mathbf{n} \qquad \text{on } \Gamma \\
&-\beta\sigma_1 n_l = \nu\sum_j (D_l u_{2,j} + D_j u_{2,l}) n_j \\
&\qquad +(\gamma - 2\nu/d)\operatorname{div}\mathbf{u}_2 n_l - \beta\sigma_2 n_l \qquad \text{on } \Gamma, \quad l = 1,\dots,d.
\end{aligned}
\tag{8.4.2}
$$

Hence, the flux is continuous across the interface, while only the normal component of the velocity is continuous across Γ.

We can formulate the interface condition in terms of the Steklov–Poincaré operator. If we summarise the operators acting in (8.4.1) in the form $L_1(\mathbf{u}_1, \sigma_1)$

and $L_2(\mathbf{u}_2, \sigma_2)$, and take for the sake of simplicity $\Gamma_N = \emptyset$, $\varphi_{1,D} = 0$, $\boldsymbol{\varphi}_{2,D} = \mathbf{0}$, we are in a position to introduce the extension operators F_1 and F_2 defined through $F_1\psi = (\mathbf{U}_1^{(0)}\psi, P_1^{(0)}\psi)$ and $F_2\boldsymbol{\eta} = (\mathbf{U}_2^{(\nu)}\boldsymbol{\eta}, P_2^{(\nu)}\boldsymbol{\eta})$, as solutions of the problems

$$\begin{cases} L_1(F_1\psi) = (\mathbf{0}, 0) & \text{in } \Omega_1 \\ \mathbf{U}_1^{(0)}\psi \cdot \mathbf{n} = \psi & \text{on } \Gamma \\ \mathbf{U}_1^{(0)}\psi \cdot \mathbf{n}^* = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \end{cases} \tag{8.4.3}$$

$$\begin{cases} L_2(F_2\boldsymbol{\eta}) = (\mathbf{0}, 0) & \text{in } \Omega_2 \\ \mathbf{U}_2^{(\nu)}\boldsymbol{\eta} = \boldsymbol{\eta} & \text{on } \Gamma \\ \mathbf{U}_2^{(\nu)}\boldsymbol{\eta} = \mathbf{0} & \text{on } \partial\Omega_2 \setminus \Gamma. \end{cases} \tag{8.4.4}$$

These operators have already been introduced in (5.5.8) and (5.4.8), respectively.

Introduce, moreover, the functions $(\mathbf{U}_{1,*}^{(0)}, P_{1,*}^{(0)})$, solutions of the problems

$$\begin{cases} L_1(\mathbf{U}_{1,*}^{(0)}, P_{1,*}^{(0)}) = (\mathbf{f}, g) & \text{in } \Omega_1 \\ \mathbf{U}_{1,*}^{(0)} \cdot \mathbf{n} = 0 & \text{on } \Gamma \\ \mathbf{U}_{1,*}^{(0)} \cdot \mathbf{n}^* = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \end{cases} \tag{8.4.5}$$

and the functions $(\mathbf{U}_{2,*}^{(\nu)}, P_{2,*}^{(\nu)})$, solutions to

$$\begin{cases} L_2(\mathbf{U}_{2,*}^{(\nu)}, P_{2,*}^{(\nu)}) = (\mathbf{f}, g) & \text{in } \Omega_2 \\ \mathbf{U}_{2,*}^{(\nu)} = \mathbf{0} & \text{on } \Gamma \\ \mathbf{U}_{2,*}^{(\nu)} = \mathbf{0} & \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases} \tag{8.4.6}$$

already defined in (5.5.9) and (5.4.9), respectively.

The Steklov–Poincaré equation is now given by

$$S\boldsymbol{\lambda} = \boldsymbol{\chi} \quad \text{on } \Gamma, \tag{8.4.7}$$

where

$$\begin{aligned} (S\boldsymbol{\eta})_l := &-\beta P_1^{(0)}(\boldsymbol{\eta} \cdot \mathbf{n})\, n_l + \beta P_2^{(\nu)}\boldsymbol{\eta}\, n_l \\ &-\nu \sum_j [D_l(\mathbf{U}_2^{(\nu)}\boldsymbol{\eta})_j + D_j(\mathbf{U}_2^{(\nu)}\boldsymbol{\eta})_l]\, n_j \\ &-(\gamma - 2\nu/d)\, \operatorname{div} \mathbf{U}_2^{(\nu)}\boldsymbol{\eta}\, n_l \quad \text{on } \Gamma \end{aligned} \tag{8.4.8}$$

$$\begin{aligned}\chi_l := \beta P_{1,*}^{(0)}\, n_l - \beta P_{2,*}^{(0)}\, n_l + \nu \sum_j [D_l(\mathbf{U}_{2,*}^{(\nu)})_j + D_j(\mathbf{U}_{2,*}^{(\nu)})_l]\, n_j \\ +(\gamma - 2\nu/d)\, \mathrm{div}\, \mathbf{U}_{2,*}^{(\nu)}\, n_l \quad \text{on } \Gamma,\end{aligned} \tag{8.4.9}$$

for $l = 1, \ldots, d$.

The solution $\boldsymbol{\lambda}$ to (8.4.7) represents the value of $\mathbf{u}_2$ on Γ and at the same time $\boldsymbol{\lambda} \cdot \mathbf{n}$ is the value of $\mathbf{u}_1 \cdot \mathbf{n}$ on Γ. In other words, we get the solution to the coupled problem (8.4.1) by setting

$$(\mathbf{u}_1, \sigma_1) = (\mathbf{U}_1^{(0)}(\boldsymbol{\lambda} \cdot \mathbf{n}), P_1^{(0)}(\boldsymbol{\lambda} \cdot \mathbf{n})) + (\mathbf{U}_{1,*}^{(0)}, P_{1,*}^{(0)}) \tag{8.4.10}$$

$$(\mathbf{u}_2, \sigma_2) = (\mathbf{U}_2^{(\nu)}\boldsymbol{\lambda}, P_2^{(\nu)}\boldsymbol{\lambda}) + (\mathbf{U}_{2,*}^{(\nu)}, P_{2,*}^{(\nu)}). \tag{8.4.11}$$

Let us now focus on the iteration-by-subdomain procedure. Let $\boldsymbol{\lambda}^0$ be a given initial vector on Γ. For $k \geq 0$ we define $(\mathbf{u}_1^{k+1}, \sigma_1^{k+1})$ and $(\mathbf{u}_2^{k+1}, \sigma_2^{k+1})$ as the solution of

$$\begin{cases} L_1(\mathbf{u}_1^{k+1}, \sigma_1^{k+1}) = (\mathbf{f}, g) & \text{in } \Omega_1 \\ \mathbf{u}_1^{k+1} \cdot \mathbf{n} = \boldsymbol{\lambda}^k \cdot \mathbf{n} & \text{on } \Gamma \\ \mathbf{u}_1^{k+1} \cdot \mathbf{n}^* = 0 & \text{on } \partial\Omega_1 \setminus \Gamma \end{cases} \tag{8.4.12}$$

$$\begin{cases} L_2(\mathbf{u}_2^{k+1}, \sigma_2^{k+1}) = (\mathbf{f}, g) & \text{in } \Omega_2 \\ \nu \sum_j (D_l u_{2,j}^{k+1} + D_j u_{2,l}^{k+1}) n_j & \\ \quad +(\gamma - 2\nu/d)\, \mathrm{div}\, \mathbf{u}_2^{k+1} n_l & \\ \quad -\beta \sigma_2^{k+1} n_l = \beta \sigma_1^{k+1} n_l & \text{on } \Gamma, \quad l = 1, \ldots, d, \\ \mathbf{u}_2^{k+1} = \mathbf{0} & \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases} \tag{8.4.13}$$

where

$$\boldsymbol{\lambda}^{k+1} := (1 - \theta)\boldsymbol{\lambda}^k + \theta \mathbf{u}_{2|\Gamma}^{k+1} \tag{8.4.14}$$

and $\theta > 0$ is again an acceleration parameter.

8.4.1 Variational formulation and finite element approximation

Referring for notation to Sections 5.4 and 5.5, the variational formulation of the Steklov–Poincaré operator S is given by

$$\langle S\boldsymbol{\eta}, \boldsymbol{\mu} \rangle := \mathcal{A}_1^{(0)}(F_1(\boldsymbol{\eta} \cdot \mathbf{n}), F_1(\boldsymbol{\mu} \cdot \mathbf{n})) + \mathcal{A}_2^{(\nu)}(F_2\boldsymbol{\eta}, F_2\boldsymbol{\mu}), \tag{8.4.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between the trace space $(\Lambda)^d$ and its dual, and, as in (5.4.10), we define

$$\begin{aligned}\mathcal{A}_2^{(\nu)}&[(\mathbf{w}_2,\rho_2),(\mathbf{v}_2,q_2)]\\ &:=\int_{\Omega_2}\Big[\beta\alpha\rho_2 q_2+\beta\,\mathrm{div}\,\mathbf{w}_2\,q_2\\ &\qquad+\alpha\mathbf{w}_2\cdot\mathbf{v}_2-\beta\rho_2\,\mathrm{div}\,\mathbf{v}_2\\ &\qquad+\frac{\nu}{2}\sum_{l,j=1}^{d}(D_l w_{2,j}+D_j w_{2,l})\,(D_l v_{2,j}+D_j v_{2,l})\\ &\qquad+\left(\gamma-\frac{2\nu}{d}\right)\mathrm{div}\,\mathbf{w}_2\;\mathrm{div}\,\mathbf{v}_2\Big]\end{aligned}\tag{8.4.16}$$

and

$$\begin{aligned}\mathcal{A}_1^{(0)}[(\mathbf{w}_1,\rho_1),(\mathbf{v}_1,q_1)]:=\int_{\Omega_1}&(\beta\alpha\rho_1 q_1+\beta\,\mathrm{div}\,\mathbf{w}_1\,q_1\\ &+\alpha\mathbf{w}_1\cdot\mathbf{v}_1-\beta\rho_1\,\mathrm{div}\,\mathbf{v}_1).\end{aligned}\tag{8.4.17}$$

We can split the Steklov–Poincaré operator as $S=S_1+S_2$, where

$$\langle S_1\boldsymbol{\eta},\boldsymbol{\mu}\rangle=\mathcal{A}_1^{(0)}(F_1(\boldsymbol{\eta}\cdot\mathbf{n}),F_1(\boldsymbol{\mu}\cdot\mathbf{n})),\tag{8.4.18}$$

which corresponds to $(S_1\boldsymbol{\eta})_l:=-\beta P_1^{(0)}(\boldsymbol{\eta}\cdot\mathbf{n})\,n_l$, $l=1,\ldots,d$, and

$$\langle S_2\boldsymbol{\eta},\boldsymbol{\mu}\rangle=\mathcal{A}_2^{(\nu)}(F_2\boldsymbol{\eta},F_2\boldsymbol{\mu}),\tag{8.4.19}$$

which corresponds to

$$\begin{aligned}(S_2\boldsymbol{\eta})_l:=\beta P_2^{(\nu)}\boldsymbol{\eta}\,n_l-\nu\sum_j(D_l(\mathbf{U}_2^{(\nu)}\boldsymbol{\eta})_j+D_j(\mathbf{U}_2^{(\nu)}\boldsymbol{\eta})_l)\,n_j\\ -(\gamma-2\nu/d)\,\mathrm{div}\,\mathbf{U}_2^{(\nu)}\boldsymbol{\eta}\,n_l,\quad l=1,\ldots,d.\end{aligned}$$

Both of the operators S_1 and S_2 is symmetric. Their continuity in $(\Lambda)^d$ is an easy consequence of elliptic regularity theorems applied to the extension operators F_1 (see (5.5.10)) and F_2. Moreover, S_1 is non-negative, because there exists a positive constant α^* such that

$$\langle S_1\boldsymbol{\eta},\boldsymbol{\eta}\rangle\geq\alpha^*||F_1(\boldsymbol{\eta}\cdot\mathbf{n})||^2_{0,\Omega_1}\geq 0\quad\forall\,\boldsymbol{\eta}\in(\Lambda)^d.$$

Finally, S_2 is coercive in $(\Lambda)^d$, owing to estimate (5.4.5) and the trace inequality (1.2.5).

It is easily verified that the iterative method (8.4.12)–(8.4.14) is equivalent to

$$S_2(\boldsymbol{\lambda}^{k+1} - \boldsymbol{\lambda}^k) = \theta(\boldsymbol{\chi} - S\boldsymbol{\lambda}^k) \qquad \text{on } \Gamma, \tag{8.4.20}$$

that is, a Richardson procedure for (8.4.7) with preconditioner S_2. Owing to the properties of S_1 and S_2, the convergence of this scheme can be established by means of Theorem 4.2.2.

In Quarteroni *et al.* (1991) the finite element approximation of the multidomain problem (8.4.1) and the iterative procedure (8.4.12)–(8.4.14) are considered.

The finite dimensional space is constructed by taking (continuous) piecewise-$\mathbf{P}_r$ elements for $\mathbf{u}_2$, piecewise-$\mathbf{P}_{r-1}$ elements for σ_1 and σ_2, whereas the Raviart–Thomas (in two dimensions) or div-conforming Nédélec elements (in three dimensions) are used for $\mathbf{u}_1$ (see Section 5.5.3).

The coupling between the finite dimensional spaces related to Ω_1 and Ω_2 occurs through the additional condition

$$p_h(\mathbf{v}_{2,h} \cdot \mathbf{n}) = \mathbf{v}_{1,h} \cdot \mathbf{n} \quad \text{on } \Gamma, \tag{8.4.21}$$

where p_h is the $L^2(\Gamma)$-orthogonal projection from $L^2(\Gamma)$ onto

$$\Psi_h := \{\psi \in L^2(\Gamma) \,|\, \psi_{|I} \in \mathbf{P}_{r-1}(I) \quad \forall\, I \in \Xi_h\}$$

(here Ξ_h is the decomposition of Γ induced by the triangulation T_h of Ω, and we make the assumption that no triangle of T_h crosses Γ).

All the procedures introduced before can be extended to the finite dimensional case. In particular, we can introduce in the natural way the operators $S_{i,h}$, $i = 1, 2$ (the finite dimensional analogue of the operators S_1 and S_2), defined on the space $(\Lambda_h)^d$ where

$$\Lambda_h := \{\lambda_h \in C^0(\Gamma) \,|\, \lambda_{h|I} \in \mathbf{P}_r(I) \quad \forall\, I \in \Xi_h\} \subset \Lambda. \tag{8.4.22}$$

The operators $S_{1,h}$ and $S_{2,h}$ are symmetric. Owing to the continuity of the bilinear forms $\mathcal{A}_2^{(\nu)}$ and $\mathcal{A}_1^{(0)}$, by proceeding as in Sections 5.4 and 5.5 we can apply Theorems 4.1.3 and 4.1.9 and obtain that they are continuous, uniformly with respect to h. Moreover, $S_{1,h}$ is non-negative and $S_{2,h}$ is coercive, uniformly with respect to h. Therefore, the Richardson method with $S_{2,h}$ as a preconditioner converges; in view of the above properties of $S_{1,h}$ and $S_{2,h}$, we could also use the preconditioned conjugate gradient iterations. In both cases, the convergence occurs at a rate independent of h.

8.4.2 An alternative formulation: variational setting and finite element approximation

An alternative approach has been proposed by Alonso and Valli (1995). The novelty resides in a different coupling, which permits us to consider two elliptic problems, one for the pressure σ_1 in Ω_1, and another for the velocity $\mathbf{u}_2$ in Ω_2 (whereas σ_2 and $\mathbf{u}_1$ are directly computed from $\mathbf{u}_2$ and σ_1, respectively). At the

finite dimensional level, this permits us to use standard piecewise-polynomial finite element spaces; moreover, no matching condition between these elements is necessary on Γ, and the meshes in Ω_1 and Ω_2 can be chosen in a completely independent fashion.

The starting point is to transform the coupled problem into one that is equivalent. Combining $(8.4.1)_1$ and $(8.4.2)_2$ we find that

$$\alpha\sigma_1 - \alpha^{-1}\beta\Delta\sigma_1 = g - \alpha^{-1}\operatorname{div}\mathbf{f}. \tag{8.4.23}$$

Similarly, we obtain

$$\alpha\mathbf{u}_2 - \nu\Delta\mathbf{u}_2 - (\gamma^* + \alpha^{-1}\beta)\nabla\operatorname{div}\mathbf{u}_2 = \mathbf{f} - \alpha^{-1}\beta\nabla g. \tag{8.4.24}$$

For deriving the new variational formulation, let us start by setting

$$\begin{aligned} V_1 &:= \{\varphi_1 \in H^1(\Omega_1) \mid \varphi_1 = 0 \text{ on } \Gamma_{1,N}\} \\ V_2 &:= \{v_2 \in H^1(\Omega_2) \mid v_2 = 0 \text{ on } \Gamma_{2,D}\} \\ \Lambda &:= \{\eta \in H^{1/2}(\Gamma) \mid \eta = \varphi_{1|\Gamma} \text{ for a suitable } \varphi_1 \in V_1\}. \end{aligned}$$

Taking into consideration the equation $(8.4.1)_1$ and the boundary condition $(8.4.1)_3$, and integrating by parts we obtain from (8.4.23)

$$\begin{aligned} \int_{\Omega_1}[\alpha\sigma_1\varphi_1 + \alpha^{-1}(\beta\nabla\sigma_1 - \mathbf{f})\cdot\nabla\varphi_1] + \int_{\Gamma_{1,D}}\varphi_{1,D}\varphi_1 + \int_\Gamma \mathbf{u}_1\cdot\mathbf{n}\,\varphi_1 \\ = \int_{\Omega_1} g\varphi_1 \qquad \forall\ \varphi_1 \in V_1. \end{aligned}$$

Similarly, using $(8.4.1)_7$ and $(8.4.1)_5$, we have from (8.4.24)

$$\begin{aligned} &\int_{\Omega_2}\Bigg[\alpha\mathbf{u}_2\cdot\mathbf{v}_2 + \frac{\nu}{2}\sum_{l,j=1}^{d}(D_l u_{2,j} + D_j u_{2,l})(D_l v_{2,j} + D_j v_{2,l}) \\ &\quad + \left(\gamma - \frac{2\nu}{d}\right)\operatorname{div}\mathbf{u}_2\operatorname{div}\mathbf{v}_2 + \alpha^{-1}\beta(\operatorname{div}\mathbf{u}_2 - g)\operatorname{div}\mathbf{v}_2\Bigg] - \int_{\Gamma_{2,N}}\boldsymbol{\varphi}_{2,N}\cdot\mathbf{v}_2 \\ &\quad + \int_\Gamma\sum_{l=1}^{d}\Bigg[-\nu\sum_{j=1}^{d}(D_l u_{2,j} + D_j u_{2,l})n_j^2 \\ &\qquad\qquad -\left(\gamma - \frac{2\nu}{d}\right)\operatorname{div}\mathbf{u}_2 n_l^2 + \beta\sigma_2 n_l^2\Bigg]v_{2,l} \\ &= \int_{\Omega_2}\mathbf{f}\cdot\mathbf{v}_2 \qquad \forall\ \mathbf{v}_2 \in (V_2)^d, \end{aligned}$$

where $\mathbf{n}^2$ is the unit outward normal vector on $\partial\Omega_2$, therefore $\mathbf{n}^2 = -\mathbf{n}$ on Γ. By means of the interface conditions (8.4.2), we can finally write

$$
\begin{aligned}
&\int_{\Omega_1} (\alpha\sigma_1\varphi_1 + \alpha^{-1}\beta\nabla\sigma_1\cdot\nabla\varphi_1) \\
&\qquad = \int_{\Omega_1} (g\varphi_1 + \alpha^{-1}\mathbf{f}\cdot\nabla\varphi_1) - \int_{\Gamma_{1,D}} \varphi_{1,D}\varphi_1 \\
&\qquad\qquad - \int_{\Gamma} \mathbf{u}_2\cdot\mathbf{n}\,\varphi_1 \qquad \forall\ \varphi_1 \in V_1 \\
&\int_{\Omega_2} \left[\alpha\mathbf{u}_2\cdot\mathbf{v}_2 + \frac{\nu}{2}\sum_{l,j=1}^{d} (D_l u_{2,j} + D_j u_{2,l})\,(D_l v_{2,j} + D_j v_{2,l})\right. \\
&\qquad\qquad \left. + \left(\gamma - \frac{2\nu}{d} + \alpha^{-1}\beta\right)\operatorname{div}\mathbf{u}_2\operatorname{div}\mathbf{v}_2\right] \\
&\qquad = \int_{\Omega_2} (\mathbf{f}\cdot\mathbf{v}_2 + \alpha^{-1}\beta g\operatorname{div}\mathbf{v}_2) + \int_{\Gamma_{2,N}} \boldsymbol{\varphi}_{2,N}\cdot\mathbf{v}_2 \\
&\qquad\qquad + \int_{\Gamma} \beta\sigma_1\mathbf{n}\cdot\mathbf{v}_2 \qquad \forall\ \mathbf{v}_2 \in (V_2)^d.
\end{aligned}
$$

Let us define in $H^1(\Omega_1)$ the bilinear form

$$
a_1^*(\psi_1,\varphi_1) := \int_{\Omega_1} (\alpha\psi_1\varphi_1 + \alpha^{-1}\beta\nabla\psi_1\cdot\nabla\varphi_1)
$$

and the linear functional

$$
\mathcal{L}_1(\varphi_1) := \int_{\Omega_1} (g\varphi_1 + \alpha^{-1}\mathbf{f}\cdot\nabla\varphi_1) - \langle\varphi_{1,D}, \varphi_{1|\Gamma_{1,D}}\rangle_{\Gamma_{1,D}},
$$

where, here and in what follows, we use the same notation $\langle\cdot,\cdot\rangle$ for the duality pairing between any couple of dual spaces. Similarly, we define in $(H^1(\Omega_2))^d$ the form

$$
\begin{aligned}
s_2^*(\mathbf{w}_2,\mathbf{v}_2) := \int_{\Omega_2} &\left[\alpha\mathbf{w}_2\cdot\mathbf{v}_2 + \frac{\nu}{2}\sum_{l,j=1}^{d} (D_l w_{2,j} + D_j w_{2,l})\,(D_l v_{2,j} + D_j v_{2,l})\right. \\
&\left. + \left(\gamma - \frac{2\nu}{d} + \alpha^{-1}\beta\right)\operatorname{div}\mathbf{w}_2\operatorname{div}\mathbf{v}_2\right]
\end{aligned}
$$

and the functional

$$
\mathcal{L}_2(\mathbf{v}_2) := \int_{\Omega_2} (\mathbf{f}\cdot\mathbf{v}_2 + \alpha^{-1}\beta g\operatorname{div}\mathbf{v}_2) + \langle\boldsymbol{\varphi}_{2,N}, \mathbf{v}_{2|\Gamma_{2,N}}\rangle_{\Gamma_{2,N}}.
$$

The variational formulation we are dealing with thus reads: find $(\sigma_1,\mathbf{u}_2) \in H^1(\Omega_1)\times(H^1(\Omega_2))^d$ such that

$$
\text{(8.4.25)}\qquad \begin{cases} a_1^*(\sigma_1,\varphi_1) = \mathcal{L}_1(\varphi_1) - \displaystyle\int_\Gamma \mathbf{u}_2\cdot\mathbf{n}\,\varphi_1 \qquad \forall\ \varphi_1 \in V_1 \\ s_2^*(\mathbf{u}_2,\mathbf{v}_2) = \mathcal{L}_2(\mathbf{v}_2) + \displaystyle\int_\Gamma \beta\sigma_1\mathbf{n}\cdot\mathbf{v}_2 \qquad \forall\ \mathbf{v}_2 \in (V_2)^d \\ \sigma_1 = \varphi_{1,N} \qquad \text{on } \Gamma_{1,N} \\ \mathbf{u}_2 = \boldsymbol{\varphi}_{2,D} \qquad \text{on } \Gamma_{2,D}. \end{cases}
$$

Note that the unknowns σ_1 and $\mathbf{u}_2$ are only coupled through the boundary integrals that appear in $(8.4.25)_1$ and $(8.4.25)_2$.

For the sake of simplicity, let us assume in what follows that $\varphi_{1,N} = 0$ and $\boldsymbol{\varphi}_{2,D} = \mathbf{0}$, so that the solution $(\sigma_1, \mathbf{u}_2)$ of problem (8.4.25) belongs to $V_1 \times (V_2)^d$. We also set $V_1^0 := \{\psi_1 \in V_1 \mid \psi_1 = 0 \text{ on } \Gamma\}$.

We define now the extension operators $E_1^* : \Lambda \to V_1$ and $\boldsymbol{\mathcal{N}}_2 : \Lambda \to (V_2)^d$ in the following way: $E_1^*\eta$ is the solution to the mixed boundary value problem (with a Dirichlet boundary condition on Γ)

$$
\text{(8.4.26)}\qquad \begin{cases} E_1^*\eta \in V_1 \ : \\ a_1^*(E_1^*\eta,\psi_1) = 0 \qquad \forall\ \psi_1 \in V_1^0 \\ E_1^*\eta = \eta \qquad \text{on } \Gamma, \end{cases}
$$

whereas $\boldsymbol{\mathcal{N}}_2\eta$ is the solution to the mixed boundary value problem (with a Neumann boundary condition on Γ)

$$
\text{(8.4.27)}\qquad \boldsymbol{\mathcal{N}}_2\eta \in (V_2)^d \ :\ s_2^*(\boldsymbol{\mathcal{N}}_2\eta,\mathbf{v}_2) = \int_\Gamma \beta\eta\,\mathbf{n}\cdot\mathbf{v}_2 \qquad \forall\ \mathbf{v}_2 \in (V_2)^d.
$$

The Steklov–Poincaré operators S_i, $i = 1, 2$, are defined as follows:

$$
\text{(8.4.28)}\qquad \langle S_1\eta,\mu\rangle := a_1^*(E_1^*\eta, E_1^*\mu) \qquad \forall\ \eta,\mu \in \Lambda
$$

and

$$
\text{(8.4.29)}\qquad \langle S_2\eta,\mu\rangle = \beta^{-1} s_2^*(\boldsymbol{\mathcal{N}}_2\eta,\boldsymbol{\mathcal{N}}_2\mu) \qquad \forall\ \eta,\mu \in \Lambda.
$$

Moreover, we set $S\eta := S_1\eta + S_2\eta$.

We also introduce the functions $\sigma_1^* \in V_1^0$ and $\hat{\mathbf{u}}_2 \in (V_2)^d$, as solutions to

$$
\text{(8.4.30)}\qquad \sigma_1^* \in V_1^0 \ :\ a_1^*(\sigma_1^*,\psi_1) = \mathcal{L}_1(\psi_1) \qquad \forall\ \psi_1 \in V_1^0
$$

$$
\text{(8.4.31)}\qquad \hat{\mathbf{u}}_2 \in (V_2)^d \ :\ s_2^*(\hat{\mathbf{u}}_2,\mathbf{v}_2) = \mathcal{L}_2(\mathbf{v}_2) \qquad \forall\ \mathbf{v}_2 \in (V_2)^d.
$$

If we define $\chi \in \Lambda'$ (the dual space of the trace space Λ) as

$$\langle \chi, \mu \rangle := \mathcal{L}_1(E_1^* \mu) - \beta^{-1} \mathcal{L}_2(\boldsymbol{\mathcal{N}}_2 \mu) \qquad \forall\, \mu \in \Lambda, \tag{8.4.32}$$

we can easily conclude that the couple $\sigma_1 = E_1^* \lambda + \sigma_1^*$, $\mathbf{u}_2 = \boldsymbol{\mathcal{N}}_2 \lambda + \hat{\mathbf{u}}_2$ is a solution to (8.4.25) if and only if $\lambda \in \Lambda$ is the solution of the Steklov–Poincaré equation

$$S\lambda = \chi. \tag{8.4.33}$$

To solve (8.4.25) we propose the following iterative algorithm: given $\lambda^0 \in \Lambda$, for each $k \geq 0$ solve

$$\mathbf{u}_2^{k+1} \in (V_2)^d \ :\ s_2^*(\mathbf{u}_2^{k+1}, \mathbf{v}_2) = \mathcal{L}_2(\mathbf{v}_2) + \int_\Gamma \beta \lambda^k \mathbf{n} \cdot \mathbf{v}_2 \qquad \forall\, \mathbf{v}_2 \in (V_2)^d, \tag{8.4.34}$$

then

$$\sigma_1^{k+1} \in V_1 \ :\ a_1^*(\sigma_1^{k+1}, \varphi_1) = \mathcal{L}_1(\varphi_1) - \int_\Gamma \mathbf{u}_2^{k+1} \cdot \mathbf{n}\, \varphi_1 \qquad \forall\, \varphi_1 \in V_1, \tag{8.4.35}$$

and finally set

$$\lambda^{k+1} = \theta \sigma_{1\,|\Gamma}^{k+1} + (1 - \theta) \lambda^k, \tag{8.4.36}$$

where $\theta > 0$ is an acceleration parameter. It must be noted that at each step we need to solve two mixed (Dirichlet–Neumann) boundary value problems, one in Ω_1 and the other in Ω_2, associated with linear elliptic operators (a scalar Helmholtz operator in Ω_1, and an elasticity-like operator in Ω_2). The boundary condition on the interface Γ is always of a Neumann type.

The proof of the convergence of this algorithm is still based on Theorem 4.2.2 (interchanging the role of S_1 and S_2). In fact, one can verify that it is equivalent to the preconditioned Richardson method:

$$\lambda^{k+1} = \lambda^k + \theta S_1^{-1}(\chi - S\lambda^k).$$

Let us show that the operators S_1 and S_2 satisfy the assumptions of Theorem 4.2.2. For brevity, we will consider directly their finite element approximations. Define for $r \geq 1$ and $s \geq 1$ the following spaces of finite elements:

$$V_{1,h} = \{\varphi_{1,h} \in C^0(\overline{\Omega}_1) \mid \varphi_{1,h|T} \in \mathbf{P}_r \ \forall\, T \in \mathcal{T}_h^1,\ \varphi_{1,h} = 0 \text{ on } \Gamma_{1,N}\}$$
$$V_{2,h} = \{v_{2,h} \in C^0(\overline{\Omega}_2) \mid v_{2,h|T} \in \mathbf{P}_s \ \forall\, T \in \mathcal{T}_h^2,\ v_{2,h} = 0 \text{ on } \Gamma_{2,D}\}.$$

Note that the polynomial degrees $r \geq 1$ and $s \geq 1$ are not related to each other, and that the triangulations $\mathcal{T}_h^1$ of Ω_1 and $\mathcal{T}_h^2$ of Ω_2 are not required to match on the interface Γ.

Finally, denoting by Ξ_h^1 the decomposition of Γ induced by $\mathcal{T}_h^1$, we define

$$\Lambda_h := \{\varphi_{h|\Gamma} \mid \varphi_h \in V_{1,h}\} = \{\eta_h \in C^0(\Gamma) \mid \eta_{h|I} \in \mathbf{P}_r(I) \ \forall\, I \in \Xi_h^1\}.$$

Note that Λ_h is clearly a finite dimensional subspace of the trace space Λ.

We consider this approximation of (8.4.25): find $(\sigma_{1,h}, \mathbf{u}_{2,h}) \in V_{1,h} \times (V_{2,h})^d$ such that

$$\text{(8.4.37)} \quad \begin{cases} a_1^*(\sigma_{1,h}, \varphi_{1,h}) = \mathcal{L}_1(\varphi_{1,h}) - \displaystyle\int_\Gamma \mathbf{u}_{2,h} \cdot \mathbf{n}\, \varphi_{1,h} & \forall\ \varphi_{1,h} \in V_{1,h} \\ s_2^*(\mathbf{u}_{2,h}, \mathbf{v}_{2,h}) = \mathcal{L}_2(\mathbf{v}_{2,h}) + \displaystyle\int_\Gamma \beta\sigma_{1,h}\mathbf{n} \cdot \mathbf{v}_{2,h} & \forall\ \mathbf{v}_{2,h} \in (V_{2,h})^d. \end{cases}$$

We can then define the finite dimensional extension operators $E_{1,h}^*$ and $\boldsymbol{\mathcal{N}}_{2,h}$ (the discrete counterpart of (8.4.26), (8.4.27)), the finite dimensional Steklov–Poincaré operators $S_{1,h}$, $S_{2,h}$ and $S_h = S_{1,h} + S_{2,h}$ (the discrete counterpart of (8.4.28), (8.4.29)), and finally the functions $\sigma_{1,h}^* \in V_{1,h}^0$, $\hat{\mathbf{u}}_{2,h} \in (V_{2,h})^d$ and $\chi_h \in \Lambda_h'$ (the discrete counterpart of (8.4.30)–(8.4.32)).

The discrete Steklov–Poincaré operators are symmetric. The coerciveness of the form a_1^* and the trace inequality (1.2.5) yield that $S_{1,h}$ is coercive, uniformly with respect to h; moreover, $S_{2,h}$ is non-negative. The uniform continuity of $S_{1,h}$ is a straightforward consequence of the continuity of the form a_1^* in $H^1(\Omega_1)$ and the uniform continuity of the extension operator $E_{1,h}^*$, proved in Section 4.1.1. Finally, the uniform continuity of $S_{2,h}$ follows from the continuity of the form s_2^* in $(H^1(\Omega_2))^d$ and the following estimate:

$$\text{(8.4.38)} \qquad ||\boldsymbol{\mathcal{N}}_{2,h}\eta_h||_{1,\Omega_2} \le C_3^* ||\eta_h||_\Lambda \qquad \forall\ \eta_h \in \Lambda_h,$$

for a suitable constant $C_3^* > 0$ independent of h. This last inequality can be verified as follows:

$$\begin{aligned} \alpha_2^* ||\boldsymbol{\mathcal{N}}_{2,h}\eta||_{1,\Omega_2}^2 \le s_2^*(\boldsymbol{\mathcal{N}}_{2,h}\eta, \boldsymbol{\mathcal{N}}_{2,h}\eta) &= \int_\Gamma \beta\eta\mathbf{n} \cdot \boldsymbol{\mathcal{N}}_{2,h}\eta \\ &\le \beta ||\eta||_{0,\Gamma} ||(\boldsymbol{\mathcal{N}}_{2,h}\eta)_{|\Gamma}||_{0,\Gamma} \\ &\le \beta ||\eta||_\Lambda C_{\Omega_2}^* ||\boldsymbol{\mathcal{N}}_{2,h}\eta||_{1,\Omega_2} \quad \forall\ \eta \in \Lambda, \end{aligned}$$

where $\alpha_2^* > 0$ is the coerciveness constant of the bilinear form $s_2^*(\cdot,\cdot)$ (see (5.4.5)), and $C_{\Omega_2}^* > 0$ is the constant of the trace inequality from $H^1(\Omega_2)$ into $H^{1/2}(\partial\Omega_2)$ (see (1.2.3)).

The convergence of the finite dimensional Dirichlet–Neumann iterative scheme corresponding to (8.4.34)–(8.4.36) is therefore a consequence of Theorem 4.2.2, because it again corresponds to a preconditioned Richardson iterative method for solving the discrete Steklov–Poincaré equation

$$\text{(8.4.39)} \qquad S_h\lambda_h = \chi_h,$$

with $S_{1,h}$ as a preconditioner. The rate of convergence is independent of h.

8.5 The coupling for the time-harmonic Maxwell equations

The derivation of the time-harmonic Maxwell equations has been carried out in Section 5.7.

Omitting from the complete set of equations (5.7.2) the terms $-i\varepsilon\alpha\mathbf{E}$ (which corresponds to considering a low-frequency problem), we are left with

$$\begin{cases} \operatorname{rot}\mathbf{H} - \sigma\mathbf{E} = \mathbf{0} & \text{in } \Omega \\ \\ \operatorname{rot}\mathbf{E} + i\mu\alpha\mathbf{H} = \mathbf{0} & \text{in } \Omega, \end{cases} \tag{8.5.1}$$

where μ is the magnetic permeability coefficient and σ is the electric conductivity. In a conductive medium, μ and σ are assumed to be symmetric matrices, depending on the space variable $\mathbf{x}$, and uniformly positive definite. On the other hand, in an insulator, μ is a constant $\mu_0 > 0$ and σ is vanishing.

On the boundary, we impose the Dirichlet boundary condition for the electric field:

$$\mathbf{n} \times \mathbf{E} = \boldsymbol{\Psi} \quad \text{on } \partial\Omega, \tag{8.5.2}$$

where $\boldsymbol{\Psi}$ is a given tangential vector field defined on $\partial\Omega$.

We are concerned here with a medium that is *heterogeneous*; namely, composed of two parts, one that is a non-homogeneous, non-isotropic conductor, and another that is a perfect insulator. In other words, this means that we are assuming that the conductivity σ is a symmetric, uniformly positive definite matrix $\hat{\sigma}$ in a subset, say Ω_2, of Ω, while it is equal to 0 in the complementary part, Ω_1.

We are thus naturally led to considering an appropriate equivalent two-domain formulation. Its main feature will be that in each subdomain we have to solve an equation of a simple type, the interaction between the two subdomains being governed by two interface conditions. This formulation yields a solver based on a domain decomposition iterative approach: one interface condition is assigned to a subdomain, the other to the complementary domain, and then an iterative procedure is employed, solving at each step the 'simple' problems in each subdomain, just by modifying step by step on the interface the value of the data that are related to the solution on the other domain.

A first remark is now in order: once we have chosen σ as we described above, for problem (8.5.1), (8.5.2) uniqueness clearly does not hold, because we can add to $\mathbf{E}$ the gradient of any function ψ having compact support in Ω_1. Therefore, we need to modify (8.5.1) by adding another equation to it. A perturbation argument suggests we add the constraint $\operatorname{div}(\mathbf{E}_{|\Omega_1}) = 0$ in Ω_1 (which corresponds to the natural physical condition that no charge is present in the insulator Ω_1).

Furthermore, it is possible to find a vector field $\mathbf{E}^*$ such that $\mathbf{n} \times \mathbf{E}^* = \boldsymbol{\Psi}$ on $\partial\Omega$, and, moreover, $\operatorname{div}\mathbf{E}^* = 0$ and $\operatorname{rot}\operatorname{rot}\mathbf{E}^* = \mathbf{0}$. Writing $\mathbf{u} := \mathbf{E} - \mathbf{E}^*$, taking the rotation of $(8.5.1)_2$ and using $(8.5.1)_1$, we can finally rewrite the problem as

$$
(8.5.3)\qquad \begin{cases} \operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{u}) + i\alpha\sigma\mathbf{u} = -\operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{E}^*) - i\alpha\sigma\mathbf{E}^* & \text{in } \Omega \\ \operatorname{div}(\mathbf{u}_{|\Omega_1}) = 0 & \text{in } \Omega_1 \\ \mathbf{n}\times\mathbf{u} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}
$$

The construction of such a vector field $\mathbf{E}^*$ relies on suitable geometric assumptions on the domain Ω (see Alonso and Valli 1996); for instance, it is true if Ω is a convex polyhedron in $\mathbf{R}^3$.

In Alonso and Valli (1997) it has been proved that this problem has a unique solution in a suitable Hilbert space, provided that the subdomain Ω_1 is also a convex polyhedron. We will not give the proof of this rather technical result here; we want only to show that (8.5.3) can be rewritten in terms of an equivalent two-domain formulation, which leads in a natural way to an iteration-by-subdomain procedure. It can also be shown that this iteration-by-subdomain procedure is equivalent to a preconditioned Richardson method applied to the Steklov–Poincaré equation on the interface.

Let us turn to the equivalent two-domain formulation. First, this requires us to determine the correct interface conditions on Γ for the solution $\mathbf{u}$ defined in Ω. As we have already noted in Remark 1.1.1, these conditions are often determined by requiring that the solution $\mathbf{u}$ belongs to a space of functions defined over the whole of Ω (this space embodies the regularity properties of $\mathbf{u}_{|\Omega_1}$ in Ω_1 and $\mathbf{u}_{|\Omega_2}$ in Ω_2, together with a suitable matching on Γ); moreover, the solution $\mathbf{u}$ satisfies the equation in the sense of distributions in the whole Ω; namely, through the interface Γ and not only in Ω_1 and Ω_2 separately.

For the problem at hand, denoting by $\mathbf{u}_i$ the restriction of $\mathbf{u}$ to Ω_i, $i = 1, 2$, this means that we have to impose the following interface conditions on Γ:

$$
(8.5.4)\qquad \begin{aligned} &\mathbf{n}\times\mathbf{u}_1 = \mathbf{n}\times\mathbf{u}_2 && \text{on } \Gamma \\ &\mathbf{n}\times(\mu_0^{-1}\operatorname{rot}\mathbf{u}_1) = \mathbf{n}\times(\mu^{-1}\operatorname{rot}\mathbf{u}_2) && \text{on } \Gamma. \end{aligned}
$$

The first condition ensures that $\mathbf{u} \in H(\operatorname{rot};\Omega)$, and the second that $\mathbf{u}$ is a solution of $(8.5.3)_1$ in Ω in the sense of distributions.

The equivalent two-domain formulation is therefore

$$
\begin{cases}
\operatorname{rot}\operatorname{rot}\mathbf{u}_1 = \mathbf{0} & \text{in } \Omega_1 \\
\operatorname{div}\mathbf{u}_1 = 0 & \text{in } \Omega_1 \\
\mathbf{n}\times\mathbf{u}_1 = \mathbf{0} & \text{on } \partial\Omega_1 \setminus \Gamma \\
\mathbf{n}\times\mathbf{u}_1 = \mathbf{n}\times\mathbf{u}_2 & \text{on } \Gamma \\
\mathbf{n}\times(\mu_0^{-1}\operatorname{rot}\mathbf{u}_1) = \mathbf{n}\times(\mu^{-1}\operatorname{rot}\mathbf{u}_2) & \text{on } \Gamma \\
\mathbf{n}\times\mathbf{u}_2 = \mathbf{0} & \text{on } \partial\Omega_2 \setminus \Gamma \\
\operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{u}_2) + i\alpha\hat{\sigma}\mathbf{u}_2 = \mathbf{F} & \text{in } \Omega_2,
\end{cases}
\tag{8.5.5}
$$

where $\mathbf{F} := -\operatorname{rot}(\mu^{-1}\operatorname{rot}\mathbf{E}^*) - i\alpha\hat{\sigma}\mathbf{E}^*$.

The equation in Ω_1 can be rewritten in a simpler way. In fact, it can be seen that $\mathbf{u}_1$ is the solution of

$$
\begin{cases}
\operatorname{rot}\mathbf{u}_1 = \nabla\omega_{1,\Gamma}((\mathbf{n}\times\mathbf{u}_2)_{|\Gamma}) & \text{in } \Omega_1 \\
\operatorname{div}\mathbf{u}_1 = 0 & \text{in } \Omega_1 \\
\mathbf{n}\times\mathbf{u}_1 = (\mathbf{n}\times\widetilde{\mathbf{u}_2})_{|\Gamma} & \text{on } \partial\Omega_1.
\end{cases}
\tag{8.5.6}
$$

Here, $\omega_{1,\Gamma}(\boldsymbol{\gamma})$ is the solution of the Neumann problem:

$$
\begin{cases}
\Delta\omega_{1,\Gamma}(\boldsymbol{\gamma}) = 0 & \text{in } \Omega_1 \\
\dfrac{\partial\omega_{1,\Gamma}(\boldsymbol{\gamma})}{\partial n} = -\operatorname{div}_\tau\widetilde{\boldsymbol{\gamma}} & \text{on } \partial\Omega_1 \\
\displaystyle\int_{\Omega_1}\omega_{1,\Gamma}(\boldsymbol{\gamma}) = 0,
\end{cases}
\tag{8.5.7}
$$

where $\boldsymbol{\gamma}$ is a tangential vector on Γ, $\widetilde{\boldsymbol{\gamma}}$ is its extension by $\mathbf{0}$ on $\partial\Omega_1\setminus\Gamma$, and $\operatorname{div}_\tau\widetilde{\boldsymbol{\gamma}}$ is the tangential divergence of $\widetilde{\boldsymbol{\gamma}}$ (for a precise definition, see, for example, Bègue *et al.* 1988; Alonso and Valli 1996).

Therefore, we can rewrite the two-domain formulation (8.5.5) as

$$
\text{(8.5.8)}\qquad \begin{cases} \Delta\varphi_1 = 0 & \text{in } \Omega_1 \\ \dfrac{\partial\varphi_1}{\partial n} = -\mathrm{div}_\tau(\mathbf{n}\times\widetilde{\mathbf{u}_2})_{|\Gamma} & \text{on } \partial\Omega_1 \\ \displaystyle\int_{\Omega_1}\varphi_1 = 0 & \\ \mathbf{n}\times\mathbf{u}_2 = \mathbf{0} & \text{on } \partial\Omega_2\setminus\Gamma \\ \mathbf{n}\times(\mu^{-1}\,\mathrm{rot}\,\mathbf{u}_2) = \mu_0^{-1}\mathbf{n}\times\nabla\varphi_1 & \text{on } \Gamma \\ \mathrm{rot}(\mu^{-1}\,\mathrm{rot}\,\mathbf{u}_2) + i\alpha\hat{\sigma}\mathbf{u}_2 = \mathbf{F} & \text{in } \Omega_2, \end{cases}
$$

and the iteration-by-subdomain procedure is easily derived as follows: given a tangential vector field $\boldsymbol{\lambda}^0$ on Γ, for each $k \geq 0$ solve

$$
\text{(8.5.9)}\qquad \begin{cases} \Delta\varphi_1^{k+1} = 0 & \text{in } \Omega_1 \\ \dfrac{\partial\varphi_1^{k+1}}{\partial n} = -\mathrm{div}_\tau\widetilde{\boldsymbol{\lambda}^k} & \text{on } \partial\Omega_1 \\ \displaystyle\int_{\Omega_1}\varphi_1^{k+1} = 0 & \end{cases}
$$

$$
\text{(8.5.10)}\qquad \begin{cases} \mathrm{rot}(\mu^{-1}\,\mathrm{rot}\,\mathbf{u}_2^{k+1}) + i\alpha\hat{\sigma}\mathbf{u}_2^{k+1} = \mathbf{F} & \text{in } \Omega_2 \\ \mathbf{n}\times(\mu^{-1}\,\mathrm{rot}\,\mathbf{u}_2^{k+1}) = \mu_0^{-1}\mathbf{n}\times\nabla\varphi_1^{k+1} & \text{on } \Gamma \\ \mathbf{n}\times\mathbf{u}_2^{k+1} = \mathbf{0} & \text{on } \partial\Omega_2\setminus\Gamma, \end{cases}
$$

where

$$
\text{(8.5.11)}\qquad \boldsymbol{\lambda}^{k+1} := \theta(\mathbf{n}_\Gamma\times\mathbf{u}_2^{k+1})_{|\Gamma} + (1-\theta)\boldsymbol{\lambda}^k.
$$

The convergence of this iterative procedure has been proved for the corresponding finite dimensional problem, using suitable finite elements for space discretisation. More precisely, standard piecewise-polynomials for the approximation in $H^1(\Omega_1)$ and the rot-conforming Nédélec finite elements for the approximation in $H(\mathrm{rot};\Omega_2) := \{\mathbf{v}\in(L^2(\Omega_2))^3 \mid \mathrm{rot}\,\mathbf{v}\in(L^2(\Omega_2))^3\}$ are employed. We refer to Alonso and Valli (1997) for a precise variational formulation and the corresponding finite element approximation. There it is proved that the assumptions of Theorem 4.2.10 are satisfied, uniformly with respect to the grid size h.

9

APPENDIX

9.1 Function spaces

In this last section we recall the definitions of some function spaces which have been often used in the book. A complete presentation of this subject can be found for instance in Yosida (1974), Brezis (1983), J.-L. Lions and Magenes (1972), and Adams (1975).

1. *Hilbert and Banach spaces*

Let V be a complex linear space. A *scalar product* on V is a map $(\cdot,\cdot) : V \times V \to \mathbf{C}$, linear in the first argument and such that $(w,v) = \overline{(v,w)}$ for each $w, v \in V$ (symmetry; the bar denotes complex conjugation); $(v,v) \geq 0$ for each $v \in V$ (positivity); and $(v,v) = 0$ if, and only if, $v = 0$. In the case of a real linear space, the map $(\cdot,\cdot)$ takes values in $\mathbf{R}$.

A *seminorm* is a map $||\cdot|| : V \to \mathbf{R}$ such that $||v|| \geq 0$ for each $v \in V$; $||cv|| = |c|\,||v||$ for each $c \in \mathbf{C}$ and $v \in V$; and $||w+v|| \leq ||w|| + ||v||$ for each $w, v \in V$ (triangular inequality).

A *norm* on V is a seminorm satisfying the additional property that $||v|| = 0$ if, and only if, $v = 0$. Two norms $||\cdot||$ and $|||\cdot|||$ on V are equivalent if there exist two positive constants M_1 and M_2 such that

$$M_1||v|| \leq |||v||| \leq M_2||v||$$

for each $v \in V$.

It is readily verified that any scalar product defines a norm by setting: $||v|| := (v,v)^{1/2}$. Moreover, any norm defines a distance: $d(w,v) := ||w-v||$.

A linear space V endowed with a scalar product (respectively, a norm) is called *pre-hilbertian* (respectively, *normed*) space. A sequence v_n is a Cauchy sequence in a normed space V if it is a Cauchy sequence with respect to the distance $d(w,v) = ||w-v||$. If any Cauchy sequence in a pre-hilbertian (respectively, normed) space V is convergent, the space V is called a *Hilbert* Hilbert space (respectively, Banach space*Banach*) space.

In a Hilbert space the *Schwarz inequality* holds:

$$|(w,v)| \leq ||w||\,||v|| \quad \text{for each } w, v \in V. \tag{9.1.1}$$

2. *Dual spaces*

If $(V, ||\cdot||_V)$ and $(W, ||\cdot||_W)$ are normed spaces, we denote by $\mathcal{L}(V;W)$ the set of linear, continuous functionals from V into W, and for $\mathcal{F} \in \mathcal{L}(V;W)$ we define the norm

$$||\mathcal{F}||_{\mathcal{L}(V;W)} := \sup_{\substack{v\in V\\ v\neq 0}} \frac{||\mathcal{F}(v)||_W}{||v||_V}. \tag{9.1.2}$$

Thus $\mathcal{L}(V;W)$ is a normed space; if W is a Banach space, then $\mathcal{L}(V;W)$ is a Banach space, too. If $W = \mathbf{C}$ (respectively, $W = \mathbf{R}$, if V is a real normed space), the space $\mathcal{L}(V;\mathbf{C})$ (respectively, $\mathcal{L}(V;\mathbf{R})$) is called the *dual space* of V and is denoted by V'. The norm in V' is indicated by $||\cdot||_{V'}$.

The bilinear form $\langle\cdot,\cdot\rangle$ from $V' \times V$ into $\mathbf{C}$ (respectively, $\mathbf{R}$) defined by $\langle\mathcal{F}, v\rangle := \mathcal{F}(v)$ is called the *duality pairing* between V' and V.

3. *Weak convergence*
In a normed space V it is possible to introduce another type of convergence, which is called *weak convergence*. It is defined as follows: a sequence v_n is called weakly convergent to $v \in V$ if $\mathcal{F}(v_n)$ converges to $\mathcal{F}(v)$ for each $\mathcal{F} \in V'$. Clearly, if the sequence v_n converges to v in V, it is also weakly convergent. The converse is not true, unless V is finite dimensional.

It can be proved that the weak limit v, if it exists, is unique. Moreover, if v_n is weakly convergent to $v \in V$, one has

$$||v|| \leq \liminf_{n\to\infty} ||v_n||.$$

4. *The Riesz theorem and the Lax–Milgram lemma*
An important result which holds in Hilbert spaces is the following one.

Theorem 9.1.1 (Riesz representation theorem) *Let V be a (real or complex) Hilbert space, endowed with the scalar product $(\cdot,\cdot)_V$. If $\mathcal{F} \in V'$, then there exists a unique $w_{\mathcal{F}} \in V$ such that*

$$\mathcal{F}(v) = (w_{\mathcal{F}}, v)_V \qquad \forall\ v \in V. \tag{9.1.3}$$

As a consequence of the Riesz representation theorem, if V is a Hilbert space, then the dual V' is a Hilbert space which can be canonically identified to V.

Another consequence of the Riesz theorem is this result.

Theorem 9.1.2 (Lax–Milgram lemma) *Assume that V is a real Hilbert space, endowed with the scalar product $(\cdot,\cdot)_V$ and the norm $||\cdot||_V$, that $\mathcal{A} : V \times V \to \mathbf{R}$ is a bilinear form, and that $\mathcal{F} : V \to \mathbf{R}$ is a linear continuous functional, namely, $\mathcal{F} \in V'$. Assume, moreover, that $\mathcal{A}$ is continuous, namely,*

$$\exists\ \gamma > 0 : |\mathcal{A}(w,v)| \leq \gamma ||w||_V\, ||v||_V \qquad \forall\ w, v \in V, \tag{9.1.4}$$

and coercive, namely,

$$\exists\ \alpha > 0 : \mathcal{A}(v,v) \geq \alpha ||v||_V^2 \qquad \forall\ v \in V. \tag{9.1.5}$$

Then there exists a unique $u \in V$ solution to

$$\mathcal{A}(u,v) = \mathcal{F}(v) \qquad \forall\ v \in V, \tag{9.1.6}$$

and, moreover,

$$||u||_V \leq \frac{1}{\alpha}||\mathcal{F}||_{V'}. \tag{9.1.7}$$

If V is a complex Hilbert space, the Lax–Milgram lemma holds provided that (9.1.5) is substituted by the assumption

$$\exists\ \alpha > 0 : |\mathcal{A}(v,v)| \geq \alpha||v||_V^2 \qquad \forall\ v \in V. \tag{9.1.8}$$

5. *L^p spaces*
Let Ω be an open set contained in $\mathbf{R}^d$, $d \geq 1$, and consider in Ω the Lebesgue measure. A very important family of Banach spaces is the following one. Let $1 \leq p \leq \infty$, and consider the set of measurable functions v such that

$$\int_\Omega |v(\mathbf{x})|^p d\mathbf{x} < \infty, \quad 1 \leq p < \infty, \tag{9.1.9}$$

or, when $p = \infty$,

$$\sup\{|v(\mathbf{x})| \,|\, \mathbf{x} \in \Omega\} < \infty. \tag{9.1.10}$$

These spaces are usually denoted by $L^p(\Omega)$ and the associated norm is

$$||v||_{L^p(\Omega)} := \left(\int_\Omega |v(\mathbf{x})|^p d\mathbf{x}\right)^{1/p}, \quad 1 \leq p < \infty, \tag{9.1.11}$$

or, when $p = \infty$,

$$||v||_{L^\infty(\Omega)} := \sup\{|v(\mathbf{x})| \,|\, \mathbf{x} \in \Omega\}. \tag{9.1.12}$$

More precisely, $L^p(\Omega)$ is indeed the space of classes of equivalence of measurable functions, satisfying (9.1.9) or (9.1.10), with respect to the equivalence relation: $w \equiv v$ if w and v are different on a subset having zero-measure. In other words, in the space $L^p(\Omega)$ two functions, different on a subset which has zero-measure, are identified to each other. Thus the definition of the space $L^\infty(\Omega)$ in (9.1.10) and of its norm in (9.1.12) should be modified in the following way: $v \in L^\infty(\Omega)$ if

$$\inf\{M \geq 0 \,|\, |v(\mathbf{x})| \leq M \text{ almost everywhere in } \Omega\} < \infty,$$

and

$$||v||_{L^\infty(\Omega)} := \inf\{M \geq 0 \,|\, |v(\mathbf{x})| \leq M \text{ almost everywhere in } \Omega\},$$

where 'almost everywhere in Ω' means 'everywhere except on a subset of Ω having zero-measure'.

The space $L^2(\Omega)$ is a Hilbert space, endowed with the scalar product

$$(w, v)_{L^2(\Omega)} := \int_\Omega w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x},$$

often indicated by $(w, v)_{0,\Omega}$ or simply (w, v).

If $1 \leq p < \infty$, the dual space of $L^p(\Omega)$ is given by $L^{p'}(\Omega)$, where $(1/p) + (1/p') = 1$ (and $p' = \infty$ if $p = 1$).

6. *Distributions*

Let us recall that $C_0^\infty(\Omega)$ (or $\mathcal{D}(\Omega)$) denotes the space of infinitely differentiable functions having compact support; that is, vanishing outside a bounded open set $\Omega' \subset \Omega$ which has a positive distance from the boundary $\partial\Omega$ of Ω.

It is useful to define the concept of convergence for sequences of $\mathcal{D}(\Omega)$. We say that $v_n \in \mathcal{D}(\Omega)$ converges to $v \in \mathcal{D}(\Omega)$ if there exists a closed bounded subset $K \in \Omega$ such that v_n vanishes outside K for each n, and for every non-negative multi-index $\boldsymbol{\alpha}$ the derivative $D^{\boldsymbol{\alpha}} v_n$ converges to $D^{\boldsymbol{\alpha}} v$ uniformly in Ω. We recall that if $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d)$, α_i non-negative integers, then

$$D^{\boldsymbol{\alpha}} v := \frac{\partial^{|\boldsymbol{\alpha}|} v}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}},$$

where $|\boldsymbol{\alpha}| := \alpha_1 + ... + \alpha_d$ is the length of $\boldsymbol{\alpha}$.

The space of linear functionals on $\mathcal{D}(\Omega)$ which are continuous with respect to the convergence introduced above is denoted by $\mathcal{D}'(\Omega)$ and its elements are called *distributions*. If $L \in \mathcal{D}'(\Omega)$ and $v \in \mathcal{D}(\Omega)$, the action of the functional L on v is usually denoted by the duality pairing $\langle L, v \rangle$.

It is easily seen that each function $w \in L^p(\Omega)$, $1 \leq p \leq \infty$, can be associated with the following distribution:

$$v \to \int_\Omega w(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x}, \quad v \in \mathcal{D}(\Omega).$$

However, setting for instance $\Omega = (-1, 1)$, the Dirac functional

$$v \to \langle \delta, v \rangle := v(0), \quad v \in \mathcal{D}(\Omega),$$

is a distribution which cannot be represented through any function belonging to $L^p(\Omega)$, $1 \leq p \leq \infty$.

We introduce now the *derivative* of a distribution. Let $\boldsymbol{\alpha}$ be a non-negative multi-index and L a distribution. Then $D^{\boldsymbol{\alpha}} L$ is the distribution defined as follows:

$$\langle D^{\boldsymbol{\alpha}} L, v \rangle := (-1)^{|\boldsymbol{\alpha}|} \langle L, D^{\boldsymbol{\alpha}} v \rangle \qquad \forall\ v \in \mathcal{D}(\Omega).$$

Note that, from this definition, a distribution turns out to be infinitely differentiable. On the other hand, when L is a smooth function, it is easily verified

by integrating by parts that its derivative in the sense of distributions coincides with the usual derivative.

Let us also recall that the Dirac distribution δ is the distributional derivative of the Heaviside function:

$$H(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

Finally, we say that the $\boldsymbol{\alpha}$-derivative of a distribution L is a function belonging to $L^p(\Omega)$, $1 \leq p \leq \infty$, if there exists a function $g_{\boldsymbol{\alpha}} \in L^p(\Omega)$ such that

$$\langle D^{\boldsymbol{\alpha}} L, v \rangle = \int_\Omega g_{\boldsymbol{\alpha}}(\mathbf{x})\, v(\mathbf{x})\, d\mathbf{x} \qquad \forall\ v \in \mathcal{D}(\Omega).$$

7. *Sobolev spaces*

We finally introduce another class of functions, which furnish the natural environment for the variational theory of partial differential equations.

The *Sobolev space* $W^{k,p}(\Omega)$, k a non-negative integer and $1 \leq p \leq \infty$, is the space of functions $v \in L^p(\Omega)$ such that all the distributional derivatives of v of order up to k belong to $L^p(\Omega)$. In short

$$W^{k,p}(\Omega) := \big\{ v \in L^p(\Omega) \,|\ D^{\boldsymbol{\alpha}} v \in L^p(\Omega) \text{ for each non-negative multi-index } \boldsymbol{\alpha} \text{ such that } |\boldsymbol{\alpha}| \leq k \big\}.$$

Clearly, for each p, $1 \leq p \leq \infty$, $W^{0,p}(\Omega) = L^p(\Omega)$ and $W^{k_2,p}(\Omega) \subset W^{k_1,p}(\Omega)$ when $k_1 \leq k_2$. For $1 \leq p < \infty$, $W^{k,p}(\Omega)$ is a Banach space with respect to the norm

$$||v||_{k,p,\Omega} := \left(\sum_{|\boldsymbol{\alpha}| \leq k} ||D^{\boldsymbol{\alpha}} v||^p_{L^p(\Omega)} \right)^{1/p} .$$

Moreover, its seminorm is defined as follows:

$$|v|_{k,p,\Omega} := \left(\sum_{|\boldsymbol{\alpha}| = k} ||D^{\boldsymbol{\alpha}} v||^p_{L^p(\Omega)} \right)^{1/p} .$$

On the other hand, $W^{k,\infty}(\Omega)$ is a Banach space with respect to the norm

$$||v||_{k,\infty,\Omega} := \max_{|\boldsymbol{\alpha}| \leq k} ||D^{\boldsymbol{\alpha}} v||_{L^\infty(\Omega)},$$

while the corresponding seminorm is denoted by

$$|v|_{k,\infty,\Omega} := \max_{|\boldsymbol{\alpha}| = k} ||D^{\boldsymbol{\alpha}} v||_{L^\infty(\Omega)}.$$

In particular, when $p = 2$ we write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$, $||\cdot||_{k,\Omega}$ and $|\cdot|_{k,\Omega}$ instead of $||\cdot||_{k,2,\Omega}$ and $|\cdot|_{k,2,\Omega}$, respectively.

Note that $H^k(\Omega)$ is a Hilbert space with respect to the scalar product

$$(w, v)_{k,\Omega} := \sum_{|\boldsymbol{\alpha}| \leq k} (D^{\boldsymbol{\alpha}} w, D^{\boldsymbol{\alpha}} v)_{0,\Omega}.$$

Finally, for $1 \leq p < \infty$ we denote by $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to the norm $|| \cdot ||_{k,p,\Omega}$, and with $W^{-k,p'}(\Omega)$ the dual space of $W_0^{k,p}(\Omega)$. As before, when $p = 2$ we write $H_0^k(\Omega)$ and $H^{-k}(\Omega)$ instead of $W_0^{k,2}(\Omega)$ and $W^{-k,2}(\Omega)$, respectively.

It can also be proved that $W_0^{0,p}(\Omega) = L^p(\Omega)$ and that, if Ω has a Lipschitz continuous boundary and $1 \leq p < \infty$, $W^{k,p}(\Omega)$ is indeed the closure of $C^\infty(\overline{\Omega})$ with respect to the norm $|| \cdot ||_{k,p,\Omega}$. In other words, $C^\infty(\overline{\Omega})$ is *dense* in $W^{k,p}(\Omega)$ for $1 \leq p < \infty$.

It is sometimes useful to consider the Sobolev space $W^{s,p}(\Omega)$, where $s \in \mathbf{R}$ and $1 \leq p \leq \infty$. For the general definition, we refer to Adams (1975). In particular, we only recall that, if $\Omega = \mathbf{R}^d$ and $p = 2$, $W^{s,2}(\mathbf{R}^d) = H^s(\mathbf{R}^d)$ can be characterised as follows by means of the Fourier transform $\hat{v}(\boldsymbol{\xi})$:

$$H^s(\mathbf{R}^d) = \left\{ v \in L^2(\mathbf{R}^d) \,|\, (1 + |\boldsymbol{\xi}|^2)^{s/2} \hat{v}(\boldsymbol{\xi}) \in L^2(\mathbf{R}^d) \right\}.$$

When considering vector-valued functions $\mathbf{v} : \Omega \to \mathbf{R}^d$, the Hilbert space

$$H(\mathrm{div}; \Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^d \,|\, \mathrm{div}\, \mathbf{v} \in L^2(\Omega) \right\}$$

is also often used. It is endowed with the graph norm

$$||\mathbf{v}||_{H(\mathrm{div};\Omega)} := (||\mathbf{v}||_{0,\Omega}^2 + ||\,\mathrm{div}\, \mathbf{v}||_{0,\Omega}^2)^{1/2}.$$

Similarly to the preceding cases, if Ω has a Lipschitz continuous boundary, it can be proved that $H(\mathrm{div}; \Omega)$ is the closure of $(C^\infty(\overline{\Omega}))^d$ with respect to the norm $|| \cdot ||_{H(\mathrm{div};\Omega)}$.

For three-dimensional vector-valued functions we also introduce the Hilbert space

$$H(\mathrm{rot}; \Omega) := \left\{ \mathbf{v} \in (L^2(\Omega))^3 \,|\, \mathrm{rot}\, \mathbf{v} \in (L^2(\Omega))^3 \right\},$$

endowed with the graph norm

$$||\mathbf{v}||_{H(\mathrm{rot};\Omega)} := (||\mathbf{v}||_{0,\Omega}^2 + ||\,\mathrm{rot}\, \mathbf{v}||_{0,\Omega}^2)^{1/2}.$$

Again, if Ω has a Lipschitz continuous boundary, then $H(\mathrm{rot}; \Omega)$ is the closure of $(C^\infty(\overline{\Omega}))^3$ with respect to the norm $|| \cdot ||_{H(\mathrm{rot};\Omega)}$.

Another important class of Sobolev spaces is given by $W^{s,p}(\Sigma)$, where $s \geq 0$, $1 \leq p \leq \infty$ and Σ is a suitable subset of the boundary $\partial\Omega$ (again, we write $H^s(\Sigma)$ instead of $W^{s,2}(\Sigma)$). Their definition needs the introduction of some technical tools, especially if Σ is a non-smooth hypersurface (for instance, the boundary

of a polygonal domain). For this we refer to Adams (1975) or Brezzi and Gilardi (1987); however, we return on a characterisation of these spaces in the following section. When $\Sigma = \partial\Omega$, the dual space of $H^s(\partial\Omega)$ is denoted by $H^{-s}(\partial\Omega)$.

9.2 Some properties of the Sobolev spaces

In this section we present without proofs some relevant properties enjoyed by functions belonging to Sobolev spaces. We mainly limit ourselves to the Hilbert spaces $H^s(\Omega)$, referring the reader to J.-L. Lions and Magenes (1972) or Adams (1975) for the general case and all the proofs.

Let us start with the so-called *trace* theorems. The trace on the boundary $\partial\Omega$ of a function $v \in H^s(\Omega)$ is, in a sense to make precise, the value of v restricted to $\partial\Omega$. Indeed, the latter statement has not even a meaning, as a function in $H^s(\Omega)$ is not univocally defined on subsets having measure equal to zero. If we denote by $C^0(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$, the precise result reads as follows.

Theorem 9.2.1 (Trace theorem) *Let Ω be a bounded open set of $\mathbf{R}^d$. Assume that the boundary $\partial\Omega$ is smooth and that $s > 1/2$.*

(a) There exists a unique linear continuous map $\gamma_0 : H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ such that $\gamma_0 v = v_{|\partial\Omega}$ for each $v \in H^s(\Omega) \cap C^0(\overline{\Omega})$.

(b) There exists a linear continuous map $\mathcal{R}_0 : H^{s-1/2}(\partial\Omega) \to H^s(\Omega)$ such that $\gamma_0 \mathcal{R}_0 \varphi = \varphi$ for each $\varphi \in H^{s-1/2}(\partial\Omega)$.

Analogous results hold true if we consider the trace $\gamma_\Sigma v$ over a smooth subset Σ of the boundary $\partial\Omega$. In particular, for $1/2 < s \leq 1$, it is enough to assume that the boundary $\partial\Omega$ or the set Σ are Lipschitz continuous.

Thus, we have seen that any function belonging to $H^{s-1/2}(\Sigma)$, $s > 1/2$ and Σ smooth, is the trace on Σ of a function in $H^s(\Omega)$. This provides a useful characterisation of the space $H^{s-1/2}(\Sigma)$.

For vector functions belonging to $H(\mathrm{div};\Omega)$ the following trace result can be proved.

Theorem 9.2.2 (Normal trace theorem) *Let Ω be a bounded open set of $\mathbf{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$.*

(a) There exists a unique linear continuous map $\gamma_n : H(\mathrm{div};\Omega) \to H^{-1/2}(\partial\Omega)$ such that $\gamma_n \mathbf{v} = (\mathbf{v} \cdot \mathbf{n}^*)_{|\partial\Omega}$ for each $\mathbf{v} \in H(\mathrm{div};\Omega) \cap (C^0(\overline{\Omega}))^d$.

(b) There exists a linear continuous map $\boldsymbol{\mathcal{R}}_n : H^{-1/2}(\partial\Omega) \to H(\mathrm{div};\Omega)$ such that $\gamma_n \boldsymbol{\mathcal{R}}_n \varphi = \varphi$ for each $\varphi \in H^{-1/2}(\partial\Omega)$.

Here we have denoted by $\mathbf{n}^*$ the unit outward normal vector on $\partial\Omega$. Let us note, moreover, that the normal trace of a vector function $\mathbf{v} \in H(\mathrm{div};\Omega)$ over a Lipschitz continuous subset Σ of $\partial\Omega$ different from the whole boundary $\partial\Omega$ does not belong in general to $H^{-1/2}(\Sigma)$, but to a larger space, which is usually denoted by $H_{00}^{-1/2}(\Sigma)$ (see, for instance, J.-L. Lions and Magenes 1972).

Now, let us introduce the space

$$\mathcal{X}_{\partial\Omega} := \{\boldsymbol{\psi} \in (H^{-1/2}(\partial\Omega))^3 \mid \boldsymbol{\psi} \cdot \mathbf{n}^* = 0,\ \mathrm{div}_\tau \boldsymbol{\psi} \in H^{-1/2}(\partial\Omega)\},$$

where $\mathrm{div}_\tau\, \boldsymbol{\psi}$ denotes the tangential divergence of $\boldsymbol{\psi}$ (see, for example, Bègue *et al.* 1988).

For three-dimensional vector functions belonging to $H(\mathrm{rot};\Omega)$ the following trace result can be proved (see Alonso and Valli 1996).

Theorem 9.2.3 (Tangential trace theorem) *Let Ω be a bounded open set of $\mathbf{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$.*

(a) There exists a unique linear continuous map $\boldsymbol{\gamma}_\tau : H(\mathrm{rot};\Omega) \to \mathcal{X}_{\partial\Omega}$ such that $\boldsymbol{\gamma}_\tau \mathbf{v} = (\mathbf{n}^* \times \mathbf{v})_{|\partial\Omega}$ for each $\mathbf{v} \in H(\mathrm{rot};\Omega) \cap (C^0(\overline{\Omega}))^3$.

(b) If either the boundary $\partial\Omega$ is smooth or Ω is a convex polyhedron, then there exists a linear continuous map $\boldsymbol{\mathcal{R}}_\tau : \mathcal{X}_{\partial\Omega} \to H(\mathrm{rot};\Omega)$ such that $\boldsymbol{\gamma}_\tau \boldsymbol{\mathcal{R}}_\tau \boldsymbol{\psi} = \boldsymbol{\psi}$ for each $\boldsymbol{\psi} \in \mathcal{X}_{\partial\Omega}$.

By means of these trace operators it is possible to characterise the spaces $H_0^1(\Omega)$, $H_0(\mathrm{div};\Omega) := \overline{(C_0^\infty(\Omega))^d}$ and $H_0(\mathrm{rot};\Omega) := \overline{(C_0^\infty(\Omega))^3}$ (here the closure has to be intended with respect to the norms $||\cdot||_{H(\mathrm{div};\Omega)}$ and $||\cdot||_{H(\mathrm{rot};\Omega)}$, respectively). As a matter of fact, if the boundary $\partial\Omega$ is Lipschitz continuous, we have:

$$\begin{aligned} H_0^1(\Omega) &= \{v \in H^1(\Omega) \,|\, \gamma_0 v = 0\} \\ H_0(\mathrm{div};\Omega) &= \{\mathbf{v} \in H(\mathrm{div};\Omega) \,|\, \gamma_n \mathbf{v} = 0\} \\ H_0(\mathrm{rot};\Omega) &= \{\mathbf{v} \in H(\mathrm{rot};\Omega) \,|\, \boldsymbol{\gamma}_\tau \mathbf{v} = \mathbf{0}\}. \end{aligned}$$

A similar characterisation holds for the space

$$H_\Sigma^1(\Omega) := \{v \in H^1(\Omega) \,|\, \gamma_\Sigma v = 0\}.$$

An important result is the so-called Poincaré inequality (see (1.2.2)).

Theorem 9.2.4 (Poincaré inequality) *Assume that Ω is a bounded connected open set of $\mathbf{R}^d$ and that Σ is a (non-empty) Lipschitz continuous subset of the boundary $\partial\Omega$. Then there exists a constant $C_\Omega > 0$ such that*

$$\int_\Omega v^2(\mathbf{x})\, d\mathbf{x} \le C_\Omega \int_\Omega |\nabla v(\mathbf{x})|^2\, d\mathbf{x} \tag{9.2.1}$$

for each $v \in H_\Sigma^1(\Omega)$.

As a consequence of the density of $C^\infty(\overline{\Omega})$ in $H^1(\Omega)$ (under the assumption that $\partial\Omega$ is Lipschitz continuous), it is easily proved that for each $w, v \in H^1(\Omega)$ the following *Green formula* holds:

$$\int_\Omega (D_j w)\, v\, d\mathbf{x} = -\int_\Omega w\, D_j v\, d\mathbf{x} + \int_{\partial\Omega} (\gamma_0 w)\, (\gamma_0 v)\, n_j^*\, d\boldsymbol{\gamma}, \quad j = 1, ..., d, \tag{9.2.2}$$

where we have denoted by D_j the partial derivative $\frac{\partial}{\partial x_j}$ and by $d\boldsymbol{\gamma}$ the surface measure on $\partial\Omega$.

Similarly, if $\mathbf{w} \in H(\mathrm{div};\Omega)$ and $v \in H^1(\Omega)$, we find that

$$\int_\Omega (\mathrm{div}\,\mathbf{w})\, v\, d\mathbf{x} = -\int_\Omega \mathbf{w}\cdot\nabla v\, d\mathbf{x} + \int_{\partial\Omega} (\gamma_n \mathbf{w})\,(\gamma_0 v)\, d\boldsymbol{\gamma}. \tag{9.2.3}$$

Finally, if $\mathbf{w} \in H(\mathrm{rot};\Omega)$ and $\mathbf{v} \in (H^1(\Omega))^3$, we find that

$$\int_\Omega (\mathrm{rot}\,\mathbf{w})\cdot\mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{w}\cdot\mathrm{rot}\,\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega} (\boldsymbol{\gamma}_\tau \mathbf{w})\cdot(\gamma_0 \mathbf{v})\, d\boldsymbol{\gamma}. \tag{9.2.4}$$

As we have already noted, the functions belonging to the Sobolev spaces $W^{s,p}(\Omega)$ are not univocally defined over subsets having measure equal to zero. However, if suitable restrictions on the indices s and p are assumed, these functions indeed turn out to be regular functions. This is made clear by the following theorem.

Theorem 9.2.5 (Sobolev embedding theorem) *Assume that Ω is a (bounded or unbounded) open set of $\mathbf{R}^d$ with a Lipschitz continuous boundary, and that $1 \le p < \infty$. Then the following embeddings are continuous.*

(a) If $0 \le sp < d$, then $W^{s,p}(\Omega) \subset L^{p^*}(\Omega)$ for $p^* = dp/(d-sp)$.
(b) If $sp = d$, then $W^{s,p}(\Omega) \subset L^q(\Omega)$ for any q such that $p \le q < \infty$.
(c) If $sp > d$, then $W^{s,p}(\Omega) \subset C^0(\overline{\Omega})$.

In the one-dimensional case, we have in particular that $H^1(\Omega) \subset C^0(\overline{\Omega})$, with continuous embedding.

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