4.

i) a)

$$f_X(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \begin{cases} x^2/2 & 0 \leq x \leq 1 \\ 1/2 + \int_1^x (2-t)\,dt & \end{cases}$$ ← 1M method

$$\int_1^x (2-t)\,dt = \left[2t - \frac{t^2}{2}\right]_1^x = 2x - \frac{x^2}{2} - \frac{3}{2}.$$

$$\Rightarrow F_X(x) = \begin{cases} 0 & x < 0 \\ x^2/2 & 0 \leq x \leq 1 \\ 2x - \frac{x^2}{2} - 1 & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$ ← 1M each $\left[\text{or } 1 - \frac{(2-x)^2}{2}\right]$

[3 marks total]

b) To invert, $U \leq 1/2$

$$U = \frac{X^2}{2} \qquad U \leq 1/2$$ 1 M

$$\Rightarrow X = \sqrt{2U} \qquad U \leq 1/2$$

$U > 1/2$

$$U = 2X - \frac{X^2}{2} - 1$$

$$\Rightarrow X^2 - 4X = -2 - 2U$$ 2M invert and correct root

$$\Rightarrow (X-2)^2 = 2(1+U)$$

$$\Rightarrow X = 2 - \sqrt{2(1+U)}$$ as $1 \leq X \leq 2$ take -ve root ~~take -ve root~~

so algorithm becomes

① Sample $U \sim U[0,1]$

1M method

② Set

$$X = \begin{cases} \sqrt{2U} & 0 \le u \le 1/2 \\ 2 - \sqrt{2(1+u)} & 1/2 < u \le 1 \end{cases}$$

[4 marks total]

ii) a) Have $f_X(x) = \frac{3}{2}(1-x^2) \quad 0 \le x \le 1$

use envelope $g_Y(y) = 1 \quad 0 \le y \le 1$

set $c = \sup \frac{f(x)}{g(x)}$

$f(x)$ maximised at $x=0$ so let

$c = \frac{f(0)}{1} = \frac{3}{2}$ 4M find c

so rejection algorithm is

① Sample $Y \sim U[0,1]$ and $U \sim U[0,1]$

② If $U \le \frac{f(Y)}{c} = \frac{3(1-Y^2)}{2c} = (1-Y^2)$

then accept $X = Y$, otherwise reject and return to step ①

2M lay out method

Given $Y = 0.5$ $U = 0.8$ 2M for working out reject

we have $1 - Y^2 = 3/4 < 0.8$ so would reject candidate Y

[8 marks total]

~~c) $\mathbb{E}[\#\text{ candidate } Y] = c = \frac{3}{2}$~~

b) For new envelope

$$\frac{f(x)}{g(x)} = \frac{\frac{3}{2}(1-x^2)}{2(1-x)} = \frac{\frac{3}{2}(1-x)(1+x)}{2(1-x)}$$

$$= \frac{3}{4}(1+x) \quad \text{2M}$$

so $\sup \frac{f(x)}{g(x)}$ as $x \to 1$ 1M

let $c^* = \frac{3}{2}$ 1M

$\Rightarrow \mathbb{E}[\#\text{ candidate } Y] = c^* = 3/2$ ← 1M

[5 mark total]

5\. Have Poisson (λ) Poisson (μ)

$$X \sim \begin{cases} \text{Poisson}(\lambda) & \text{if } Y=0 \\ \text{Poisson}(\mu) & \text{if } Y=1 \end{cases}$$

$$Y = \begin{cases} 0 & \text{w.p. } 1-\omega \\ 1 & \text{w.p. } \omega \end{cases}$$

$$P(X,Y|\theta) = P(Y)P(X|Y,\theta)$$

i)

2M method

$$= \prod_{Y_i=0} (1-\omega)\frac{\lambda^{x_i}}{x_i!}e^{-\lambda} \prod_{Y_i=1} \omega \frac{\mu^{x_i}}{x_i!}e^{-\mu}$$

$$\Rightarrow \quad \ell(\lambda,\mu,\omega;X,Y) = \sum_{Y_i=0} \log(1-\omega) + x_i \log\lambda - \log x_i! - \lambda$$

$$+ \sum_{Y_i=1} \log\omega + x_i\log\mu - \log x_i! - \mu$$

3M corr. sol.^{n}

$$= \sum (1-Y_i)\left[\log(1-\omega) + x_i\log\lambda - \log x_i! - \lambda\right]$$

$$+ Y_i\left[\log\omega + x_i\log\mu - \log x_i! - \mu\right]$$

[5 marks total]

ii)

Mle

$\hat{\omega}$

$$\frac{\partial \ell}{\partial \omega} = -\sum \frac{(1-Y_i)}{1-\omega} + \frac{Y_i}{\omega}$$

1M method

$$\Rightarrow \quad \hat{\omega} = \frac{\sum Y_i}{n}$$

1M each sol.^{n}

$\hat{\lambda}$ $\quad \dfrac{\partial \ell}{\partial \lambda} = \sum_i (1-y_i)\left[\dfrac{x_i}{\lambda} - 1\right] = 0$

$$\Rightarrow \frac{1}{\lambda} \sum_i (1-y_i)x_i - \sum_i (1-y_i) = 0$$

$$\Rightarrow \hat{\lambda} = \frac{\sum_i (1-y_i)x_i}{\sum_i (1-y_i)}$$

$$\hat{\mu} = \frac{\sum_i y_i x_i}{\sum_i y_i}$$

[4 marks total]

~~iii) Look at terms in log-likelihood involving λ, μ, ω and data $\underline{x}, \underline{y}$.~~

~~$\hat{\mu}$ now~~ ~~Involve $\sum x_i$, $\sum y_i$, $\sum x_i y_i$~~

iii) Using Bayes

$$P(y_i = 1 \mid x_i) = \frac{P(x_i \mid y_i = 1)\, P(y_i = 1)}{P(x_i)}$$ ↖ 2M method

$$= \frac{\omega \dfrac{\mu^{x_i}}{x!} e^{-\mu}}{\omega \dfrac{\mu^{x_i}}{x_i!} e^{-\mu} + (1-\omega) \dfrac{\lambda^{x_i}}{x_i!} e^{-\lambda}}$$

↙ 2M sol.^{n}

$$= \frac{\omega\, \mu^{x_i} e^{-\mu}}{\omega \mu^{x_i} e^{-\mu} + (1-\omega) \lambda^{x_i} e^{-\lambda}} = p_i$$

[4 marks total]

iv) $Q(\Theta|\Theta_{old}) = \mathbb{E}\left[\ell(\Theta; X, Y) \mid X, \Theta = \Theta_{old}\right]$

Maximise at

$$\omega_{new} = \frac{\sum_{i=1}^{n} \mathbb{E}[Y_i|X_i, \Theta_{old}]}{n}$$

$$\lambda_{new} = \frac{\sum X_i \mathbb{E}[(1-Y_i)|X_i, \Theta_{old}]}{\sum \mathbb{E}[1-Y_i|X_i, \Theta_{old}]}$$ ← 3M for formulae

$$\mu_{new} = \frac{\sum_i X_i \, \mathbb{E}[Y_i|X_i, \Theta_{old}]}{\sum_i \mathbb{E}[Y_i|X_i, \Theta_{old}]}$$

Now $\mathbb{E}[Y_i|X_i] = p_i$ ← 1M

$$\Rightarrow \omega_{new} = \frac{\sum_i p_i}{n}$$

$$\lambda_{new} = \frac{\sum_i x_i(1-p_i)}{\sum_i (1-p_i)}$$ ← 1M each

$$\mu_{new} = \frac{\sum_i x_i p_i}{\sum_i p_i}$$

[7 marks total]

6.

i) a) $$P(X > c_i) = \int_{c_i}^{\infty} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k} \, dx$$

$$= \left[e^{-(x/\lambda)^k} \right]_{\infty}^{c_i}$$

$$= e^{-(c_i/\lambda)^k}$$

[2 marks - 1 method + 1 sol.ⁿ]

b) $$L(\lambda, k; y) = \prod_{i=1}^{n} f(x_i; \lambda, k) \prod_{i=n+1}^{n+m} P(X > c_i)$$

$$= \prod_{i=1}^{n} k \lambda^{-k} x_i^{k-1} e^{-(x_i/\lambda)^k} \prod_{i=n+1}^{n+m} e^{-(c_i/\lambda)^k}$$

← 2 for method

$$\Rightarrow \quad \ell(\lambda, k; y) = \sum_{i=1}^{n} \left(\log k - k \log \lambda + (k-1) \log x_i - \lambda^{-k} x_i^{k} \right) + \sum_{i=n+1}^{n+m} \left(- \lambda^{-k} c_i^{k} \right)$$

$$= n \log k - nk \log \lambda + (k-1) \sum_{i=1}^{n} \log x_i - \frac{1}{\lambda^k} \left[\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k \right]$$

← 3 for sol.ⁿ

[5 mark total]

c) $$\frac{\partial \ell}{\partial \lambda} = -\frac{nk}{\lambda} + \frac{k}{\lambda^{k+1}}\left[\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k\right]$$

Solve
$$\frac{nk}{\hat{\lambda}} = \frac{k}{\hat{\lambda}^{k+1}} C \qquad \text{where} \quad C = \sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k$$

$$\Rightarrow \hat{\lambda}^k = \frac{C}{n}$$

$$\Rightarrow \hat{\lambda}^k = \frac{\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k}{n}$$

$$\Rightarrow \ell_p(k) = n \log k - n \log\left(\frac{\sum_{i=1}^{n} x_i^k + \sum_{i=n+1}^{n+m} c_i^k}{n}\right)$$
$$+ (k-1)\sum_{i=1}^{n} \log x_i - n$$

[6 marks total – 3 find $\hat{\lambda}$
3 plug-in]

ii) Have Weibull (λ, k)

Probably easier to use Weibull(4,6) as opposed to a general (k,λ). Either is fine

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

$$\Rightarrow \quad \ell(x) = \log f(x) = \log k - k\log\lambda + (k-1)\log x - \left(\frac{x}{\lambda}\right)^k$$

1M take logs

To find mode, consider

$$\ell'(x) = \frac{k-1}{x} - \frac{k x^{k-1}}{\lambda^k} = 0 \qquad [k>1]$$

$$\Rightarrow \quad \bar{x}^k = \lambda^k \frac{k-1}{k}$$

$$\Rightarrow \quad \bar{x} = \lambda\left(\frac{k-1}{k}\right)^{1/k}$$

2M find $\bar{x}$

To find variance, need

$$M = \ell''(x)\Big|_{x=\bar{x}} = \frac{-(k-1)}{\bar{x}^2} - \frac{k(k-1)\bar{x}^{k-2}}{\lambda^k}$$

$$= -\left[\frac{k-1}{\lambda^2\left(\frac{k-1}{k}\right)^{2/k}} + \frac{k(k-1)\lambda^{k-2}\left(\frac{k-1}{k}\right)^{k-2/k}}{\lambda^k}\right]$$

$$= \frac{-1}{\lambda^2}\left[(k-1)\left(k/_{k-1}\right)^{2/k} + k(k-1)\left(\frac{k-1}{k}\right)^{\frac{k-2}{k}}\right]$$

3M find M and evaluate at $\bar{x}$

∴ Now set $\lambda = 4$

and $k = 6$ to find

$$\bar{x} = 4\left(5/6\right)^{1/6}$$

$$M = -\frac{1}{16}\left[5\left(6/5\right)^{1/3} + 6.5\left(5/6\right)^{2/3}\right]$$

So use normal approximation with

$$\mu = 4.\left(5/6\right)^{1/6}$$ 1M state sol^n

$$\sigma^2 = -M^{-1}$$

[7 marks total]