Steps taken so far with PNLSS modeling

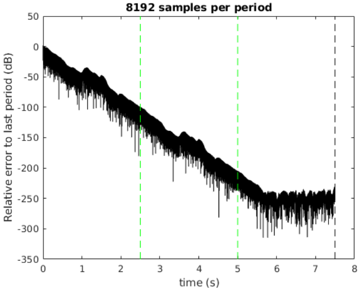
Generate data
Consistency
Nonparametric analysis
Parametric modeling

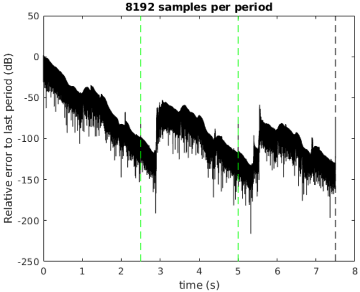
Avoid variable time step solvers

To ensure periodicity, a fixed time step integrator is used

- Depending on forcing level, `ode45` at times gives nonperiodic output.
- Ex: low A; nonperiodic / high A; periodic guess: Low A allows for large step $\rightarrow$ interpolation destroys periodicity.
- Why RK5? Why not RK8/9/...?
  - Higher order methods requires more function evaluations per step.
  - Lower order have higher rounding errors.
  - In celestial mechanics(oscillating over long time), at least RK8 should be used. `https://doi.org/10.1007/BF00049361`
  - Instead of increasing fs, RK8 might be used (next slide)
- Alternative: 2-order Newmark-β; Not clear how to be used with benchmark4/5.

Example: Same system/forcing.
Upper: fixed step RK5, lower: ode45



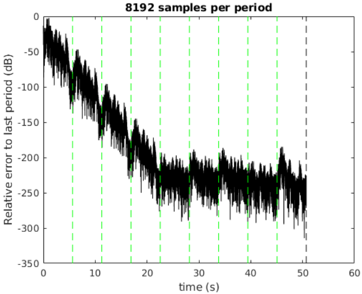


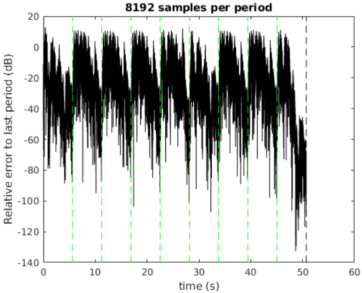
Increase sampling frequency

Example: Same system/forcing.
Upper: $fs_{\text{int}} = \text{upsamp} \cdot fs$

To ensure numerical correct integration of highly nonlinear systems, we use a higher "integration frequency", fx `upsamp=20`

- Select integration `fsint` as multiple of desired `fs`. $fs_{\text{int}} = \text{upsamp} \cdot fs$
- downsample forcing, `u=u(1:upsamp:end)`. Since the Nyquist frequency for the downsampled fs is still above f2, the last excited frequency.
- decimate output. The nonlinear system might generate higher harmonics in the output: y should be decimated; ie. first low-pass filtered and then downsampled to avoid aliasing.

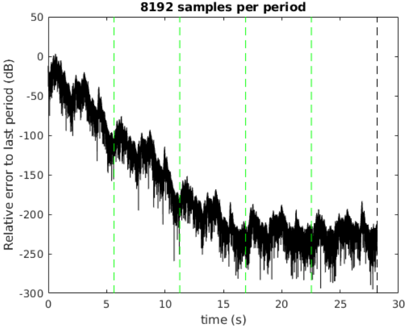


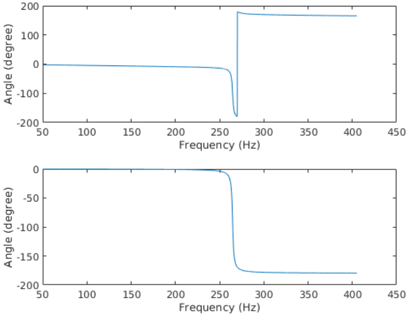


Use RK8 - maybe

RK8 with fixed time step.

- RK8 steady state is reached after 3 periods. High A.
- RK5 with upsamp: SS after 4 periods (previous slide). High A.
- Not clear why different
- But BLA(angle) is not estimated properly for RK8 data. Low A
  Upper: RK8; lower: RK5
- ... but estimated PNLSS models are similar.





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Discrete model

The EOM for the duffing oscillator in benchmark1 can be written as

$$\ddot{y} + 2\beta\omega_0\dot{y} + \omega_0^2 y + \gamma y^3 = q\cos(\Omega t) \tag{1}$$

Using the state vector $\mathbf{u} = [y\ \dot{y}]^T$, the duffing eq. is written in state space

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = u(t) - 2\beta\omega_0 x_2(t) - \omega_0^2 x_1 - \gamma x_1^3 \end{cases} \tag{2}$$

where $u(t) = q\cos(\Omega t)$. The continuous-time model is converted into discrete-time using a forward Euler discretization, ie. $\dot{x} = \frac{x(t+h) - x(t)}{h}$

$$\begin{cases} \mathbf{x}(t \dot{+} h) = \begin{bmatrix} 1 & h \\ -\omega_0^2 h & 1 - \beta\omega_2 h \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ h \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ -\gamma h \end{bmatrix} x_2^3(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases} \tag{3}$$

Compare this to the general PNLSS form

$$\begin{cases} \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{E}\mathbf{g}(x, u) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) + \mathbf{F}\mathbf{h}(x, u) \end{cases} \tag{4}$$

This is only a good approximation for small h due to the simplicity of the differentiation rule.

Approaches for comparing the linear part of PNLSS(A,B,C,D)

PNLSS identify discrete time state space matrices. $\mathbf{A}_k$ and $\mathbf{B}_k$(discrete time) is converted to continuous time (no conversion needed for C and D). We assume a correct identified nonlinear model is needed for the linear parameters to be correct.

- SS models are not unique. Use a similarity transform $\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}$ (linear change of state variable coordinates) to get physical coordinates $\mathbf{x}$.[1]

$$\mathbf{A} = \mathbf{T}\hat{\mathbf{A}}\,\mathbf{T}^{-1}, \quad \mathbf{B} = \mathbf{T}\hat{\mathbf{B}}, \quad \mathbf{C} = \hat{\mathbf{C}}\mathbf{T}^{-1}, \quad \mathbf{D} = \hat{\mathbf{D}}, \quad \mathbf{T} = \begin{bmatrix} \hat{\mathbf{C}} \\ \hat{\mathbf{C}}\hat{\mathbf{A}} \end{bmatrix} \tag{5}$$

- Compare invariant parameters.
  - The transfer function $\mathbf{G}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$.
  - or eigenvalues of $\mathbf{A}$(poles)

[1] Etienne Gourc, J.P. Noël, et.al: Obtaining Nonlinear Frequency Responses from Broadband Testing
https://orbi.uliege.be/bitstream/2268/190671/1/294_gou.pdf

Compare nonlinear model with true system.

- Compute RMS error of the PNLSS simulated signal compared with the 'true' simulated system.
- Simulate the PNLSS model with NLvib; obtaining a frequency response directly comparable
- We are not aware of any way to compare the identified E/F matrices(coefficients of the nonlinear state-dependent polynomials) with the true system.

general PNLSS settings around 1 mode

Ensure that there is no modal interaction by setting the multisine amplitude lower than the level where we expect tongues and ensuring high enough damping of higher modes. For benchmark 1-3 (polynomial nl):

```
n = 2;
whichtermsx = 'statesonly';
whichtermsy = 'empty';
nx = [3]      % benchmark 1/2
nx = [2,3]    % benchmark 3
```

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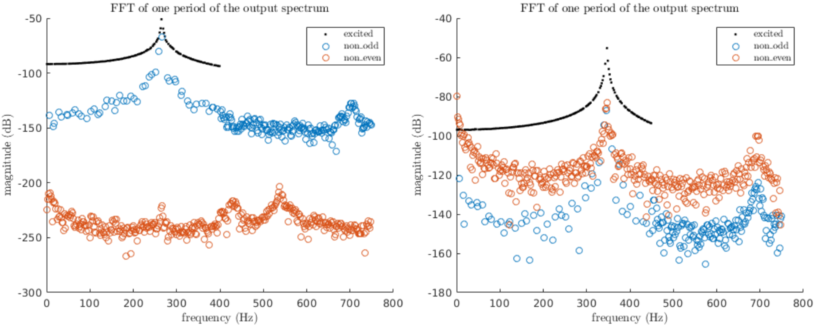
Characterization of nonlinearity

Use random-odd multisine. Only odd frequency lines are excited. For each group of four lines, one is randomly set to zero.

$$u(t) = U \sum_{n=1}^{N} A_n \sin(2\pi n f_0 t + \phi) / \sqrt{\sum_n A_n} \tag{6}$$

Normalized to ensure constant power as the number of included harmonics N is varied. A_n is a boolean, f_0 frequency resolution(or fundamental frequency). Number of periods is chosen by end time $t_2 = P/f_0$.

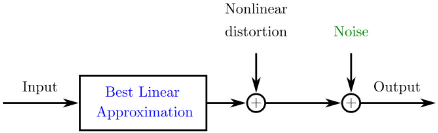
A general conclusion: The only way to increase the freq. resolution is to increase the measurement time. Increasing the sampling frequency does not help.

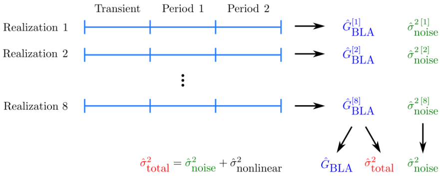


left: straight beam; right: curved beam

Estimating distortion levels

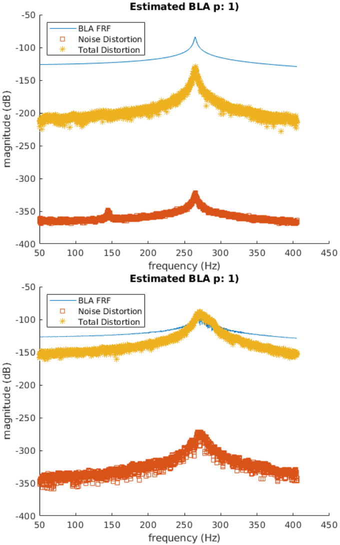
Use full multisine. Average over periods and realizations to estimate noise and nonlinear distortion → is NL-model necessary?





Increasing NL distortion with increasing A

upper: $A = 1$; lower $A = 30$



Note: dB for power: $L_p = 10\log_{10}(P)$. dB for field quantity: $L_F = 10\log_{10}(F^2)$. Thus when comparing distortion(variance, proportional to power) with $|G|$, do either (matlab pseudo-syntax) (Also remember to multiply distortion with number of realizations): `plot(freq,[db(abs(G)), db(var*R,'power')])` or `plot(freq,[db(abs(G)), db(sqrt(var*R))])`.

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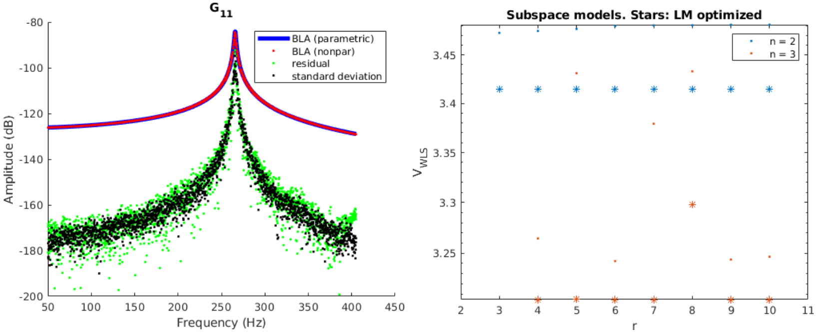
Subspace model

- Linear model of good quality
  Model error and standard deviation of BLA coincides
- Error seems to increase around 400 Hz.
  First thought: Maybe a second order discrete model is insufficient to model the continuous system. But third order shows same behavior

left: BLA and subspace($n = 2$); right: cost function for different subspace models

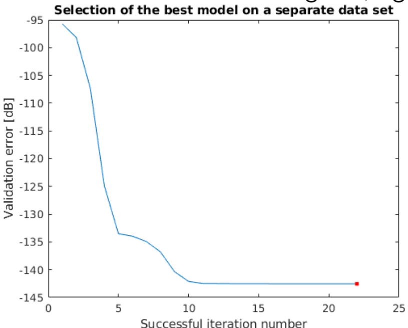


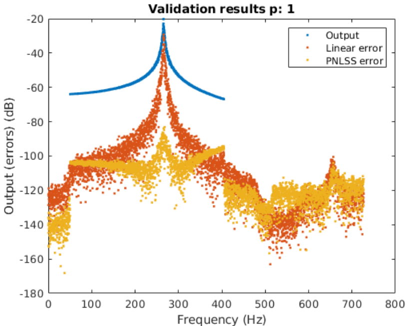
Full optimized model

- Linear error significant around resonance(larger amplitudes)
- PNLSS RMS error decreased 3 orders of magnitude
- Not much difference in ω_0 or ζ.
- For fully correct model, pnlss error should be equal to noise floor (no noise, ie. error should be as low as integration precision).
  Here: 80 dB $\rightarrow 10^{-4}$.

	ω_0 Hz	ζ %	RMS error (new data)
duffing	264.72	0.38	
linear	265.76	0.37	$1.64 \cdot 10^{-5}$
pnlss	264.72	0.38	$7.46 \cdot 10^{-8}$

left: convergence; right: Error on new data





Noise

- SNR?
- Type/color?

```
noise = 1e-3*std(y(:,end,end))*randn(size(y));
% Do some filtering
noise(1:end-1,:,:) = noise(1:end-1,:,:) + noise(2:end,:,:);
```