Hypothesis testing

Example fixed effects only

$H_0: \quad Y_i = \alpha + \varepsilon_i$ 'reduced model'

$H_1: \quad Y_i = \alpha + \beta x_i + \varepsilon_i$ 'full model'

$(\equiv \; H_0: \beta = 0 \quad \text{vs} \quad H_1: \beta \neq 0)$

F-test $\quad f = \dfrac{(D_0 - D_1)/q}{D_1/n-p}$

D_0 = deviance of reduced model. $D_1 = $ deviance of full model ($\approx$ RSS)

q = # parameter constraints $= 1$

p = # parameters in full model $= 2$

n = # data points

Under $H_0 \qquad f \sim F_{q,n-p}$

Or alternatively...

Generalized likelihood ratio test (GLRT)

Data $\underline{y} = (y_1, \dots, y_n)$

Full model has parameters $\underline{\varphi}$, likelihood $L(\underline{\varphi}|\underline{y}) = \prod_{i=1}^{n} f_i(y_i|\underline{\varphi})$

loglikelihood $\quad l(\underline{\varphi}|\underline{y}) = \sum_{i=1}^{n} \log f_i(y_i|\underline{\varphi})$

Compare with the reduced model

$\hookrightarrow$ parameters φ_s = obtained by applying r constraints to φ, i.e. reduced model is nested within full model.

Reduced model loglikelihood is

$$l(\underline{\varphi}_s|\underline{y}) = \sum_{i=1}^{n} \log f_i(y_i|\underline{\varphi}_s)$$

Eg ctd
Constraint is $\beta = 0$
So $r = 1$

Likelihood ratio

$$\Lambda(\underline{y}) = \frac{L(\underline{\hat{\varphi}}_s \mid \underline{y})}{L(\underline{\hat{\varphi}} \mid \underline{y})}$$

where $\underline{\hat{\varphi}}$ & $\underline{\hat{\varphi}}_s$ are the maximum likelihood estimates of $\underline{\varphi}$ & $\underline{\varphi}_s$

Note $0 \leq \Lambda \leq 1$ as full model will always fit better than the reduced model i.e. $L(\underline{\hat{\varphi}}_s \mid \underline{y}) \leq L(\underline{\hat{\varphi}} \mid \underline{y})$

Λ expresses how much more likely the data are under H_1 rather than H_0.

Large values of Λ (i.e. $\Lambda \simeq 1$) suggest the reduced model is as good as the full model

Small values of Λ (ie. $\Lambda \simeq 0$) suggest the full model is better than the reduced model.

GLRT says reject H_0 in favour of H_1 if Λ is smaller than expected under H_0.

GLRT 1) Find the MLEs $\underline{\hat{\varphi}}$ & $\underline{\hat{\varphi}}_s$

2) Calculate the test statistic

$$L = -2 \log \Lambda = -2\left(l(\underline{\hat{\varphi}}_s \mid \underline{y}) - l(\underline{\hat{\varphi}} \mid \underline{y})\right)$$

3) Calculate the distribution of L under H_0 & reject H_0 if $\mathbb{P}(L \geq L_{obs})$ is small.

[Λ is small if H_1 is better, thus $L = -2 \log \Lambda$ will be large if H_1 is better]

Wilks' Theorem: says that as $n \to \infty$, (
then (under a bunch of restrictions)

$$L \dot{\sim} \chi^2_r$$

Example ctd $\varphi = (\alpha, \beta, \sigma^2)$ $\quad \varphi_s = (\alpha, \sigma^2)$

1 constraint $\beta = 0$ so $r = 1$

Since $Y_i \sim N(\alpha + \beta x_i, \sigma^2)$

$$f_i(y_i | \varphi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(y_i - \alpha - \beta x_i\right)^2\right)$$

Exercise

We find $\underline{\hat{\varphi}} = (\hat{\alpha}, \hat{\beta}, \hat{\sigma^2})$

$$= \left(\bar{y} - \hat{\beta}\bar{x}, \ \frac{S_{xy}}{S_{xx}}, \ \frac{1}{n}\sum\left(y_i - \hat{\alpha}_i - \hat{\beta} x_i\right)^2\right)$$

↖ note $\frac{1}{n}$ not $\frac{1}{n-1}$

& $\underline{\hat{\varphi}}_s = (\hat{\alpha}, \hat{\sigma^2}) = \left(\bar{y}, \frac{1}{n}\sum(y_i - \bar{y})^2\right)$

So find $L_{obs} = -2\left(l(\underline{\hat{\varphi}}_s | \underline{y}) - l(\underline{\hat{\varphi}} | \underline{y})\right)$

& compare with a χ^2_1 random variable.