1. i) a)

$$f_X(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} x^2/2 & 0 \leq x \leq 1 \\ 1/2 + \int_1^x (2-t)\,dt & \end{cases}$$

1 mark method

$$\int_1^x (2-t)\,dt = \left[2t - \frac{t^2}{2}\right]_1^x = 2x - \frac{x^2}{2} - \frac{3}{2}$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1 & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$

1M

2M $\left[\text{or } 1 - \frac{(2-x)^2}{2}\right]$

[4 marks]

b) To invert, $U \leq 1/2$

$$U = \frac{x^2}{2} \qquad U \leq 1/2$$

$$\Rightarrow x = \sqrt{2U} \qquad U \leq 1/2$$

1M for split

1M invert

~~1 mark method~~

$U > 1/2$

$$U = 2X - \frac{X^2}{2} - 1$$

-ve root as $1 \leq X \leq 2$

$$\Rightarrow X^2 - 4X = -2 - 2U$$

$$\Rightarrow (X-2)^2 = 2(1+U)$$

$$\Rightarrow X = 2 - \sqrt{2(1+U)}$$

3M invert and state - root

~~$1 \leq X \leq 2$~~

so algorithm becomes

1. Sample $U \sim U[0,1]$
2. Set

$$X = \begin{cases} \sqrt{2U} & 0 \leq U \leq 1/2 \\ 2 - \sqrt{2(1+U)} & 1/2 < U \leq 1 \end{cases}$$

1M set out algorithm

[6 marks]

ii) Have $f_X(x) = \frac{3}{2}(1-x^2) \qquad 0 \leq x \leq 1$

use envelope $g_Y(y) = 1 \qquad 0 \leq y \leq 1$

set $c = \sup \frac{f(x)}{g(x)}$

$f(x)$ maximised at $x = 0$ so let

$$c = \frac{f(0)}{1} = \frac{3}{2}$$

4 marks find c

so rejection algorithm is

1. Sample $Y \sim U[0,1]$ and $U \sim U[0,1]$
2. If $U \leq \frac{f(Y)}{c} = \frac{3(1-Y^2)}{2c} = (1-Y^2)$

then accept $X = Y$, otherwise reject and return to step 1

4 marks lay out method

Given $Y = 0.5$ $\quad U = 0.8$

we have $1 - Y^2 = 3/4 < 0.8$ so would reject candidate Y $\quad$ – 4 marks for working out reject

[12 marks]

c) $\mathbb{E}[\#\text{ candidate } Y] = c = \frac{3}{2}$

[2 marks]

d) For new envelope

$$\frac{f(x)}{g(x)} = \frac{\frac{3}{2}(1-x^2)}{2(1-x)} = \frac{\frac{3}{2}(1-x)(1+x)}{2(1-x)}$$

$$= \frac{3}{4}(1+x)$$ ← 3 marks

so $\sup \frac{f(x)}{g(x)}$ as $x \to 1$ ← 1 mark

let $c^* = \frac{3}{2}$ ← 1 mark

$\Rightarrow \mathbb{E}[\#\text{ candidate } Y] = c^* = 3/2$ ← 1 mark

2\. i) a) Estimate of $\mathbb{E}[M] = £243{,}000$ ← 1 mark

CI is $0.243 \pm 1.96\sqrt{\dfrac{8.176}{400}}$ + 1 mark units

2 marks

i.e. £ $(-0.037, 0.523)$ million

[4 marks]

b) Need

$$2 \times 1.96\sqrt{\frac{8.176}{n}} < 0.1$$ - 2 for method
+2 for soln

i.e. $n > 12550$ [4 marks]

c) Write

$$M^* = \iint c(\Theta, \phi)\, p(\Theta)\, g(\phi)\, d\phi\, d\Theta$$

<u>Note</u> pdf of $\text{Beta}(\alpha, \beta) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$, $g(\phi)$ is pdf $\text{Beta}(3,5)$

$$M^* = \iint c(\Theta, \phi)\, \frac{g(\phi)}{p(\phi)}\, p(\Theta)\, p(\phi)\, d\phi\, d\Theta$$

3 marks method

where $p(\phi)$ is pdf of $\text{Beta}(4,2)$

<u>So</u> under original dist. for Θ and ϕ, we want expectation of

$$c(\Theta, \phi)\, \frac{g(\phi)}{p(\phi)}$$ ← 3 marks

2 marks formula

Hence estimate is $\hat{M}^* = \dfrac{1}{400}\displaystyle\sum_{i=1}^{400} a_i \dfrac{g(\phi_i)}{p(\phi_i)}$ where $\phi_1, \dots, \phi_{400}$ are original sampled ϕ values

[8 marks total]

ii) a) Testing if means are equal using F statistic

$H_0: \mu_A = \mu_B = \mu_C$

2 marks

H_1: Means not equal

Method 1

- Randomisation Test
- Reallocate students to groups (as under H_0)
- See how extreme observed test stat is compared with other random reallocations   2 marks

Method II

- Non-parametric bootstrap   2 marks [non-p + bootstrap]
- Draw with replacement exam marks from ecdf

[6 marks total]

b) Method I: p-value 0.0229

Strong evidence to reject null in favour that group means not equal

Method II: p-value 0.0221

Same conclusion as above

[3 marks - 2 p-val, 1 for strength]

c) Allocation of students to groups was performed at random

[2 marks]

d) Incorrect. Bootstrap relies also on good approx. to ecdf

↑ N will reduce Monte Carlo error but will not effect ability to approximate ecdf as this depends on original sample size i.e. 24   [3 marks - 1 sol^n + 2 exp.]

3. Have Poisson(λ) Poisson(μ)

$$X \sim \begin{cases} \text{Poisson}(\lambda) & \text{if } Y=0 \\ \text{Poisson}(\mu) & \text{if } Y=1 \end{cases}$$

$$Y = \begin{cases} 0 & \text{w.p. } 1-\omega \\ 1 & \text{w.p. } \omega \end{cases}$$

$$P(X,Y|\Theta) = P(Y)P(X|Y,\Theta)$$

i)

3M method

$$= \prod_{Y_i=0} (1-\omega)\frac{\lambda^{x_i}}{x_i!} e^{-\lambda} \prod_{Y_i=1} \omega \frac{\mu^{x_i}}{x_i!} e^{-\mu}$$

$$\Rightarrow \quad \ell(\lambda,\mu,\omega; X,Y) = \sum_{Y_i=0} \log(1-\omega) + x_i \log\lambda - \log x_i! - \lambda$$

$$+ \sum_{Y_i=1} \log\omega + x_i\log\mu - \log x_i! - \mu$$

4M for correct sol.^{n}

$$= \sum (1-Y_i)\left[\log(1-\omega) + x_i\log\lambda - \log x_i! - \lambda\right]$$

$$+ Y_i\left[\log\omega + x_i\log\mu - \log x_i! - \mu\right]$$

[7 marks total]

ii)

<u>Mle</u>

$$\hat{\omega} \qquad \frac{\partial \ell}{\partial \omega} = -\sum \frac{(1-Y_i)}{1-\omega} + \frac{Y_i}{\omega}$$

$$\Rightarrow \quad \hat{\omega} = \frac{\sum Y_i}{n}$$

$$\hat{\lambda} \qquad \frac{\partial \ell}{\partial \lambda} = \sum_i (1-Y_i)\left[\frac{x_i}{\lambda} - 1\right] = 0$$

$$\Rightarrow \frac{1}{\lambda}\sum_i (1-Y_i)X_i - \sum_i (1-Y_i) = 0$$

$$\Rightarrow \hat{\lambda} = \frac{\sum_i (1-Y_i)X_i}{\sum_i (1-Y_i)}$$

$$\hat{\mu} = \frac{\sum_i Y_i X_i}{\sum_i Y_i}$$ [6 marks total - 2 for each]

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iii) Look at terms in log-likelihood involving λ, μ, ω and data $\underline{X}, \underline{Y}$.

$\cancel{\hat{\mu}_{new}}$ Involve $\sum X_i$, $\sum Y_i$, $\sum X_i Y_i$ [3 marks - 1 each]

iv) Using Bayes

$$P(Y_i = 1 \mid X_i) = \frac{P(X_i \mid Y_i = 1)\, P(Y_i = 1)}{P(X_i)}$$ ← 2M for method

$$= \frac{\omega \frac{\mu^{x_i}}{x!} e^{-\mu}}{\omega \frac{\mu^{x_i}}{x_i!} e^{-\mu} + (1-\omega)\frac{\lambda^{x_i}}{x_i!} e^{-\lambda}}$$

$$= \frac{\omega \mu^{x_i} e^{-\mu}}{\omega \mu^{x_i} e^{-\mu} + (1-\omega)\lambda^{x_i} e^{-\lambda}} = p_i$$ ↙ 3M for sol.^{n}

[5 marks total]

v) $Q(\Theta|\Theta_{old}) = \mathbb{E}\left[\ell(\Theta; X, Y) \mid X, \Theta = \Theta_{old} \right]$

Maximise at

$$\omega_{new} = \frac{\sum_{i=1}^{n} \mathbb{E}[Y_i | X_i, \Theta_{old}]}{n}$$

$$\lambda_{new} = \frac{\sum X_i \mathbb{E}[(1-Y_i)|X_i, \Theta_{old}]}{\sum \mathbb{E}[1-Y_i|X_i, \Theta_{old}]}$$ ← 5 marks for formula

$$\mu_{new} = \frac{\sum X_i \; \mathbb{E}[Y_i|X_i, \Theta_{old}]}{\sum \mathbb{E}[Y_i|X_i, \Theta_{old}]}$$

Now $\mathbb{E}[Y_i|X_i] = p_i$ ← 1 mark

$$\Rightarrow \quad \omega_{new} = \frac{\sum p_i}{n}$$

$$\lambda_{new} = \frac{\sum x_i(1-p_i)}{\sum (1-p_i)}$$ 1 mark each

$$\mu_{new} = \frac{\sum x_i p_i}{\sum p_i}$$

[9 marks total]

4.

i) a) $$P(X > c_i) = \int_{c_i}^{\infty} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} \, dx$$

$$= \left[e^{-(x/\lambda)^k} \right]_{\infty}^{c_i}$$

$$= e^{-(c_i/\lambda)^k}$$

[3 marks - 1 method + 2 sol.^{n}]

b) $$L(\lambda, k; y) = \prod_{i=1}^{n} f(x_i; \lambda, k) \prod_{i=n+1}^{n+m} P(X > c_i) \leftarrow 2 \text{ for method}$$

$$= \prod_{i=1}^{n} k \lambda^{-k} x_i^{k-1} e^{-(x_i/\lambda)^k} \prod_{i=n+1}^{n+m} e^{-(c_i/\lambda)^k}$$

3 for plug-in

$$\Rightarrow \ell(\lambda, k; y) = \sum_{i=1}^{n} \left(\log k - k \log \lambda + (k-1) \log x_i - \lambda^{-k} x_i^k \right) + \sum_{i=n+1}^{n+m} \left(-\lambda^{-k} c_i^k \right)$$

$$= n \log k - nk \log \lambda + (k-1) \sum_{i=1}^{n} \log x_i - \frac{1}{\lambda^k} \left[\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k \right] \leftarrow 3 \text{ for sol.}^n$$

[8 marks total]

c) $$\frac{\partial \ell}{\partial \lambda} = -\frac{nk}{\lambda} + \frac{k}{\lambda^{k+1}}\left[\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k\right]$$

Solve

$$\frac{nk}{\hat{\lambda}} = \frac{k}{\hat{\lambda}^{k+1}} C \qquad \text{where } C = \sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k$$

$$\Rightarrow \hat{\lambda}^k = \frac{C}{n}$$

$$\Rightarrow \hat{\lambda}^k = \frac{\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k}{n}$$

$$\Rightarrow \ell_p(k) = n \log k - n \log\left(\frac{\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k}{n}\right)$$

$$+ (k-1)\sum_{i=1}^{n} \log x_i - n$$

[7 marks - 4 find $\hat{\lambda}$
3 plug-in to get ℓ_p]

ii) Have Weibull (λ, k)

May use actual Weibull$(4,6)$ as opposed to general (λ, k) ..

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

$$\Rightarrow \quad \ell(x) = \log f(x) = \log k - k\log\lambda + (k-1)\log x - \left(\frac{x}{\lambda}\right)^k$$

1M find $\ell(\cdot)$

To find mode, consider

$$\ell'(x) = \frac{k-1}{x} - \frac{k x^{k-1}}{\lambda^k} = 0 \qquad [k>1]$$

$$\Rightarrow \quad \bar{x}^k = \lambda^k \frac{k-1}{k}$$

$$\Rightarrow \quad \bar{x} = \lambda\left(\frac{k-1}{k}\right)^{1/k}$$

4 marks find $\bar{x}$

To find variance, need

$$M = \ell''(x)\Big|_{x=\bar{x}} = \frac{-(k-1)}{\bar{x}^2} - \frac{k(k-1)\bar{x}^{k-2}}{\lambda^k}$$

$$= -\left[\frac{k-1}{\lambda^2\left(\frac{k-1}{k}\right)^{2/k}} + \frac{k(k-1)\lambda^{k-2}\left(\frac{k-1}{k}\right)^{k-2/k}}{\lambda^k}\right]$$

$$= \frac{-1}{\lambda^2}\left[(k-1)\left(k/_{k-1}\right)^{2/k} + k(k-1)\left(\frac{k-1}{k}\right)^{\frac{k-2}{k}}\right]$$

5 marks find M

Now set $\lambda = 4$

and $k = 6$ to find

$$\bar{x} = 4 \left(5/6\right)^{1/6}$$

$$M = -\frac{1}{16}\left[5 \left(6/5\right)^{1/3} + 6.5 \left(5/6\right)^{2/3} \right]$$

so use normal approximation with

$$\mu = 4 \left(5/6\right)^{1/6}$$ ← 2 marks sol.^{n}

$$\sigma^2 = -M^{-1}$$

[12 marks]