Computer class 4 exercises

Richard Wilkinson

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Question 1

The density of the standard Cauchy distribution is

$$f(x) = \frac{1}{\pi(1+x^2)}.$$

By using the substitution $x = \tan(u)$ or otherwise show that

$$F(x) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}.$$

Use the inversion method to derive an algorithm for generating a Cauchy random variable.

Implement this in R and use it to generate 10^6 random variables. Check your estimated values of

$$\mathbb{P}(X \leq -10), \mathbb{P}(X \leq -5), \mathbb{P}(X \leq 0), \mathbb{P}(X \leq 5), \text{ and } \mathbb{P}(X \leq 10)$$

against the true values using the built in CDF in R (`pcauchy`).

Question 2

Consider the density function

$$g(x) = \tfrac{1}{2}\mathrm{e}^{-|x|}$$

for $-\infty < x < \infty$. Show how $g(x)$ may be sampled from by considering it to be the mixture of two exponential distributions (hint: you may find point (d) on slide 17 useful).

(a) Derive a rejection sampling algorithm for sampling a standard normal random variable using g as the proposal distribution. Implement your method, and simulate 10^5 N(0,1) rvs. Check your answer.

(b) What is the acceptance probability of a single random draw from $g(x)$ for your algorithm? Check this numerically using your code.

Question 3

We will now consider using rejection sampling to solve a Bayesian inference problem. To begin with, we will consider a problem for which a conjugate analysis exists.

Suppose that we want to learn about the unknown parameter p, to which we assign a U[0,1] distribution. We collect data $x_1, \ldots, x_{10}$ which are independent $Bin(20, p)$ random variables. Show that the likelihood times the prior for this problem is proportional to

$$f_1(p) = p^{\sum x_i}(1-p)^{200-\sum_i x_i}$$

We are told that $\sum_{i=1}^{10} x_i = 50$. Describe a rejection sampling algorithm for sampling from the posterior distribution using a $U[0,1]$ distribution as the proposal density g and use it to draw a histogram of the posterior distribution.

In this case, we can calculate the posterior distribution analytically using a conjugate analysis. Show that the posterior distribution is

$$\pi(p|x_1, \ldots, x_{10}) = \text{Beta}(51, 151).$$

Check your code by plotting the pdf of this distribution on top of a histogram of samples you generated using the rejection algorithm.