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A Hyper-Catalan Series Solution to Polynomial Equations, and the Geode

N. J. Wildberger and Dean Rubine

Abstract. The Catalan numbers C_m count the number of subdivisions of a polygon into m triangles, and it is well known that their generating series is a solution to a particular quadratic equation. Analogously, the hyper-Catalan numbers $C_{\mathbf{m}}$ count the number of subdivisions of a polygon into a given number of triangles, quadrilaterals, pentagons, etc. (its type $\mathbf{m}$), and we show that their generating series solves a polynomial equation of a particular *geometric* form. This solution is straightforwardly extended to solve the general univariate polynomial equation.

A layering of this series by numbers of faces yields a remarkable factorization that reveals the *Geode*, a mysterious array that appears to underlie Catalan numerics.

1. INTRODUCTION. Four millennia ago, the Babylonians could solve the system of equations $x + y = s,\ xy = p$, equivalent to a quadratic equation. Our modern quadratic formula involves a square root operation, which even in ancient times was known to generally not terminate, yielding only approximate solutions. In the sixteenth century, Scipione del Ferro, Niccolò Tartaglia, and Gerolamo Cardano gave a similar but more complicated expression for the solution of a cubic equation, involving both square and cube roots, and with Lodovico Ferrari found an even more complicated formula in radicals for the quartic equation.

After Joseph-Louis Lagrange's important theoretical advances bringing in nascent ideas of group theory, the Abel/Ruffini Theorem, expanded upon by Évariste Galois, showed that there is no solution in radicals to a general polynomial equation of degree five or higher. It is our impression that students of mathematics hence perceive a major division between the quartic and quintic cases, separating the merely complicated from the impossible. Further investigation into these matters is now generally only of a historical nature, as Galois theory has gone in different directions [1].

We would like to resurrect this topic and show that it has intimate connections to a remarkable combinatorial geometry, which allows us to dramatically reconsider what it really means to solve an algebraic equation, sidestepping the classical work on solutions in radicals and Galois theory.

After all, if we're permitted nested unending nth root calculations, why not a simpler ongoing sum that actually solves polynomials beyond degree four? Instead of needing to find a new solution for each degree, why not write one formula that solves all degrees?

Here is our geometric polynomial formula, Theorem 6. The equation:

$$0 = 1 - \alpha + t_2\alpha^2 + t_3\alpha^3 + t_4\alpha^4 + t_5\alpha^5 + \ \ldots$$

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has a formal power series solution

$$\alpha = \mathbf{S}[t_2, t_3, \ldots] \equiv \sum_{m_2, m_3, \cdots \geq 0} \frac{(2m_2 + 3m_3 + 4m_4 + \ldots)!}{(1 + m_2 + 2m_3 + \ldots)!\, m_2!\, m_3! \cdots} t_2^{m_2} t_3^{m_3} \cdots .$$

Arthur Erdélyi and Ivor Etherington [2,3] and others found the above coefficient as the hyper-Catalan number $C_{\mathbf{m}}$, which we re-express in terms of the number of vertices and edges of a subdivided polygon of type $\mathbf{m}$ in Section 7. In Section 8 we use $\mathbf{S}$ to approximate a root of John Wallis's famous cubic equation, employing a natural bootstrapping method.

A power series solution to a polynomial equation is not a new idea. In 1844, Gotthold Eisenstein found such a solution to $x^5 + x - t = 0$, the simplest polynomial known not to have a general zero expressible in radicals [4, 5]. We write down the general quintic solution and use it to recover Eisenstein's series for the Bring radical in Section 9, and discuss the natural connections with series reversion, also going back to Lagrange, in Section 10.

The central algebraic object that we wish to emphasize is the hyper-Catalan generating series $\mathbf{S}[t_2, t_3, t_4, \ldots]$ in an unbounded number of variables (see Section 5). This is connected with the Online Encyclopedia of Integer Sequences (OEIS) [6] in many interesting ways (see Sections 11 and 12). A curious factorization involving $\mathbf{S}$ reveals the mysterious *Geode array* which algebraically encodes the hyper-Catalan numbers. In Section 11, we offer several compelling conjectures about this remarkable new algebraic object.

2. HISTORY.

The literature on solving polynomials is vast. Here we touch on developments that led to the series solution we present, including combinatorial, algebraic, and analytic histories covering series reversion and hyper-Catalan objects.

In 1693, Gottfried Wilhelm Leibniz used the method of undetermined coefficients of a power series to solve differential equations, and in 1712, Isaac Newton showed how to revert series by that method [1,7,8]. (In 2021, the first author began his exploration of polynomial solutions in much the same way [9], greatly aided by advances in symbolic computation [10–12].) In 1770, Lagrange gave us a general solution to series reversions in the form of the Lagrange Inversion Theorem [13–15]. The traditional results of Lagrange, Niels Henrik Abel, Paolo Ruffini, and Galois take us to the 1830's.

The combinatorial path [16,17] begins in 1751 [18] with Leonhard Euler and Christian Goldbach counting triangulations of polygons, continued by Johann Andreas Segner in 1758. In 1771 Nicolas Fuss, prompted by Johann Friedrich Pfaff, generalized the problem to polygon dissections into d-gons, which give the first slices of the hyper-Catalan array beyond the Catalan numbers. In the 1830's, Gabriel Lamé, Eugéne Charles Catalan, Jacques Philippe Marie Binet, Johann August Grunert, and Olinde Rodrigues continued the research, with Catalan publishing the modern formula in 1838, and Binet using Lagrange Inversion to obtain the formula for the two-parameter Fuss numbers in 1843. In 1870, Ernst Schroeder produced the little Schroeder numbers, perhaps the first one-dimensional sum over the entire hyper-Catalan array, coefficients of $\mathbf{S}[v, v^2, v^3, \ldots]$. In 1890, Arthur Cayley [19] considered dissections of a polygon, publishing an array that is a 2D projection of the entire hyper-Catalan array (summed over vertices and faces, $\mathbf{S}[vf, v^2 f, v^3 f, \ldots]$).

The algebraic path begins in 1855, when Joseph B. Mott solves a general polynomial equation using undetermined coefficients in his remarkable self-published book [20] with the subtitle "... and a New Discovery of One General Root Theorem for the Solution of Equations of All Degrees ...," and in an 1882 journal article [21], writing 29 terms of **S** that he uses to approximate roots. In 1894, James McMahon [22] uses Lagrange series reversion to obtain a general form for the hyper-Catalan coefficient $C_{\mathbf{m}}$, which he exasperatingly doesn't quite write all together despite the title of his short note. This is clarified in 1910 by Charles Edwin Van Orstrand [23] and absorbed into the combinatorics literature by John Riordan in 1968 [15].

The analytic path perhaps begins in 1907 with Alfredo Capelli [24, 25], who writes expressions containing a hyper-Catalan coefficient while studying the behavior of roots when polynomial coefficients are perturbed. Research is continued in 1920 by Richard Birkeland [26], in 1921 by Giuseppe Belardinelli [27] and Hjalmar Mellin [28], and in 1936 by Karl Mayr [29], among others (and nicely summarized by Belardinelli in 1960 [30]). Citing this work in 1940, Erdélyi and Etherington [2, 3] count polygonal subdivisions of a given type, and appear to be the first to give a combinatorial interpretation of the hyper-Catalan coefficient. In 1978, Arthur D. Sands [31] names what we call the hyper-Catalan numbers the 'generalized Catalan numbers,' repeated in 1987 by Wenchang Chu [32]. In 1992, later on the analytic path, Günter Lettl [33] comes very close to one of our formulas.

The combinatorial story continues with two articles that seem unaware of the work of Erdélyi and Etherington and each other. In 1960, George Raney [34] counts types of well-formed expressions of compositions of functions of arbitrary arity, producing an array slightly more general than the hyper-Catalan numbers (as unary functions exist but 2-gons do not). In 1964, William T. Tutte [35] counts plane trees of a given type, producing the same array, notably recognizing the connection of the Euler characteristic involving vertices, edges and faces. This is followed up in 1975 by Ihor Zinovie Chorneyko and S. G. Mohanty [36], in 1979 by Dana S. Richards [37], and appears in the 1983 textbook by Ian Goulden and David Jackson [38].

In 1972, Germain Kreweras [39, 40] counts non-crossing partitions of a cyclic graph, independently deriving the more general array. The hyper-Catalan numbers appear when he considers non-crossing partitions without singletons. He reproduces Cayley's 1890 array, describing it as, "the coefficients (or more correctly sums of the coefficients of the terms of the same 'weight') of the expressions giving [a series reversion]," citing Cayley [19], Raney, 1960 [34], and Riordan, 1968 [15]. Rodica Simion [41] offered a survey of the resulting non-crossing partition research in 2000; see also Richard P. Stanley, 1999 [42] and Drew Armstrong, 2006 [43].

Of the literature reviewed, Mott [20, 21] and Lettl [33] come closest to our results, with the series reversions discussed in Section 10 not far behind. In the category of papers that initially appear relevant but are less so upon closer examination, we list [44–46]; there are many others.

We now present our solution, beginning with the quadratic case.

3. CATALAN NUMBERS AND QUADRATIC EQUATIONS.

The Catalan numbers were introduced by Euler [18] in 1751 to count subdivisions into n triangles of a fixed planar convex $(n+2)$-gon, for a natural number n, with the convention that $C_0 = 1$. In 1838, Catalan [16] obtained the familiar modern formula:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}. \tag{1}$$

We call this expression **sub-multinomial**, as the sum of the factorial inputs in the denominator is one more than that of the numerator.

The Catalan sequence A000108: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, $\cdots$ has the longest entry in the OEIS [6], and is the subject of three relatively recent books [47–49], as well as numerous research articles, for example [41, 43, 50–52]. Figure 1 illustrates $C_4 = 14$, the 14 ways to subdivide a roofed hexagon into four triangles.

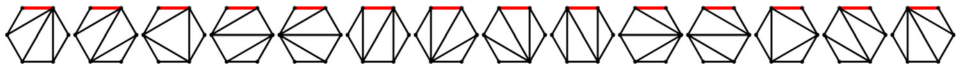

Figure 1. $C_4 = 14$ ways to subdivide a roofed hexagon into four triangles.

To see how the Catalan numbers relate to quadratic equations, consider

$$1 - \alpha + t\alpha^2 = 0 \tag{2}$$

which according to the usual quadratic formula has the solution in radicals:

$$\alpha = \frac{1}{2t}(1 \pm \sqrt{1-4t}).$$

Applying Newton's binomial expansion to the minus sign solution [42, 53] reveals the generating series of the Catalan numbers as the formal power series solution:

$$\alpha = \sum_{n\geq 0} \frac{(2n)!}{n!\,(n+1)!}\, t^n = \sum_{n\geq 0} C_n t^n.$$

The result justifies the lovely identity:

$$(1 + t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + \cdots)^2 = 1 + 2t + 5t^2 + 14t^3 + 42t^4 + \cdots .$$

An instructive argument for why this particular α works is suggested by the insightful Exercise 7.22 of *Concrete Mathematics* [54], which Ronald Graham, Donald Knuth, and Oren Patashnik further attribute to George Pólya's *On Picture Writing* [55]. We describe that now.

4. MULTISET ALGEBRA WITH TRIAGONS.

Define a **triagon** to be a planar convex polygon with a distinguished side, referred to as the **roof**, with the polygon subdivided into triangular faces by non-crossing diagonals. We regard this as a combinatorial object, so the orientation and indeed the shape of the triagon is secondary to the relative position of its roof and its various triangular faces. The vertical bar | denotes the **null triagon**, a degenerate polygon with two vertices, no faces, and a single side, its roof.

For a natural number $n = 0, 1, 2, \ldots$ let $\mathcal{T}_n$ denote the multiset of triagons with exactly n faces. The size of $\mathcal{T}_n$ is clearly the Catalan number C_n.

Of course $\mathcal{T}_n$ could also be viewed as a set, but we want to work in the more flexible arena of multisets where repetitions are allowed and crucially where *addition* is a natural operation: just combine all the elements of the multisets together into a bigger multiset. This is the fundamental operation underpinning the usefulness of multisets for the foundations of arithmetic [56]. We denote multisets with square brackets [. . .], without commas, as order is unimportant [57].

We now introduce the unbounded **triagon multiset** $\mathcal{T}$, which is naturally layered by the number of faces n:

$$\mathcal{T} \equiv \sum_{n\geq 0} \mathcal{T}_n.$$

Following *Concrete Mathematics*, for triagons r_1 and r_2, we define $\overline{\nabla}(r_1, r_2)$ as the triagon formed by taking a new roofed triangle and adjoining r_1 by its roof to the left side of the new triangle and r_2 by its roof to the right side. We think about this as hinging together the roofs of r_1 and r_2, and erecting a new roof on the combination. Hence we refer to $\overline{\nabla}$ as the **panelling operator**. This binary operation is extended to multisets M_1 and M_2 of triagons in the obvious way:

$$\overline{\nabla}(M_1, M_2) \equiv \left[\ \overline{\nabla}\ (r_1, r_2) :\ r_1 \in M_1, r_2 \in M_2\ \right].$$

Since any triagon r is either the null triagon $|$ or of the form $\overline{\nabla}(r_1, r_2)$ for unique triagons r_1 and r_2, we deduce the multiset identity:

$$\mathcal{T} = \left[\ |\ \right] + \overline{\nabla}(\mathcal{T}, \mathcal{T}). \tag{3}$$

This is to be interpreted in a similar way to a formal power series identity: if we consistently truncate all multiset expressions up to some fixed level, some number of faces, then the resulting finite multiset identity holds.

We now map the algebra of multisets of triagons onto the usual algebra of polynomials, via the function ψ that maps a triagon r to its **accounting monomial** t^n, where n is the number of faces in the triangulation of r. As the null triagon contains no triangles, we have $\psi(|) = t^0 = 1$.

Since the $\overline{\nabla}$ panelling operator adds a triangle, we have:

$$\psi(\overline{\nabla}(r, s)) = t\ \psi(r)\psi(s).$$

Subsequently, we define Ψ on multisets of triagons as the sum of ψ applied to each element. For a multiset M of triagons, we have the polynomial $\Psi(M) \equiv \sum_{r \in M} \psi(r)$. Then by linearity, or respecting addition:

$$\Psi(\overline{\nabla}(M_1, M_2)) = t\ \Psi(M_1)\Psi(M_2).$$

Figure 2 illustrates the $\overline{\nabla}$ panelling operator on triagons and multisets of triagons, the Ψ map and its associated identity.

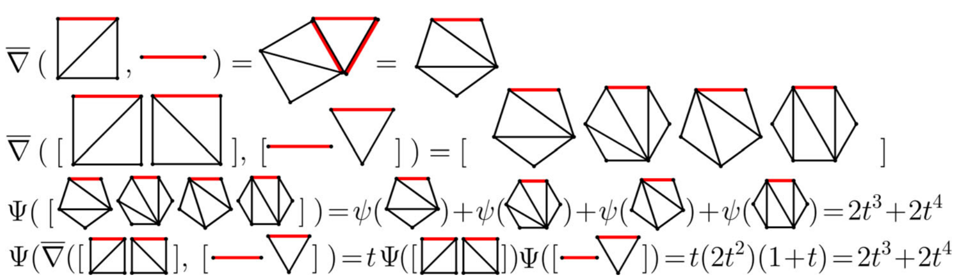

Figure 2. Panelling of triagons and their multisets, and the Ψ map.

We now extend Ψ to the triagon multiset $\mathcal{T}$ and introduce the **triagon polyseries**, which is the generating series for the Catalan numbers:

$$\mathbf{T} = \mathbf{T}[t] \equiv \Psi(\mathcal{T}) = \sum_{n \geq 0} \Psi(\mathcal{T}_n) = \sum_{n \geq 0} C_n t^n.$$

Applying Ψ to both sides of equation (3) we get $\Psi(\mathcal{T}) = \psi(|) + \Psi(\overline{\nabla}(\mathcal{T}, \mathcal{T}))$ or:

$$\mathbf{T} = 1 + t\mathbf{T}^2.$$

Let's agree that a polynomial with constant term 1 and linear coefficient -1 is in **geometric form**, so that $1 - \alpha + t\alpha^2$ is the **general geometric quadratic**. Then we have demonstrated the following result, independently of the quadratic formula, Newton's binomial series, and Catalan's formula.

Theorem 1 (The soft geometric quadratic formula). The equation $1 - \alpha + t\alpha^2 = 0$ has a formal power series solution:

$$\alpha = \mathbf{T}[t] = \sum_{n \geq 0} C_n t^n$$

where C_n counts the number of triagons with n faces.

We can now quickly deduce a more general formula involving variable coefficients; the result is a multivariate power series in c_0, c_1^{-1}, and c_2. Please note the minus sign on the linear term of the equation.

Theorem 2 (The soft quadratic formula). The equation $c_0 - c_1 x + c_2 x^2 = 0$ has a series solution:

$$x = \sum_{n \geq 0} C_n \, \frac{c_0^{1+n} c_2^n}{c_1^{1+2n}}.$$

Proof. If we set $x = \dfrac{c_0}{c_1}\alpha$ then the equation becomes $c_0 \left(1 - \alpha + \dfrac{c_0 c_2}{c_1^2}\alpha^2\right) = 0$. We apply Theorem 1 with $t = \dfrac{c_0 c_2}{c_1^2}$ to solve for α and then write down x. ∎

The formula, like the general case below, fails when $c_1 = 0$, meaning that e.g., $x^2 = 2$ is not handled by our series solution. It's curious that the equation with the easiest solution in radicals is the least accessible with this method. A simple change of variables lets us skirt around the problem.

5. SUBDIGONS AND HYPER-CATALAN NUMBERS.

Our aim is now to suitably extend the above argument to higher degrees. The primary generalization is to consider multisets of more generally subdivided roofed planar polygons, whose types are counted with an extension of the Catalan numbers called the hyper-Catalan numbers, and then to consider the algebra obtained by panelling these *subdigons* together to form larger ones.

We find the explicit formulas for the hyper-Catalan numbers in the combinatorics literature (see Section 7), with precursors going back to 19th century work on reversion of series (see Section 10). However, we wish to emphasize that the soft approach that algebraically solves the general equation is formulated completely independently of these explicit formulas.

Define a **subdigon** to be a planar convex roofed polygon s which is subdivided by non-crossing diagonals into polygonal faces. If s is subdivided into m_2 triangles, m_3 quadrilaterals, m_4 pentagons, and so on, then we say it is a subdigon of **type** $\mathbf{m} = [m_2, m_3, m_4, \ldots]$. Necessarily, there are only a finite number of nonzero m_k, and we agree that appending zeros does not change the type. The null subdigon | is a subdigon, with one side, its roof, two vertices, no faces and type $\mathbf{m} = [\ ]$, all $m_k = 0$.

Define the **hyper-Catalan number** $C_{\mathbf{m}} \equiv C[m_2, m_3, m_4, \ldots]$ to be the number of subdigons of type $\mathbf{m}$. For the special case when m_3 and up are all zeros, we recover the usual Catalan numbers $C[m] = C_m$. Figure 3 illustrates $C[2, 0, 1] = 28$.

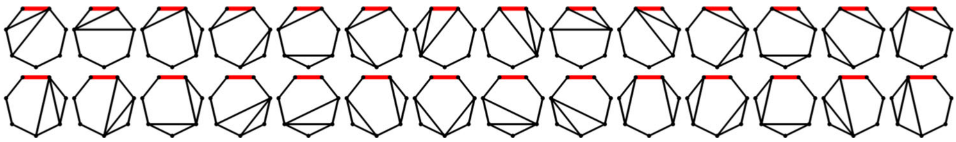

Figure 3. $C[2, 0, 1] = 28$ subdigons with two triangles and a pentagon.

The number of faces in a subdigon of type $\mathbf{m}$ is:

$$F = F_{\mathbf{m}} = m_2 + m_3 + m_4 + \ldots = \sum_{k\geq 2} m_k. \tag{4}$$

To count the number of edges, we note that the null subdigon $|$ consists of a single edge, and adjoining a triangle to a subdigon adds two edges, adjoining a quadrilateral adds three edges, and so on, therefore:

$$E = E_{\mathbf{m}} = 1 + 2m_2 + 3m_3 + 4m_4 + \ldots = 1 + \sum_{k\geq 2} k\, m_k. \tag{5}$$

The Euler polytope formula $V - E + F = 1$ (for polygons, we don't count the surrounding region as a face) then yields the number of vertices as:

$$V = V_{\mathbf{m}} = 2 + m_2 + 2m_3 + 3m_4 + \ldots = 2 + \sum_{k\geq 2} (k-1)m_k. \tag{6}$$

As a type contains only a finite number of nonzero m_k, we know V, E, and F are always finite natural numbers.

For a given type $\mathbf{m}$, we let $\mathcal{S}_{\mathbf{m}}$ denote the finite multiset of subdigons of type $\mathbf{m}$. Then we can introduce the unbounded **subdigon multiset** $\mathcal{S}$, layered by type:

$$\mathcal{S} \equiv \sum_{\mathbf{m}\geq\mathbf{0}} \mathcal{S}_{\mathbf{m}}.$$

Let ψ be the function which maps a subdigon s of type $\mathbf{m} = [m_2, m_3, m_4, \ldots]$ to its **accounting monomial**

$$\psi(s) \equiv t_2^{m_2} t_3^{m_3} t_4^{m_4} \cdots \equiv \mathbf{t}^{\mathbf{m}}$$

where as before the null subdigon has $\psi(|) = 1$.

For M, a multiset of subdigons, let $\Psi(M)$ be the polynomial which is the sum of $\psi(s)$ for each element s of M:

$$\Psi(M) \equiv \sum_{s\in M} \psi(s).$$

Applying Ψ to the unbounded multiset $\mathcal{S}$, we get the generating series for the hyper-Catalan numbers, an unbounded polynomial, or power series, in the variables t_2, t_3, $t_4, \ldots$ which we call the **subdigon polyseries** and denote:

$$\mathbf{S} = \mathbf{S}[t_2, t_3, \ldots] \equiv \Psi(\mathcal{S}) = \sum_{s\in\mathcal{S}} \psi(s) = \sum_{\mathbf{m}\geq\mathbf{0}} C_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}.$$

We claim that these two objects $\mathcal{S}$ and $\mathbf{S}$ give us a vast extension of the Catalan story, and yield a simple but powerful way of solving general polynomial equations, independent of any explicit counting formulas for $C_{\mathbf{m}}$.

6. SOLVING GENERAL POLYNOMIAL EQUATIONS SOFTLY. The subdigon multiset $\mathcal{S}$ supports a family of **k-ary panelling operators** $\overline{\nabla}_k$ for $k = 2, 3, 4, \dots$ that extend the operation $\overline{\nabla}$ that we considered in the quadratic case. These allow us to express an essentially geometric identity for $\mathcal{S}$, giving then a corresponding algebraic identity for **S** as a solution of the general geometric polynomial equation.

For subdigons s_1 and s_2, $\overline{\nabla}_2(s_1, s_2)$ consists of a central roofed triangle to which are adjoined s_1 by its roof along the first (left) side, and s_2 by its roof along the second (right) side. Analogously, $\overline{\nabla}_3(s_1, s_2, s_3)$ consists of a central roofed quadrilateral, to which is adjoined s_1 to its first side, s_2 to its second side and s_3 to its third side, each adjoined along its roof. Here we again number the non-roof sides of the central quadrilateral counterclockwise from its roof. In general, $\overline{\nabla}_k(s_1, s_2, \dots, s_k)$ consists of a central roofed $(k+1)$-gon, with attached subdigons $s_1, s_2, \dots, s_k$, each adjoined along its sides in a sequential order from the roof in a counterclockwise fashion.

Figure 4 illustrates the panelling operator $\overline{\nabla}_4$ applied to four subdigons.

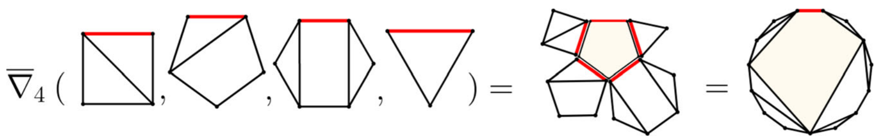

Figure 4. The operator $\overline{\nabla}_4$ creates a subdigon with a central pentagon.

In general, since $\overline{\nabla}_k$ adds a $(k+1)$-gon we have the identity:

$$\psi(\overline{\nabla}_k(s_1, s_2, \dots, s_k)) = t_k\, \psi(s_1)\psi(s_2)\cdots\psi(s_k).$$

What makes this work is that multiplication of the $\psi(s_i)$ is essentially addition of the type vectors of the s_i, and the t_k factor is another vector addition, adding one to the k-th component. When we naturally extend $\overline{\nabla}_k$ to multisets of subdigons,

$$\overline{\nabla}_k(M_1, M_2, \dots, M_k) \equiv \left[\, \overline{\nabla}_k\ (s_1, s_2, \dots, s_k) :\ s_1 \in M_1, s_2 \in M_2, \dots, s_k \in M_k \right]$$

per linearity we get the identity:

$$\Psi(\overline{\nabla}_k(M_1, M_2, \dots, M_k)) = t_k\, \Psi(M_1)\Psi(M_2)\cdots\Psi(M_k).$$

Every subdigon s is either the null subdigon or it has a central roofed $(k+1)$-gon for some natural $k \geq 2$. In the latter case it is necessarily of the form

$$s = \overline{\nabla}_k(s_1, s_2, \dots, s_k)$$

for unique subdigons $s_1, s_2, \dots, s_k$. It follows that we get the multiset equation

$$\mathcal{S} = [\ |\ ] + \overline{\nabla}_2(\mathcal{S}, \mathcal{S}) + \overline{\nabla}_3(\mathcal{S}, \mathcal{S}, \mathcal{S}) + \overline{\nabla}_4(\mathcal{S}, \mathcal{S}, \mathcal{S}, \mathcal{S}) + \dots \tag{7}$$

which Graham et al. [54] address in Bonus Problem 7.50. Their result differs from ours, as they count the subdivisions of a polygon as a power of a single variable z.

We obtain an algebraic version of equation (7) by applying Ψ to both sides:

$$\Psi(\mathcal{S}) = \psi(|) + \Psi(\overline{\nabla}_2(\mathcal{S}, \mathcal{S})) + \Psi(\overline{\nabla}_3(\mathcal{S}, \mathcal{S}, \mathcal{S})) + \Psi(\overline{\nabla}_4(\mathcal{S}, \mathcal{S}, \mathcal{S}, \mathcal{S})) + \dots$$

giving an equation in $\mathbf{S} = \Psi(\mathcal{S})$ that's quite general: the $t_k,\ k \geq 2$ become the coefficients of a power series equation

$$\mathbf{S} = 1 + t_2\mathbf{S}^2 + t_3\mathbf{S}^3 + t_4\mathbf{S}^4 + \dots$$

whose constant is 1 and whose linear coefficient is -1. We conclude:

Theorem 3 (The soft geometric polynomial formula). The polynomial or power series equation

$$0 = 1 - \alpha + t_2\alpha^2 + t_3\alpha^3 + t_4\alpha^4 + \dots$$

has a formal power series solution:

$$\alpha = \mathbf{S} = \mathbf{S}[t_2, t_3, t_4, \dots] = \sum_{m_2, m_3, m_4, \dots \geq 0} C[m_2, m_3, m_4, \dots]\, t_2^{m_2} t_3^{m_3} t_4^{m_4} \cdots = \sum_{\mathbf{m} \geq \mathbf{0}} C_{\mathbf{m}} \mathbf{t}^{\mathbf{m}}$$

where the hyper-Catalan number $C_{\mathbf{m}} = C[m_2, m_3, m_4, \dots]$ counts the number of subdigons with m_2 triangles, m_3 quadrilaterals, m_4 pentagons, etc.

We can now, with the same change of variable that we used in the quadratic case, give a series solution to the general polynomial or polyseries equation. Please note again that the coefficient of the linear term has a minus sign.

Theorem 4 (The soft polynomial formula). The polynomial or power series equation

$$0 = c_0 - c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

has a formal series solution:

$$\begin{aligned} x &= \frac{c_0}{c_1}\, \mathbf{S}\left[\, \frac{c_0 c_2}{c_1^2},\ \frac{c_0^2 c_3}{c_1^3},\ \frac{c_0^3 c_4}{c_1^4},\ \dots \right] \\ &= \sum_{m_2, m_3, m_4, \dots \geq 0} C[m_2, m_3, m_4, \dots]\, \frac{c_0^{1+m_2+2m_3+3m_4+\dots}}{c_1^{1+2m_2+3m_3+4m_4+\dots}} c_2^{m_2} c_3^{m_3} c_4^{m_4} \cdots . \end{aligned}$$

Proof. Setting $x = \frac{c_0}{c_1}\alpha$ the equation to solve becomes

$$c_0 \left(1 - \alpha + \frac{c_0 c_2}{c_1^2}\alpha^2 + \frac{c_0^2 c_3}{c_1^3}\alpha^3 + \frac{c_0^3 c_4}{c_1^4}\alpha^4 + \dots\right) = 0.$$

We apply Theorem 3 with $t_k = \dfrac{c_0^{k-1} c_k}{c_1^k}$ for $k = 2, 3, \dots$ to solve for α and then write down x. ∎

If we introduce the notation $\mathbf{c}^{\mathbf{m}} \equiv c_2^{m_2} c_3^{m_3} c_4^{m_4} \cdots$, which does not involve c_0 or c_1, and use our previous expressions for the number of vertices and edges in a subdigon of type $\mathbf{m}$, then we can further abbreviate the soft general polynomial zero as:

$$x = \sum_{\mathbf{m} \geq \mathbf{0}} C_{\mathbf{m}}\, \frac{c_0^{V_{\mathbf{m}}-1}}{c_1^{E_{\mathbf{m}}}}\, \mathbf{c}^{\mathbf{m}}. \tag{8}$$

This simple looking formula effectively solves the most important problem in algebra. It hides a huge amount of structure and complexity, massively extending the Catalan story.

7. FORMULAS FOR THE HYPER-CATALAN NUMBERS. To make our soft formulas explicit we want to write down concrete formulas for the hyper-Catalan array. Alison Schuetz and Gwyn Whieldon, 2016 [51] gave us the formula we sought in terms of subdivided polygons as their Lemma 3.2. They cite Brendon Rhoades, 2011 [52] and both papers cite Kreweras, 1972 [39,40] as the source of the formula. We've since found earlier results from Erdélyi and Etherington, 1940 [2], Raney, 1960 [34] and Tutte, 1964 [35].

Theorem 5 (Erdélyi and Etherington, Raney, Tutte, Kreweras, Rhoades, Schuetz and Whieldon). The number of subdigons of type $\mathbf{m} = [m_2, m_3, m_4, \dots]$ is

$$C_{\mathbf{m}} = \frac{(2m_2 + 3m_3 + 4m_4 + \dots)!}{(1 + m_2 + 2m_3 + 3m_4 + \dots)!\, m_2!\, m_3!\, m_4! \cdots}.$$

Equivalently, we can write this explicitly as a sub-multinomial coefficient:

$$C_{\mathbf{m}} = \frac{1}{1 + 2m_2 + 3m_3 + \dots} \binom{1 + 2m_2 + 3m_3 + \dots}{1 + m_2 + 2m_3 + \dots,\ m_2,\ m_3,\ m_4,\ \dots}$$

which *a priori* is not necessarily a natural number. There is some magic here, just as with the Catalan numbers, giving us integers because we are counting something.

Using the notation $\mathbf{m}! \equiv m_2! m_3! m_4! \cdots$ and equations (5) and (6), we may further rewrite this formula involving the Euler polygonal relation as:

$$C_{\mathbf{m}} = \frac{(E_{\mathbf{m}} - 1)!}{(V_{\mathbf{m}} - 1)!\, \mathbf{m}!}. \tag{9}$$

Note that the sum of the m_i is $F_{\mathbf{m}}$, the number of faces of a subdigon of type $\mathbf{m}$ (equation (4)), corresponding to the $\mathbf{m}!$ factor. When combined with equation (8) we can rewrite the soft formulas of Theorems 3 and 4 with an explicit coefficient:

Theorem 6 (The geometric polynomial formula). The equation

$$0 = 1 - \alpha + t_2\alpha^2 + t_3\alpha^3 + t_4\alpha^4 + t_5\alpha^5 \ + \ \dots$$

has a formal power series solution:

$$\alpha = \sum_{m_k \geq 0} \frac{(2m_2 + 3m_3 + 4m_4 + \cdots)!}{(1 + m_2 + 2m_3 + \dots)!\, m_2!\, m_3! \cdots} t_2^{m_2} t_3^{m_3} \cdots = \sum_{\mathbf{m} \geq \mathbf{0}} \frac{(E_{\mathbf{m}} - 1)!}{(V_{\mathbf{m}} - 1)!\, \mathbf{m}!} \mathbf{t}^{\mathbf{m}}.$$

Theorem 7 (The polynomial formula). The polynomial or power series equation

$$0 = c_0 - c_1\alpha + c_2\alpha^2 + c_3\alpha^3 + c_4\alpha^4 + \dots$$

has a formal series solution:

$$\begin{aligned}\alpha &= \sum_{m_2, m_3, \dots \geq 0} \frac{(2m_2 + 3m_3 + 4m_4 + \dots)!}{(1 + m_2 + 2m_3 + 3m_4 + \dots)! m_2! m_3! \cdots} \frac{c_0^{1 + m_2 + 2m_3 + \dots} c_2^{m_2} c_3^{m_3} \cdots}{c_1^{1 + 2m_2 + 3m_3 + 4m_4 + \dots}} \\ &= \sum_{\mathbf{m} \geq \mathbf{0}} \frac{c_0^{V_{\mathbf{m}} - 1}}{(V_{\mathbf{m}} - 1)!} \left(\frac{c_1^{E_{\mathbf{m}}}}{(E_{\mathbf{m}} - 1)!} \right)^{-1} \frac{\mathbf{c}^{\mathbf{m}}}{\mathbf{m}!}.\end{aligned}$$

Euler would probably like this formula, which combines a great extension of his polygon subdivision work with his polytope formula $V - E + F = 1$ appearing here quite noticeably once we observe the degree of $\mathbf{c^m}$ is the number of faces F. As the original equation is homogeneous in the c_i, the total degree of each term in the variables $c_0, c_1, c_2, \ldots$ is 0.

The forms spur us to define, for a natural number n, the **factorial power quotient**

$$\underline{x}^n \equiv \frac{x^n}{n!}$$

which we use to rewrite our two main results more succinctly, giving insight into the structure and relation of our series zeros.

Theorem 8 (The geometric polynomial formula, factorial power quotient form). The equation $0 = 1 - \alpha + \sum_{k\geq 2} t_k \alpha^k$ has a series solution:

$$\alpha = \mathbf{S}[t_2, t_3, \ldots] = \sum_{m_k \geq 0} \underline{1}^{1+m_2+2m_3+\ldots} \left(\underline{1}^{2m_2+3m_3+\ldots}\right)^{-1} \underline{t_2}^{m_2} \, \underline{t_3}^{m_3} \cdots$$

$$= \sum_{\mathbf{m}\geq \mathbf{0}} \underline{1}^{V_\mathbf{m}-1} \left(\underline{1}^{E_\mathbf{m}-1}\right)^{-1} \underline{\mathbf{t}}^\mathbf{m}, \quad \text{where} \quad \underline{\mathbf{t}}^\mathbf{m} \equiv \frac{\mathbf{t^m}}{\mathbf{m}!} = \underline{t_2}^{m_2} \, \underline{t_3}^{m_3} \cdots .$$

Theorem 9 (The polynomial formula, factorial power quotient form). The equation $0 = c_0 - c_1 x + \sum_{k\geq 2} c_k x^k$ has a series solution:

$$x = c_1^{-1} \sum_{m_k\geq 0} \underline{c_0}^{1+m_2+2m_3+\ldots} \left(\underline{c_1}^{2m_2+3m_3+\ldots}\right)^{-1} \underline{c_2}^{m_2} \, \underline{c_3}^{m_3} \cdots$$

$$= c_1^{-1} \sum_{\mathbf{m}\geq\mathbf{0}} \underline{c_0}^{V_\mathbf{m}-1} \left(\underline{c_1}^{E_\mathbf{m}-1}\right)^{-1} \underline{\mathbf{c}}^\mathbf{m}, \quad \text{where} \quad \underline{\mathbf{c}}^\mathbf{m} \equiv \frac{\mathbf{c^m}}{\mathbf{m}!} = \underline{c_2}^{m_2} \, \underline{c_3}^{m_3} \cdots .$$

8. THE BI-TRI ARRAY AND A ONE LINE CUBIC APPROXIMATION.

Let's give an algebraic alternative to the Tartaglia / Cardano solution of the cubic, and use it to straightforwardly compute approximate numerical solutions. Specializing the geometric polynomial formula (Theorem 6):

Theorem 10 (The geometric cubic formula). The equation

$$1 - \alpha + t_2\alpha^2 + t_3\alpha^3 = 0$$

has a formal power series solution:

$$\alpha = \sum_{m_2, m_3 \geq 0} C[m_2, m_3] \, t_2^{m_2} t_3^{m_3} = \sum_{m_2\geq 0} \sum_{m_3 \geq 0} \frac{(2m_2 + 3m_3)!}{(1 + m_2 + 2m_3)! \, m_2! \, m_3!} \, t_2^{m_2} t_3^{m_3}.$$

We'll call $C[m_2, m_3]$ the **Bi-Tri array**, as it counts not only subdigons with m_2 triangles and m_3 quadrilaterals (and no other kinds of faces), but also the number of full plane trees with exactly m_2 binary nodes and m_3 ternary nodes (and no other arities except leaf nodes).

Here is an initial part of the array, also described as A104978 in OEIS, where it is recognized as having a generating function satisfying a general geometric cubic equation.

$m_2 \backslash m_3$	0	1	2	3	4	5
	The Bi-Tri array $C[m_2, m_3] = \frac{(2m_2+3m_3)!}{(1+m_2+2m_3)!\,m_2!\,m_3!}$					
0	1	1	3	12	55	273
1	1	5	28	165	1001	6188
2	2	21	180	1430	10920	81396
3	5	84	990	10010	92820	813960
4	14	330	5005	61880	678300	6864396
5	42	1287	24024	352716	4476780	51482970
6	132	5005	111384	1899240	27457584	354323970
7	429	19448	503880	9806280	159352050	2283421140

The first column, $C[m_2, 0]$, contains of course the Catalan numbers. The first row, $C[0, m_3]$, contains the quadrilateral Fuss numbers, A001764, which count the number of subdigons with exactly m_3 quadrilateral faces.

A surprising property is that the alternating sum of cross diagonals, with the exception of the first, is always 0. That is, $0 = 1 - 1 = 2 - 5 + 3 = 5 - 21 + 28 - 12$ and so on. This relates to the Geode factorization in Section 11.

Let's see how we can use even just a modest part of this array to get numerical solutions to the famous cubic equation

$$f(x) = x^3 - 2x - 5 = 0$$

used by Wallis to illustrate Newton's method [58]. This will also be an opportunity to introduce the idea of bootstrapping. We begin by taking just the first few cross diagonals of the array, terms up to degree 3:

$$Q(t_2, t_3) = 1 + (t_2 + t_3) + (2t_2^2 + 5t_2t_3 + 3t_3^2) + (5t_2^3 + 21t_2^2t_3 + 28t_2t_3^2 + 12t_3^3). \qquad (10)$$

To solve a general cubic $c_0 - c_1x + c_2x^2 + c_3x^3 = 0$, the soft polynomial formula (Theorem 4) tells us that we should also define:

$$K\,(c_0, c_1, c_2, c_3) \equiv \frac{c_0}{c_1} Q\left(\frac{c_0c_2}{c_1^2}, \frac{c_0^2c_3}{c_1^3}\right). \qquad (11)$$

We calculate $K(-5, 2, 0, 1) \approx -999.082031$, clearly a fail. But we cannot expect a single formula to automatically generate accurate numerical zeros; after all, most such algorithms, such as Newton-Raphson, are sensitive to their starting guess.

For Wallis's equation, $x = 2$ is a reasonable first approximation, so we look at $g(x) = f(x + 2) = -1 + 10x + 6x^2 + x^3$ and calculate $K(-1,-10,6,1) = 0.0945345708$ which gives an approximate zero 2.0945345708.

In comparison, our computer determines two (necessarily approximate) complex zeros and one approximate rational one, $x \approx 2.0945514815423265915$ to 19 decimal places. So we have agreement up to 4 decimal places.

To increase the accuracy, we use bootstrapping: just input the obtained value as a new initial guess. For illustration, we truncate the guess to four decimal places. So we calculate

$$f(x + 2.0945) = -0.000574591374999045 + 11.16079075x + 6.2835x^2 + x^3$$

and adding 2.0945 to K applied to these coefficients gives 2.09455148154232650 98, agreeing with our computer to 16 decimal places.

Of course this has been a pretty modest example, but it shows that, even just using a small portion of the full subdigon polyseries, we can obtain impressive results. And now we have a one line cubic approximation formula, namely equation (10), our expression for $Q(t_2, t_3)$ above!

9. EISENSTEIN'S EXAMPLE. An immediate corollary to Theorem 7 is the algebraic solution to the general quintic equation: it has one once we let go of radicals.

Theorem 11 (The quintic formula). The quintic equation

$$c_0 - c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 = 0$$

has a formal series solution:

$$x = \sum_{\substack{m_2, m_3, \\ m_4, m_5 \geq 0}} \frac{(2m_2 + 3m_3 + 4m_4 + 5m_5)!\, c_0^{1+m_2+2m_3+3m_4+4m_5} c_2^{m_2} c_3^{m_3} c_4^{m_4} c_5^{m_5}}{(1 + m_2 + 2m_3 + 3m_4 + 4m_5)!\, m_2! m_3! m_4! m_5! c_1^{1+2m_2+3m_3+4m_4+5m_5}}.$$

This also contains a solution to the general quadratic, cubic, and quartic equations.

Eisenstein [4, 5] determined a power series for the Bring radical of t, solving $-t + x + x^5 = 0$. That's $c_0 = -t$, $c_1 = -1$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$, $c_5 = 1$ in Theorem 11. The zero values for c_2, c_3, and c_4 mean that we're only summing over $m_5 = m$ so we have:

$$x = \sum_{m \geq 0} \frac{(5m)!\, (-1)^m t^{4m+1}}{(4m+1)!\, m!} = t - t^5 + \frac{10}{2!} t^9 - \frac{15 \cdot 14}{3!} t^{13} + \frac{20 \cdot 19 \cdot 18}{4!} t^{17} - \dots$$

which is indeed Eisenstein's series solution.

Theorem 7 lets us just as easily solve the more general $-t + x + x^d = 0$:

$$x = \sum_{m \geq 0} \frac{(dm)!\, (-1)^m\, t^{(d-1)m+1}}{((d-1)m + 1)!\, m!}$$

which essentially agrees with Theorem 2.3 of Schuetz and Whieldon [51]; the coefficients are the two-parameter Fuss numbers [50].

10. REVERSION OF SERIES AND LAGRANGE. Although we have been unable to find any reference for an explicit series zero of the general polynomial with a closed-form hyper-Catalan coefficient, we've found formulas for the coefficients of series reversions that are similar to our polynomial zero result as early as McMahon, 1894 [15, 22, 23, 51].

Suppose we have an ongoing polyseries $y = p(x) = x - a_2 x^2 - a_3 x^3 - a_4 x^4 - \dots$ and we want to revert the series, that is, express x as a power series in y, say $x = q(y) = y + b_2 y^2 + b_3 y^3 + b_4 y^4 + \dots$.

When we compute $p(q(y))$ the result is a power series in y that has coefficients which are finite sums of terms involving the a_i and b_j variables, starting with

$$\begin{aligned} p(q(y)) = y &+ y^2(b_2 - a_2) + y^3(b_3 - a_3 - 2a_2 b_2) + y^4(b_4 - 3a_3 b_2 - a_4 - a_2 b_2^2 \\ &- 2a_2 b_3) + y^5(b_5 - a_5 - 3a_3 b_2^2 - 2a_2 b_4 - 3a_3 b_3 - 4a_4 b_2 - 2a_2 b_2 b_3) + \dots . \end{aligned}$$

Setting this to y, we sequentially find that $b_2 = a_2$, $b_3 = 2a_2^2 + a_3$, $b_4 = 5a_2^3 + 5a_2a_3 + a_4$, $b_5 = 14a_2^4 + 21a_2^2a_3 + 3a_3^2 + 6a_2a_4 + a_5$ and onwards. While the pattern is not obvious, Lagrange found a general formula, which in this case takes the form

$$b_n = \frac{1}{n!} \lim_{y\to 0} \frac{d^{n-1}}{dy^{n-1}} \left(\frac{y}{p\left(y\right)}\right)^n.$$

So for example to find b_3, we expand

$$\left(\frac{y}{p\left(y\right)}\right)^3 = \left(\frac{1}{1 - a_2y - a_3y^2 + \ldots}\right)^3 = 1 + 3a_2y + \left(6a_2^2 + 3a_3\right)y^2 + \cdots$$

up to the power y^2, and then take $\frac{1}{3}$ times that coefficient, giving $b_3 = 2a_2^2 + a_3$.

This series reversion is highly relevant to our problem, since if we set $y = 1$ then the original equation reduces to $1 = x - a_2x^2 - a_3x^3 - a_4x^4 + \ldots$, exactly our geometric polynomial equation, and the inversion gives $x = 1 + b_2 + b_3 + b_4 + \ldots$, which can be seen to be our solution $\mathbf{S}$. In fact we will see exactly this particular presentation of our solution in the next section, when we discuss vertex layerings of $\mathbf{S}$ (and in Mott, 1855 [**20**], etc.).

Proofs of Lagrange's formula were given by Lagrange himself in 1770 [**13**], by Pierre-Simon Laplace and by Charles Hermite using contour integration in 1865. Since then many people have given alternate or more explicit versions of Lagrange's formula [**14**]. Riordan's 1968 version [**15**] of McMahon's [**22**] formulation of Lagrange Inversion is, using our notation:

$$b_n = \sum_{\substack{m_2+m_3+m_4+\ldots=k \\ m_2+2m_3+3m_4+\ldots=n}} \frac{1}{n+1}\binom{n+k}{k}\frac{k!}{m_2!\,m_3!\cdots m_{n+1}!}a_2^{m_2}a_3^{m_3}\cdots a_{n+1}^{m_{n+1}}$$

where the sum is over natural numbers $m_2, m_3, \ldots \geq 0$. This solution for a series reversion appears very similar to our solution for the series zero. Indeed, if we note that $n = V_{\mathbf{m}} - 2$, $n + k = E_{\mathbf{m}} - 1$, and $k = F_{\mathbf{m}}$ (equations (6), (5), and (4)), then we get

$$\frac{1}{n+1}\frac{(n+k)!}{n!\,k!}\frac{k!}{\mathbf{m}!} = \frac{(n+k)!}{(n+1)!\,\mathbf{m}!} = \frac{(E_{\mathbf{m}} - 1)!}{(V_{\mathbf{m}} - 1)!\,\mathbf{m}!} = C_{\mathbf{m}}.$$

Collectively the two derivations above, namely the soft geometric polynomial formula Theorem 3, and the derivation based on Lagrange Inversion, together give us the explicit closed form of the hyper-Catalan numbers. There is some irony that Lagrange unknowingly found a passage to the secret of solving polynomial equations with his reversion of series formula, but this connection would lay hidden for centuries more.

11. LAYERING, FACE FACTORIZATION, AND THE GEODE. The subdigon polyseries $\mathbf{S} = \mathbf{S}\left[t_2, t_3, t_4, \ldots\right] \equiv \Psi(\mathcal{S})$ is the key algebraic object in the theory, so it's worthwhile to try to come to better grips with it. We do this by judicious layerings, and as we do so another surprising and even more mysterious algebraic object emerges: the *Geode*.

Vertex layering of S. To organize $\mathbf{S}$ according to the number of vertices $V = 2 + m_2 + 2m_3 + 3m_4 + \ldots$ (equation (6)), we introduce the auxiliary variable v and define $\mathbf{S}_V \equiv v^2\mathbf{S}\left[vt_2, v^2t_3, v^3t_4, \ldots\right]$. Setting $v = 1$ recovers $\mathbf{S}$, but by expanding in powers of v we obtain a layering into (finite) polynomials which count subdigons with a given number of vertices:

$$\mathbf{S}_V = v^2 + t_2v^3 + \left(2t_2^2 + t_3\right)v^4 + \left(5t_2^3 + 5t_2t_3 + t_4\right)v^5 \\ + \left(14t_2^4 + 21t_2^2t_3 + 3t_3^2 + 6t_2t_4 + t_5\right)v^6 + \cdots .$$

Note that the coefficients here are the same as the quantities $1, b_2, b_3, \ldots$ which appeared in the reversion of series in the previous section. Figure 5 shows subdigons that are hexagons, corresponding to the v^6 term.

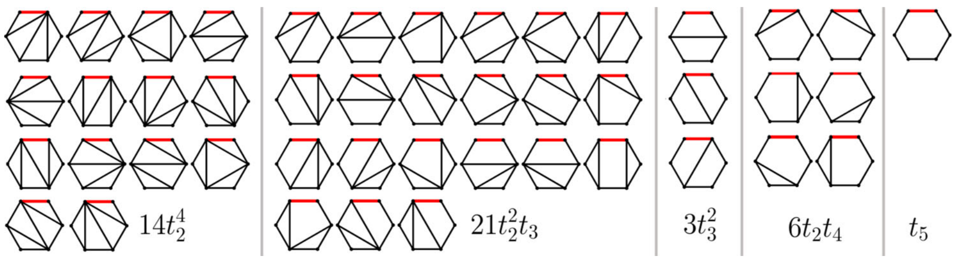

Figure 5. 45 subdigons with six vertices, $[v^6]\mathbf{S}_V = 14t_2^4 + 21t_2^2t_3 + 3t_3^2 + 6t_2t_4 + t_5$.

Setting all $t_k = 1$ in $\mathbf{S}_V$ gives $v^2 + v^3 + 3v^4 + 11v^5 + 45v^6 + 197v^7 + 903v^8 + 4278v^9 + \cdots$ whose coefficient sequence is A001003, the little Schroeder numbers.

We thank an anonymous referee for pointing out that setting $t_i = t$ in $\mathbf{S}_V$ gives A033282 and the resulting connection to associahedra. We note that A033282, "the number of diagonal dissections of a convex n-gon into $k + 1$ regions," is the Cayley array, rediscovered by Kreweras, which we discussed in Section 2.

Edge layering of S. To organize $\mathbf{S}$ according to the number of edges $E = 1 + 2m_2 + 3m_3 + 4m_4 + \ldots$ (equation (5)), we introduce the auxiliary variable e and set: $\mathbf{S}_E \equiv e\mathbf{S}\left[e^2t_2, e^3t_3, e^4t_4, \ldots\right]$. Expanding, we then get a layering of $\mathbf{S}$ into (finite) polynomials which account for subdigons with a given number of edges:

$$\mathbf{S}_E = e + t_2e^3 + t_3e^4 + \left(2t_2^2 + t_4\right)e^5 + \left(5t_2t_3 + t_5\right)e^6 \\ + \left(5t_2^3 + 3t_3^2 + 6t_2t_4 + t_6\right)e^7 + \left(21t_2^2t_3 + 7t_3t_4 + 7t_2t_5 + t_7\right)e^8 + \cdots .$$

Figure 6 shows the subdigons with exactly 7 edges, corresponding to the e^7 term. Setting $t_k = 1$, we get the series $e + 0e^2 + e^3 + e^4 + 3e^5 + 6e^6 + 15e^7 + 36e^8 + 91e^9 + \cdots$ whose coefficient sequence is A005043, the Riordan numbers.

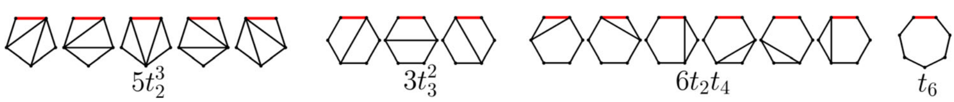

Figure 6. 15 subdigons with seven edges, $[e^7]\mathbf{S}_E = 5t_2^3 + 3t_3^2 + 6t_2t_4 + t_6$.

Face layering of S and the Geode factorization. To organize $\mathbf{S}$ according to the number of faces $F = m_2 + m_3 + m_4 + \ldots$, we introduce the auxiliary variable f and set $\mathbf{S}_F \equiv \mathbf{S}[f t_2, f t_3, f t_4, \ldots]$. Expanding, we get a layering of $\mathbf{S}$ into (unbounded) polyseries of a given total degree, accounting for subdigons with a given number of faces:

$$\begin{aligned}\mathbf{S}_F &= 1 + (t_2 + t_3 + t_4 + \ldots)\, f \\ &\quad + \left(2t_2^2 + 5t_2t_3 + 3t_3^2 + 6t_2t_4 + 7t_3t_4 + 4t_4^2 + \cdots\right) f^2 \\ &\quad + \begin{pmatrix} 5t_2^3 + 21t_2^2t_3 + 28t_2t_3^2 + 12t_3^3 + 28t_2^2t_4 + 72t_2t_3t_4 \\ + 45t_2t_4^2 + 45t_3^2t_4 + 55t_3t_4^2 + 22t_4^3 + \cdots \end{pmatrix} f^3 \\ &\quad + \begin{pmatrix} 14t_2^4 + 84t_2^3t_3 + 180t_2^2t_3^2 + 165t_2t_3^3 + 55t_3^4 + 120t_2^3t_4 \\ + 495t_2^2t_3t_4 + 660t_2t_3^2t_4 + 286t_3^3t_4 + 330t_2^2t_4^2 + 858t_2t_3t_4^2 \\ + 546t_3^2t_4^2 + 364t_2t_4^3 + 455t_3t_4^3 + 140t_4^4 + \cdots \end{pmatrix} f^4 + \cdots .\end{aligned}$$

Defining $\mathbf{S}_1 \equiv t_2 + t_3 + t_4 + \ldots$ we notice what appears to be a remarkable factorization:

$$\begin{aligned}\mathbf{S}_F &= 1 + \mathbf{S}_1 f + \mathbf{S}_1\,(2t_2 + 3t_3 + 4t_4 + \ldots)\, f^2 \\ &\quad + \mathbf{S}_1\left(5t_2^2 + 16t_2t_3 + 12t_3^2 + 23t_2t_4 + 33t_3t_4 + 22t_4^2 + \cdots\right) f^3 \\ &\quad + \mathbf{S}_1 \begin{pmatrix} 14t_2^3 + 70t_2^2t_3 + 110t_2t_3^2 + 55t_3^3 + 106t_2^2t_4 + 319t_2t_3t_4 \\ + 224t_2t_4^2 + 231t_3^2t_4 + 315t_3t_4^2 + 140t_4^3 + \cdots \end{pmatrix} f^4 + \cdots .\end{aligned}$$

Theorem 12 (Subdigon polyseries factorization). There is a unique polyseries $\mathbf{G}$ for which we have the identity $\mathbf{S} - 1 = \mathbf{S}_1\mathbf{G}$.

We note that $\mathbf{S}_1$ is not a power series beginning with a nonzero constant, hence is not automatically invertible, so this is a nontrivial statement.

Proof. Let $\mathcal{S}_n$ be the multiset of subdigons with n faces. We have $\mathcal{S}_0 = [\,|\,]$, the singleton with the null subdigon, and $\mathbf{S}_0 = \Psi(\mathcal{S}_0) = 1$. The rest, $\mathcal{S}_n,\ n \geq 1$, are necessarily formed by applying $\overline{\nabla}_k$ to k-tuples of multisets of smaller subdigons,

$$\mathcal{S}_n = \sum_{k\geq 2} \sum_{1+\sum_{i=1}^{k} j_i = n} \overline{\nabla}_k(\mathcal{S}_{j_1}, \mathcal{S}_{j_2}, \ldots, \mathcal{S}_{j_k})$$

where $0 \leq j_i < n$, strictly less as $\overline{\nabla}_k$ adds one face. The term for each k where all the j_i are zero, $\overline{\nabla}_k(\mathcal{S}_0, \mathcal{S}_0, \ldots)$, gives a singleton with the unsubdivided $(k+1)$-gon, exactly one face. Collectively those subdigons form $\mathcal{S}_1$, so $\mathbf{S}_1 = \Psi(\mathcal{S}_1) = t_2 + t_3 + t_4 + \ldots$, agreeing with the definition above. For $\mathcal{S}_2$, to make $1 + \sum_{i=1}^{k} j_i = 2$, one of the $j_i = 1$ and the rest are zero, so $\mathcal{S}_1$ appears once and $\mathcal{S}_0$ appears $k-1$ times. Applying Ψ to a term of $\mathcal{S}_2$ gives:

$$\Psi(\overline{\nabla}_k(\mathcal{S}_{j_1}, \mathcal{S}_{j_2}, \ldots, \mathcal{S}_{j_k})) = t_k\Psi(\mathcal{S}_{j_1})\cdots\Psi(\mathcal{S}_{j_k}) = t_k\Psi(\mathcal{S}_1)(\Psi(\mathcal{S}_0))^{k-1} = t_k\mathbf{S}_1.$$

We see that each term of $\mathbf{S}_2 = \Psi(\mathcal{S}_2)$ is a multiple of $\mathbf{S}_1$; we conclude that $\mathbf{S}_1$ is a factor of $\mathbf{S}_2$. Similarly, each term of $\mathbf{S}_n$ for $n \geq 3$ has at least one nonzero j_i, so each term has at least one of $\mathbf{S}_1$, $\mathbf{S}_2$, … $\mathbf{S}_{n-1}$ as a factor, so $\mathbf{S}_1$ as a factor. For $n \geq 1$ we conclude $\mathbf{S}_n$ has $\mathbf{S}_1$ as a factor, so $\mathbf{S}_1$ divides $\mathbf{S} - 1 = \sum_{n\geq 1} \mathbf{S}_n$. ■

As an application, here is a simplification of equation (10), our one line approximate cubic formula:

$$Q\left(t_2, t_3\right) = 1 + \left(t_2 + t_3\right)\left(1 + 2t_2 + 3t_3 + 5t_2^2 + 16t_2t_3 + 12t_3^2\right). \tag{12}$$

The factorization suggests to us that the factor **G**, which we call the **Geode**, encodes structure underlying the hyper-Catalan numbers which is perhaps even more fundamental. Here are a few of the initial terms of **G**:

$$\begin{aligned}\mathbf{G} = 1 + 2t_2 + 3t_3 + 4t_4 + \ldots + 5t_2^2 + 16t_2t_3 + 12t_3^2 + 23t_2t_4 + 33t_3t_4 + 22t_4^2 \\ + 14t_2^3 + 70t_2^2t_3 + 110t_2t_3^2 + 55t_3^3 + 106t_2^2t_4 + 319t_2t_3t_4 + 224t_2t_4^2 + \cdots .\end{aligned}$$

The coefficients form the **Geode array**: $G_{\mathbf{m}} = G[m_2, m_3, \ldots]$. Even in the cubic case we see something new; the first two dimensional slice, the **Geode Bi-Tri array**:

m_2 \ m_3	0	1	2	3	4	5
	The Geode Bi-Tri array $G[m_2, m_3]$					
0	1	3	12	55	273	1428
1	2	16	110	728	4760	31008
2	5	70	702	6160	50388	395010
3	14	288	3850	42432	418950	3853696
4	42	1155	19448	259350	3010700	31870410
5	132	4576	93366	1466080	19612560	235282320
6	429	18018	433160	7845024	119041650	1598394798
7	1430	70720	1961256	40310400	685026342	10189625600

The first column is the shifted Catalan numbers, and the first row is the shifted quadrilateral Fuss numbers, A001764. We didn't find other rows, columns, or the dovetailed array in the OEIS. We conjecture the following form for its entries:

$$G[m_2, m_3] \stackrel{?}{=} \frac{1}{(2m_2 + 2m_3 + 3)(m_2 + m_3 + 1)} \frac{(2m_2 + 3m_3 + 3)!}{(m_2 + 2m_3 + 2)!\, m_2!\, m_3!}$$

and further conjecture that these numbers are counting ordered incomplete trees with m_2 binary nodes, m_3 ternary nodes, and a single additional leaf node. We conjecture that the natural generalization of this explains all the entries of the Geode array $G_{\mathbf{m}}$.

We note that our above conjectured formula for the Geode Bi-Tri array does not extend in an obvious way more generally, on account of large prime factors appearing early in the array, e.g., $G[1, 0, 1] = 23$, $G[2, 0, 2] = 1549$, and $G[3, 0, 3] = 145687$.

With $2k$ parameters, we conjecture $\mathbf{G}[-f, f, \ldots, -f, f] \stackrel{?}{=} \sum_n k^n f^n$.

With $k - 2$ leading zeros, we conjecture that $G[0, \ldots, 0, m_k]$ is a two-parameter Fuss number [50], and that our proposed formula for the Geode Bi-Tri entries generalizes naturally to entries of the form $G[0, \ldots, 0, m_k, m_{k+1}] \stackrel{?}{=}$

$$\frac{(km_k + (k+1)(m_{k+1} + 1))!}{(k(m_k + m_{k+1} + 1) + 1)(m_k + m_{k+1} + 1)((k-1)m_k + k(m_{k+1} + 1))!\, m_k!\, m_{k+1}!}.$$

Obtaining an explicit formula for the general entry appears to be a major challenge.

12. FURTHER DIRECTIONS. The hyper-Catalan numbers form a vast edifice which naturally extends the ubiquitous Catalan sequence, so they are worth serious exploration. This suggests a broad enlargement of the OEIS to incorporate arbitrary sequences found within the hyper-Catalan array, as we've already seen, or for example, by fixing all but one of the variables in $C[m_2, m_3, m_4, \ldots]$.

We've found that $C[n]$ is A000108, $C[0, n]$ is A001764, $C[0, 0, n]$ is A002293, $C[0, 0, 0, n]$ is A002294, $C[n, 1]$ is A002054, $C[1, n-1]$ is A025174, $C[n-3, 2]$ is A074922, $C[1, 0, n]$ is A257633, $C[0, 1, n]$ is A224274, $C[n, 0, 1]$ is A002694, and $C[0, 0, 1, n]$ is A163456. Likely, there are many more. We might have to enlist the help of some AI friends here!

An obvious question for investigation is: What besides subdigons do hyper-Catalan numbers $C_{\mathbf{m}}$ count, and can we give direct bijective correspondences between these different incarnations? Schuetz and Whieldon [**51**] and Rhoades [**52**] have found interesting other such structures, but there are likely many more. How does the surprising connection with Euler's polytope formula manifest itself in these other situations?

Our subdigon polyseries factorization theorem (Theorem 12) shows that the Geode array $G_{\mathbf{m}}$ directly encodes the hyper-Catalan array, hence it is now also an object of considerable interest. Can we establish our conjectures for its entries, and investigate the significance of the kinds of ordered incomplete trees that are possibly being counted here? What is the combinatorial significance of these curious objects in other Catalan-type situations? Further, the suggestion we made above for incorporating the hyper-Catalan array systematically into the OEIS also applies to the Geode array.

As for solving polynomial equations, we have given one solution, but are there others, and what do they look like? How does Galois theory connect with this power series approach? What about more traditional extension fields: how are they involved, and in particular how do we generate complex solutions? (We have observed that bootstrapping works to approximate complex solutions.)

On the applied side, we can ask about the utility of the solution for numerical approximation. In particular, the conditions for convergence and optimization of bootstrapping could be further studied.

Finally, in other areas of pure mathematics, formal power series give algebraic and combinatorially explicit alternatives to functions which cannot actually be concretely evaluated (such as nth root functions). Hence they ought to assume a more central position. This is a solid, logical way of removing many of the infinities which currently abound in our mathematical landscape.

The combinatorial and computational orientation is full of power, and we ought to harness it more fully, opening up new landscapes with the aid of our symbolic computation machines.

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