16/04

EXAMPLE

exchange rate

foreign asset in. $

1) $$\text{Payoff}_T = \left(X_T S_f(T) - K\right)^+$$ strike in. €

$$= \left(\hat{S}_f(T) - k\right)^+$$

↳ $\sigma_{\hat{f}} = \|\sigma_f + \sigma_x\|$ volatility sum of volatility

(domestic risk-free measure)

$$\text{Price}_t(\text{call}) = \hat{S}_f\,\Phi(d_1) - e^{-r_d(T-t)} K\,\Phi(d_2)$$

$$d_{1,2} = \frac{1}{\|\sigma_f + \sigma_x\|\sqrt{T-t}}\left[\ln\left(\frac{\tilde{S}_f}{K}\right) - \left(r_d \pm \frac{1}{2}\|\sigma_f + \sigma_x\|^2\right)\cdot(T-t)\right]$$

we want to price in a for. country.

€ $\quad$ \$ $\quad$ \$

2) $$\text{Payoff}_T = X(T)\left(S_f(T) - K\right)^+ =$$

$$= \left(S_f X - K X\right)^+ = \left(\tilde{S}_f - XK\right)^+$$

domestic $\quad$ foreign

$$\text{Price}^d_t = X_t\,\text{Price}^f_t$$

$$\text{Price}_t^f = S_f\,\Phi(d_1) - e^{-r_f(T-t)}\,\Phi(d_2)$$

$$d_{1,2} = \frac{\ln(S_f/K) + \left(r_f \pm \frac{1}{2}\|\sigma_f\|^2\right)(T-t)}{\|\sigma_f\|\sqrt{T-t}}$$

foreign risk-free measure

EXAMPLE $S_d, S_f, X$ (uncorrelated, $\sigma$ diag.)

comparison between two different asset in different markets

$$\text{Payoff}_T = \left(X_T S_f(T) - S_d\right)^+ = \left(\hat{S}_f - S_d\right)^+$$

IN GENERAL $\mathbb{Q}^f \neq \mathbb{Q}^d$: we have two different martingale measure in different markets (SIEGEL PARADOX)

exchange rate

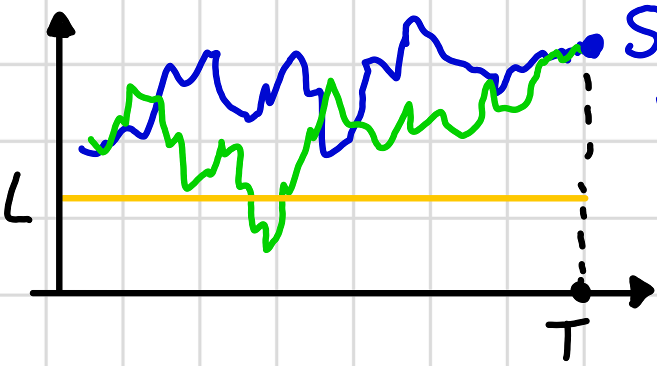
$$\mathbb{Q}^f = \mathbb{Q}^d \iff X \text{ is deterministic}$$

# Barrier Options

ex. DOC : Down-and-out-call

$Payoff_T = (S_T - K)^+ \mathbb{1}_{S_t > L \ \forall t \le T}$

If we touch the barrier the payoff become zero.

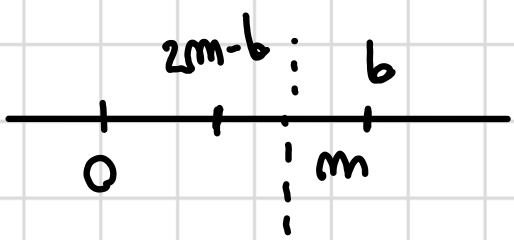



$S_T = S_T \hookrightarrow Payoff_T = 0$

$\hookrightarrow Payoff_T = (S_T - K)^+$

Why buy it? Barrier options are cheaper (but more risky)

## Reflection Principle

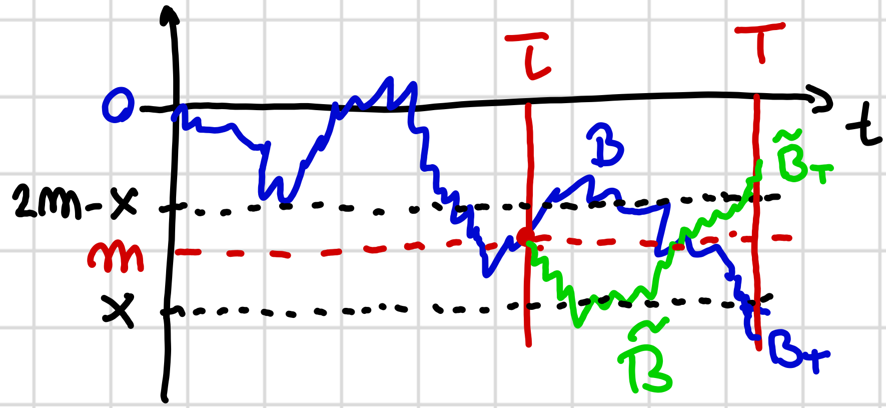



reflecting b w.r.t. a mirror m

$$\tilde{B}_t = \begin{cases} B_t & t < \tau \\ 2m - B_t & T \ge t \ge \tau \end{cases}$$

MIRROR REFLECTION OF A BROWNIAN MOTION AT THE LEVEL m IN [t,T]

$\mathbb{P}(B_T > x) = \mathbb{P}(\tilde{B}_T < 2m - x)$ provided that B touches the barrier m




Brownian motion is reflected when m = B

Let $M(t) := \sup_{0 \le r \le T} B_r$, then

$T_a = \inf\{t : B_t = a\}$ the first time when the price reach the barrier.

Because of markovianity, also $B(t+T) - B(t)$ is B.m. This means that

$$B(t+T_a) - B(T_a) = B(t+T_a) - a$$

THEOREM (Reflection principle)

$$\boxed{\mathbb{P}(M(t) \geq a) = 2\,\mathbb{P}(B_t \geq a)}$$

simpler gaussian density.

$$2\frac{1}{\sqrt{2\pi t}}\int_a^{+\infty} e^{-\frac{x^2}{2t}}\,dx$$

**Proof:** $\mathbb{P}(A) = \mathbb{P}(A \cap \Omega) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c)$

$$\mathbb{P}(B(t) \geq a) = \mathbb{P}(B(t) \geq a, M(t) \geq a) + \underbrace{+\ \mathbb{P}(B(t) \geq a, M(t) < a)}_{=0 \text{ we don't touch}} =$$

$$= \mathbb{P}(B(t) \geq a \mid M(t) \geq a)\, \mathbb{P}(M(t) \geq a) =$$

$$= \mathbb{P}(\underbrace{B(T_a + (t - T_a)) - a}_{\text{this is a B.M. due to condit.}} \geq 0 \mid \overset{\text{Has touched}}{T_a \leq t}) \cdot \mathbb{P}(M \geq a) =$$

Prob B.M is greater than zero ($p = 1/2$)

$$= \frac{1}{2} \mathbb{P}(M(t) \geq a)$$

$\square$

## JOINT DISTRIBUTION

$$\boxed{\mathbb{P}(M(t) \geq a, B(t) \leq a - y) = \mathbb{P}(B(t) > a + y)}$$

or:

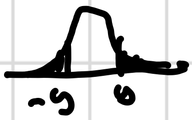
$$\boxed{\mathbb{P}(M(t) \geq m, B(t) < b) = \mathbb{P}(B(t) > 2m - b)}$$

Proof

$$\mathbb{P}(B(t) > a+y) = \mathbb{P}(B(t) > a+y,\ M(t) \geq a) +$$

$$+ \mathbb{P}(\cancel{B(t) > a+y,\ M(t) < a}) =$$

$$= \mathbb{P}(\underbrace{B(T_a + (t - T_a)) - a}_{B.m} > y \mid M(t) \geq a)\, \mathbb{P}(M(t) \geq a)$$




$$= \mathbb{P}(B(t) - a < -y \mid M(t) \geq a)\, \mathbb{P}(M(t) \geq a)$$

$\square$

An corollary, for $a \geq 0$, $y \geq 0$

$$\mathbb{P}(M(t) \leq a,\ B(t) \leq a - y) = \Phi\left(\frac{a-y}{\sqrt{t}}\right) - \Phi\left(\frac{-y-a}{\sqrt{t}}\right)$$

$\Rightarrow$ JOINT DENSITY

$$\mathbb{P}(M(t) \in dm,\ B(t) \in db) =$$

$$= -\frac{\partial^2}{\partial m \partial b} \mathbb{P}(B(t) > 2m - b) = \frac{2(2m-b)}{\sqrt{2\pi t^3}} e^{-\frac{(2m-b)^2}{2t}} \cdot \mathbb{1}_{m \geq b} \mathbb{1}_{m \geq 0}$$

Proof.

$$-\frac{\partial^2}{\partial m \partial b} \frac{1}{\sqrt{2\pi t}} \int_{2m-b}^{+\infty} e^{-x^2/2t}\, dx =$$

$$= -\frac{\partial}{\partial m} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(2m-b)^2}{2t}} = \frac{2(2m-b)}{\sqrt{2\pi t^3}} e^{-\frac{(2m-b)^2}{2t}}$$

# Hitting time for Brownian Motion

We have seen that $T_a = \inf\{t \geq 0 \mid B(t) = a\}$

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(M(t) \geq a) = 2\mathbb{P}(B(t) \geq a) =$$

$$= \frac{2}{\sqrt{2\pi t}} \int_a^{+\infty} e^{-\frac{x^2}{2t}} dx =$$

$$= 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$$

So the density is:

$$f_{T_a}(t) = \frac{\partial}{\partial t} \mathbb{P}(T_a \leq t) = \frac{a}{\sqrt{2\pi t^3}} e^{-a^2/2t}$$

Then we can compute the moment generating function:

$$\mathbb{E}[e^{-\lambda T_a}] = \int_0^{+\infty} e^{-\lambda t} \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} dt =$$

$$= e^{-\sqrt{2\lambda}\, a}$$

Alternatively we use: $\forall \alpha \in \mathbb{R}$

$e^{\alpha B_t - 1/2 \alpha^2 t}$ is martingale, so

$$\mathbb{E}[e^{\alpha B_t - 1/2 \alpha^2 t}] = 1$$

We extend using $t = T_a$:

$$\mathbb{E}\left[e^{\alpha B_t - 1/2\,\alpha^2 T_a}\right] = e^{+\alpha a}\,\mathbb{E}\left[e^{-1/2\,\alpha^2 T_a}\right] \overset{!}{=} 1$$

$$\mathbb{E}\left[e^{-1/2\,\alpha^2 T_a}\right] = e^{-\alpha a} \qquad \frac{1}{2}\alpha^2 =: \lambda$$

true only if bounded

$$\Longrightarrow \quad \mathbb{E}\left[e^{-\lambda T_a}\right] = e^{-\sqrt{2\lambda}\, a}$$

It can be shown that $\mathbb{P}[T_a < +\infty] = 1$

But $\mathbb{E}[T_a] = +\infty$.

$$\frac{\partial}{\partial\lambda}\mathbb{E}\left[e^{-\lambda T_a}\right]\Big|_{\lambda=0} = \mathbb{E}[T_a] = +\infty$$

17/04

## MAXIMUM OF B-M WITH DRIFT

$$\begin{cases} dX_t = \mu\, dt + \sigma\, dW_t \\ X(0) = \alpha \end{cases}$$

$$X_t = \alpha + \mu t + \sigma W_t \qquad \vartheta := \mu/\sigma$$

$$= \alpha + \sigma \underbrace{\left( W_t + \vartheta t \right)}_{\text{B.M under other prob. } \tilde{\mathbb{P}}}$$

Remembering that

$$\tilde{W}_t = W_t + \lambda t \Rightarrow \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_t = e^{-\int \lambda\, dW - \frac{1}{2}\int \lambda^2 ds}$$

$$\Rightarrow \left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_t = \exp\left\{ -\vartheta W_t - \frac{1}{2}\vartheta^2 t \right\}$$

We call $T_\beta$ the first time X reach the value $\beta$:

$$T_\beta := \inf\{t : X_t = \beta\} = \inf\left\{ t : \tilde{W}_t = \frac{\beta - \alpha}{\sigma} \right\}$$

$$\mathbb{P}\left(\max_{s\le t} X_s \ge \beta\right) = \mathbb{E}^{\mathbb{P}}\left[\mathbb{1}_{T_\beta \le t}\right] =$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{\vartheta W_t + \frac{\vartheta^2}{2}t}\,\mathbb{1}_{T_\beta \le t}\right] =$$

$$\mathbb{E}[g] = \int g\, d\mathbb{P} = \int g \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\, d\tilde{\mathbb{P}} = \mathbb{E}\left[g \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\right]$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{\vartheta \tilde{W}_t - \frac{\vartheta^2}{2}t}\,\mathbb{1}_{T_\beta \le t}\right] =$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}\left[\mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{\vartheta \tilde{W}_t - \vartheta^2/2\, t}\,\mathbb{1}_{T_\beta \le t} \,\middle|\, \mathcal{F}_{t\wedge T_\beta}\right]\right] =$$

$$= \mathbb{E}^{\tilde{\mathbb{P}}}\left[e^{\vartheta\left(\frac{\beta-\alpha}{\sigma}\right) - \frac{\vartheta^2}{2}T_\beta}\,\mathbb{1}_{T_\beta \le t}\right] =$$

$$= \int_0^t e^{\vartheta\left(\frac{\beta-\alpha}{\sigma}\right) - \frac{\vartheta^2}{2}y}\, \mathbb{P}(T_\beta \in dy) =$$

$$= \int_0^t e^{\vartheta\left(\frac{\beta-\alpha}{\sigma}\right) - \frac{\vartheta^2}{2}y}\, \frac{\frac{\beta-\alpha}{\sigma}}{\sqrt{2\pi y^3}}\, e^{-\frac{(\beta-\alpha)^2}{2\sigma^2 t}}\, dy =$$

$$= \int_0^t \frac{\beta-\alpha}{\sigma} \frac{1}{\sqrt{2\pi y^3}} \exp\left\{-\left[\frac{\left(\frac{\beta-\alpha}{\sigma}\right)-\vartheta y}{2y}\right]^2\right\} dy$$

This is a classical approach to price barrier options.

A more general framework is the following:

From B&S:

$$S_t = S_0 e^{(r - 1/2\sigma^2)t + \sigma W_t} =$$

$$= S_0 \exp\left\{\sigma\left(W_t + \underbrace{\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)}_{=:\,\theta} t\right)\right.$$

$\Rightarrow$ $\tilde{W}_t =: W_t + \theta t$ new Brownian motion.

The max of B-m is: $M_t^S := S_0 e^{\sigma M_t^{\beta}}$.

The prob. under $\mathbb{Q}$ in:

$$\mathbb{Q}(M_t^S \le L,\ S_t \ge K) = \mathbb{E}^{\mathbb{Q}}\left(\mathbb{1}_{\substack{M^{\hat{\beta}} \le \frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right) \\ \tilde{\beta} \ge \frac{1}{\sigma}\ln(K/S_0)}}\right) =$$

We have to move to $\tilde{\mathbb{Q}}$

$$\left.\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}}\right|_t = e^{-\theta t - 1/2\theta^2 t}$$

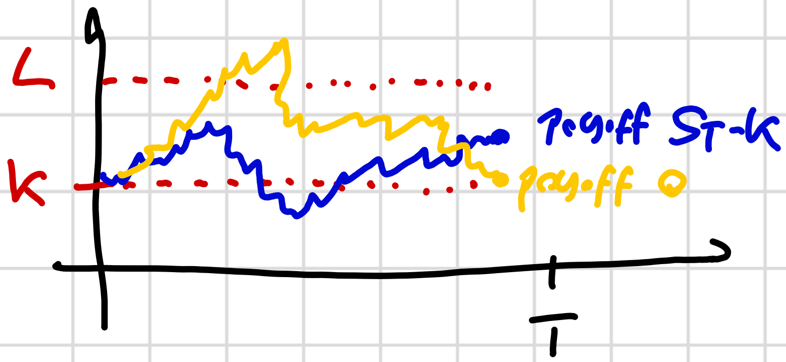
$$= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{\theta W_t + 1/2\theta^2 t}\ \mathbb{1}_{\dots}\right] =$$

$$= \mathbb{E}^{\tilde{\mathbb{Q}}}\left[e^{\theta \tilde{W}_t - 1/2\theta^2 t}\ \mathbb{1}_{\dots}\right]$$

## • EXAMPLE UOC (Up-and-out call)

$$\text{Payoff}_T = (S(T)-K)^+ \mathbb{1}_{\sup_{t\le T} S_t < L}$$

If underlying reach $L$, then the payoff is zero.




max. underl. price

$$\text{Price}_0(\text{UOC}) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_T-K)^+ \mathbb{1}_{M_T^S < L}\right] =$$

we have two stochastic processes $S_t$ and $M_t^S$.

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}\Big[\underbrace{S_0\exp\{\sigma\tilde{B}_T - K) \mathbb{1}_{\tilde{B}_T > \frac{1}{\sigma}\ln\left(\frac{K}{S_0}\right)}}_{\text{for the call}}$$

$$\cdot \mathbb{1}_{M_T^{\tilde{B}} < \frac{1}{\sigma}\ln\left(\frac{L}{S_0}\right)}\Big] =$$

$$= e^{-rT}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big[(S_0\exp\{\sigma\tilde{B}_T - K) \cdot$$

$$\cdot e^{\vartheta\tilde{B}_T - \frac{1}{2}\vartheta^2 T} \cdot \mathbb{1}_{\dots}\Big] =$$

$$= e^{-rT} \int_{1/\sigma \ln(K/S_0)}^{1/\sigma \ln(L/S_0) =: \tilde{m}} db \int_b^{1/\sigma \ln(L/S_0) = \tilde{m}} S_0 \exp\{\sigma b - K\} e^{\theta b - \frac{1}{2}\theta^2 T} \cdot$$

$\tilde{b}$ !! ↳ BM integral in the corridor $[K, L]$ ↳ integration of the maximum

$$\cdot \frac{2(2m-b)}{\sqrt{2\pi T^3}} e^{-\frac{(2m-b)^2}{2T}} dm =$$

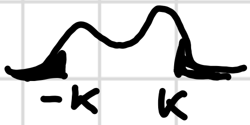
$$= e^{-rT} \int_{\tilde{b}}^{\tilde{m}} db \left( S_0 e^{\sigma b} - K \right) e^{\theta b - \frac{1}{2}\theta^2 T} \cdot$$

$$\cdot \left( e^{-\frac{(2\tilde{m}-b)^2}{2T}} - e^{-\frac{(2b-b)^2}{2T}} \right) = \dots$$

→ possibility to move limit inside expected value.

## EXAMPLE: (uniform integrability)

?x

expected value of the tail 

on $k \to +\infty$ (sup for all times)

$$\lim_{k\to+\infty}\sup_{t\geq 0} \mathbb{E}\left[|X_t| \mathbb{1}_{|X_t|>k}\right] = 0$$

Considering a GBM, which is martingale,

$$y_t = e^{\tilde{W}_t - 1/2\, t}$$

$\tau_m = \inf\{t : y_t = m\}$ first time $y$ reach $m$. Now what is the prob. $y_t = m$ for some time?

$y_t$ is martingale but not uniform integrable: $y_\infty = \lim_{t\to+\infty} y_t = 0$ but

$$\lim_{t\to+\infty}\mathbb{E}[y_t] = 1 \quad \text{because martingale}$$

$$\neq$$

$$\mathbb{E}\left[\lim_{t\to\infty} y_t\right] = 0 \quad \to \text{no U.I.}$$

So we can restrict the time interval bounded $y_{t\wedge\tau_m}$ (which is before $\tau_m$, so $y_{t\wedge\tau_m} \leq m$)

$y_{t \wedge \tau_m}$ is Bounded martingale and so it's also uniform integrable U.I..

Since is U.I.

$$y_\infty^{\tau_m} = \begin{cases} 1 & \text{if } \exists\, t \mid y_t = m \\ 0 & \text{if } \nexists\, t \mid y_t = m \end{cases}$$

↗ if reach bound $m$

Then it's simpler to compute averages.

$$1 = \mathbb{E}\left[y_\infty^{\tau_m}\right] = m \cdot \mathbb{P}(y_t = m \text{ for some time}) + $$
$$+ 0 \cdot \cancel{(1 - \mathbb{P}(\dots))}$$

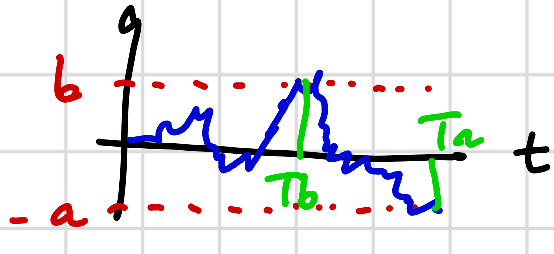
$$\Rightarrow \boxed{\mathbb{P}(y_t = m \text{ for some time}) = \frac{1}{m}}$$

## EXAMPLE:

Hitting times $a, b > 0$ where $T_{-a}$ is the first time $B_t$ reach $-a$ and $T_b$ when it reach $b$.

$$T_{-a} = \inf\{t : B_t = -a\}$$

$$T_b = \inf\{t : B_t = b\}$$

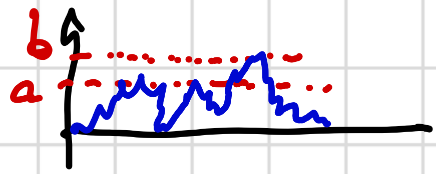



Which is more probable? $T_a$ or $T_b$?
(RUIN PROBABILITY OF A GAMBLER)

$$\mathbb{P}(T_{-a} < T_b) = ?$$

We call $T = T_{-a} \wedge T_b$ which is also unbounded.

$$|B_t| \leq \max(a, b)$$




$B_{t \wedge T}$ is a Bounded martingale $\Rightarrow$ U.I.

$$\lim_{t \to \infty} \mathbb{E}[B_{t\wedge T}] = \mathbb{E}[\lim_{t \to \infty} B_{t \wedge T}]$$

$$\mathbb{E}[B_T] = -a\,\mathbb{P}(B_T = -a) + b\,\mathbb{P}(B_T = b) +$$

$$+ (?)\,\underbrace{\mathbb{P}(\cancel{B_T \in (a,b)})}_{0}$$

reach intermediate state but we choose T.

For Martingale $\mathbb{E}[B_T] = \mathbb{E}[B_0] = 0$.

Also $\mathbb{P}(B_T = -a) + \mathbb{P}(B_T = b) = 1$, so:

$$\mathbb{P}(B_T = b) = \frac{a}{b}\mathbb{P}(B_T = -a) =$$

$$= \frac{a}{b}(1 - \mathbb{P}(B_T = b)) =$$

$$= \frac{a}{b}\left(\frac{1}{1 + a/b}\right) = \frac{a}{a+b}$$

GAIN PROBABILITY $\mathbb{P}(B_T = b) = \dfrac{a}{a+b}$

RUIN PROBABILITY $\mathbb{P}(B_T = -a) = \dfrac{b}{b+a}$

Now $B_t^2 - t$ is mg, so $(B_{t\wedge T} - t\wedge T)$ is mg

$$\mathbb{E}[B_{t\wedge T}^2] = \mathbb{E}[t\wedge T]$$

$$\mathbb{E}[B_T^2] = \mathbb{E}[T] = a\cdot b$$

$$\parallel$$

$$a^2\,\mathbb{P}(B_t = -a) + b^2\,\mathbb{P}(B_t = b) =$$

$$= \frac{a^2 b}{a+b} + \frac{ab^2}{a+b} = a\cdot b \checkmark$$

23/04

!

## Barrier Options

$$dX = \mu dt + \sigma dW$$

The hitting time $T_\beta = \inf\{t : X_t = \beta\}$.
We introduce the ABSORBED PROCESS:

$$X^\beta = X_{t \wedge T_\beta} = \begin{cases} X_t & t < T_\beta \\ \beta & t > T_\beta \end{cases}$$ (freeze the price at $\beta$)


$$\begin{cases} dX = \mu t + \sigma dW \\ X_0 = \alpha \end{cases} \longrightarrow X(t) = \alpha + \mu t$$

$$f_{X^\beta} = \left[ \varphi(x;\, \mu t + \alpha,\, \sigma\sqrt{t}) - e^{-\frac{2\mu(\alpha-\beta)}{\sigma^2}} \cdot \varphi(x;\, \mu t - \alpha + 2\beta,\, \sigma\sqrt{t}) \right] \cdot \mathbb{1}_{(\beta,+\infty)}$$

In the B&S market $\frac{dS}{S} = r\,dt + \sigma dW^{\mathbb{Q}}$.
We consider an european-style payoff $Z := \Phi(S_T)$ (some function of underlying at time $T$).

Now we consider a DOWN-AND-OUT CONTRACT with payoff:

$$Z_L = \Phi(S_T)\,\mathbb{1}_{S_t > L \ \forall t \in [0,T]}$$

So, the payoff is 0 if $S_t \leq L$ for any time.

↓

THEOREM: (Price of a Down-knock out contract)
The price of the Barrier contract is:

$$\text{Price}_{LO}(t, \overset{\overset{s := S_t}{\uparrow}}{s}, \Phi) = \Big[\overbrace{\text{Price}(t, s, \Phi_L)}^{\text{vanilla option}} - \left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2} \text{Price}\left(t, \frac{L^2}{s}, \Phi_L\right)\Big]\mathbb{1}_{s > L}$$

where $\tilde{r} = r - \frac{1}{2}\sigma^2$ where the payoff $\Phi_L$ is:

$$\Phi_L(x) = \Phi(x)\,\mathbb{1}_{x > L}$$

european-style contract.

Proof:

$$\text{Price}_{L_0}(0, s, \Phi) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T)\mathbb{1}_{S_t > L\ \forall t}\right] =$$

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T)\mathbb{1}_{\inf S_t > L}\right] =$$

obtained process coincide to $S_T$ if $L$ is not reached

$$= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi_L(S_T^L)\right]$$

$$= e^{-rT}\int_L^{+\infty}\Phi_L(x)\, h(x)\, dx =$$

we need to know the distribution of $S_T^L$

$$S_T = s\exp\left\{\underbrace{\left(r - \tfrac{1}{2}\sigma^2\right)}_{\tilde{r}}T + \sigma W_T^{\mathbb{Q}}\right\} =$$

$$= \exp\left\{\ln(s) + \tilde{r}T + \sigma W_T^{\mathbb{Q}}\right\} = e^{X_T}$$

we reparametrize

$$\Rightarrow \begin{cases} dX_t = \tilde{r}\,dt + \sigma\, dW^{\mathbb{Q}} \\ X_0 = \ln(s) \end{cases}$$

then $S_t^L = \exp\left\{X_t^{\ln L}\right\}$. We change variables:

dist of $X_T^{\ln L}$

$$= e^{-rT}\int_{\ln L}^{+\infty}\Phi_L(e^x)\, f(x)\, dx =$$

$$f(x) = \varphi(x; \ln s + \tilde{r}T, \sigma\sqrt{T}) -$$

$$- e^{-2\tilde{r}(\ln s - \ln L)/\sigma^2}\varphi(x; \tilde{r}T - \ln s + 2\ln L, \sigma\sqrt{T})$$

now we can split the integral:

$$= e^{-rT}\int_{\ln L}^{+\infty} \Phi_L(e^x)\,\varphi(x;\ln s+\tilde{r}T,\sigma\sqrt{T})\,dx -$$

$$- e^{-rT}\left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2}\int_{\ln L}^{+\infty} \Phi_L(e^x)\,\varphi\left(x;\tilde{r}T+\ln\left(\frac{L^2}{s}\right),\sigma\sqrt{T}\right)dx$$

We can also integrate in $\mathbb{R}$ because $\int_{-\infty}^{L}(\dots)=0$

$$= e^{-rT}\int_{-\infty}^{+\infty} \Phi_L(e^x)\,\varphi(x;\ln s+\tilde{r}T,\sigma\sqrt{T})\,dx -$$

$$- e^{-rT}\left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2}\int_{-\infty}^{+\infty} \Phi_L(e^x)\,\varphi\left(x;\tilde{r}T+\ln\left(\frac{L^2}{s}\right),\sigma\sqrt{T}\right)dx =$$

$$= e^{-rT}\,\mathbb{E}^{Q}_{0,s}\left[\Phi_L(S_T)\right] -$$

$$- e^{-rT}\left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2}\mathbb{E}^{Q}_{0,\frac{L^2}{s}}\left[\Phi_L(S_T)\right] =$$

$$= \text{Price}(0,s,\Phi_L) - \left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2}\text{Price}\left(0,\frac{L^2}{s},\Phi_L\right)$$

$\square$

**Proposition:** Linearity of barrier options price

$$\text{Price}_{LO}(t, s, \alpha\Phi + \beta\Psi) =$$
$$= \alpha\,\text{Price}_{LO}(t, s, \Phi) + \beta\,\text{Price}_{LO}(t, s, \Psi)$$

---

For up-and-out contract the reasoning in the same:

$$Z^{LO} = \Phi(S_T)\,\mathbb{1}_{S_t < L \;\forall t \in [0,T]}$$

$$\text{price}^{LO}(t, s, \Phi) = \Big[\text{price}(t, s, \Phi^L) -$$
$$- \left(\frac{L}{s}\right)^{2\tilde{r}/\sigma^2} \text{price}\left(t, \frac{L^2}{s}, \Phi^L\right)\Big]\,\mathbb{1}_{s<L}$$

where: $\Phi^L(x) = \Phi(x)\,\mathbb{1}_{x<L}$

---

EXAMPLES :

1. $ST(x) = x$ the payoff is a share of the underlying (go long in the stock)

$$ST(x,s) = s \quad \text{price}$$

2. $BO(x) = 1$ Zero coupon bond

$$BO(t,s) = e^{-r(T-t)} \quad \text{price}$$

3. $H(x,L) = \mathbb{1}_{x>L} = \begin{cases} 1 & x>L \\ 0 & x \leq L \end{cases}$ digital call

$$\underset{\text{digital option price}}{H(t,s,\overset{\text{strike}}{L})} = e^{-r(T-t)} \overset{\text{cum gaussian}}{\Phi}\Bigg(\overbrace{\frac{\tilde{r}(T-t) + \ln(s/L)}{\sigma\sqrt{T-t}}}^{d_2}\Bigg)$$

4. $C(x,K) = (x-K)^+$ call option

Price is B&S formula

How to write the price of a basic option written in these payoff? We need to modify the payoff $\Phi_L$.

!

2) $BO_L(x) = BO(x)\,\mathbb{1}_{x>L} = H(x,L)$

become a digital option!

(DEFAULTABLE BOND)

$$\boxed{B_{L_0}(t,S) = \left[H(t,r,L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} H\left(t,\frac{L^2}{S},L\right)\right]\mathbb{1}_{r>L}}$$

---

DOWN-AND-OUT ON STOCK

$$ST_{L_0}(t,S) = \left[L\,H(t,r,L) - L\left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} H\left(t,\frac{L^2}{r},L\right) + \right.$$
$$\left. + \,\mathrm{Call}(t,r,L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathrm{Call}\left(t,\frac{L^2}{r},L\right)\right]\mathbb{1}_{S>L}$$

Proof:

$$ST_{L_0}(t,S) = \left[\mathrm{price}(t,r,ST_L) - \left(\frac{L}{S}\right)^{\frac{2\tilde{r}}{\sigma^2}} \mathrm{price}\left(t,\frac{L^2}{r},ST_L\right)\right]\mathbb{1}_{S>L}$$

$$ST_L(x) = x\,\mathbb{1}_{x>L} = (x-L+L)\,\mathbb{1}_{x>L} =$$
$$= \underbrace{(x-L)^+}_{\text{call}} + \underbrace{L\,\mathbb{1}_{x>L}}_{\text{Digital option}}$$

exploiting the linearity we find exactly the price as before $\square$

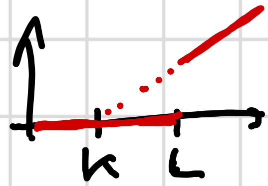
---

# DOWN-AND-OUT CALL (DOC)

$$\mathrm{Call}_L(x,K) = (x-K)^+ \mathbb{1}_{x>L} =$$

$$= (x-K)\,\mathbb{1}_{x>K}\,\mathbb{1}_{x>L} =$$

$$= \begin{cases} (x-K)\,\mathbb{1}_{x>L} & K<L \\ (x-K)\,\mathbb{1}_{x>K} & K>L \end{cases}$$





$$= \begin{cases} \overbrace{(x-L)^+}^{\text{call}} + \overbrace{(L-K)\,\mathbb{1}_{x>L}}^{\text{digitale}} & \text{if } K<L \\ \mathrm{Call}_L(X,K) & \text{if } K>L \end{cases}$$

$\hookrightarrow$ it's not B&S, we [illegible] additional term $-\left(\frac{L}{S}\right)^{2\tilde{r}/\sigma}$ ....
so the price is [illegible]

- $K < L$

$$\mathrm{Call}_{LO}(t,r,K) = \mathrm{Call}(t,r,L) + (L-K)\,H(t,r,L) -$$
$$- \left(\frac{L}{S}\right)^{2\tilde{r}/\sigma^2}\left[\mathrm{Call}\left(t,\frac{L^2}{r},L\right) + (L-K)\,H\left(t,\frac{L^2}{S},L\right)\right]$$

- $K > L$

$$\mathrm{Call}_{LO}(t,r,K) = \mathrm{Call}(t,r,K) - \left(\frac{L}{S}\right)^{2\tilde{r}/\sigma^2}\mathrm{Call}\left(t,\frac{L^2}{S},K\right)$$

$(\cdot\,\mathbb{1}_{r>L})$ for both

24/04

!

## DOC - DOP Parity

Down and out put and call:

$$P_u(x,K) = K\,BO(x) - ST(x) + Call(x,K)$$

$$(K-x)^+ = K - x + (x-K)^+$$

$$\boxed{Put_{Lo}(t,r,K) = K\,BO_{Lo}(t,r) - ST_{Lo}(t,r) + Call_{Lo}(t,r,K)}$$

---

## Up-and-out contract

UOP (up and out put). We start with the put

$$Put^L(x,t) = (K-x)^+ \mathbb{1}_{x<L} =$$

$$= (K-x)\mathbb{1}_{x<K}\mathbb{1}_{x<L} =$$

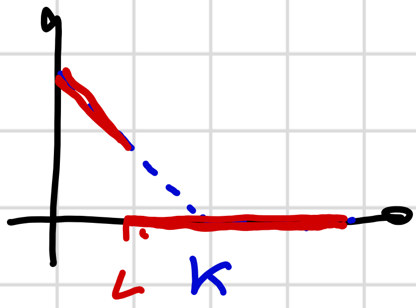
1) If $K<L \rightsquigarrow Put^L(x,t) = (K-x)^+$

2) If $K>L$

$$Put^L(x,t) = (K-L)\mathbb{1}_{x<L} + (L-x)\mathbb{1}_{x<L} =$$

$$= Put(X,L) + (L-K)\underbrace{BO(X)}_{=1} - (L-K)\mathbb{1}_{X>L} =$$

$$= \underbrace{Put(X,L)}_{\text{Put option}} + (L-K)\big[\underbrace{BO(X)}_{\text{Zero coupon bond}} - \underbrace{H(X,L)}_{\text{digital}}\big]$$




$\Downarrow$

If $K<L$

$$Put^{Lo}(t,r,K) = \left[Put(t,r,K) - \left(\frac{L}{r}\right)^{\frac{2\tilde{r}}{\sigma^2}} Put\left(t,\frac{L^2}{s},K\right)\right]\mathbb{1}_{s<L}$$

If $K>L$

$$Put^{Lo}(t,r,K) = \Big[Put(t,r,L) - (K-L)\,H(t,r,L) -$$
$$- \left(\frac{L}{r}\right)^{\frac{2\tilde{r}}{\sigma^2}}\left[Put\left(t,\frac{L^2}{r},L\right) - (K-L)\,H\left(t,\frac{L^2}{s},L\right)\right] +$$
$$+ \left(1-\left(\frac{L}{s}\right)^{\frac{2\tilde{r}}{\sigma^2}}\right)(K-L)\,e^{-r(T-t)}\Big]\mathbb{1}_{s<L}$$

For the UOC we exploit the parity.
Due to the digital around the barrier L, we have difficulties to edging.

# Knock-in contracts

## ① Down-and-in (touch a specific barrier to have the payoff)

$$Z_{LI} = \begin{cases} \Phi(S_T) & \text{if } \overset{\nearrow \text{touch}}{S(t) \leq L} \ \forall t \in [0,T] \\ 0 & \text{if do not touch} \end{cases}$$

We have a relation between IN and OUT barrier options: IN-OUT PARITY

$$\boxed{Price_{LI}(t,s,\Phi) = Price(t,s,\Phi) - Price_{LO}(t,s,\Phi)}$$

this because $Z_{LI} + Z_{LO} = \Phi$ :

$$\Phi_L + \Phi^L = \Phi \mathbb{1}_{x>L} + \Phi \mathbb{1}_{x<L} = \Phi .$$

We get

$$Price_{LI}(t,s,\Phi) = price(t,s,\Phi^L) + \left(\frac{L}{s}\right)^{\frac{2\bar{r}}{\sigma^2}} price\left(t, \frac{L^2}{s}, \Phi_L\right)$$

## ② UP-AND-IN

$$Price^{LI}(t,s,\Phi) = price(t,s,\Phi_L) + \left(\frac{L}{s}\right)^{\frac{2\bar{r}}{\sigma^2}} price\left(t, \frac{L^2}{s}, \Phi^L\right)$$

IN-OUT-UP-PARITY

$$Price^{LI} = Price - Price^{LO}$$

## EXAMPLE : DIC (Down-and-in call)

$L < K$ : $\quad Call_{LI}(t,n,K) = \left(\frac{L}{S}\right)^{2\tilde{n}/\sigma^2} Call\left(t, \frac{L^2}{n}, K\right)$

$L > K$ : $\quad Call_{LI}(t,n,K) = \left(\frac{L}{S}\right)^{2\tilde{n}/\sigma^2} \Big[ Call\left(t, \frac{L^2}{n}, K\right) +$

$+ (L-K) H\left(t, \frac{L^2}{n}, K\right) - (L-K) H(t,n,L) +$

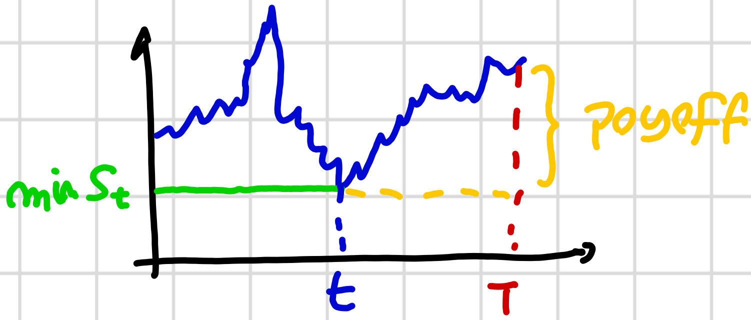
$+ Call(t,K) - Call(t,L) \Big]$

# Look Back Options

!

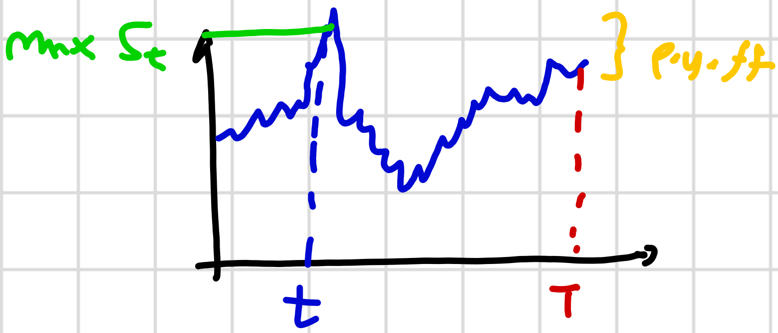
## Look Back Call

$$\Phi = S_T - \min_{t \leq T} S_t$$




## Look Back Put

$$\Phi = \max_{t \leq T} S_t - S_T$$




## Forward Looking

- Call $\quad \Phi = \left(\max_{t \leq T} S_t - K\right)^+$
- Put $\quad \Phi = \left(K - \max_{t \leq T} S_t\right)^+$

$$M_S(T) = \max_{t \leq T} S(t) \qquad m_S(T) = \min_{t \leq T} S(t)$$

## (1) Lookback Call

$$\text{Price}_0 = -r\Phi(-d) + re^{-rT}\Phi(-d+\sigma\sqrt{T}) r - r\frac{\sigma^2}{2r}\Phi(d) - r e^{-rT}\frac{\sigma^2}{2r}\Phi(-d+\sigma\sqrt{T})$$

where $d = (rT + 1/2\sigma^2 T)/\sigma\sqrt{T}$.

This can be found from the distribution of the maximum

$$\begin{cases} dX = \mu dt + \sigma dW \\ X_0 = \alpha \end{cases}$$

$$F_{\max_{s\le T} X_s}(x) = \left[\Phi\left(\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right) - e^{-\frac{2\mu(x-\alpha)}{\sigma^2}}\Phi\left(-\frac{x-\alpha-\mu t}{\sigma\sqrt{t}}\right)\right]\mathbb{1}_{x\ge\alpha} =$$

change of variable with $S(t)$

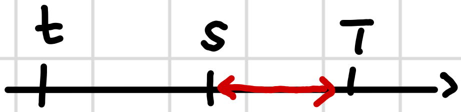
$$= \left[\Phi\left(\frac{x-\ln r-\tilde{r}T}{\sigma\sqrt{T}}\right) - e^{2\tilde{r}\left(\frac{x-\ln r}{\sigma^2}\right)}\Phi\left(-\frac{x-\ln r+\tilde{r}T}{\sigma\sqrt{T}}\right)\right]\cdot$$

$$\cdot\,\mathbb{1}_{x\ge\ln r}$$

# BOND AND INTEREST RATES

**Def.** The **ZERO COUPON BOND** has the following assumptions:

1) $\exists$ bonds $\forall\, T>0$
2) $P(t,t)=1$ almost sure $\forall\, t\geq 0$
3) $P(t,T)$ is differentiable w.r.t. $(T-t)$ time to maturity




## INTEREST RATES

How can we fix a deterministic return from $s \to T$ at time $t$? Yes

1) short zero coupon bond at time $t$ to $s$
   ↳ $P(t,s)$
2) long in another instrument $P(t,T)$ with an amount $P(t,s)/P(t,T)$

Let's look the cash-flow:

| | $t$ | $s$ | $T$ |
|---|---|---|---|
| 1) | $+P(t,s)$ | $-1$ | $0$ |
| 2) | $-P(t,T)\left[\dfrac{P(t,s)}{P(t,T)}\right]$ | $0$ | $+1\cdot\dfrac{P(t,s)}{P(t,T)}$ |
| | $0$ | $-1$ | $+1\, P(t,s)/P(t,T)$ |

I pay nothing at time $t$, I pay 1 at time $S$, and get a deterministic gain $P(t,S)/P(t,T)$ at time $T$

$\Rightarrow$ RISKLESS RETURN ON $[S,T]$

At time $t$ I can compute the SIMPLE FORWARD RATE $F(t,S,T)$

$$1 + F(t,S,T)(T-S) = \frac{P(t,S)}{P(t,T)}$$

$$F(t,S,T) = \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right)$$

If our window start today $S=t$
SIMPLE SPOT RATE (YIELD)

$$F(S,S,T) = \frac{1}{T-S}\left(\frac{1}{P(S,T)} - 1\right)$$

The FORWARD RATE AGREEMENT (FRA)

$$\text{Payoff}_{FRA} = (T-S)\left(F(S,S,T) - R\right)$$

payoff will be a bet to future interest rate. Now we want to price this contract.

$$\text{Price}_{FRA}(t,S,T,R) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}\left[(T-S)\left(F(S,S,T) - R\right)\right]$$

we change numeraire :

$$= P(t,T)\,(T-S)\,\mathbb{E}_t^{Q^T}\left[ F(S,S,T) - R \right] =$$

$$= P(t,T)\,(T-S)\,\mathbb{E}_t^{Q^T}\left[ \frac{1}{T-S}\left(\frac{1}{P(S,T)} - 1\right) - R \right] =$$

$$= P(t,T)\left\{ \mathbb{E}_t^{Q^T}\left[ \frac{P(S,S)}{P(S,T)} - 1 \right] - (T-S)R \right\} =$$

↪ is $Q^T$ martingale

$$= P(t,T)\left\{ \frac{P(t,S)}{P(t,T)} - 1 - (T-S)R \right\}$$

But we know $\text{Price}_{FRA} = 0$ , so :

$$R = \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right) = F(t,S,T)$$

$$F(t,S,T) = \mathbb{E}_t^{Q^T}\left[F(S,S,T)\right] = \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right)$$

(This was before the financial crisis)

30/04

Before the crisis LIBOR:

$$L(S,S,T) = F(S,S,T) = \frac{1}{T-S}\left(\frac{1}{P(S,T)} - 1\right)$$

↓

$$P(S,T) = \left[1 + (T-S)\underbrace{L(S,S,T)}\right]^{-1}$$

Libor was used as discount factor

↳ (LONDON INTERBANK OFFER RATE) administrated by the BBA (British Bankers association) until 2014. After is managed by ICE (intercontinental exchange)

For Europe we have EURIBOR and it is managed by EMMI (European money market instit.)

↓ They can be subjected by manipulation => 2021 FCA declare that LIBOR is not more reliable. (discontinuation of LIBOR).

ICE take SOFR (Secured Overnight financial rate) as benchmark to build a new discounting rate.

This is based by real transactions, so is more protected by manipulation.

This is more tractable because is based on treasury bonds.

In the EU we get the transition from EONIA (euro overnight index rate) to €STER (euro short term rate) published by ECB from 2019.
But there is no plan to discontinue EURIBOR.

After the crisis the LIBOR cannot be considered as discounting rate

$$L(S,S,T) > \frac{1}{T-S}\left(\frac{1}{P(S,T)}-1\right)$$

$\hookrightarrow$ libor is more risky

The forward libor rate $L(t,S,T) := \mathbb{E}_t^{Q^T}(L(S,S,T))$
This means that:

$$L(t,S,T) = \mathbb{E}_t^{Q^T}[L(S,S,T)] > \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)}-1\right) = F(t,S,T)$$

So, we define the SPREAD

$$L(t,S,T) - F(t,S,T) = r(t, \underbrace{T-S}_{\text{Tenor } \Delta})$$

$r(t, 3\text{ Months}) \leq r(t, 6\text{ months}) \leq r(t, 12\text{ Months})$

So we cannot jump from different time window, because of the stochasticity of the spread.

What can we use as a proxy to the riskless? => OIS (overnight indexed swap rate) is a swap where the floating rate is obtained by compounding the overnight rates over the corresponding interval between 2 subsequential rates

$$\begin{array}{c}\text{LIBOR}\\ \text{SWAP}\\ \text{RATE}\end{array} - \text{OIS} = \text{LIBOR-OIS swap spread}$$

$$\begin{array}{c}\text{EURIBOR}\\ \text{SWAP}\\ \text{RATE}\end{array} - \begin{array}{c}\text{EONIA}\\ \text{SWAP}\\ \text{RATE}\end{array} = \text{EURIBOR-EONIA swap spread}$$

Before the crisis this was almost zero

$$\begin{pmatrix}\text{LIBOR}^{6M}\\ \text{SWAP}\end{pmatrix} - \begin{pmatrix}\text{LIBOR}^{3M}\\ \text{SWAP}\end{pmatrix} = \text{BASIS SWAP SPREAD}$$

As proxy for the risk free rate we take the LIBOR overnight OIS:

$$F(t,S,T) = \underbrace{L^{OIS}}_{1\text{ DAY}}(t,S,T) = \frac{1}{T-S}\left(\frac{P(t,S)}{P(t,T)} - 1\right)$$

SPOT SHORT RATE $= \lim\limits_{\Delta \to 0} L(t, t, t+\Delta)$

But is not practical, only theoretical.

---

## CONTINUOUSLY COMPOUNDED FORWARD RATE

SIMPLE : $1 + F(t,S,T)(T-S) = \dfrac{P(t,S)}{P(t,T)}$

CONTINUOUS : $e^{R(t,S,T)\cdot(T-S)} = \dfrac{P(t,S)}{P(t,T)}$

$$R(t,S,T) = \frac{1}{T-S} \log\left(\frac{P(t,S)}{P(t,T)}\right)$$

When $t \to S$ , $R(S,T) = -\dfrac{1}{T-S} \log P(S,T)$

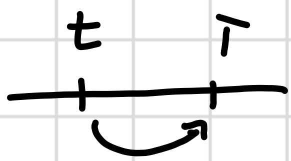
The istantaneous forward rate $S \to T$

$$f(t,T) = -\frac{\partial}{\partial T} \ln P(t,T)$$

↳ $\lim\limits_{S \to T} \dfrac{\ln P(t,T) - \ln P(t,T)}{T-S}$

$\Downarrow$

$$P(t,T) = \exp\left\{-\int_t^T f(t,u)\, du\right\}$$



more manageable to start from the forward rate $f$ instead of the short rate $r$:

$$r(t) := f(t,t)$$

$t+\varepsilon$

Therefore the Bank account will be:

$$B_t = e^{\int_0^t r(s)\,ds}$$

$$P(t,T) = \mathbb{E}_t^{Q}\left[e^{-\int_t^T r(s)\,ds}\right] = \underbrace{e^{-\int_t^T f(t,u)\,du}}_{\text{become deterministic using forward rate.}}$$

↓ zero coupon bond

like a rolling over trading strategy $\forall t \; P(t, t+dt)$

We now want to understand the dynamic:

BOND

$$\begin{cases} dr(t) = a(t)\,dt + b(t)\,dW \\ \dfrac{dP(t,T)}{P(t,T)} = m(t,T)\,dt + \nu(t,T)\,dW \\ df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW \end{cases}$$

GBM because always positive

↳ istant. forward

Interest rate is not an asset so we don't have to require AAO

9/05

THEOREM:

1) bond $\Rightarrow$ forward : $\alpha$ and $\sigma$ solve:

$$\begin{cases} \alpha(t,T) = \dfrac{\partial v}{\partial t}(t,T)\cdot v(t,T) - \dfrac{\partial m}{\partial T}(t,T) \\ \sigma(t,T) = -\dfrac{\partial v}{\partial T}(t,T) \end{cases}$$

(HJM condition '92)

2) forward $\Rightarrow$ short : $a, b$ solve

$$\begin{cases} a(t) = \dfrac{\partial f}{\partial t}(t,t) + \alpha(t,t) \\ b(t) = v(t,t) \end{cases}$$

3) forward $\Rightarrow$ bond : $P(t,T)$ solves

$$\frac{dP(t,T)}{P(t,T)} = \left(r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right)dt + S(t,T)\,dW$$

where
$$\begin{cases} A(t,T) = -\int_t^T \alpha(t,s)\,ds \\ S(t,T) = -\int_t^T \sigma(t,s)\,ds \end{cases}$$

PROOF :

(1) $\frac{dP}{P} = m\,dt + v\,dW$

$$df = -\frac{\partial \ln P}{\partial T} = -\frac{\partial}{\partial T}\left( d\ln P(T,t)\right) =$$

$$= -\frac{\partial}{\partial T}\left[\left(m - \frac{1}{2}v^2\right)dt + v\,dW\right] =$$

$$= \underbrace{\left(-\frac{\partial m}{\partial T} + v\frac{\partial v}{\partial T}\right)}_{\alpha} dt - \underbrace{\frac{\partial v}{\partial T}}_{\sigma} dW =$$

(2) from $df = \alpha\,dt + \sigma\,dW$ we integrate from 0 to t

$$f(t,T) - f(0,T) = \int_0^t \alpha\,ds + \int_0^t \sigma\,dW_s$$

now $t = T$

$$\underbrace{f(t,t)}_{r(t)\text{ short rate}} - f(0,t) = \int_0^t \alpha\,ds + \int_0^t \sigma\,dW_s$$

Now $\alpha(s,t) = \alpha(s,s) - \int_s^t \frac{\partial \alpha}{\partial T}\,du$
(Taylor expansion)
Same for $\sigma(s,t) = \sigma(s,s) - \int_s^t \frac{\partial \alpha}{\partial T}\,du$

$$r(t) = f(0,t) + \int_0^t \alpha\, ds - \int_0^t ds \int_s^t \frac{\partial \alpha}{\partial T}\, du \; +$$

$$+ \int_0^t \sigma(s,s)\, dW_s - \int_0^t ds \int_s^t \frac{\partial \sigma}{\partial T}\, dW_u =$$

Fubini

$$= f(0,t) + \int_0^t \alpha\, ds + \int_0^t ds \int_0^u \frac{\partial \alpha}{\partial T}\, ds \; +$$

$$+ \int_0^t \sigma(s,s)\, dW_s + \int_0^t du \int_0^u \frac{\partial \sigma}{\partial T}\, dW_s$$

taking the differential:

$$dr = \frac{\partial f(0,t)}{\partial T}\, dt \; +$$

$$+ \alpha(t,t)\, dt + \sigma(t,t)\, dW_t \; +$$

$$+ \left( \int_0^t \frac{\partial \alpha}{\partial T}\, ds + \int_0^t \frac{\partial \sigma}{\partial T}\, dW_s \right) dt$$

from the definition of $f$

$$\frac{\partial f}{\partial T}(t,t) = \frac{\partial f}{\partial T}(0,t) + \int_0^t \frac{\partial \alpha}{\partial T}\, ds + \int_0^t \frac{\partial \sigma}{\partial T}\, dW_s \Rightarrow$$

$$df = \left[ \frac{\partial}{\partial T} f(t,t) + \alpha(t,t) \right] dt + \sigma(t,t)\, dW_t$$

③ from $df = \alpha dt + \sigma dW$

$$P(t,T) = e^{-\int_t^T f(t,s)\,ds} := e^{y(t,T)}$$

using Itō: $\frac{dP}{P} = dy + \frac{1}{2} d\langle y \rangle$

$$dy = d\left(-\int_t^T f(t,s)\,ds\right) =$$

$$= -\frac{\partial}{\partial t}\left[\int_t^T f(\hat{t},s)\,ds\right] dt - \int_t^T df(t,s)\,ds =$$

$$= f(t,t)\,dt - \int_t^T (\alpha(t,s)\,dt + \sigma(t,s)\,dW_t)\,ds =$$

$$= r(t)\,dt - \Big(\underbrace{\int_t^T \alpha(t,s)\,ds}_{A}\Big)dt - \Big(\underbrace{\int_t^T \sigma(t,s)\,ds}_{S}\Big)dW$$

$$d\langle y \rangle = \Big(\underbrace{\int_t^T \sigma(t,r)\,dr}_{S}\Big)^2 dt$$

$\Downarrow$

$$\frac{dP}{P} = r(t)\,dt + \left(-A + \frac{1}{2}\|S\|^2\right)dt - S\,dW$$

$\square$

If AAO $\Rightarrow$ $A = \frac{1}{2}\|S\|^2$

drift of $P$ equal drift $r$

14/5

## Linear fixed income products

- Zero Coupon Bond (ZCB)
- Forward rate agreement (FRA)
- Fixed Coupon Bond (FCB)

$\downarrow$

This product deliver coupon $\{c_1, c_2, \dots, c_n\}$ at fixed time $\{T_1, T_2, \dots, T_n\}$.

$$\text{Price}_t^{FIX} = K\,p(t,T_n) + \sum_{i=1}^{n} c_i\,p(t,T_i)$$

In the typical case $T_i - T_{i-1} = \delta$, then

$$c_i = r(T_i - T_{i-1})K$$ i-th coupon return

If the rate $r$ is floating (FRN)

$$c_i = (T_i - T_{i-1})\underbrace{L(T_{i-1},T_i)}_{\text{es. forward LIBOR } L(T_{i-1},T_{i-1},T_i)}K$$

Paid in advance. Use interest rate at $T_{i-1}$

Before the crisis ($K=1$, $T_i - T_{i-1} = \delta$)

$$c_i = \delta\,\frac{1-p(T_{i-1},T_i)}{\delta\,p(T_{i-1},T_i)} = \frac{1}{p(T_{i-1},T_i)} - 1$$

$c_i$ is delivered at time $t=T_i$ using the $T_{i-1}$ rate. To find $q(T_{i-1},T_i)$ :

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^{T_i} \overbrace{r_s}^{\text{short-term interest rate}} ds}\, c_i\right] = \leftarrow \text{change of numeraire}$$

$$= P(t,T_i)\, \mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[c_i\right] =$$

$$= P(t,T_i)\, \mathbb{E}_t^{\mathbb{Q}^{T_i}}\left[\frac{\overset{\nearrow 1}{P(T_{i-1},T_{i-1})}}{\underbrace{P(T_{i-1},T_i)}_{\text{this is the Numeraire!}}} - 1\right] =$$

$$\Downarrow$$

$$\Rightarrow X_{t_{i-1}} := \frac{P(T_{i-1},T_{i-1})}{P(T_{i-1},T_i)} \text{ is martingale}$$

$$= P(t,T_i)\left[\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right] =$$

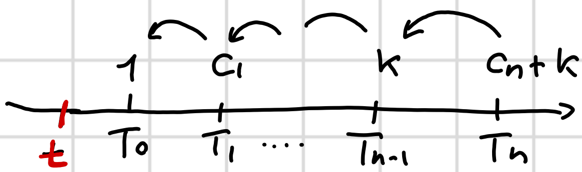
$$= P(t,T_{i-1}) - P(t,T_i) \qquad (\text{telescopic series})$$

$$\text{Price}_t^{\text{FLOAT}} = P(t,T_n) + \sum_{i=1}^{N}\left[P(t,T_{i-1}) - P(t,T_i)\right] =$$

$$= \cancel{P(t,T_n)} + P(t,T_0) - \cancel{P(t,T_n)} =$$

$$= P(t,T_0)$$

$$\text{Price}_t^{\text{FLOAT}} = P(t,T_0)$$



After the crisis we cannot discount using the Libor.

$$\text{Price}_t^{FLOAT} = P(t,T_n) - \sum_{i=1}^{n} P(t,T_i)\,\delta\,\underbrace{\mathbb{E}_t^{Q^{T_i}}\left[L(T_{i-1},T_{i-1},T_i)\right]}_{=:\,L(t,T_{i-1},T_i)}$$

we have to see the evolution of the rates.

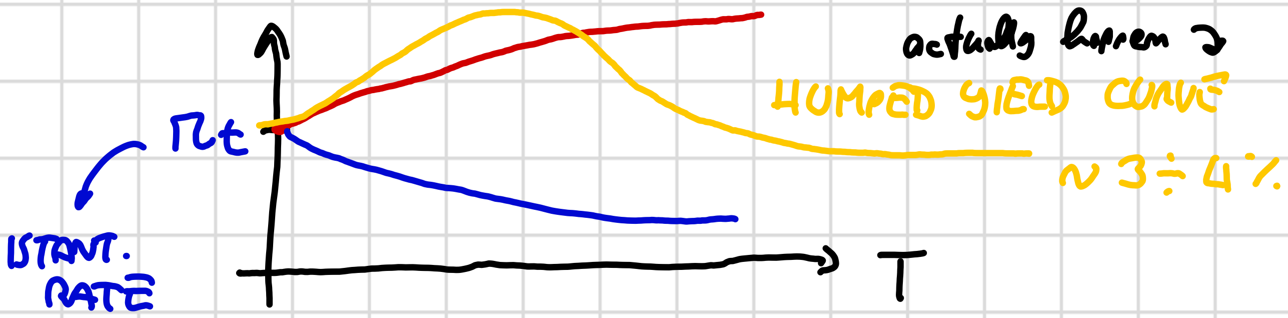
---

## YIELD & DURATION

Given a ZCB $P(t,T) = e^{-\overset{\text{YIELD}}{y}(T-t)}$, $y$ is the yield to maturity (continuously compounded) for a ZCB

$$y(t,T) = -\frac{\ln P(t,T)}{T-t}$$

$T \mapsto y(t,T)$ is the yield curve



In the case of a coupon bond, the yield is the solution of

$$P^{coupon}(t) = \sum_{i=1}^{n} c_i \, e^{-(T_i - t)\, y(t,T)}$$

same interest rate s.t. constant return

non analytical invertable.

**Def.** given a fix coupon bond with price $P$ at $t=0$, and yield to maturity $y$. The DURATION

$$D := \frac{\sum_{i=1}^{n} T_i c_i e^{-y \cdot T_i}}{P_{ZCB}}$$

it express the mean time of the coupon payment.

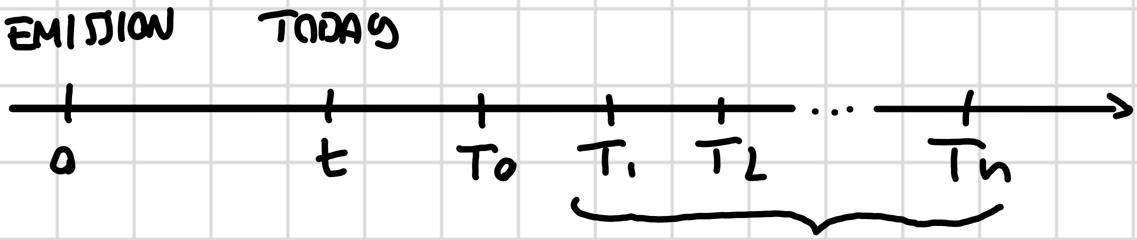
$$\frac{dP}{dy} = \frac{d}{dy}\left(\sum c_i e^{-yT_i}\right) = -\sum c_i T_i e^{-yT_i} = -P_{ZCB} D$$

D express the change of ZCB price w.r.t. shock of the interest rates $y$ (yield)

$$D = -\% \text{ sensitivity to a shift of yield curve}$$

# FORWARD SWAPS
(settled in arrears)

?




At $T_1, T_2, \dots, T_n$ are paid "FIXED LEG" for "FLOATING LEG"

At time $t = T_i$ the RECEIVER swap gives:

$$K\,\delta\big(\underbrace{L(T_{i-1}, T_{i-1}, T_i)}_{\text{FLOATING BOND}} - \underset{\text{FIXED BOND}}{\underset{\downarrow}{K}}\big)$$

$$\text{Price}_t^{\text{SWAP}} = \text{Price}_t^{\text{FLOAT}} - \text{Price}_t^{\text{FIX}}$$

Before the crisis:

$$\text{Price}_t^{\text{SWAP}} = K P(t, T_0) - K\Big(P(t, T_n) + \sum_i^n \overset{\text{SWAP RATE}}{\overset{\uparrow}{R}}\,\delta P(t, T_i)\Big)$$

Swap rate is the rate s.t. make zero the price:

$$R = \frac{P(0, T_0) - P(0, T_n)}{\delta \sum_{i=1}^n P(0, T_i)}$$

After the crisis

$$Price_t^{SWAP} = k \sum_{i=1}^{n} P(t,T_i)\, \delta \left( L(t,T_{i-1},T_i) - R \right)$$

$$R = \sum_{i=1}^{n} \frac{P(t,T_i)}{\sum_{j=1}^{n} P(t,T_j)} L(t,T_{i-1},T_i)$$

convex combination of the forward libor rates.

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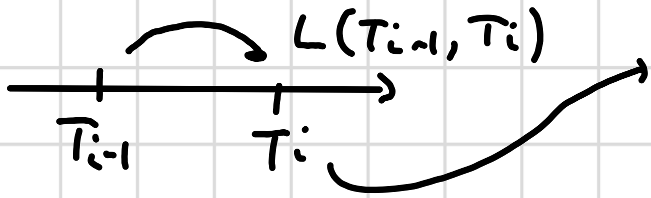
## NON LINEAR CONTRACT

?

- **CAPS & FLOORS**

  - CAP: protect the payment of an high interest rate, more than the CAP RATE (FIXED).




At $T_i$ the payment is

$$k\delta\left(L(T_{i-1},T_i) - R\right)^+$$

"CAPLET"

(FLOOR for the put)

Before the crisis

$$\delta\left(L(T_{i-1},T_i) - R\right)^+ = \delta\left(\frac{1-P(T_{i-1},T_i)}{\delta P(T_{i-1},T_i)} - R\right)^+ =$$

$$= \frac{1+\delta R}{P(T_{i-1},T_i)}\left(\frac{1}{1+\delta R} - P(T_{i-1},T_i)\right)^+$$

We know at $T_i$. At time $T_{i-1}$ i can discount using ZCB and get

$$(1+\delta R)\left(\frac{1}{1+\delta R} - P(T_{i-1},T_i)\right)^+$$

which is a put option on the zero coupon bond

We, therefore, need a model for the term structure.

$\downarrow$

## SHORT INTEREST RATE MODELS

We start with pricing the ZCB $P(t,T)$, which is related to the dynamic of interest rate $r$. ZCB is then a derivative written on $r$.

$$\begin{cases} \frac{\partial F}{\partial t} + \mathcal{A}F - r(t,x)F = 0 \\ F(T,x) = 1 \end{cases}$$

ZCB dynamics.

By Feynman Kač : $\mathbb{E}\left[1 \cdot e^{-\int_t^T r(s,x)\,ds}\right]$.

In general $\mathcal{A} = k\frac{\partial}{\partial x} + \frac{1}{2}H^2\frac{\partial^2}{\partial x^2}$ the infinitesimal generator

EXAMPLE: $dr_t = \mu(t, r_t)\,dt + \sigma(t, r_t)\,dW$

① VASIČEK $dr_t = (b - ar)\,dt + \sigma\,dW_t$

(GAUSSIAN, ORNSTEIN-UHLENBECK)

② COX-INGERSOLL-ROSS ('85)

$$dr_t = a(b - r)\,dt + \sigma\sqrt{r}\,dW_t$$

($\chi^2$, $P(r_t > 0\ \forall t) = 1$, if $ab > \sigma^2/2$ FELLER CONDITION)
not quite well because often rmin r can be negative

③ DOTHAN $dr = ar\,dt + \sigma r\,dW$

(LOG-NORMAL)

④ BLACK-DERMAN-TOY

$$dr = \theta(t)\,r\,dt + \sigma(t)\,r\,dW$$

⑤ HO-LEE

$$dr = \theta(t)\,dt + \sigma\,dW$$

(special case of HJR Model)

⑥ HULL-WHITE ('90)

$$dr = (g(t) - a(t)r)\,dt + \sigma(t)\,dW$$

21/05

We have seen that $P(t,T) = \mathbb{E}_t^{Q}\left[1 e^{-\int_t^T r_s ds}\right]$ can be written as a linear affine expression

$$P(t,T) = e^{A(t,T) - r_t B(t,T)}$$

with A, B deterministic functions, then the short-rate model is said to possess the AFFINE TERM STRUCTURE (ATS).
ATS give information about ~~A~~ ?

Using $F(t,r) = e^{A(t,T) - r B(t,T)}$ in the PDE, we get

$$\begin{cases} \frac{\partial A}{\partial t} - \frac{\partial B}{\partial t} r - B\mu + \frac{1}{2} B^2 \sigma^2 - r = 0 \\ A(T,T) = B(T,T) = 0 \end{cases}$$

The only choice s.t. $A = B = 0$ at $t = T$, we need to impose constraints on $\mu$ and $\sigma$:

$$\begin{cases} \mu(t,r) = \alpha(t) r + \beta(t) \\ \sigma(t,r) = \sqrt{\gamma(t) r + \delta(t)} \end{cases}$$

$$\underbrace{\frac{\partial A}{\partial t} - \beta B + \frac{1}{2}\delta B^2}_{=0} - \underbrace{\left(1 + \frac{\partial B}{\partial t} + \alpha B - \frac{1}{2}\gamma B^2\right) r}_{=0} = 0$$

$$\begin{cases} \dfrac{\partial B}{\partial t} + \alpha B - \dfrac{1}{2}\gamma B^2 = -1 \\ \dfrac{\partial A}{\partial t} = \beta B - \dfrac{1}{2}\delta B^2 \end{cases}$$

$\Rightarrow$ we first solve for B (RICCATI ODE)

use the solution to find A, simply integrating.

## EXAMPLE:

### ① Vašíček model $dr_t = (b - ar_t)dt + \sigma dW$

$$\begin{cases} \mu = b - ar & \Rightarrow \quad \alpha = -a, \quad \beta = b \\ \sigma = \sigma & \Rightarrow \quad \gamma = 0, \quad \delta = \sigma^2 \end{cases}$$

then the ODEs are:

$$\begin{cases} \dfrac{\partial B}{\partial t} - aB = -1 \\ B(T,T) \end{cases} \Rightarrow B(t,T) = \frac{1}{a}\left(1 - e^{-a(T-t)}\right)$$

$$\frac{\partial A}{\partial t} = \frac{b}{a}\left(1 - e^{-a(T-t)}\right) - \frac{1}{2}\sigma^2 \frac{1}{a^2}\left(1 - e^{-a(T-t)}\right)^2$$

$$\begin{cases} B(t,T) = \frac{1}{a}\left(1 - e^{-r(T-t)}\right) \\ A(t,T) = \frac{(B(t,T) - T + t)(ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a} \end{cases}$$

② Ho-Lee model $dr = \theta(t)\,dt + \sigma\,dW$

$$\begin{cases} \frac{\partial B}{\partial t} = -1 \\ B(T,T) = 0 \end{cases} \Rightarrow B(t,T) = T - t$$

$$\frac{\partial A}{\partial t} = \theta(t)(T-t) - \frac{1}{2}\sigma^2 (T-t)^2$$

$$A(t,T) = \int_t^T \theta(t)(T-t)\,dt + \frac{\sigma^2}{6}(T-t)$$

③ Cox-Ingersoll-Ross CIR ('85)

$$dr_t = a(b - r_t)\,dt + \hat{\sigma}\sqrt{r_t}\,dW$$

$$\begin{cases} \mu = a(b - r_t) = ab - ar_t \\ \sigma = \hat{\sigma}\sqrt{r} \stackrel{!}{=} \sqrt{\gamma r + \delta} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \frac{\partial B}{\partial t} - aB - \frac{1}{2}a^2 B^2 = -1 \\ B(T,T) = 0 \end{cases} \qquad \begin{cases} \frac{\partial A}{\partial t} = abB \\ A(T,T) = 0 \end{cases}$$

We can linearize the Ricati equation and we find the equation:

$$B(t,T) = \frac{2\left(e^{\varepsilon(T-t)}-1\right)}{(\varepsilon+a)\left(e^{\varepsilon(T-t)}-1\right)+2\varepsilon}$$

where $\varepsilon = \sqrt{a^2+2\sigma^2}$.

---

# STOCHASTIC VOLATILITY

$$\frac{dS}{S} = r\,dt + \sigma_t\,dW_t^{Q} =$$

↳ stochastic volatility reproduce the vol. smile. We use CIR model

$$= r\,dt + \sqrt{V_t}\,dW_t^{Q}$$

we define $V_t = \sigma_t^2$ the ISTANTANEUS VARIANCE and we modelle it on the following SDE

## HESTON MODEL ('93)

$$dV_t = K(V_\infty - V_t)\,dt + \eta\sqrt{V_t}\left(\rho\,dW^{Q} + \sqrt{1-\rho^2}\,dW^{Q_2}\right)$$

$V_t$ at $\infty$, mean reversion

volatility of volatility

correlation with S

The correlation of the underlying and the volatility is:

$$\left\langle \frac{dS}{S}, dV \right\rangle \propto \rho\,\eta\,V$$

$V_t$ is called TIME CHANGED BESSEL SQUARED PROCESS

(FELLER CONDITION: If $2kV_\infty \geq \eta^2$, then
$\mathbb{P}(V_t > 0 \ \forall t) = 1$ )

- BESSEL SQUARE PROCESSES

$$\mathcal{A} = 2x \frac{d^2}{dx^2} + \delta \frac{d}{dx}$$

$BESQ^d$ : $\begin{cases} d\rho(t) = \delta dt + 2\sqrt{\rho(t)}\, dW_t \\ \rho(0) = x \geq 0 \end{cases}$

$\delta \geq 0$ is called direction. $\nu = \frac{\delta}{2} - 1$ is called INDEX

If $\delta = n \in \mathbb{N}$, then

$$BESQ^\delta_x = \| (W_1, W_2, \dots, W_n) \| = \sqrt{\sum_{i=1}^n W_i^2}$$

in fact, if $R_t = \|W_t\| \Rightarrow R_t^2 = \sum W_i^2$, then using Itô:

$$dR_t^2 = n\,dt + \sum_{i=1}^n 2W_i\, dW_i = n\,dt + 2R_t\, d\overline{W}_t$$

If $\delta \geq 2$ ($\nu \geq 0$), then $BESQ^\delta_x$ never reach zero (Markovian process is transient)
If $0 \leq \delta < 2$ ($-1 \leq \nu < 0$) $BESQ^\delta_x$ may reach zero

On the CIR model

$$\begin{cases} dr_t = k(\vartheta - r_t)\,dt + \sigma\sqrt{r}\,dW \\ r_0 = x > 0 \end{cases}$$

$$\Rightarrow \quad r_t = e^{-kt}\underbrace{\rho}_{\text{BESQ}}\left(\frac{\sigma^2}{4k}(e^{kt}-1)\right)$$

where $\rho(r) \sim BESQ^{\delta}_{x}$ with $\delta = \frac{4k\vartheta}{\sigma^2}$. Then the Feller condition is

$$2k\vartheta \geq \sigma^2 \Rightarrow r_t > 0 \ \forall t \text{ a.s.}$$

---

## PRICING WITH STOCHASTIC VOLATILITY

$$\frac{dS}{S} = r\,dt + \sqrt{V_t}\,dW_t$$

We define $X_t = \log S_t$

$$\text{price}_t = e^{-r(T-t)}\,\mathbb{E}^{\mathbb{Q}}_t[\text{payoff}] \overset{\text{e.g. call}}{=}$$

$$= e^{-r(T-t)}\int_{\mathbb{R}} (S_T - K)^+ f(X_T | X_t)\,dX_T$$

In the B&S we know $S_t$ is lognormal (then $X$ is normal). How to know the distrib.?

We use fourier transform :

$$\hat{\mathcal{F}}[f](z) = \mathbb{E}[e^{izx}] = \int_{\mathbb{R}} e^{izx} f(x)\, dx = \hat{f}(z)$$

from the FT we can recovery the original function

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx} \hat{f}(z)\, dz$$

$$= e^{-r(T-t)} \int_{\mathbb{R}} (e^{x_T} - K)^+ \left( \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix_T z} \hat{f}(z)\, dz \right) dx_T$$

D integrate first w.r.t. x and after with z.

$$= \frac{1}{2\pi} e^{-r(T-t)} \int_{\mathbb{R}} dz\, \hat{f}(z) \underbrace{\int_{\mathbb{R}} (e^{x_T} - K)^+ e^{-ix_T z} dx_T}_{\text{fourier transfer of the } \widehat{Payoff}(-z)}$$

$$\boxed{\text{Price}_t(\text{PAYOFF}) = e^{-r(T-t)} \int_{\mathbb{R}} dz\, \tilde{f}(z)\, \widehat{\text{PAYOFF}}(-z)}$$

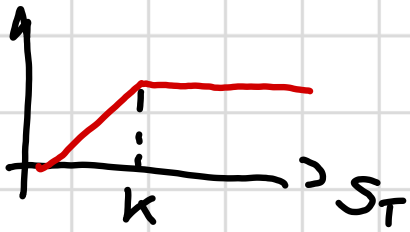
22/05

## FFT PRICING, example of payoff:

① COVERED CALL (because call is unbounded)

$$Payoff_T = h(X_T) = \min(e^{X_T}, K)$$

($X_T = \log S_T$), in this case is bounded.




K is the strike. Can be computed with

$$S_T - \underbrace{(S_T - K)^+}_{\text{call}}$$

The fourier transform of the payoff is:

$$\tilde{h}(z) = \int_{\mathbb{R}} e^{izx} \min(e^x, K)\, dx =$$

$$= \int_{-\infty}^{\ln K} e^{izx} e^x \, dx + \int_{\ln K}^{+\infty} K e^{izx} \, dx =$$

$$= \int_{-\infty}^{\ln K} e^{(iz+1)x} \, dx + K \int_{\ln K}^{+\infty} e^{izx} \, dx =$$

We assume $z = Re(z) + i\, Im(z)$. To have convergence of the integral, $Im(z) > 0$ and $-Im(z) + 1 > 0$

second integral $\Rightarrow$ $0 < Im(z) < 1$ first integral

$$= \frac{K^{iz+1}}{iz+1} - \frac{K^{iz+1}}{iz} = K^{iz+1}\left(\frac{1}{iz+1} - \frac{1}{iz}\right) =$$

$$= K^{iz+1} \frac{1}{z^2 - iz}$$

$$\Rightarrow \quad \tilde{h}(z) = \frac{K^{iz+1}}{z^2 - iz}$$

with domain of convergence
$\mathcal{Z} = \{z \in \mathbb{C} : -1 < \Im m(z) < +1\}$ because we have to compute $\int \hat{f}_X(t)\, \text{payoff}(-z)$.

② CALL $g(X_T) = (e^{X_T} - K)^+$

$$\tilde{g}(x) = \int_{\mathbb{R}} e^{izx}(e^x - K)^+ dx = \dots =$$

$$= -\frac{K^{iz+1}}{z^2 - iz} \quad \text{with } \mathcal{Z} = \{z \in \mathbb{C} : \Im m(z) > 1\}$$

Now, for the underlying the FT in
$$\mathbb{E}[e^{izX_T}] = \mathbb{E}[e^{i(\mathrm{Re}(z) + i\,\Im m(z))X_T}] =$$

$$= \mathbb{E}[e^{i\,\mathrm{Re}(z) X_T} \underbrace{e^{-\Im m(z) X_T}}_{S_t^{-\Im m(z)}}]$$

$S_t^{-\Im m(z)}$ $>0$ $\Rightarrow$ potential moment explosion

③ PUT $g(X_T) = (K - X_T)^+$

$$\hat{g}(z) = -\frac{K^{iz+1}}{z^2 - iz} \quad \text{with } \mathcal{Z} = \{z \in \mathbb{C} : \Im m(z) < 0\}$$

and we don't have moment explosion issue

## HESTON MODEL C.F.

The characteristic function $\phi(z) = \mathbb{E}^{\mathbb{Q}}_{t,x}[e^{izX_T}]$.
$X_T$ is not simply markovian, because depends on the stochastic volatility (unknown distribution).
We use Feynman-Kac, assuming linearly affine.

$$\begin{cases} \frac{dS}{S} = r\,dt + \sqrt{V}\,dW \\ dV = k(\theta - V)\,dt + \eta\sqrt{V}\left(\rho\,dW + \sqrt{1-\rho^2}\,dW^{\perp}\right) \end{cases}$$

$$dx = d\ln S = \left(r - \frac{1}{2}V_t\right)dt + \sqrt{V_t}\,dW$$

$$\Rightarrow \mathcal{A}_x = \left(r - \frac{1}{2}V_t\right)\frac{\partial}{\partial x} + \frac{1}{2}V_t\frac{\partial^2}{\partial x^2}$$

$$\Rightarrow \mathcal{A}_V = \left[k(\theta - V)\right]\frac{\partial}{\partial V} + \frac{1}{2}\eta^2 V\frac{\partial^2}{\partial V^2}$$

Separately they are affine. The joint is:

$$\mathcal{A}_{xv} = \mathcal{A}_x + \mathcal{A}_v + \eta \rho V \frac{\partial^2}{\partial x \partial v}$$

is again affine. This means that a solution candidate could be:

$$\mathbb{E}^{\mathbb{Q}}_{t,x,v}\left[e^{izX_T}\right] = e^{A(t,T) + B(t,T)V_t + C(t,T)X_t}$$

$$=: g(\tau, z, x, v)$$

$\hookrightarrow$ T-t time to maturity

$\tau = T - t$

$$\boxed{-\frac{\partial g}{\partial \tau} + \mathcal{A}_{xv} g = 0}$$

with initial condition $\tau = 0$ (on $t = T$)

$g(0, z, x, v) = e^{izx}$.

$$e^{A(0) + B(0)V + C(0)x} = e^{izx} \Rightarrow \begin{cases} A(0) = 0 \\ B(0) = 0 \\ C(0) = iz \end{cases}$$

Plugging this to the PDE

$$-\left(\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \tau} V + \frac{\partial C}{\partial \tau} x\right) + \left(r - \frac{1}{2}V\right)C +$$

$$+ \frac{1}{2}VC^2 + k(\vartheta - V)B + \frac{1}{2}\eta^2 V B^2 + \eta \rho V C B = 0$$

$$\begin{cases} \dfrac{\partial C}{\partial \tau} = 0 \\ C(0) = iz \end{cases} \Rightarrow \boxed{C = iz}$$

$$\frac{\partial B}{\partial t} = -\frac{1}{2}C + \frac{1}{2}C^2 - kB + \frac{1}{2}\eta^2 B^2 + \eta\rho C B$$

$$\begin{cases} \dfrac{\partial B}{\partial t} = -\dfrac{1}{2} iz(1-iz) - kB + \dfrac{1}{2}\eta^2 B^2 + \eta\rho iz B \\ B(0) = 0 \end{cases} \quad \text{(RICCATI EQUATION)}$$

(does not depends on A! We solve it and get B, which make possible to plug to find A)

$$\begin{cases} \dfrac{\partial A}{\partial t} = iz r + k\vartheta B \\ A(0) = 0 \end{cases}$$

SOLUTION FOR B

$$\frac{\partial B}{\partial t} = -\frac{1}{2} iz(1-iz) - kB + \frac{1}{2}\eta^2 B^2 + \eta\rho iz B$$

We guess the solution in the form:

$$B(\tau) := -\frac{2}{\eta^2}\frac{\dot{E}(\tau)}{E(\tau)}$$

this transform to second order differential equation.

(One other way is assuming $B(\tau) = E(t)/F(\tau)$ with $E, F$ whatever function, this give a linear system $\begin{pmatrix}\dot E\\ \dot F\end{pmatrix} = A\begin{pmatrix}E\\ F\end{pmatrix} \Rightarrow \begin{pmatrix}E\\ F\end{pmatrix} = \begin{pmatrix}f_0\\ \dot f_0\end{pmatrix}\exp(A)$
Useful for WISHART MODEL (generalize Heston))

$$\frac{\partial^2 E}{\partial \tau^2} + (k - \rho\eta i z)\frac{\partial E}{\partial t} + \frac{\eta^2}{4}(iz + z^2)E = 0$$

we use Laplace transform:

$$\lambda^2 + (k - \rho\eta i z)\lambda + \frac{\eta^2}{4}(iz + z^2) = 0$$

$$\Delta = (k - \rho\eta i z)^2 - \eta^2(iz + z^2)$$

$$\psi^{\pm} := -(k - \rho\eta i z) \pm \sqrt{\Delta}$$

$$\begin{cases}\psi^+ - \psi^- = 2\sqrt{\Delta}\\ \psi^+\psi^- = -\eta^2(1z + z^2)\end{cases}$$

$\Downarrow$

$$E(\tau) = \alpha_1 e^{\frac{1}{2}\psi^+\tau} + \alpha_2 e^{\frac{1}{2}\psi^-\tau}$$

$$\dot E(\tau) = \frac{1}{2}\alpha_1\psi^+ e^{1/2\psi^+\tau} + 1/2\,\alpha_2\psi^- e^{1/2\psi^-\tau}$$

$$E(0) = \alpha_1 + \alpha_2$$

$$\dot E(0) = 1/2\,\alpha_1\psi^+ + 1/2\,\alpha_2\psi^-$$

$$\alpha_1 = \frac{\psi^- E(0)}{2\sqrt{\Delta}} \qquad \alpha_2 = -\frac{\psi^+ E(0)}{2\sqrt{\Delta}}$$

$$B(\tau) = -\frac{\dot{E}(\tau)}{\eta^2/2\, E(\tau)} = -\frac{\psi^+\psi^-\left(e^{1/2\psi^+\tau} - e^{1/2\psi^-\tau}\right)}{\eta^2\left[\psi^- e^{1/2\psi^+\tau} - \psi^+ e^{1/2\psi^-\tau}\right]} =$$

$$= -(it+t^2)\frac{\psi^+\psi^-\left(e^{1/2\psi^+\tau} - e^{1/2\psi^-\tau}\right)}{\eta^2\left[\psi^- e^{1/2\psi^+\tau} - \psi^+ e^{1/2\psi^-\tau}\right]}$$

Now we want to find $A$

$$\frac{\partial A}{\partial t} = K\theta B + r i t$$

$$A(\tau) = \int_0^\tau K\theta B(\tau)\, d\tau + r i t \tau =$$

$$= \int_0^\tau K\theta\left(-\frac{2}{\eta^2}\frac{\dot{E}(\tau)}{E(\tau)}\right) d\tau + r i t \tau =$$

$$= -\frac{2K\theta}{\eta^2}\ln\left(\frac{E(\tau)}{E(0)}\right) + r i t \tau = \dots =$$

$$= -\frac{K\theta}{\eta^2}\left[2\ln\left(\frac{-\psi^- + \psi^+ e^{-\sqrt{\Delta}\tau}}{2\sqrt{\Delta}}\right) + \psi^+\tau\right] + r i t \tau$$

Now that we have A, B and C :

$$\text{Price}_t(\text{PAYOFF}) = \int_Z \hat{f}(z)\, \widehat{\text{PAYOFF}}(-z)\, dz$$

$$\Big\downarrow \text{ HESTON MODEL}$$

$$e^{A(\tau) + B(\tau)V_t + C(\tau)X_t}$$

During the calculation we have to be careful because of log can produce multiple reuelts (complex log) : LITTLE HESTON TRAP

CARR - MADAN ('99) FFT to compute numerically

23/5

# 2-FACTOR HESTON MODEL

$$\begin{cases} \frac{dS}{S} = r\,dt + \sqrt{V_1}\,dZ_1 + \sqrt{V_2}\,dZ_2 \\ dV_1 = k_1(\vartheta_1 - V_1)\,dt + \xi_1\sqrt{V_1}\,dW_1 \\ dV_2 = k_2(\vartheta_2 - V_2)\,dt + \xi_2\sqrt{V_2}\,dW_2 \end{cases}$$

Because of cross terms like $\sqrt{V_1 V_2}$, we cannot use an affine structure for the expected value. Then we have to:

$$\langle Z_1, W_1 \rangle = \rho_1 \qquad \langle Z_2\, W_2 \rangle = \rho_2 \quad \longrightarrow \quad \begin{cases} \langle Z_1, Z_2 \rangle = 0 \\ \langle W_1, W_2 \rangle = 0 \\ \langle Z_1, W_2 \rangle = 0 \\ \langle Z_2, W_1 \rangle = 0 \end{cases}$$

(1) VOLATILITY FACTORS ARE INDEPENDENT

Then the Fourier pricing is

$$\mathbb{E}_t^{Q}[e^{iz \ln S_T}] =$$

$$= e^{iz \ln S_t + A(\tau) + B_1(\tau)V_{1t} + B_2(\tau)V_{2t} + \dots}$$

As in 1D case, $B_1$ and $B_2$ solves indep. Riccati equations.

We define $y_t := \ln S_t$ :

$$dy = \left[r - \tfrac{1}{2}(V_1+V_2)\right]dt + \underbrace{\sqrt{V_1}\,dz_1 + \sqrt{V_2}\,dz_2}_{= \sqrt{V_1+V_2}\,d\hat{z}}$$

$$\mathrm{Corr}\left(\mathrm{Noise}(dy), \mathrm{Noise}(\mathrm{Vol}(dy))\right) = \frac{d\langle y, V_1+V_2\rangle}{\sqrt{d\langle y_1\rangle\, d\langle V_1+V_2\rangle}} =$$

$$= \frac{\rho_1 \xi_1 V_1 + \rho_2 \xi_2 V_2}{\sqrt{V_1+V_2}\,\sqrt{\xi_1^2 V_1 + \xi_2^2 V_2}}$$

this is stochastic!

This model is limited by the constrain on the conditions.

We need a more sofisticated model.

$\downarrow$

## WISHART PROCESSES

Squared Bessel process of dimension $n>1$ are $dX_t = 2\sqrt{X_t}\,dW_t + n\,dt$, $X_t = B_t^T B_t$, $B_t \in \mathbb{R}^n$.

For $\delta \in \mathbb{R}$ we have

$$BESQ^\delta:\ dX_t = 2\sqrt{X_t}\,dW_t + \delta\,dt \qquad \delta \geq 0$$

$$\mathcal{A} = 2x\frac{\partial^2}{\partial x} + \delta\frac{\partial}{\partial x}$$

We fix $t$, the matrix analog of $X_t$ has a WISHART DISTRIBUTION

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1d} \\ \vdots & \ddots & \vdots \\ B_{n1} & \cdots & B_{nd} \end{pmatrix}_{n \times d}$$

$$B^T B_{d \times d} \sim \text{WISHART}$$

(matrix extension of $\chi^2$)

EXAMPLE:

$d=1$ : $B^T B = B_{11}^2 + \ldots + B_{n1}^2 = BESQ^T$

$n=1$ : $B^T B = \begin{pmatrix} B_{11} \\ \vdots \\ B_{d1} \end{pmatrix} (B_{11} \cdots B_{d1}) = \begin{pmatrix} & \end{pmatrix}_{d \times d}$

but has rk 1, is degenerate

$\Rightarrow$ We will consider $\boxed{n \geq d}$

Def: A WISHART PROCESS of dimension $d \geq 1$, index $n \geq d$ and initial state $s_0 = C^T C \in S_d^*$

$$WIS(n, d, s_0) : \quad S_t = N_t^T N_t$$

Where $N_t$ is a Brownian motion $M_{n \times d}$, with $N_0 = C$

## Theorem:

1) The dynamic of this process is

$$dS_t = \sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t} + n \mathbb{1}_{d\times d}\, dt$$

where $B_t$ is a brownian motion $\in M_{d\times d}$

2) The infinitesimal generator is

$$A = \mathrm{Tr}[nD + 2SD^2]$$

where $D = (\partial / \partial s_{ij})_{i,j}$

**proof:** $dS_t = (dN_t^T) N_t + N_t^T dN_t + d[N_t^T N]_t =$

$$= dN_t^T \frac{N_t}{\sqrt{N_t^T N_t}} \sqrt{N_t^T N_t} + \sqrt{N_t^T N_t} \frac{N_t^T}{\sqrt{N_t^T N_t}} dN_t + n I\, dt$$

BM BM

we consider $\alpha, \beta \in \mathbb{R}^d$ vectors

$$\mathrm{Cov}_t(dN_t \cdot \alpha, dN_t \cdot \beta) = \mathbb{E}[(dN\alpha)(dN\beta)^T] =$$

$$= \alpha^T \beta\, \mathbb{1}_{n\times n}\, dt$$

because $\mathrm{Cov}_t\left( \frac{N_t^T dN_t}{\sqrt{N_t^T N_t}} \cdot \alpha, \frac{N_t^T dN_t}{\sqrt{N_t^T N_t}} \cdot \beta \right) =$

$$= \mathbb{E}\left[ \frac{N_t^T dN_t \, \alpha \, (dN_\beta)^T N_t}{N_t^T N_t} \right] = \alpha^T \beta \, \mathbb{1}_n \, dt$$

$$\Rightarrow \quad \frac{N_t^T dN_t}{\sqrt{N_t^T N_t}} = dB$$ Brownian motion and Itō works

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## EXTENSION TO MEAN REVERTION

We consider $(X_t)_{n\times d}$ matrix with dynamics

$$dX_t = \gamma \, dB_t + \beta X_t \, dt$$

ORNSTAIN-UHENBECK MATRIX PROCESS

$X_0 \in M_{n\times d}$

$(B_t)_{n\times d}$ Brownian motion

theorem: set $S_t = X_t^T X_t$, then

$$dS_t = \gamma\left(\sqrt{S_t}\, dB_t + dB_t^T \sqrt{S_t}\right) + 2\beta S_t + n\gamma^2 \mathbb{1}_{d\times d} \, dt$$

with $S_0 = X_0^T X_0$

Now, because the infinitesimal generator has the trace (sums), then I have an affine terms:

$$d\Sigma_t = (\beta Q^T Q + M\Sigma + \Sigma M^T)dt + \\ + \sqrt{\Sigma}\, dW Q + Q^T (dW_t^T)\sqrt{\Sigma}$$

$\beta \geq d-1$ (FELLER CONDITION)

$$A_\Sigma = Tr\left[(\beta Q^T Q + M\Sigma + \Sigma M^T) D + \\ + 2\Sigma D Q^T Q D\right]$$

EXAMPLE $d=2$

The dynamic of the upper left $\Sigma^{11}$ in

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \sqrt{\Sigma}$$

$$d\Sigma^{11} = (\dots)dt + 2\sigma_{11}(Q_{11}\, dW^{11} + Q_{21}\, dW^{12}) + \\ + 2\sigma_{12}(Q_{11}\, dW^{21} + Q_{21}\, dW^{22})$$

...

$$d\langle \Sigma^{11}, \Sigma^{22} \rangle = 4\underbrace{(\sigma^{11}\sigma^{12} + \sigma^{12}\sigma^{22})}_{\Sigma_{12}}(Q_{11}Q_{12} + Q_{21}Q_{22})dt$$

And now we have correlation between the two dynamics and is still affine (if $Q$ not dyagonal)

**Theorem:** (LAPLACE TRANSFORM)

$$\mathbb{E}_t\left[e^{Tr(\Lambda \Sigma_T)}\right] = e^{-Tr[A(\tau)\Sigma_t] + a(\tau)}$$

where $A(\tau)$, $a(\tau)$ deterministic functions solving the ODE (element by element)

$$\begin{cases} \dot{A}(\tau) = A(\tau)M + M^T A(\tau) - 2A(\tau)Q^T Q A(\tau) \\ A(0) = \Lambda \end{cases}$$

(MATRIX RICCATI EQUATION)

$$\begin{cases} \dot{a}(\tau) = -Tr[\beta Q^T Q A(\tau)] \\ a(0) = 0 \end{cases}$$

To solve the Riccati $A(\tau) = F^{-1}(\tau)G(\tau)$

$$\frac{d}{dt}(F \cdot A) = \dot{F}A + F\dot{A}$$ Riccati

$$\dot{G} - \dot{F}A = GM + (FM^T - 2GQ^TQ)A$$

$$\begin{cases} \dot{G} = GM \\ \dot{F} = -FM^T + 2GQ^TQ \end{cases}$$

$$\frac{d}{dt}(G \quad F) = (G \quad F)\begin{pmatrix} M & 2Q^TQ \\ 0 & -M^T \end{pmatrix}$$

$$\left(G(\tau), F(\tau)\right) = \left(G(0), F(0)\right) \exp\left[\tau \begin{pmatrix} M & 2Q^TQ \\ 0 & -M^T \end{pmatrix}\right]$$

We find $G(\tau)$ and $F(\tau)$, and then $A(\tau)$.
If the expon. give $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, then:

$$A(\tau) = \left(\Lambda A_{12}(\tau) + A_{22}(\tau)\right)^{-1} \left(\Lambda A_{11}(\tau) + A_{21}(\tau)\right)$$

$$\dot{a}(\tau) = -\beta \operatorname{Tr}\left[Q^TQ\, A(\tau)\right] =$$

$$= -\beta \operatorname{Tr}\left[Q^TQ\, F^{-1} G\right]$$

$$a(\tau) = -\beta/2 \operatorname{Tr}\left[\log(F(\tau)) + \Lambda^T \tau\right]$$