Computer class 4 solutions

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Solution 1

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+x'^2)} dx' = \int_{-\frac{\pi}{2}}^{\tan^{-1}(x)} \frac{\sec^2(u)}{\pi(1+\tan^2(u))} du \tag{1}$$

$$= \int_{-\frac{\pi}{2}}^{\tan^{-1}(x)} \frac{1}{\pi} du \tag{2}$$

$$= \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2} \tag{3}$$

Thus, by rearranging $U = F(X)$ we get

$$X = \tan\left(\pi(U - \frac{1}{2})\right)$$

To implement this, we can do the following:

```
U <- runif(10^6)
X <- tan(pi*(U-0.5))
```

To check this as requested

```
xx <- c(-10,-5,0,5,10)
sapply(xx,  function(x) sum(X<=x)/length(X) )
```

```
## [1] 0.031591 0.062311 0.500319 0.937105 0.968155
```

```
pcauchy(xx)
```

```
## [1] 0.03172552 0.06283296 0.50000000 0.93716704 0.96827448
```

Solution 2

$$g(x) = \frac{1}{2}g_1(x) + \frac{1}{2}g_2(x)$$

where

$$g_1(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$g_2(x) = \begin{cases} e^{x} & \text{if } x \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

So we can sample from $g(x)$ by sampling $Y \sim \text{Exp}(1)$ and then setting

$$X = \begin{cases} Y & \text{with probability } \frac{1}{2} \\ -Y & \text{otherwise.} \end{cases}$$

Let's start by creating a function to sample from g.

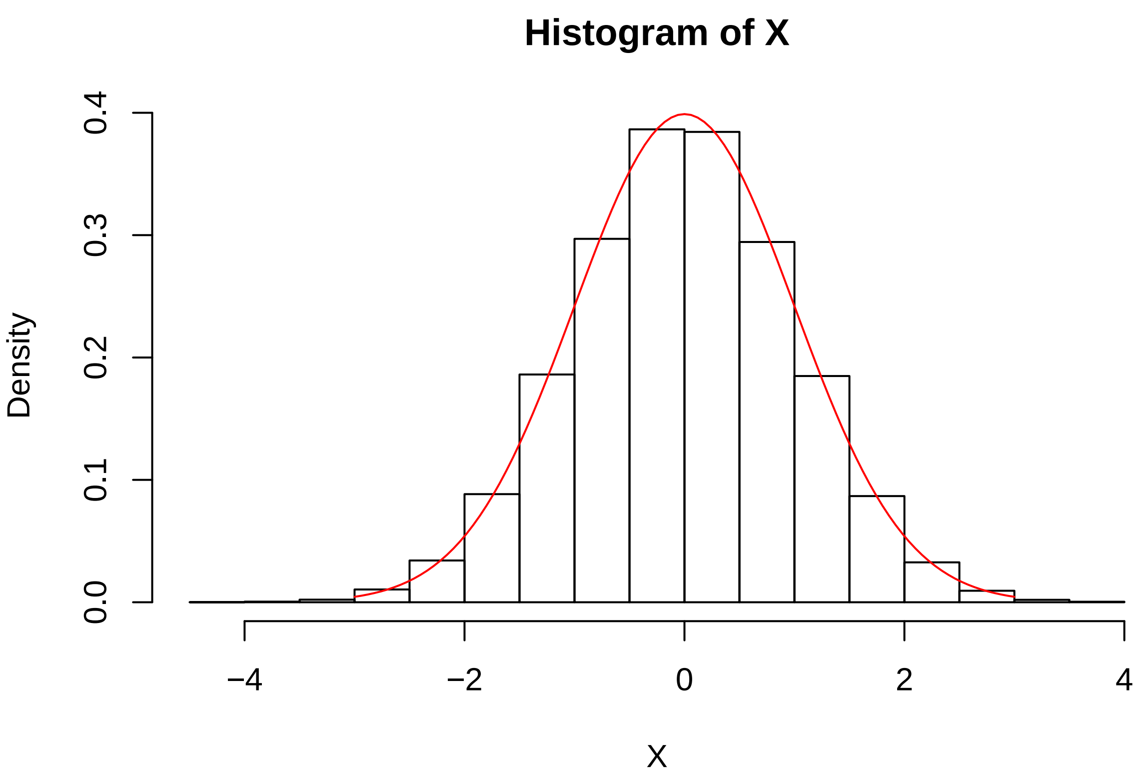
```
rg <- function(n){
  Y <- rexp(n,1)
  U <- sample(c(-1,1), n, replace=TRUE)
  return(Y*U)
}
```

We can write another function to calculate the acceptance probability:

```
acceptanceProb <- function(x){
  exp(abs(x) - x^2/2 -1/2)
}
```

We can then write a loop to do rejection sampling.

```
nacc<-0
X <- c()
while(nacc < 10^5){
  Y <- rg(1)
  if(runif(1)< acceptanceProb(Y)){
    nacc <- nacc+1
    X[nacc] <-Y
  }
}
hist(X, probability=TRUE)
curve(dnorm, -3,3, col=2, add=TRUE)
```



This shows how slow R is at doing loops. In this case it is quicker to do the vectorized calculation, even if this means we simulate more than the 10^5 random variables requested

```
Y<- rg(10^6)
p <- acceptanceProb(Y)
```

```r
accept <- runif(10^6)<=p
X2 <- Y[accept]
length(X2)
```

```
## [1] 760465
```

The accepance probability is

$$\frac{1}{M} = \sqrt{\frac{\pi}{2e}}$$

which we can check with

```r
sqrt(pi/(2*exp(1)))
```

```
## [1] 0.7601735
```

```r
length(X2)/10^6
```

```
## [1] 0.760465
```

Solution 3

The prior is

$$\pi(p) = \begin{cases} 1 & \text{if } p \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

and the likelihood is

$$\begin{aligned} \pi(X|p) &= \prod_{i=1}^{10} \binom{n}{x_i} p^{x_i}(1-p)^{n-x_i} \\ &\propto p^{\sum x_i}(1-p)^{10n-\sum x_i} \end{aligned}$$

To do rejection sampling, we are told to use $g(x) = 1$ on $[0,1]$. We need to calculate

$$M = \sup \frac{f(p)}{g(p)}$$

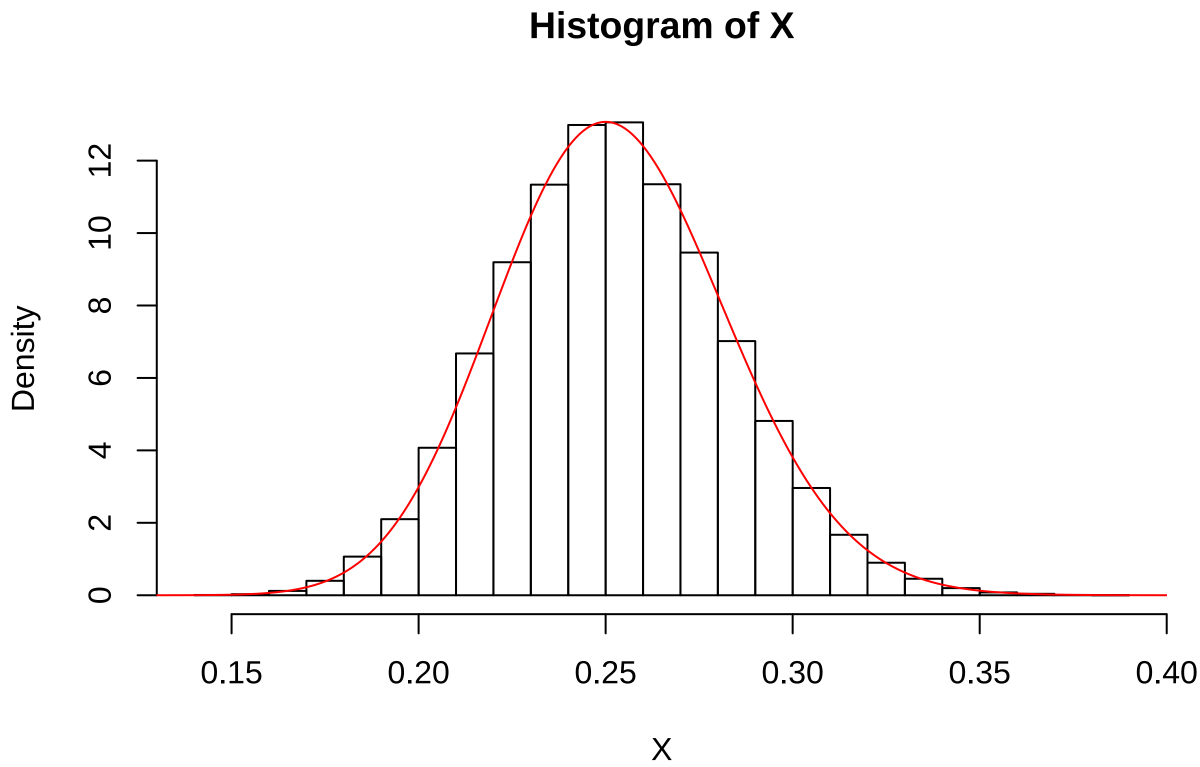
which obviously occurs at the MLE $\hat{p} = \frac{\sum x_i}{200}$ giving

$$M = \hat{p}^{50}(1-\hat{p})^{150}$$

```r
phat <- 50/200
M <- phat^{50}*(1-phat)^{150}
```

```r
Y <- runif(10^6)
U <- runif(10^6)
keep <- U <= Y^50*(1-Y)^150/M
X <- Y[keep]
```

```r
hist(X, probability=TRUE)
xvals <- seq(0,1,0.001)
lines(xvals, dbeta(xvals, shape1=51, shape2=151), col=2)
```


Histogram of X
Density
0
2
4
6
8
10
12
0.15
0.20
0.25
0.30
0.35
0.40
X