CS2100 COMPUTER ORGANISATION

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Lecture #13

Boolean Algebra

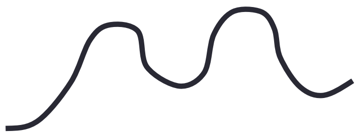
NUS National University of Singapore | School of Computing

Lecture #13: Boolean Algebra

1. Digital Circuits (1/2)

- Two voltage levels
  - High/true/1/asserted
  - Low/false/0/deasserted

Signals in digital circuit



Signals in analog circuit

- Advantages of digital circuits over analog circuits
  - More reliable (simpler circuits, less noise-prone)
  - Specified accuracy (determinable)
  - Abstraction can be applied using simple mathematical model – Boolean Algebra
  - Ease design, analysis and simplification of digital circuit – Digital Logic Design

1. Digital Circuits (2/2)

- Combinational: no memory, output depends solely on the input
  - Gates
  - Decoders, multiplexers
  - Adders, multipliers
- Sequential: with memory, output depends on both input and current state
  - Counters, registers
  - Memories

2. Boolean Algebra

- Boolean values:
  - True (T or 1)
  - False (F or 0)
- Connectives
  - Conjunction (AND)
    - A · B; A ∧ B
  - Disjunction (OR)
    - A + B; A ∨ B
  - Negation (NOT)
    - A'; $\overline{A}$; ¬A;

In CS2100, we use the symbols 1 for true, 0 for false, · for AND, + for OR, and ' for negation (you may use the accent bar). Please follow.

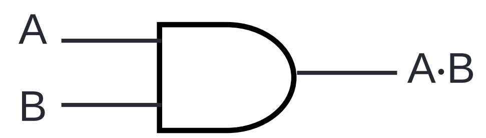
- Truth tables

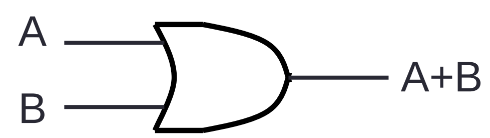
A	B	A · B
0	0	0
0	1	0
1	0	0
1	1	1

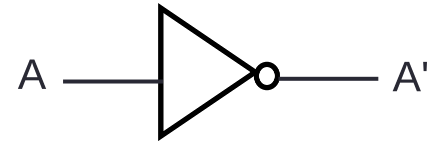
A	B	A + B
0	0	0
0	1	1
1	0	1
1	1	1

A	A'
0	1
1	0

- Logic gates







2. Boolean Algebra: AND

- Do write the AND operator · (instead of omitting it)
  - Example: Write ***a·b*** instead of ***ab***
  - Why? Writing ***ab*** could mean that it is a 2-bit value.

3. Truth Table

- Provide a listing of every possible combination of inputs and its corresponding outputs.
  - Inputs are usually listed in binary sequence.
- Example
  - Truth table with 3 inputs x, y, z and 2 outputs (y + z) and (x · (y + z)).

x	y	z	y + z	x · (y + z)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

3. Proof using Truth Table

- Prove: **x · (y + z) = (x · y) + (x · z)**
  - Construct truth table for LHS and RHS

x	y	z	y + z	**x · (y + z)**	x · y	x · z	**(x · y) + (x · z)**
0	0	0	0	**0**	0	0	**0**
0	0	1	1	**0**	0	0	**0**
0	1	0	1	**0**	0	0	**0**
0	1	1	1	**0**	0	0	**0**
1	0	0	0	**0**	0	0	**0**
1	0	1	1	**1**	0	1	**1**
1	1	0	1	**1**	1	0	**1**
1	1	1	1	**1**	1	1	**1**

  - Check that column for LHS = column for RHS
- DLD page 59 Quick Review Questions Question 3-1.

4. Precedence of Operators

- Precedence from highest to lowest
  - Not (')
  - And (·)
  - Or (+)

Note the difference with CS1231/CS1231S. Here in CS2100, AND has higher precedence than OR.

- Examples:
  - A · B + C = (A · B) + C
  - X + Y' = X + (Y')
  - P + Q' · R = P + ((Q') · R)

Hence, A · B + C is <u>not</u> ambiguous in CS2100.

- Use parenthesis to overwrite precedence. Examples:
  - A · (B + C) [Without parenthesis, it means A·B+C or (A·B)+C]
  - (P + Q)' · R [Without parenthesis, it means P+Q'·R or P+(Q'·R)]

5. Laws of Boolean Algebra

Identity laws	
$A + 0 = 0 + A = A$	$A \cdot 1 = 1 \cdot A = A$
Inverse/complement laws	
$A + A' = A' + A = 1$	$A \cdot A' = A' \cdot A = 0$
Commutative laws	
$A + B = B + A$	$A \cdot B = B \cdot A$
Associative laws *	
$A + (B + C) = (A + B) + C$	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$
Distributive laws	
$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$	$A + (B \cdot C) = (A + B) \cdot (A + C)$

* Due to the associative laws, $A + B + C$ is unambiguous. It may be evaluated as $A + (B + C)$ or $(A + B) + C$. Likewise for $A \cdot B \cdot C$.

6. Duality

- If the AND/OR operators and identity elements 0/1 in a Boolean equation are interchanged, it remains valid.
- Example:
  - The dual equation of $a+(b \cdot c)=(a+b) \cdot (a+c)$ is $a \cdot (b+c)=(a \cdot b)+(a \cdot c)$.
- Duality gives free theorems – “two for the price of one”, as a Boolean equation is logically equivalent to its dual. So, you prove one theorem and the other comes for free!
- Examples:
  - If $(x+y+z)' = x' \cdot y' \cdot z'$ is valid, then its dual $(x \cdot y \cdot z)' = x'+y'+z'$ is also valid.
  - If $x+1 = 1$ is valid, then its dual $x \cdot 0 = 0$ is also valid.

Do not confuse duality with negation!

7. Theorems

Idempotency	
$X + X = X$	$X \cdot X = X$
One element / Zero element	
$X + 1 = 1 + X = 1$	$X \cdot 0 = 0 \cdot X = 0$
Involution	
$(X')' = X$	
Absorption 1	
$X + X \cdot Y = X$	$X \cdot (X + Y) = X$
Absorption 2	
$X + X' \cdot Y = X + Y$	$X \cdot (X' + Y) = X \cdot Y$
DeMorgans' (can be generalised to more than 2 variables)	
$(X + Y)' = X' \cdot Y'$	$(X \cdot Y)' = X' + Y'$
Consensus	
$X \cdot Y + X' \cdot Z + Y \cdot Z = X \cdot Y + X' \cdot Z$	$(X+Y) \cdot (X'+Z) \cdot (Y+Z) = (X+Y) \cdot (X'+Z)$

7. Proving a Theorem

- Theorems can be proved using truth table, or by algebraic manipulation using other theorems/laws.
- Example: Prove absorption theorem $X + X \cdot Y = X$

$$\begin{aligned} X + X \cdot Y &= X \cdot 1 + X \cdot Y \text{ (by identity law)} \\ &= X \cdot (1+Y) \text{ (by distributivity)} \\ &= X \cdot 1 \text{ (by one element law)} \\ &= X \text{ (by identity law)} \end{aligned}$$

- By the principle of duality, we may also cite (<u>without proof</u>) that $X \cdot (X+Y) = X$.

8. Boolean Functions

- Examples of Boolean functions (logic equations):

  F1(x,y,z) = x·y·z'

  F2(x,y,z) = x + y'·z

  F3(x,y,z) = x'·y'·z + x'·y·z + x·y'

  F4(x,y,z) = x·y' + x'·z

x	y	z	F1	F2	F3	F4
0	0	0	0	0	0	0
0	0	1	0	1	1	1
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	1	0	0
1	1	1	0	1	0	0

From the truth table, F3 = F4.

Can you prove F3 = F4 by using Boolean Algebra?

9. Complement Functions

- Given a Boolean function F, the complement of F, denoted as F', is obtained by interchanging 1 with 0 in the function's output values.
- Example: $F1 = x \cdot y \cdot z'$
- What is F1' ?
  - $F1' = (x \cdot y \cdot z')'$
    $= x' + y' + (z')'$ (DeMorgan's)
    $= x' + y' + z$ (Involution)

x	y	z	F1	F1'
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	0	1

10. Standard Forms (1/2)

- Certain types of Boolean expressions lead to circuits that are desirable from an implementation viewpoint.
- Two standard forms:
  - Sum-of-Products (SOP)
  - Product-of-Sums (POS)
- Literals
  - A Boolean variable on its own or in its complemented form
  - Examples: (1) x, (2) x', (3) y, (4) y'
- Product term
  - A single literal or a logical product (AND) of several literals
  - Examples: (1) x, (2) x·y·z', (3) A'·B, (4) A·B, (5) d·g'·v·w

10. Standard Forms (2/2)

- Sum term
  - A single literal or a logical sum (OR) of several literals
  - Examples: (1) x, (2) $x+y+z'$, (3) $A'+B$, (4) $A+B$, (5) $c+d+h'+j$
- Sum-of-Products (SOP) expression
  - A product term or a logical sum (OR) of several product terms
  - Examples: (1) x, (2) $x + y\cdot z'$, (3) $x\cdot y' + x'\cdot y\cdot z$, (4) $A\cdot B + A'\cdot B'$,
    (5) $A + B'\cdot C + A\cdot C' + C\cdot D$
- Product-of-Sums (POS) expression
  - A sum term or a logical product (AND) of several sum terms
  - Examples: (1) x, (2) $x\cdot(y+z')$, (3) $(x+y')\cdot(x'+y+z)$,
    (4) $(A+B)\cdot(A'+B')$, (5) $(A+B+C)\cdot D'\cdot(B'+D+E')$
- Every Boolean expression can be expressed in SOP or POS form.
  - DLD page 59 Quick Review Questions Questions 3-2 to 3-5.

Quiz Time!

SOP expr: A product term or a logical sum (OR) of several product terms.
POS expr: A sum term or a logical product (AND) of several sum terms.

- Put the right ticks in the following table.

	Expression	***SOP?***	***POS?***
(1)	X'·Y + X·Y' + X·Y·Z	✓	✗
(2)	(X+Y')·(X'+Y)·(X'+Z')	✗	✓
(3)	X' + Y + Z	✓	✓
(4)	X·(W' + Y·Z)	✗	✗
(5)	X·Y·Z'	✓	✓
(6)	W·X'·Y + V·(X·Z + W')	✗	✗

11. Minterms and Maxterms (1/2)

- A minterm of *n* variables is a product term that contains *n* literals from all the variables.
  - Example: On 2 variables x and y, the minterms are: x'·y', x'·y, x·y' and x·y
- A maxterm of *n* variables is a sum term that contains *n* literals from all the variables.
  - Example: On 2 variables x and y, the maxterms are: x'+y', x'+y, x+y' and x+y
- In general, with *n* variables we have up to 2^n minterms and 2^n maxterms.

11. Minterms and Maxterms (2/2)

- The minterms and maxterms on 2 variables are denoted by m0 to m3 and M0 to M3 respectively.

x	y	Minterms		Maxterms	
		Term	Notation	Term	Notation
0	0	x'·y'	m0	x+y	M0
0	1	x'·y	m1	x+y'	M1
1	0	x·y'	m2	x'+y	M2
1	1	x·y	m3	x'+y'	M3

- Important fact: Each minterm is the <u>complement</u> of its corresponding maxterm. Likwise, each maxterm is the complement of its corresponding minterm.
  - Example: m2 = x·y'
    m2' = (x·y')' = x' + (y')' = x' + y = M2

Quiz Time Again!

- Ability to convert minterms and maxterms from its Boolean expression to its notation (and vice versa) is important.
- Test yourself with the following quiz, assuming that you are given a Boolean function on 4 variables A, B, C, D.

Minterm

	Boolean expression	***Minterm notation***
(1)	A'·B'·C·D	m3
(2)	**A·B'·C·D'**	m10
(3)	**A·B'·C·D**	m11
(4)	A·B·C·D'	**m14**
(5)	A·B'·C'·D	**m9**

Maxterm

	Boolean expression	***Maxterm notation***
(1)	A+B+C'+D'	M3
(2)	**A'+B'+C+D'**	M13
(3)	**A+B+C+D**	M0
(4)	A+B+C'+D	**M2**
(5)	A'+B+C+D'	**M9**

12. Canonical Forms

- Canonical/normal form: a unique form of representation.
  - Sum-of-minterms = Canonical sum-of-products
  - Product-of-maxterms = Canonical product-of-sums

12.1 Sum-of-Minterms

- Given a truth table, example:

x	y	z	F1	F2	F3
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	0	1	0

- Obtain sum-of-minterms expression by gathering the minterms of the function (where output is 1).

F1 = x·y·z' = m6

F2 = x'·y'·z + x·y'·z' + x·y'·z + x·y·z' + x·y·z
= m1 + m4 + m5 + m6 + m7 = Σm(1,4,5,6,7) or Σm(1,4 – 7)

F3 = x'·y'·z + x'·y·z + x·y'·z' + x·y'·z
= m1 + m3 + m4 + m5 = Σm(1,3,4,5) or Σm(1,3 – 5)

12.2 Product-of-Maxterms

- Given a truth table, example:

x	y	z	F1	F2	F3
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0
1	1	1	0	1	0

- Obtain product-of-maxterms expression by gathering the maxterms of the function (where output is 0).

$$F2 = (x+y+z) \cdot (x+y'+z) \cdot (x+y'+z')$$
$$= M0 \cdot M2 \cdot M3 = \Pi M(0,2,3)$$

$$F3 = (x+y+z) \cdot (x+y'+z) \cdot (x'+y'+z) \cdot (x'+y'+z')$$
$$= M0 \cdot M2 \cdot M6 \cdot M7 = \Pi M(0,2,6,7)$$

12.3 Conversion of Standard Forms

- We can convert between sum-of-minterms and product-of-maxterms easily
- Example: F2 = $\Sigma m(1,4,5,6,7) = \Pi M(0,2,3)$
- Why? See F2' in truth table.

x	y	z	F2	F2'
0	0	0	0	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

- F2' = m0 + m2 + m3
  Therefore,
  F2 = (m0 + m2 + m3)'
  = m0' · m2' · m3' (by DeMorgan's)
  = M0 · M2 · M3 (as m*x*' = M*x*)

- Read up DLD section 3.4, pg 57 – 58.
- Quick Review Questions: pg 60 – 61, Q3-6 to 3-13.

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