Report 4

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Contents

1 Introduction

The dynamics of asset prices has a large influence in the calculations of various financial mathematical instruments. One of the principal approaches to study these dynamics is the Black Scholes model, this model is based on the assumption that the trajectory of the price of an option follows a Geometric Brownian Motion. This stochastic process returns an estimate of the final price of the option as well as an estimate of the volatility and drift of the asset.

However, for a large class of derivatives, especially those that are path-dependent, such as Asian, lookback, or barrier options, the BS model does not constitute an accurate approach. In such cases, it is optimal to implement methods such as Monte Carlo (MC) simulation. These techniques allows to numerically simulate the possible evolution of asset prices and compute expected payoffs under the risk-neutral measure.

The following report explores various applications of Monte Carlo simulations in the context of financial derivative pricing.

2 Monte Carlo Trajectories Simulation

As mentioned, asset pricing is often done by appling the Black Scholes model. The related GBM generic formulation can be summarized by the following stochastic differential equation, and its analytical solution:

$$dS_t = rS_t dt \; + \; \sigma S_t dW_t \qquad\qquad S_t = S_0 \cdot e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \tag{1}$$

Having: S_t as the asset price at time t, r as the risk free interest rate, σ as the volatility of the asset and dW_t as the Wiener process infinitesimal increment.

Geometric Brownian Motion pricing can be interpreted by imagining that the value of the underlying varies randomly over time. This variation which is governed by the stochastic equation of Eq:[(1)] is often called random walk. It is our interest to actively visualize these ”random walks”. This is done by calculating the price of the option for many discrete steps using the discrete time version of the precedent formula.

$$S_{t+\Delta t} = S_t \cdot e^{\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t} Z} \tag{2}$$

Where $Z \sim N(0, 1)$ is a normal random variable.

These equations are thus implemented in a VBA script to simulate a total of 100 GBM paths. The parameters used for the option price are:

S_0	σ	r	T (Years)	dt (Years)
100	20%	1%	1	1/252

Table 1: Model parameters used for the simulations

In particular, we can see that the time steps are the fraction of active market days in years.

By running the macro we obtain a clear visualization of the trajectories of the prices, as expected we experience a random walks behaviour.

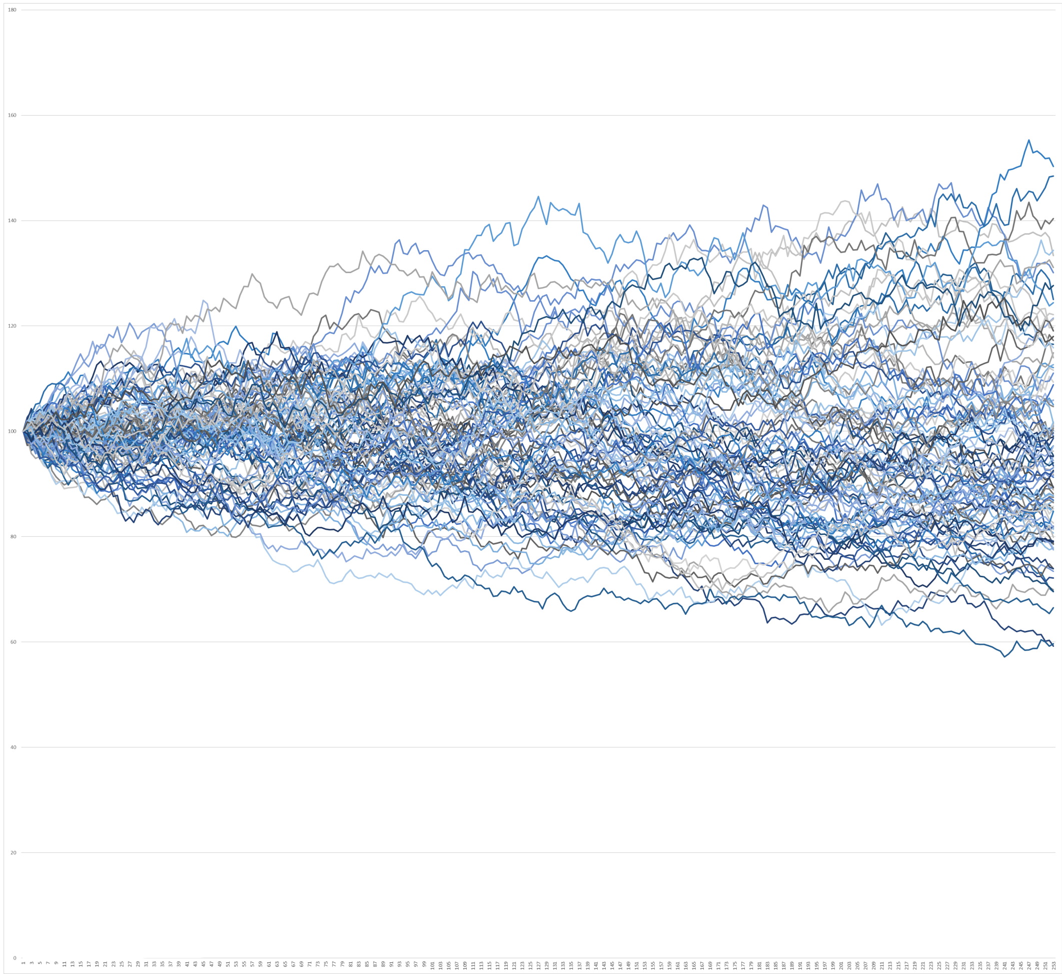

Figure 1: Simulated trajectories for 100 prices in the Black Scholes framework

3 Monte Carlo Pricing of Vanilla Options

Our main interest is to obtain the final result for the payoff of an option. This calculation is done by applying the previously mentioned formula in Eq:(1) using the parameters of Tab:[1] to find the asset price at maturity and apply the Black Scholes formula.

The focus on this analysis is that it is not necessary to perform the computationally expensive operation previously done, and it is sufficient to do a one step simulation. The reason why this is possible is that the asset price at maturity is normally logarithmically distributed. Specifically, the Z term in the exponent of the GBM equation Eq:(1) (which represents a random shock) follows a normal distribution, making the price at maturity log normal. The log normality of the price at maturity implies a strong influence of the price on the risk-free interest rate and the volatility of the asset. This implies that the price at maturity is path-independent and can be calculated directly because the distribution of S_T is known and determined by the above factors. Therefore, we do not need to simulate the entire asset price path. Instead, we only need to simulate a single random variable for S_T, and use that to compute the payoff. In conclusion, this allows to directly apply the well known Black Scholes formula for a discounted payoff.

$$payoff_{Call} = max(S_T - K, 0) \qquad payoff_{Put} = max(K - S_T, 0) \tag{3}$$

For the case being we will use as a strike $K = 99$.

We are now able to calculate the prices of the options using Monte Carlo simulations. To do so we simulate N independent values of S_T and compute the average payoff, which will get discounted to the present value. The price can be written as:

$$Price_{Call} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} max(S_T^{(i)} - K, 0) \qquad Price_{Put} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} max(K - S_T^{(i)}, 0) \tag{4}$$

Where T is the time to maturity.

By simulating many possible outcomes for the asset price at maturity, Monte Carlo methods allow for the calculation of expected payoffs. This provides an approximate but accurate solution to the pricing problem for vanilla options. The results obtained for the price are summarized in the following table.

Call, Price	Put, Price
8,637	6,917

Table 2: Price of the Vanilla Options Using Monte Carlo simulation

4 Monte Carlo Pricing Using Euler Scheme

We have priced vanilla options using Monte Carlo simulation doing a one step calculation for the price of the asset. In practice, many financial derivatives have payoffs that depend on the entire path of the underlying asset. Simulating the whole path becomes therefore an important part of the calculations.

To perform these analysis we will implement the Euler method that allows to simulate the entire path of the asset. This numerical method will provide an alternative to the closed form Black Scholes solutions, especially when that solution is not available, as for non constant volatility or interest rates.

The main idea behind the Euler method stands in the discretization of Eq:(1) similarly to what was showcased in Eq:(2):

$$S_{t+\Delta t} \simeq S_t \cdot \left(1 + r\Delta t + \sigma\sqrt{\Delta t} Z\right) \tag{5}$$

By means of this discretization is possible to simulate each intermediate step up to maturity in the time interval $[0, T]$.

Once the path is simulated and the final value is obtained it is possible to compute the payoff and calculate the price of the options via the discounting, exactly as done in Eq:(3) and Eq:(4).

The results obtained will turn out to be less accurate due to the discretization, but more flexible and suitable to multiple situations. The Prices of the options are:

Call, Price	Put, Price
8,955	6,922

Table 3: Price of the Vanilla Options With Euler Scheme Approximation

We can now summarize the analysis done by directly comparing the results with the prediction of the Black Scholes formula.

B-S Call	B-S Put	Monte Carlo Call	Monte Carlo Put	Euler Call	Euler Put
8,918	6,933	8,637	6,917	8,955	6,922

Table 4: Pricing of the Vanilla Options

As expected all the results are coherent with the Black Scholes formula prediction. Interestingly, the Euler based Monte Carlo simulation yields option prices slightly closer to the Black-Scholes benchmark compared to the one-step simulation. However, this should not be interpreted as the Euler method being more accurate, effect such as random sampling effects might be the main cause of these accuracies. Nonetheless, the key aspect of these results is the confirmation that both methods are accurate and consititute valid alternatives for pricing vanilla options.

4.1 Monte Carlo Pricing of Asian Options

Asian options are a type of options, whose pricing depends on the average price over time. These options are particularly useful in real markets where prices can be volatile and subject to manipulation near maturity. The averaging of the price makes the option less sensitive to fluctuations granting a lower risk.

Since Asian Options are path dependent it is necessary to simulate the full trajectory of the asset using the Euler scheme. The payoff of an Asian option can be written in its discrete form as:

$$Payoff_{Asian} = max\left(\frac{1}{M}\sum_{i=1}^{M} S_t - K, 0\right) \tag{6}$$

Here $\frac{1}{M}\sum_{i=1}^{M} S_t$ represent the discrete average over M steps.

This is implemented in a loop that simulates at each step S_t, using the Euler scheme of Eq:(5). Ultimately, after simulating the paths and computed the discrete average, is possible to calculate the discounted payoff of the Asian Call and Put options:

$$Price_{Call} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} max(\frac{1}{M} \sum_{i=1}^{M} S_t^{(i)} - K, 0) \qquad Price_{Put} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} max(K - \frac{1}{M} \sum_{i=1}^{M} S_t^{(i)}, 0) \tag{7}$$

What showcased is a clear application of how a Monte Carlo methods with path discretization represent an important tool to effectively price options for which no closed form solution exists. So, even if the Euler based simulation might result not optimal, their relevance in pricing and their flexibility, set the base for multiple applications.

The pricing of the Asian Call and Put options for the initial parameter indicated in Tab:[1] are:

Asian, Call	Asian, Put
5,369	3,755

Table 5: Asian Option Pricing, Using Monte Carlo Simulation

As expected, the prices of these options are lower than those of the Vanilla option, Tab:[4]. This difference is due to the averaging effect in the Asian option's payoff, which reduces the impact of extreme fluctuations in the underlying asset. Averaging smoothens the volatility, reducing the risk but also limiting the potential for high payoffs. In the end, the lower price reflects the reduced risk/reward profile which is an intrinsic property of the Asian options.

4.2 Monte Carlo Pricing of Lookback Options

As for Asian option, Lookback options are path dependent derivatives, and their payoff depends on the entire path of the underlying asset. The formulation of the payoff is far more complex than the one of vanilla options, as it depends on a floating strike:

$$Payoff_{Call} = S_T - min_{0 \leq t \leq T}\ S_t \qquad Payoff_{Put} = max_{0 \leq t \leq T}\ S_t - S_T \tag{8}$$

As we can see, the payoff of these options depends on the path of the price of the underlying, in the sense that it is possible to choose the favorable strike between the available ones. Basically these options allows the holder to "look back" over the life of the option to choose the lowest or highest underlying price, that will constitute the strike. This valuable feature often results in an higher option price compared to vanilla options.

The pricing of the lookback options is done by means of the Euler discretization of the GBM over multiple steps. By keeping track, during the simulation, of the minimum and maximum value of the asset, it was possible to compute the corresponding payoff. Once discounted to the present value it was finally possible to average over all simulation to obtain the expected price.

For the initial parameters used in Table [1], we obtain the following prices:

Lookback, Call	Lookback, Put
14,865	15,695

Table 6: Lookback Option Pricing Using Monte Carlo Simulation

As expected these values are significantly higher than their vanilla counterparts. This is due to the optimization of the strike which translates to additional benefits, hence, more valuable options.

5 Pricing a Certificate on a Worst-Of Option Written on 3 Assets

A Worst-Of Option is a type of multi-asset derivative where the payoff depends on the worst performing asset among a set of two or more underlyings over a specified period. These are usually embedded in certificates sold by banks and are a complex kind of derivative. Assuming that the option is written on three stocks S_1, S_2, S_3, the payoff is based on the behaviour of the worst performing asset.

$$Worst_T = \min\left(S_{1,T}, S_{2,T}, S_{3,T}\right) \tag{9}$$

$$Payoff = \max\left(\text{Worst}_T - K, 0\right) \tag{10}$$

To be clear, this is not the exact formulation for the payoff of the certificate considered, but a schematic expression indicating the general behaviour. In practice, what happens is that the holder of the derivative receives a return only if the worst performing asset ends up above a certain threshold (typically a percentage of the initial level), otherwise the capital protection is lost and the payoff is reduced accordingly.

The price of the option today is obtained by discounting the expected value of the payoff:

$$Price = e^{-rT} \cdot \mathbb{E}^{\mathbb{Q}}\left[max(min(S_{1,T}, S_{2,T}, S_{3,T}) - K, 0)\right] \tag{11}$$

Where $\mathbb{E}^{\mathbb{Q}}[\cdot]$ represent the expected value under the risk neutral measure $\mathbb{Q}$, which can be estimated as the average over multiple Monte Carlo iterations.

A key peculiarity of the underlying assets is that they belong to the same market sector or are influenced by similar macroeconomic factors, which means they are not statistically independent. This implies that their Brownian motions are correlated.

It is therefore necessary to adjust the computations by introducing the correlation matrix ρ between the assets.

$$\rho_{i,j} = Corr(dW_i(t), dW_j(t)), \qquad \forall i \neq i \tag{12}$$

The correlation matrix is introduced to apply the Cholesky decomposition. This decomposition allows the generation of correlated standard normal variables. These variables (Z_i, for $i = 1, 2, 3$) are necessary to compute the step-by-step evolution of the prices as shown in Eq:[(2)]. Once the correlated prices are simulated, it becomes possible to proceed with the Monte Carlo evaluation, as previously done for simpler cases.

The Certificate considered is a **Cash Collect Worst Of** issued by UniCredit (ISIN: IT0005640815). It is structured on a basket of three underlyings: Coca Cola Co, Amazon.com Inc., and Netflix Inc., with a fixed maturity of 4 years (from 24.04.2025 to 24.04.2029).

We first proceed by downloading the data for these assets and by calculating the correlation matrix and the annual volatility of the asset. The relevant data is resumed in the following table:

Asset	S_0	σ_{annual}	ρ_{AMZN}	ρ_{KO}	ρ_{NFLX}
AMZN	180,6	0,335	1	-0,121	0,534
KO	73,3	0,165	-0,121	1	-0,002
NFLX	1049,6	0,321	0,534	-0,002	1

Table 7: Properties of the Assets: AMZN, KO, NFLX

The interest rate for these option is $r = 4,30\%$ while the time to maturity is $T = 4$ Years. The results obtained for the price of this Certificate of Worst of is:

Simulation	Worst-Of Call Price (VBA Result)	Unicredit Pricing
10000 paths	€618,55	€961,08

Table 8: Pricing of Worst-Of Certificate via Monte Carlo

It is evident how the result obtained is clearly wrong and the analysis just proposed suffers from an underpricing of the value. The reason behind this inaccuracy is that the certificate considered is a Conditional Coupon paying certificate. The certificate pays each three months a Conditional Coupon of €23,50 if, on each observation date, all three underlyings are above 60% of their initial level. On top of this even if some coupons are skipped, if the condition is satisfied in a future quarter, all missed coupons are paid retroactively. In coclusion, this specific certificate works by the following logic: at maturity, the capital is returned if all underlyings are above 60% of their initial values. Otherwise, the investor suffers a loss proportionally to the worst-performing asset.

By adjusting the VBA script by taking into account these specific properties the result obtained for the price becomes:

Simulation	Worst-Of Call Price (VBA Result)	Unicredit Pricing
10000 paths	€934,03	€961,08

Table 9: Specific Pricing of Worst-Of Certificate via Monte Carlo

The result is significantly improved, almost exactly reproducing the pricing proposed by Unicredit, with an error of $\approx$ 2,81%. Given the complexity of the product studied and the use of an approximate Monte Carlo simulation, the error is reasonably small and the result are satisfactory. Increasing the number of simulations could potentially improve the results, reducing this error.

6 Conclusion

We have been able to apply the Monte Carlo approach across multiple practical situations, exploring its flexibility and adaptability to different types of derivatives. The results obtained confirmed our expectations and helped us understand, from a practical perspective, exotic options such as Asian and Lookback options.

In conclusion, the study of the Worst-Of certificate has opened the way to analyzing more complex financial products, involving multiple parameters and dependencies. The combination of theoretical understanding and practical implementation in this report has enhanced our ability to approach option pricing using new tools, which grants higher adaptability to different context and requests.