Report 3

April 4, 2025

Università degli Studi di Padova
Dipartimento di Fisica e Astronomia "Galileo Galilei"

Riccardo Corte

Contents

1 Introduction

This report aims to investigate the pricing of options, with particular focus on European call options. The analysis will directly compare discrete-time models such as the Binomial model and the more refined Leisen-Reimer model with continous-time models such as the Black-Scholes method. The key objective is to examine how these discrete approximations converge toward the continuous-time solution provided by the Black-Scholes formula. By studying the convergence behavior as the number of time steps increases, we will highlight how the Leisen-Reimer approach achieves improved accuracy and faster convergence compared to the standard Binomial method.

2 Methods

2.1 Binomial Model

The price of an option can be estimated using the binomial model. This model approximates market behavior by assuming that, at each discrete time step, the price of the underlying asset can move either up or down, according to a risk-neutral (arbitrage-free) probability q. The pricing process involves determining the upward and downward movement factors, u and d, which represent how much the asset price may increase or decrease at each time step.

Ultimately, the option price is derived based on given inputs such as the asset's volatility, the risk-free interest rate, the number of time steps in the binomial tree, the strike price, and the current price of the underlying asset.

Practically speaking, starting from n equal time intervals $\Delta t = \frac{T}{n}$. The up and down movement factors are computed using the asset's volatility σ:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

From these element is now possible to obtain the risk-neutral probability q:

$$q = \frac{e^{r\Delta t} - d}{u - d}$$

Pursuing the binomial approach, from the stock prices at maturity, S_j, we obtain the corresponding option payoffs:

$$C_j^{(n)} = \max(S_j - K, 0)$$

To conclude we procede with backward to compute the option price at earlier nodes using the discounting.

$$C_j^{(i)} = e^{-r\Delta t}\left(p \cdot C_{j+1}^{(i+1)} + (1-p) \cdot C_j^{(i+1)}\right)$$

this is done by iteration, where j is an indicator for the number of up and down movement, meanwhile i indicates which step we are referring to.

2.2 Leisen Reimer Binomial Model

The Leisen Reimer model consitutes an improvement over the classical binomial model. The new idea is that the underlying price binomial tree is centered around the option's strike price at expiration. From a practical standpoint the application is the same as for the Binomial model and the only difference is constituted by how the up and down parameters are calculated. First we start from calculating the parameters $d1$ and $d2$ which will be later mentioned in the Black-Scholes model. These parameters will be used to calculate the probability of an up move in a Leisen-Reimer tree:

$$p = h^{-1}(d_2)$$

Where $h^{-1}(z)$ is the Peizer-Pratt inversion function, a function used to obtain the discrete binomial estimates for the continuous normal cumulative distribution function.

$$h^{-1}(z) = \frac{1}{2} + \frac{sgn(z)}{2}\sqrt{1 - e^{\left[-\left(\frac{z}{n+\frac{1}{3}+\frac{0.1}{n+1}}\right)^2(n+\frac{1}{6})\right]}}$$

we have that n is number of steps of the binomial model. One restriction of Leisen-Reimer model is that number of steps must be odd to avoid imprecise option prices.

The Leisen-Reimer up and down move formulas are:

$$u = e^{r\Delta t} \cdot \frac{p'}{p}$$

$$d = e^{r\Delta t} \cdot \frac{1-p'}{1-p}$$

Where the exponent term $e^{r\Delta T}$ can be interpreted as net cost of holding the underlying security over one step. In the formulas it is present a ratio of two probabilities, p' and p for up move, and 1–p' and 1–p for down move, in particular p' is obtained applying the Peizer-Pratt formula using $d1$ as the argument.

We have now derived the up and down move sizes and probabilities using the Leisen-Reimer model. Starting from this value the price of the option will be easily calculated following the same procedure used for the binomial model.

2.3 Black-Scholes Model

The Black-Scholes model has a pivotal importance in financial mathematics, providing a theoretical definitions for pricing options. It is based on the assumption that the underlying asset follows a geometric Brownian motion. The derivation of the pricing formulas for European call and put options relies on several key assumptions: constant volatility σ, a constant risk-free interest rate r, and the assumption that the asset's future prices are log-normally distributed. The key idea of computing a price using the Black-Scholes formula is to compute two intermediate quantities:

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

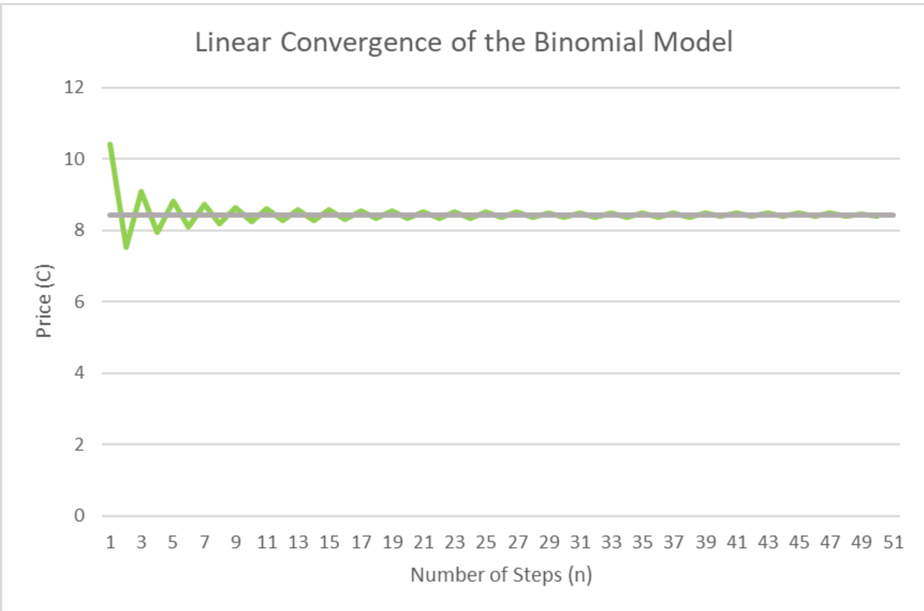
These values are used in the cumulative distribution function of the standard normal distribution, $N(d)$, to evaluate the option price:

$$C = S \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2)$$

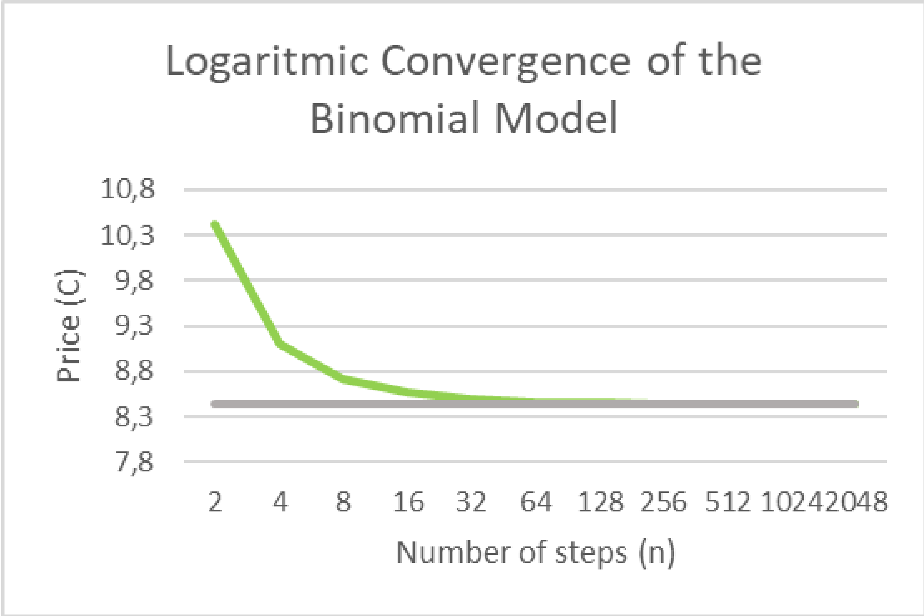
Therefore we can define a function that takes the current stock price S, strike price K, risk-free rate r, volatility σ, and time to maturity, and returns the theoretical price of the call option C. It is important to note that this model is valid under the assumption of European-style options, meaning the option can only be exercised at maturity.

3 Results

We can now observe the results.
First we study the plots obtained for the binomial model.



We see that the results of the binomial model oscillates around the correct value, which is obtained from the Black-Scholes formula. As expected, as the number n of steps increases, the pricing is more accurate and the convergence is evident. To verify the convergence at infinity we study the logarithmic convergence which confirmes our expectations.



We can now examine how the Leisen Reimer model performs. As expected the value converges for $n \longrightarrow \infty$, in contrast to the binomial model we have that the prices does not oscillates around the correct value. Instead, it gets accurate very soon, already starting from a close value.

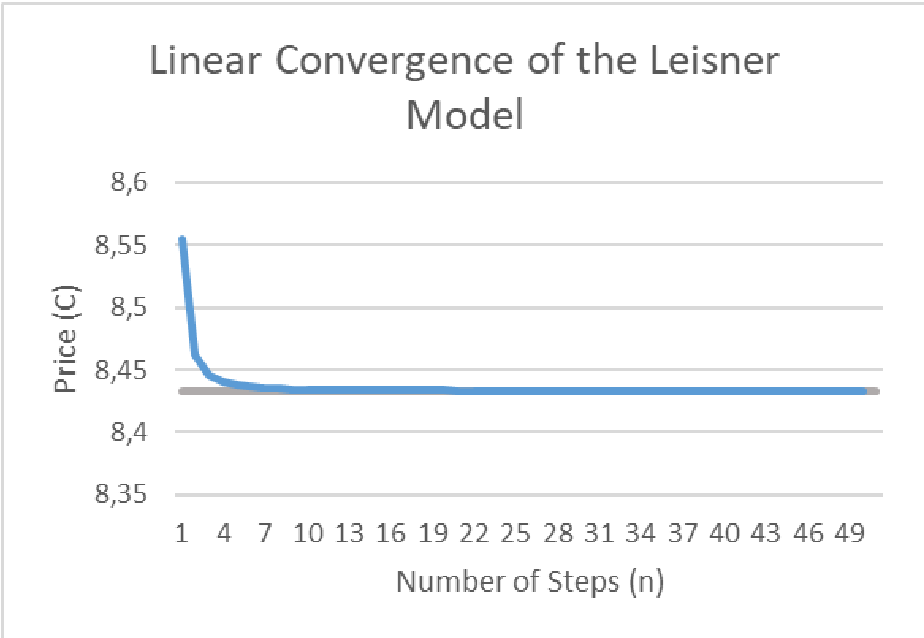


Figure 1: Enter Caption

The logaritmic plot confirmes the result.

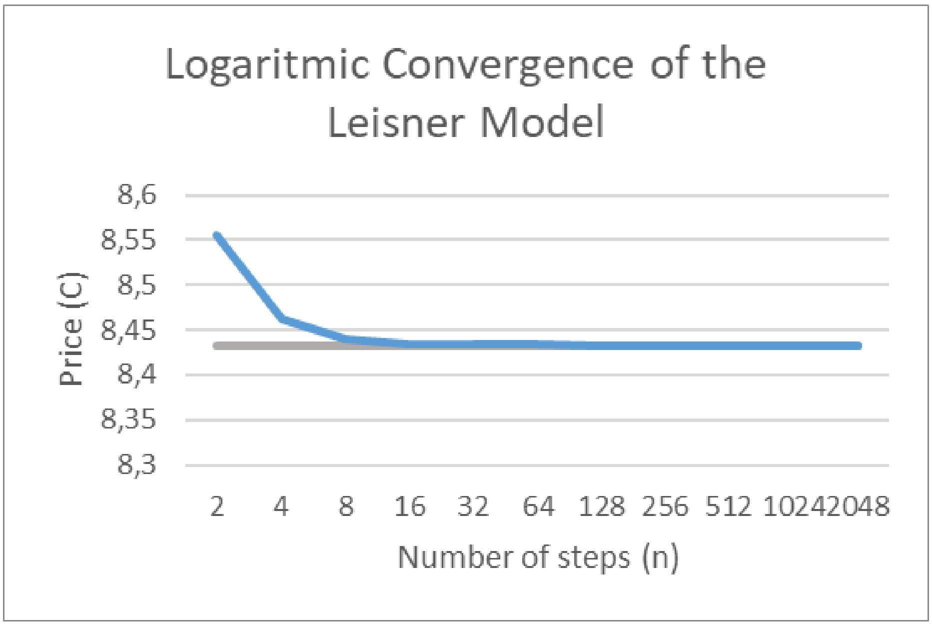


Figure 2: Enter Caption

It is now interesting to directly confront the two results. By directly comparing the two methods it is evident how the Leisen Reimer method yields significantly better values by converging more rapidly to the Black Scholes formula results. Nonetheless both methods converges at infinity.

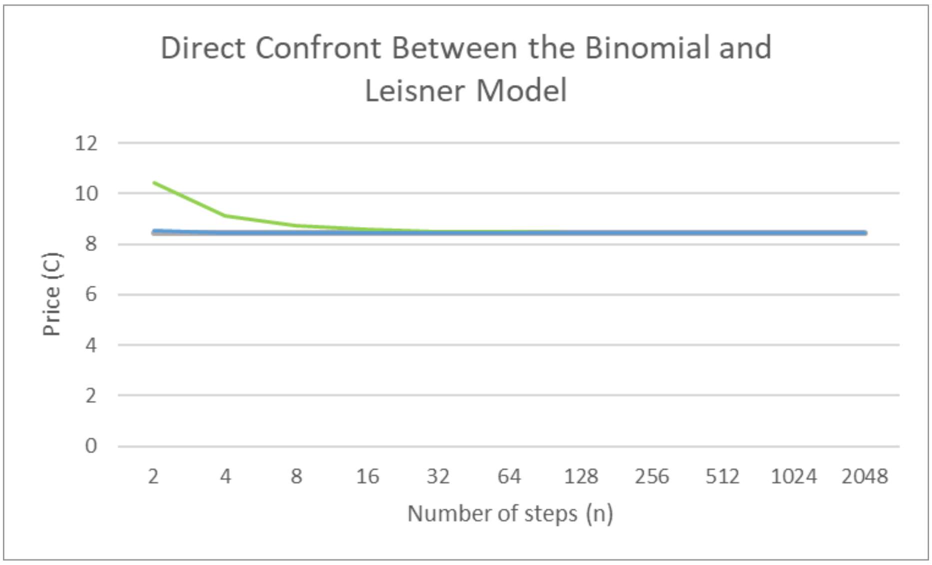


Figure 3: Enter Caption

4 Conclusion

In conclusion we have successfully demonstrated the binomial option pricing convergence theorem, which states that the price obtained from the binomial model converges to the result of the Black Scholes formula. In particular given a Call option we have confronted the binomial model with the Leisen Reimer binomial method. It is evident how the latter is more advanced, allowing for faster convergence to the result, necessitating only a restricted number of steps. Nonetheless both methods yields similar results as the number of steps increases, demonstrating that both of them represents valid models when pricing an option.