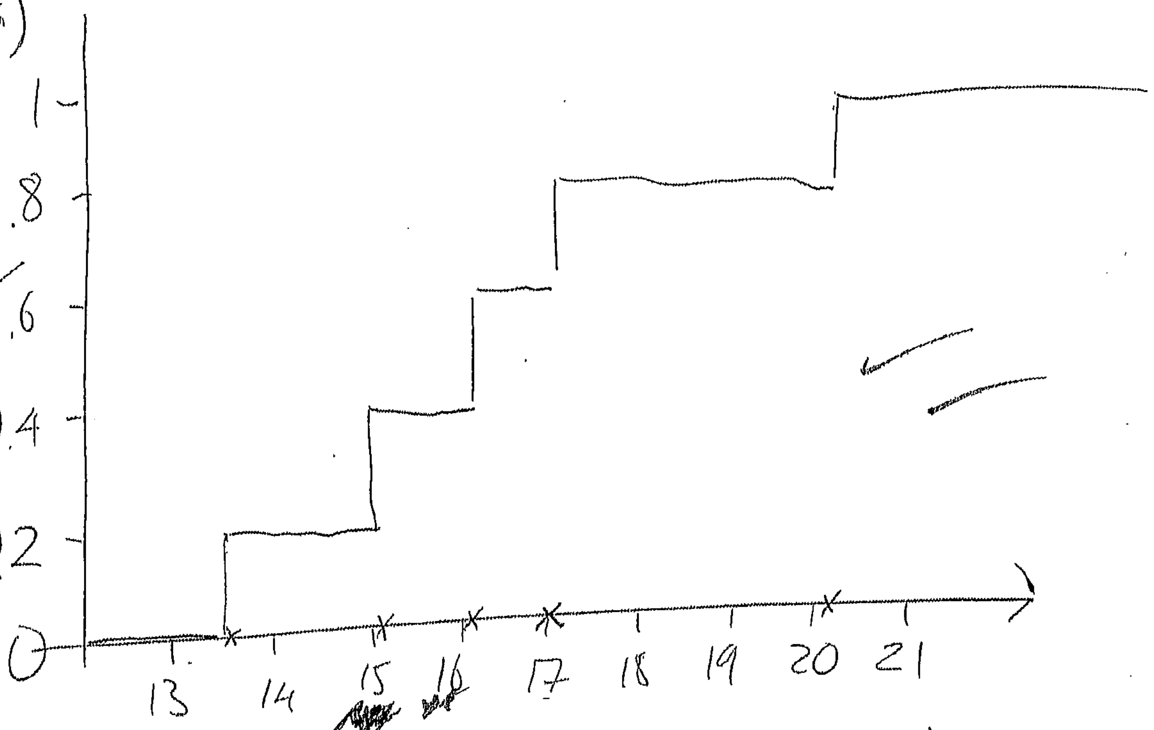
1/ (i) a)



b) Pick X_1 if U_1 lies in $[0, 0.2)$
X_2 .. U_2 . . . . $[0.2, 0.4)$ etc

[NB - other possible approaches here]

giving 13.6, 20.2, 20.2, 16.1, 16.1

& so the bootstrap median is 16.1

c) To estimate the std error of the median, we would repeat this to get B bootstrap estimates $m_1, .., m_B$ say & then calculate the observed / sample std. deviation

$$s = \left(\frac{1}{B-1} \sum (m_i - \bar{m})^2\right)^{1/2}$$

1) c) d) (iii) a) Use ~~$T = \mathrm{Var}(t_{ij}) = \frac{1}{5-1} \sum (t_i - \bar{t})^2$~~

~~as a test statistic.~~

To conduct the MC test, we first fit the exponential dist^n to the data & estimate

$$\hat{\lambda} = \frac{1}{\bar{t}}$$

We then do

for $i = 1, \dots, B$

- Simulate $\tilde{t}_1, \dots, \tilde{t}_{81} \sim \mathrm{Exp}(\hat{\lambda})$
- Calculate $\tilde{T}_i = \frac{1}{81-1} \sum_{j=1}^{81} (\tilde{t}_j - \bar{\tilde{t}})^2$

& estimate the p-value by

$$\frac{1}{B} \left(\sum \mathbb{I}_{\tilde{T}_i \leq T_{obs}} + 1 \right)$$

~~as we expect~~

If the null hypothesis of independence is incorrect, we expect the variance to be smaller & so we are looking to see if T_{obs} is smaller compared to what is expected under H_0

b) We could generate a p-value of 0.05 with as few as 19 ~~test~~ statistics.

This would not be advisable ∵ the test is random & so even if the true p-value is >0.05 there is a reasonable probability we would not observe any simulated values less than T_{obs} if we only used 19 samples.

2 (i)(a)

$$G(x) = \mathbb{P}(X \leq x) = \int_0^x g(x')\,dx'$$

$$= \int_0^x \frac{2}{\pi(1+x'^2)}\,dx'$$

Let $x' = \tan u$

$$\frac{dx'}{du} = \sec^2 u$$

$$= \int_0^{\tan^{-1}(x)} \frac{2}{\pi(1+\tan^2 u)} \sec^2 u\,du$$

$$= \int_0^{\tan^{-1}(x)} \frac{2}{\pi}\,du = \frac{2}{\pi}\tan^{-1}(x)$$

(b) $G^{-1}(u) = \tan\left(\frac{\pi u}{2}\right)$

So $X_1 = \tan\left(\frac{0.2256 \times \pi}{2}\right) = 6.4716$

$X_2 = 1.1228$

$X_3 = 0.1997$

(c)

(c) To sample from $f(\cdot)$ using rejection, we can sample from $g(\cdot)$ & then accept ~~[illegible]~~ the simulated value w.p.

$$w = \frac{f(\cdot)}{cg(\cdot)} = \frac{\frac{2}{(1+x)^3}}{c\frac{2}{\pi(1+x^2)}} = \frac{\pi(1+x^2)}{c(1+x)^3}$$

Where we choose c so that $w \leq 1$

i.e. $$c = \max\left(\pi\frac{(1+x^2)}{(1+x)^3}\right)$$

This is clearly maximized at $x=0$ & so we set $c = \pi$

So the rejection algorithm we can use is

Set $X_i = \tan\left(\frac{\pi U_i}{2}\right)$

Accept X_i w.p. $\frac{1+X_i^2}{(1+X_i)^3}$

~~4~~ (ii) Sample $X_1, \dots, X_n$ from the prior $\pi(x)$.

Set $$w_i = \frac{f(X_i)}{\pi(y \mid X_i)}$$

~~We require~~

~~The appro~~ Set $$\hat{w}_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$$

Approximate Q by
$$\sum_{i=1}^{n} \hat{w}_i h(X_i)$$

~~Q2~~

Q3

(ii) $$\mathrm{Var}(\overline{X}_{2n}) = \mathrm{Var}\left(\frac{1}{2n}\sum_{i=1}^{2n} X_i\right)$$

$$= \frac{1}{4n^2}\sum \mathrm{Var}(X_i)$$

$$= \frac{1}{2n}\mathrm{Var}(X_1)$$

$$\mathrm{Var}(X_1) = \mathbb{E}X_1^2 - (\mathbb{E}X_1)^2 = \frac{1}{I^2} = 1$$ (can quote)

Q3 ctd (i)(a)

Thus $Var(\bar{X}_{2n}) = \frac{1}{2n}$

(b) $Var\left(\frac{\bar{X}_n + \bar{Y}_n}{2}\right) = \frac{1}{4} Var\left(\frac{\sum_{i=1}^{n}(X_i + Y_i)}{n}\right)$

$= \frac{n}{4n^2} Var(X_1 + Y_1)$ as $X_i, Y_i \perp\!\!\!\perp X_j, Y_j$ for $j \neq i$

$Var(X_1 + Y_1) = Var(X_1) + Var(Y_1) + 2\,Cov(X_1, Y_1)$

$= 1 + 1 + 2\left(\mathbb{E}(X_1 Y_1) - \mathbb{E}X_1 \mathbb{E}Y_1\right)$

$= 2\mathbb{E}(X_1 Y_1)$

$= 2\int_0^1 (-\log u)(-\log(1-u))\, du$

$= 2\int_0^1 \log x \log(1-x)\, dx = 2I$

Thus $Var\left(\frac{\bar{X}_n + \bar{Y}_n}{2}\right) = \frac{I}{2n}$ as required.

(c) $\mathbb{E}(\overline{X_{2n}}) = \mathbb{E}\left(\frac{\overline{X_n} + \overline{Y_n}}{2}\right) = 1$

so both $\overline{X_{2n}}$ & $\frac{\overline{X_n} + \overline{Y_n}}{2}$ estimate the mean & they both use $2n$ random draws.

But $Var\left(\frac{\overline{X_n} + \overline{Y_n}}{2}\right) = \frac{1}{2n}\left(2 - \frac{\pi^2}{6}\right) \leq \frac{1}{2n} = Var(\overline{X_{2n}})$

as $1 < \frac{\pi^2}{6} < 2$

& so $\frac{\overline{X_n} + \overline{Y_n}}{2}$ is more accurate than $\overline{X_{2n}}$ (lower variance)

This is an example of the use of antithetic variables.

(iii) $L(t \mid \alpha, \beta) = \prod_{i=1}^{4} (\alpha\beta)(\beta t_i)^{\alpha-1} e^{-(\beta t_i)^\alpha}$

$= (\alpha\beta)^4 \beta^{4\alpha-4} \left(\prod t_i\right)^{\alpha-1} e^{-\sum(\beta t_i)^\alpha}$

So $l(t \mid \alpha, \beta) = 4\log\alpha + 4\alpha\log\beta - (\alpha-1)\sum\log t_i - \sum(\beta t_i)^\alpha$

$\frac{dl}{d\beta} = \frac{4\alpha}{\beta} - \alpha\beta^{\alpha-1}\sum t_i^\alpha$

& thus $\hat{\beta} = \left(\frac{4}{\sum t_i^\alpha}\right)^{1/\alpha}$

Hence $l(p) = 4\log\alpha + 4\log\left(\frac{4}{\sum t_i^{\alpha}}\right) + (\alpha-1)\sum \log t_i - 4$

(ii) $l_p(\hat{\alpha}) = -12.2$

$l_p(1) = 0 + 4\log\left(\frac{4}{42}\right) + 0 - 4$

$= -13.41$

So $D_p(1) = -2\left(l_p(1) - l_p(\hat{\alpha})\right)$

$= -2(-13.41 - -12.2)$

$= 2.42$

Compare to a χ_1^2 distribution

$2.42 < \chi_1^2(0.95) = 3.84$ so no evidence to reject H_0 at 5% level.

Q4

~~(i)~~ (ia)

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} \prod_{j=1}^{m} \lambda e^{-\lambda s_j} = \lambda^{m+n} \exp\left(-\lambda\left(\sum t_i + \sum s_j\right)\right)$$

So $\log L(\lambda) = (m+n)\log \lambda - \lambda\left(\sum_{i=1}^{n} t_i + \sum_{j=1}^{m} s_j\right)$

(ii) (b) $\mathbb{E}(S \mid S \leq h, \lambda)$ requires us to calculate the pdf of S given $S \leq h$.

$$\mathbb{P}(S \leq t \mid S \leq h) = \frac{\mathbb{P}(S \leq t)}{\mathbb{P}(S \leq h)} = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda h}}$$

Then $\mathbb{E}(S \mid S \leq h, \lambda) = \int_0^h \mathbb{P}(S > t \mid S \leq h)\, dt$

$$= \int_0^h 1 - \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda h}}\, dt$$

$$= h - \frac{1}{1 - e^{-\lambda h}}\left[t + \frac{e^{-\lambda t}}{\lambda}\right]_0^h$$

$$= h - \frac{1}{1 - e^{-\lambda h}}\left(h + \frac{e^{-\lambda h}}{\lambda} - \frac{1}{\lambda}\right)$$

$$= \frac{1}{\lambda} - \frac{h e^{-\lambda h}}{1 - e^{-\lambda h}}$$ as req^{d}.

[Or we can find pdf $\frac{dG(t)}{dt} = \frac{\lambda e^{-\lambda t}}{1-e^{-\lambda h}}$

& use $\int_0^h t\,g(t)\,dt = \mathbb{E}(S \mid S \le h)$]

(iii) $Q(\lambda \mid \lambda^{(r)}) = \mathbb{E}_{\lambda^{(r)}}\left(\log L(\lambda \mid t_1, \dots, t_n, S_1, \dots, S_m) \,\middle|\, \begin{array}{l} t_1, \dots, t_n \\ r \text{ of the } S_i < h \\ m-r \text{ of the } S_i > h, \lambda^{(r)} \end{array}\right)$

$$= (m+n)\log\lambda - \lambda\left(\sum_{i=1}^{n} t_i + (m-r)\,\mathbb{E}(S \mid S>h) + r\,\mathbb{E}(S \mid S \le h)\right)$$

$$= (m+n)\log\lambda - \lambda\left(n\bar{t} + (m-r)\left(h + \frac{1}{\lambda^{(r)}}\right) + r\left(\frac{1}{\lambda^{(r)}} - \frac{h e^{-h\lambda^{(r)}}}{1-e^{-h\lambda^{(r)}}}\right)\right)$$

as $\mathbb{E}(S \mid S>h) = h + \frac{1}{\lambda^{(r)}}$ because S is exponential & has the memoryless property

So that $\pi(s \mid S>h) = \begin{cases} e^{-\lambda(s-h)} & s \ge h \\ 0 & \text{otherwise} \end{cases}$

(iii) Thus to find $\lambda^{(r+1)}$ we minimize $Q(\lambda|\lambda^{(r)})$

(iv) $$\frac{dQ}{d\lambda} = \frac{m+n}{\lambda} - \left(n\bar{t} + (m-r)\left(h + \frac{1}{\lambda^{(r)}}\right) + r\left(\frac{1}{\lambda^{(r)}} - \frac{he^{-h\lambda^{(r)}}}{1-e^{-h\lambda^{(r)}}}\right)\right)$$

& so $$\lambda^{(r+1)} = \frac{m+n}{n\bar{t} + (m-r)\left(h + \frac{1}{\lambda^{(r)}}\right) + r\left(\frac{1}{\lambda^{(r)}} - \frac{he^{-\lambda h^{(r)}}}{1-e^{-h\lambda^{(r)}}}\right)}$$