Report 3

Valentina Zonta

April 5, 2024

1 Introduction

In this homework, I am going to study three different approaches to price a call option: the Binomial model, Leisen-Reimer model, and Black-Scholes formula. The first two models are defined in discrete time, whereas the last lives the continuum time, in particular, it is the continuum limit of the discrete models.
Then I am going to investigate their convergence, letting N, the number of steps, go to infinity.
In the first part, i describe theoretically the three approaches, and in the second one, i show our analysis and results.

2 Methods

2.1 N steps binomial model

The N steps binomial model is a mathematical framework used in finance to model the dynamics of asset prices over discrete time periods. It is particularly useful for pricing derivatives such as options. In this model, the price of the underlying asset evolves over time in a series of steps, with each step representing a discrete time interval.

The key components of the N steps binomial model include the price movement of the underlying asset, the calculation of probabilities for up and down movements, and the valuation of options.

At each step, the price of the underlying asset can either move up or down by certain factors u and d, respectively. These factors are determined based on market conditions and volatility.

The probabilities of up and down movements, denoted by p and q respectively, are calculated using a risk-neutral probability approach. These probabilities ensure that the model is consistent with no-arbitrage principles. They are calculated as follows:

$$p = \frac{e^{r\cdot\Delta t} - d}{u - d}, \quad q = 1 - p$$

where r is the risk-free interest rate and Δt is the length of each time step.

The future prices of the asset at each step are then calculated based on the current price, up and down factors, and probabilities of each movement.

If the model is being used to price call options, the option value at each step can be calculated recursively using the risk-neutral valuation principle. The final option value corresponds to the option's value at the initial step.

The price of a call option C with varying asset prices S can be determined using the N steps binomial model. The option value at each step is calculated based on the difference between the asset price and the strike price, with the final option value being the maximum of zero and the difference between the asset price and the strike price.

2.2 Blach-Scholes

The Black-Scholes formula provides a closed-form solution for pricing European-style options. It is based on key assumptions such as constant volatility (σ), risk-free interest rates (r), and log-normal distribution of asset prices. The formula calculates the price (C) of a call option as:

$$C = S_0 N(d_1) - Xe^{-rt} N(d_2)$$

where S_0 is the current price of the underlying asset, X is the strike price of the option, t is the time to expiration, and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution. The parameters d_1 and d_2 are given by:

$$d_1 = \frac{\ln(S_0/X) + (r + \sigma^2/2)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

As the number of steps (N) in the binomial model approaches infinity, its option prices converge to those generated by the Black-Scholes formula. This convergence, known as the binomial option pricing convergence theorem, underscores the robustness and accuracy of the Black-Scholes model in estimating option prices, despite being derived from different assumptions and methodologies compared to the binomial model.

2.3 Leisen and Reimer model

The Leisen-Reimer binomial model is an enhancement of the original binomial model, designed to improve accuracy, particularly for options with American-style exercise. It incorporates a correction factor to adjust the probabilities of up and down movements, resulting in better convergence with the Black-Scholes model This model, similar to the Binomial model, employs an underlying price binomial tree centered around the option's strike price at expiration. The methodology for computing tree nodes and option prices aligns with other binomial models. However, the Leisen-Reimer model differs in the calculation of tree parameters, specifically u, d, and q. Notably, probabilities are computed before move sizes since they serve as inputs for the latter. The risk-neutral weight q is determined using the Peizer-Pratt inversion function $h^{-1}(d_2)$, where d_2 mirrors the Black-Scholes formula's d_2, providing discrete binomial approximations for the continuous normal cumulative distribution function. Once q is established, u and d are computed using the following formulas:

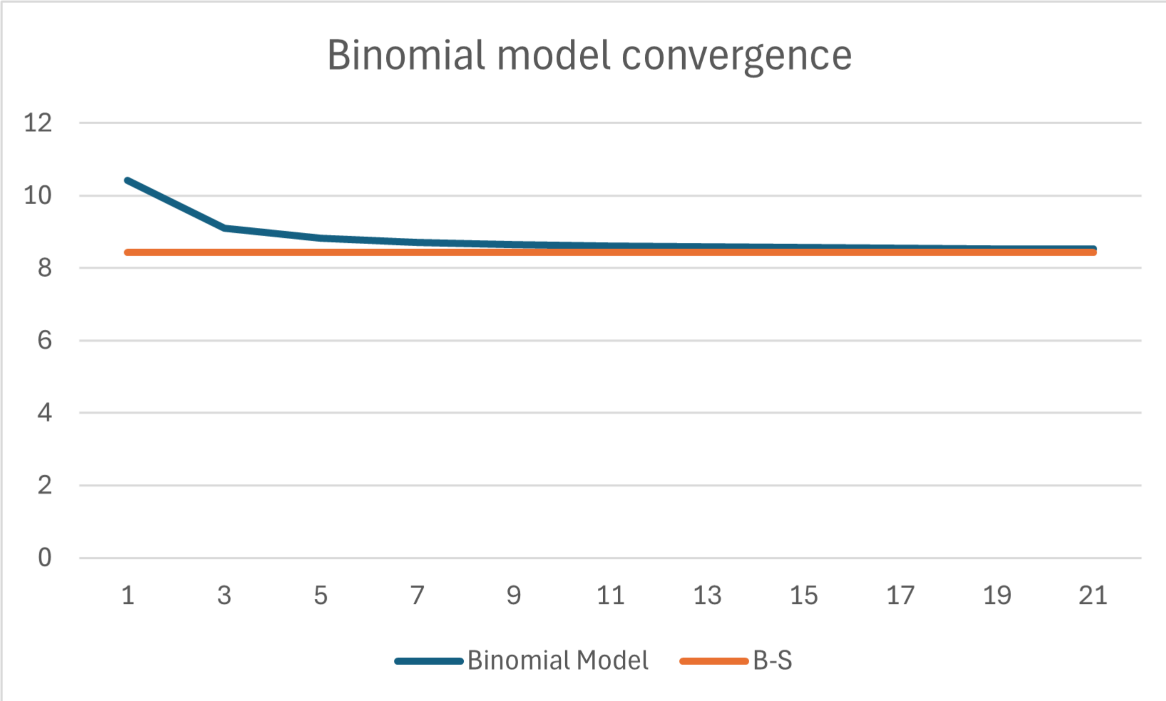
$$u = e^{r\cdot\Delta T} \cdot \frac{q'}{q}$$

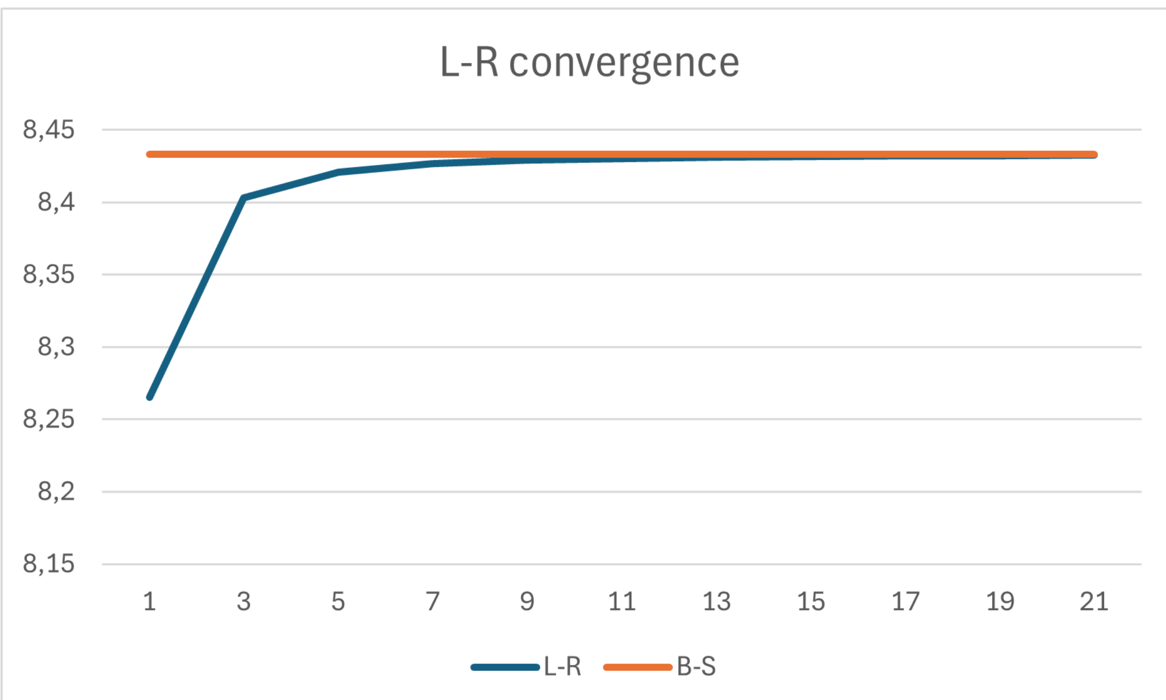
$$d = e^{r\cdot\Delta T} \cdot \frac{1-q'}{1-q}$$

where ΔT represents the maturity time, calculated as T/n, and $q' = h^{-1}(d_1)$, with d_1 calculated as in the Black-Scholes framework. It's essential to note that in this context, the continuous dividend yield is assumed to be zero.

3 Analysis and results

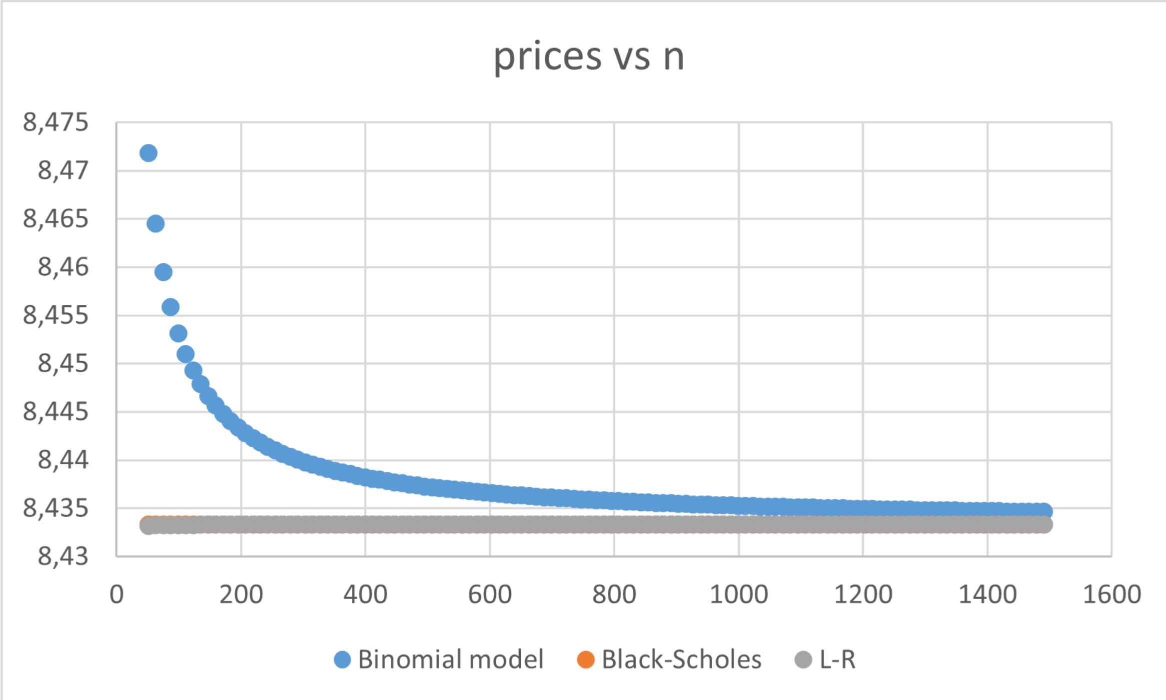
For each of these models, I wrote a VBA script that can compute the price of the option given the values of initial asset price S, interest rate r, strike price K , maturity time T, volatility and the number of steps n; in particular in this report I'm going to assume S = 100, K = 100, r = 0.01, T = 1 year = 0.2.
The results, prices of the option on the y-axis, and steps on the x axis are reported in the graph below (My hypothesis is that the Black-Sholes formula is the exact one):





It is evident from the price scale that the L-R method converges more rapidly to Blach-Sholes formula results.

For a large number of steps, we can see this difference.(L-R and B-S are overlapping in the graph)



Lastly from the paper given in the exercise:

Definition. European call options, computed with a lattice approach converge with order $r > 0$ if there exists a constant $C > 0$ such that

$$\text{for every } n : e_n \leq \frac{C}{n^r} \tag{1}$$

So is possible to use a log-log plot to identify the rate of convergence. In fact given a monomial equation $y = ax^k$, taking the logarithm of the equation (with any base) yields:

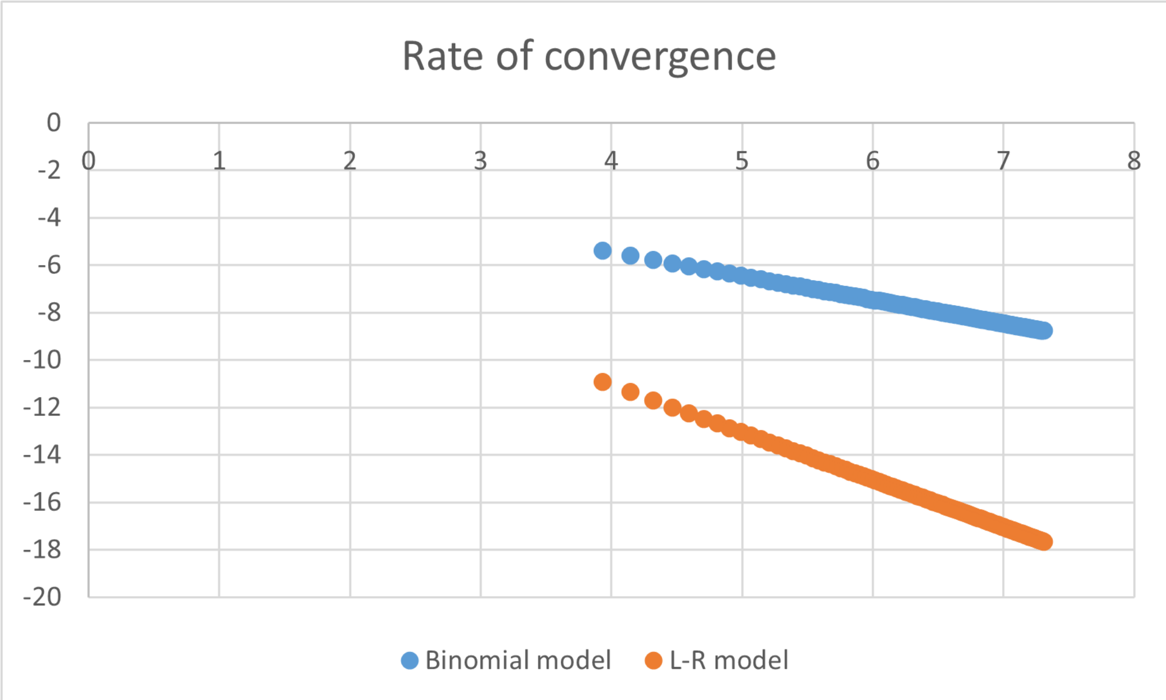
$$\log y = k \log x + \log a.$$

Setting $X = \log x$ and $Y = \log y$, which corresponds to using a log-log graph, yields the equation

$$Y = mX + b$$

where $m = k$ is the slope of the line.

In the following log-log plot, it's evident that the slope, so the rate of convergence, is bigger in the L-R model.



4 Conclution

Upon comparing the two methods, it becomes evident that the Leisen-Reimer method offers superior accuracy compared to the Binomial model. Notably, its primary advantage lies in its ability to achieve greater precision with a smaller number of steps, in contrast to the Binomial model.

As the number of steps (n) increases, both the Binomial Model and the Leisen-Reimer method converge towards the Black-Scholes formula. However, the Binomial Model consistently exhibits slightly lower accuracy compared to the Leisen-Reimer method. For instance, even at n=1000 steps, the Binomial Model and Leisen-Reimer method relative error differ by several significant figure.

Nevertheless, both methods demonstrate convergence towards the Black-Scholes formula for large values of n