09/04

EXCHANGE OPTION

Possibility to change a stock S_1 with another S_2 at maturity T.

The Payoff is $(S_1(T) - S_2(T))^+$. Then the price is, as always:

$$Price_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[Payoff_T]$$

But this is a CALL with strike $S_2(T)$, so we can think to use B&S:

$$Price_t = S_1(t)\Phi(d_1) - S_2(T)e^{-r(T-t)}\Phi(d_2)$$

$S_2(T)$ is not measurable at time t!
We use as strike $S_2(t)$

$$\Rightarrow Price_t = S_1(t)\Phi(d_1) - S_2(t)\Phi(d_2)$$

But in d_1, d_2 what volatility should I choose? This answer is WRONG $\Rightarrow$ we expect dependence on the correlation between the two assets.

We rewrite the Payoff as:

$$\text{Payoff}_T = S_2(T)\left(\frac{S_1(T)}{S_2(T)} - 1\right)^+$$

the transformation $S_1(T)/S_2(T)$ embed both the volatility.
More generally:

$$F(T, x_1, x_2) = x_2\, g\left(T, \frac{x_1}{x_2}\right) = x_2\, G(T, z)$$

So our PDE will have a change of variable with z: $F(t, x_1, x_2) = x_2 G(t, z)$.

We can extend the Feynman-Kač formula for a multidimensional case:

$$\begin{cases} \dfrac{\partial F}{\partial t} + \nabla F \begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix} + \dfrac{1}{2} (dx)^T \underbrace{H_F}_{\text{Hessian}} (dx) - rF = 0 \\[2ex] F(T, x) = \text{Payoff}(T, x) \end{cases}$$

then, the solution is:

$$F(t,x) = e^{-r(T-t)}\,\mathbb{E}_t^{\mathbb{Q}}\left[\text{Payoff}_T(T,x)\right]$$

where
$$\underset{n\times 1}{dx} = \underset{n\times n}{\text{diag}(x)} \underset{n\times 1}{\begin{pmatrix} r \\ r \\ \vdots \\ r \end{pmatrix}} dt + \underset{n\times n}{\sigma}\, \underset{n\times 1}{dW}$$

(we also need $(\nabla F \sigma) \in \mathcal{H}$).

We start with the 2-dimensional case:

$$\frac{\partial F}{\partial t} + r\left(x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} \right) + \frac{1}{2}\left(x_1^2 c_{11} \frac{\partial^2 F}{\partial x_1^2} + \right.$$
$$\left. + x_2^2 c_{22} \frac{\partial^2 F}{\partial x_2^2} + 2 x_1 x_2 c_{12} \frac{\partial^2 F}{\partial x_1 \partial x_2} \right) - rF = 0$$

where the coefficients are

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \sigma \sigma^T = \begin{pmatrix} \sigma_{11}^2 + \sigma_{22}^2 & \sigma_{11}\sigma_{12} + \sigma_{21}\sigma_{22} \\ \shortparallel & \sigma_{11}^2 + \sigma_{22}^2 \end{pmatrix}$$

If the solution is in the form $F(t, x_1, x_2) = x_2\, g(t, z)$ we can compute the derivative

$$\frac{\partial F}{\partial t} = x_2 \frac{\partial g}{\partial t} \qquad \frac{\partial F}{\partial x_1} = \frac{\partial g}{\partial z} \qquad \frac{\partial F}{\partial x_2} = g - \frac{x_1}{x_2}\frac{\partial g}{\partial z}$$

$$\frac{\partial^2 F}{\partial x_1^2} = \frac{\partial^2 g}{\partial z^2}\,\frac{1}{x_1}$$

$$\frac{\partial^2 F}{\partial x_2^2} = -\cancel{\frac{x_1}{x_2^2}\frac{\partial g}{\partial z}} + \cancel{\frac{x_1}{x_2^2}\frac{\partial g}{\partial z}} - \frac{x_1}{x_2}\frac{\partial^2 g}{\partial z^2}\left(-\frac{x_1}{x_2^2}\right)$$

$$= + \frac{x_1^2}{x_2^3}\frac{\partial^2 g}{\partial z^2}$$

$$\frac{\partial^2 F}{\partial x_1 \partial x_2} = -\frac{x_1}{x_2^2}\frac{\partial^2 g}{\partial z^2}$$

The original PDE became:

$$x_2 \frac{\partial g}{\partial t} + r\left(\cancel{x_1 \frac{\partial g}{\partial z}} + \cancel{x_2}\left(\cancel{g} - \cancel{\frac{x_1}{x_2}\frac{\partial g}{\partial z}}\right)\right) +$$

$$+ \frac{1}{2}\left(x_1^2 c_{11}\frac{\partial^2 g}{\partial z^2}\frac{1}{x_2} + x_2^{\cancel{2}} c_{22}\frac{x_1^2}{x_2^{\cancel{3}}}\frac{\partial^2 g}{\partial z^2} +\right.$$

$$\left. + 2 c_{12} x_1 \cancel{x_2}\left(-\frac{x_1}{x_2^{\cancel{2}}}\frac{\partial^2 g}{\partial z^2}\right)\right) - \cancel{r x_2 g} = 0$$

$$x_2\left[\frac{\partial g}{\partial t} + \frac{1}{2}\left(z^2 c_{11}\frac{\partial^2 g}{\partial z^2} + c_{22} z^2 \frac{\partial^2 g}{\partial z^2} - 2 c_{12} z^2 \frac{\partial^2 g}{\partial z^2}\right)\right] = 0$$

$\downarrow$ x_2 is positive so we can simplify (also payoff)

$$\begin{cases} \dfrac{\partial G}{\partial t} + \dfrac{1}{2}\left(C_{11} + C_{22} - 2C_{12}\right) z^2 \dfrac{\partial^2 G}{\partial z^2} = 0 \\ G(T, z) = (z-1)^+ \end{cases}$$

The SDE is $\dfrac{dz}{z} = \sigma_z \, dW$ which is a 1D Brownian motion. $\left(\sigma_z = \sqrt{C_{11} + C_{22} - 2C_{12}}\right)$

This correspond to a call with strike 1 and with no r.

$$G(t, z) = z\,\Phi(d_1^{(t)}) - \Phi(d_2^{(t)})$$

We now multiply by X_2 to find the actual price:

$$\boxed{\begin{aligned} F(t, X_1, X_2) &= X_2 G(t, z) = \\ &= X_1 \Phi(d_1^{(t)}) - X_2 \Phi(d_2^{(t)}) \end{aligned}}$$

where: $d_{1,2}^{(t)} = \dfrac{\ln(z) \pm \frac{1}{2}\sigma_z^2 (T-t)}{\sigma_z \sqrt{T-t}} =$

$$= \frac{\ln(X_1/X_2) \pm \frac{1}{2}\sigma_z^2 (T-t)}{\sigma_z \sqrt{T-t}}$$

Intuitively, we don't have interest rate because we are comparing assets. This is the same solution we have naively supposed but with the correlations embedded to $d_{1,2}$ (so in σ_2).

We have mapped a 2D problem in a 1D one.

An alternative approach is based on CHANGE OF NUMERAIRE;

↓

Under $\mathbb{P}$: $\begin{cases} \frac{dS}{S} = \mu dt + \sigma dW_t^{\mathbb{P}} \\ \frac{dS_0}{S_0} = r dt \end{cases}$

Under $\mathbb{Q}$: $\begin{cases} \frac{dS}{S} = r dt + \sigma dW_t^{\mathbb{Q}} \\ \frac{dS_0}{S_0} = r dt \end{cases}$

with the property of $e^{-rt} S = \frac{S_t}{S_{t,0}}$ be a True martingale.

This means that we can transform:

$$(S_t, S_t^0) \longrightarrow \left(\underbrace{\frac{S_t}{S_t^0}}_{\text{martingale}}, \underbrace{1}_{\text{trivial martingale}}\right)$$

If the NUMERAIRE Num is a traded asset whose price is always positive, then $\exists$ $\mathbb{Q}^{Num}$ s.t.

1) $\dfrac{\text{ASSETS}}{Num} = \mathbb{Q}^{Num}$ (martingale under $\mathbb{Q}^{N}$)

2) $\dfrac{\text{PORTFOLIO}}{Num} = \mathbb{Q}^{Num}$

3) $\dfrac{\text{OPTION PRICES}}{Num} = \mathbb{Q}^{Num}$

$\Rightarrow$ We can choose a Numeraire to exploit martingality

$$\frac{Price_t}{Num} = \mathbb{E}_t^{\mathbb{Q}^{Num}}\left[\frac{Payoff_T}{Num_T}\right]$$

EXAMPE (Application to Exchange option)

$$Payoff_T = (S_1(T) - S_2(T))^+$$

$$\mathbb{P}: \begin{cases} \dfrac{dS_1}{S_1} = \mu_1 dt + \sigma_{11} dW_1^{\mathbb{P}} + \sigma_{12} dW_2^{\mathbb{P}} \\ \dfrac{dS_2}{S_2} = \mu_2 dt + \sigma_{22} dW_1^{\mathbb{P}} + \sigma_{21} dW_2^{\mathbb{P}} \end{cases}$$

We choose S_2 as Numeraire.

$$(S_1, S_2) \longmapsto \left(\frac{S_1}{S_2}, 1\right) := (Z, 1)$$

then z is $\mathbb{Q}^{S_2}$. So the asset z dynamics under this new martingale measure $\mathbb{Q}^{S_2}$. Using Itō:

$$dz_t = d\left(\frac{S_1}{S_2}\right) = \overbrace{0}^{\text{Because martingale!}} \cdot dt + (\dots)\, dW^{\mathbb{Q}^{S_2}}$$

Drift is zero, we can skip calculation with time.

$$d\left(S_1 \cdot \frac{1}{S_2}\right) = \frac{1}{S_2} dS_1 + S_1\, d\left(\frac{1}{S_2}\right) + (\dots)dt =$$

$$= \frac{S_1}{S_2}\left(\sigma_{11} dW_1^{\mathbb{P}} + \sigma_{12} dW_2^{\mathbb{P}}\right) -$$

$$- \frac{S_1}{S_2}\left(\sigma_{21} dW_1^{\mathbb{P}} + \sigma_{22} dW_2^{\mathbb{P}}\right) + (\dots)dt =$$

$$= \frac{S_1}{S_2}\left((\sigma_{11} - \sigma_{21})\, dW_1^{\mathbb{Q}^{S_2}} + (\sigma_{12} - \sigma_{22})\, dW_2^{\mathbb{Q}^{S_2}} \right) =$$

$$\Rightarrow \boxed{dz = z\,\sigma^z\, dW^z}$$

where $\sigma_z = \sqrt{(\sigma_{11} - \sigma_{21})^2 + (\sigma_{12} - \sigma_{22})^2} =$
$= \sqrt{c_{11} + c_{22} - 2c_{12}}$ as Before!

dW^z is a $\mathbb{Q}^{S_2}$- Brownian motion.

$$\frac{\text{Price}_t}{S_2} = \mathbb{E}_t^{Q^{S_2}}\left[\frac{(S_1(T)-S_2(T))^+}{S_2(T)}\right] =$$

$$= \mathbb{E}_t^{Q^{S_2}}\left[\left(\frac{S_1(T)}{S_2(T)}-1\right)^+\right] =$$

$$= \mathbb{E}_t^{Q^{Z}}\left[\left(Z(T)-1\right)^+\right] \qquad \frac{dZ_t}{Z_t} = \sigma^Z dW^Z$$

Which is a call with $r=0$ and strike 1.

10/04

Application of the change of NUMERAIRE to:

STOCHASTIC INTEREST RATE

$$\begin{cases} \dfrac{dS}{S} = \mu\, dt + \sigma\, dW^{\mathbb{P}} \\ \dfrac{dS_0}{S_0} = r_t\, dt \quad \text{(Locally deterministic)} \\ \qquad\qquad\quad r_t \text{ is a random variable} \end{cases}$$

From AAO $\exists\, \mathbb{Q}$ s.t. discounted asset $S_t \exp\{-\int_0^t r_s\, ds\}$ are true martingale (if r_t is constant we get the same as B&S)

$$\frac{dS_t}{S_t} = r_t\, dt + \sigma \overbrace{\left(dW^{\mathbb{P}} - \frac{r_t - \mu}{\sigma}\, dt\right)}^{dW^{\mathbb{Q}}}$$

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_t = \exp\left\{\int_0^t \frac{r_s - \mu}{\sigma}\, dW^{\mathbb{P}} - \frac{1}{2}\int_0^t \frac{r_s - \mu}{\sigma}\, ds\right\}$$

Since discounted asset is mg., then we can compute the price as usual:

$$\cancel{e^{-\int_0^t r_s ds}}\, \text{Price}_t = \mathbb{E}_t^{\mathbb{Q}}\Big[\underbrace{e^{-\int_0^T r_s\, ds}}\, \text{Payoff}_T\Big]$$

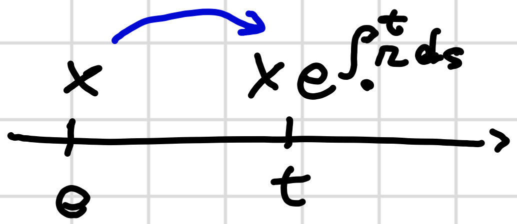
take outside and semplify $\quad \left\{\cancel{e^{-\int_0^t r_s ds}} - \int_t^T r_s\, ds\right.$

$$\text{Price}_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \, \text{PAYOFF}_T \right]$$

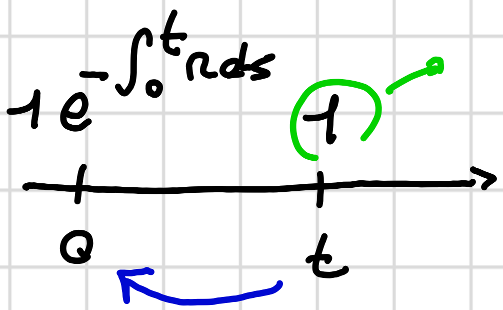
interest rate and Price are connected, then we cannot use the independence to find the expected value.
We'd like to use a numeraire s.t. it cancel the discounting.

We need something always positive and connected to interest rate $\Rightarrow$ ZERO COUPON BOND

$B(t,T)$



Another possibility is looking backward using the discounted price



it's measurable at time t
(FORWARD LOOKING)

$$\left.\begin{aligned} B(T,T) &= 1 \\ B(t,T) &= 1 \cdot e^{-\int_0^t r \, ds} \end{aligned}\right\} \text{ it's a contract, no:}$$

$$B(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \right] > 0$$

We choose $B(t,T)$ as numeraire, then $S_t / B(t,T)$ become a true martingale under $\mathbb{Q}^T$ (FORWARD MEASURE).

$$\frac{\text{Price}_t}{B(t,T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{\text{Payoff}_T}{\underbrace{B(T,T)}_{=1}}\right] = \mathbb{E}^{\mathbb{Q}^T}[\text{Payoff}_T]$$

$$\boxed{\text{Price}_t = B(t,T)\, \mathbb{E}^{\mathbb{Q}^T}[\text{Payoff}_T]}$$

price of zero coupon Bond

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]$$

under $\mathbb{Q}^T$ is like $r=0$

under $\mathbb{Q}^T$ we are able to split the expected value

if r is deterministic $\mathbb{Q} \equiv \mathbb{Q}^T$

This is valid for any interest rate type.

$$\text{Price}_t = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \text{Payoff}_T\right] =$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right] \mathbb{E}^{\mathbb{Q}^T}[\text{Payoff}_T]$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[\frac{e^{-\int_t^T r_s ds}\,\mathrm{Payoff}_T}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]}\right] = \mathbb{E}_t^{\mathbb{Q}^T}\left[\mathrm{PAyoff}_T\right]$$

$$\Downarrow$$

$$\left.\frac{d\mathbb{Q}^T}{d\mathbb{Q}}\right|_t = \frac{e^{-\int_t^T r_s ds}}{\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]}$$

The B&S-like formula in the presence of stochastic interest rate for a call can be find from:

$$Call_t = \mathbb{E}_t^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\underbrace{(S_T-K)^+}_{S_T \mathbb{1}_{S_T \geq K} - K \mathbb{1}_{S_T \geq K}}\Big] =$$

$$= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} S_T \mathbb{1}_{S_T \geq K}\right] -$$

$$- K\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mathbb{1}_{S_T \geq K}\right]$$

The last term in eq.:

$$K\,\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds} \mathbb{1}_{S_T \geq K}\right] =$$

$$= K\,B(t,T)\,\mathbb{E}_t^{\mathbb{Q}^T}\left[\mathbb{1}_{S_T \geq K}\right] =$$

$$= K B(t,T)\, \mathbb{Q}^T_t (S_T \geq K)$$

we should know the distribution for r.

For the first term is not more convenient to work with $\mathbb{Q}^T$. We can change the numeraire. We could use S as numeraire $\Rightarrow \mathbb{Q}^S$

$$\mathbb{E}^{\mathbb{Q}}_t \left[e^{-\int_t^T r_s ds} S_T \mathbb{1}_{S_T \geq K} \right] =$$

$$= S_t\, \mathbb{E}^{\mathbb{Q}^S}_t \left[\mathbb{1}_{S_T \geq K} \right] = S_t \mathbb{Q}^S_t (S_T \geq K)$$

$$\boxed{Call_t = S_t \mathbb{Q}^S_t (S_T \geq K) - K B(t,T) \mathbb{Q}^T_t (S_T \geq K)}$$

without modeling interest rate we cannot get a better form. We need to do some assumptions.

↓

We suppose $Vol\left(\frac{S_t}{B(t,T)} := z\right)$ $=: \sigma_t^z$ to be deterministic.

z_t is martingale under $\mathbb{Q}^T$ (because $B(t,T)$ is numeraire)

$$\frac{dz}{z} = \cancel{0\, dt} + \sigma_t^z \, dW^{\mathbb{Q}^T}$$

$$Z_T = Z_0 \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_s^z\|^2 ds + \underbrace{\int_0^T \sigma_s^z dW_s^{\mathbb{Q}^T}}_{N(0, \int_0^T \|\sigma\|^2 ds)}\right\}$$

We know the distribution of Z under $\mathbb{Q}^T$:

$$Z_T = \frac{S_0}{B(0,T)} \exp\left\{ N\left(-\frac{1}{2}\int_0^T \|\sigma_s^z\|^2 ds, \int_0^T \|\sigma_s^z\|^2 ds\right)\right\}$$

$$Call_t = S_t \mathbb{Q}_t^S(S_T \geq K) - B(t,T) K \mathbb{Q}_t^T(S_T \geq K)$$

we now know the last term:

$$\mathbb{Q}_t^T(S_T \geq K) = \mathbb{Q}_t^T\left(\frac{S_T}{B(T,T)} \geq K\right) =$$

$$= \mathbb{Q}_t^T(Z_T \geq K) = \Phi(d_2)$$

where:

$$d_2 = \frac{\ln\left(S_t / B(t,T) K\right) - \frac{1}{2}\int_t^T \|\sigma_s^z\|^2 ds}{\sqrt{\int_t^T \|\sigma_z^s\|^2 ds}}$$

And for the first term? We need to compute $\mathbb{Q}_t^S(S_T \geq K)$. To do so we need the numeraire at the denominator

$$\mathbb{Q}_t^S(S_T \geq K) = \mathbb{Q}_t^S\left(\frac{1}{S_T} \leq \frac{1}{K}\right) =$$

$$= \mathbb{Q}_t^S\left(\frac{B(T,T)}{S_T} \leq \frac{1}{K}\right) =$$

$$= \mathbb{Q}_t^S\left(\frac{1}{Z_T} \leq \frac{1}{K}\right)$$

Z is martingale under $\mathbb{Q}^T$ but $1/Z_T$ is a martingale under $\mathbb{Q}^S$.
We use Itô to find the dynamics of $1/Z$.

$$\frac{d\left(\frac{1}{Z}\right)}{1/Z} = \cancel{0\,dt} - \sigma_Z^t \, dW^{\mathbb{Q}^S}$$

$$\frac{1}{Z_T} = \frac{1}{Z_0} \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_Z^s\|^2 ds - \int_0^T \sigma_Z^s \, dW_s^{\mathbb{Q}^S}\right\}$$

$$= \frac{B(0,T)}{S_0} \exp\left\{N\left(-\frac{1}{2}\int_0^T \|\sigma\|^2 ds, \int_0^T \|\sigma\|^2 ds\right\}\right.$$

$$\mathbb{Q}_t^S\left(\frac{1}{Z_T} \leq \frac{1}{K}\right) = \Phi(d_1)$$

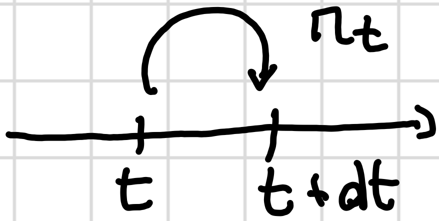
where: $d_1 = d_2 + \sqrt{\int_t^T \|\sigma_s^Z\|^2 ds}$

If $\sigma_t = Vol(S_t / B(t,T))$, then:

$$\boxed{Call_t = S_t \Phi(d_1) - B(t,T) K \Phi(d_2)}$$

This is independent with respect to interest rate dynamics.

EXAMPLE: (VASIĆEK INTEREST RATE '97)



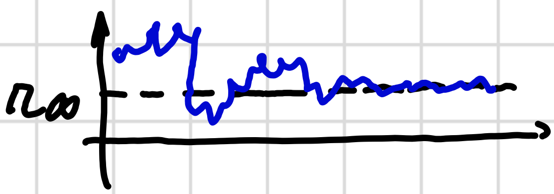
Evolution of short interest rate.
r_t has a MEAN REVERSION dynamic

$$dr_t = (a - b r_t)\,dt + \sigma_r\, dW^Q \qquad a, b, \sigma > 0$$

$$= b\left(\frac{a}{b} - r_t\right)dt + \sigma_r\, dW^Q$$

[ORNSTAIN UHLEMBECK EQUATION

$\frac{a}{b}$ ↳ long term interest rate (FED/BCE target)



(can also be negative!)

b → strenght of mean reversion

To solve this we use an integrating factor

$$d(r_t e^{bt}) = e^{bt}\,dr + r b e^{bt}\,dt =$$

$$= e^{bt}\big((a-br)\,dt + \sigma_r\,dW\big) + r b e^{bt}\,dt$$

$$= e^{bt}\,dt + \sigma_r e^{bt}\,dW$$

$$r_t e^{bt} - r_0 = \frac{1}{b}(e^{bt}-1) + \sigma_r \underbrace{\int_0^t e^{bs}\,dW_s}_{N\left(0,\ \int_0^t e^{2bs}\,ds\right)}$$

$$\Downarrow$$

$$\underbrace{r_t = r_0 e^{-bt} + \frac{1}{b}(1-e^{-bt})}_{N\left(\text{average},\right.} + \underbrace{\sigma_r \int_0^t e^{b(s-t)}\,ds}_{\left.\sigma_r^2 \int_0^t e^{2b(s-t)}\,ds\right)}$$

$\mathbb{Q}(r_t < 0) > 0$ there is always possibility to get negative interest rates.