Recap

effect of $i^{th}$ drug on $j^{th}$ patient on replicate k
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$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$$ fixed effects

CI for $\alpha_i$ $\quad \hat{\alpha}_i \pm \frac{1.96\,\hat{\sigma}}{\sqrt{2Jk}}$

$\alpha_1$ is not the effect of drug 1 on the pop^n

but the effect averaged over the patients in the study!

↳ who cares!

$$Y_{ijk} = \underbrace{\mu + \alpha_i^P}_{\text{Fixed}} + \underbrace{b_j + b_{ij} + \varepsilon_{ijk}}_{\text{Random}}$$

Mixed effect model

$b_j \overset{iid}{\sim} N(0, \sigma_{b_1}^2)$ — random effect for subject j

$b_{ij} \overset{iid}{\sim} N(0, \sigma_{b_2}^2)$ — random interaction btwn drug i & subject j

$\varepsilon_{ijk} \overset{iid}{\sim} N(0, \sigma^2)$ — random error: variation due to a single subject repeatedly taking the same drug

$\alpha_1^P$ = effect of drug 1 for the population of subjects.

A 95% CI for $\alpha_1^P$ is

$$\hat{\alpha}_1^P \pm 1.96\sqrt{\frac{\sigma_{b_2}^2}{2J} + \frac{\sigma^2}{2Jk}}$$

$Y_{ij}$ = effort for subject $i$, stool type $j$.

What model would you use for $Y_{ij}$?

$$Y_{ij} = \underbrace{\beta_j}_{\text{fixed effect for stool type}} + \underbrace{b_i}_{\text{Random effect (subject}} + \underbrace{\varepsilon_{ij}}_{\text{random error}}$$

We're not interested in the 9 individuals.

$i = 1, \ldots, 9, \quad j = 1, \ldots, 4$

$$b_i \sim N(0, \sigma_b^2)$$

$$\varepsilon_{ij} \sim N(0, \sigma^2)$$

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Alternatively

$$Y_{ij} = \mu + \beta_j + b_i + \varepsilon_{ij}$$

where $\sum \beta_j = 0$

or $\beta_1 = 0$

Equivalent models – reparametrized versions of

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$$Y_{ij} \sim N\left(\beta_j, \sigma^2 + \sigma_b^2\right)$$

Parameters $\beta_1, \ldots, \beta_4, \sigma^2, \sigma_b^2$

We don't think of the $b_i$ as parameters.

Unlike in a fixed effects model, observations are correlated:

$$\mathrm{Cov}(Y_{ij}, Y_{ij'}) = \mathrm{Cov}(\beta_j + b_i + \varepsilon_{ij}, \beta_{j'} + b_i + \varepsilon_{ij'})$$

$$= \cancel{\mathrm{Cov}(\beta_j, \beta_{j'})} + \cancel{\mathrm{Cov}(\beta_j, b_i)} + \cancel{\mathrm{Cov}(\beta_j, \varepsilon_{ij'})}$$

$$\cancel{\mathrm{Cov}(b_i, \beta_{j'})} + \ldots + \cancel{\mathrm{Cov}(\varepsilon_{ij}, \varepsilon_{ij'})}$$

$$= \mathrm{Cov}(b_i, b_i) = \mathrm{Var}(b_i)$$

$$= \sigma_b^2$$

So obs's from the same <u>subject</u> are correlated.

$b_i \sim N(0, \hat{\sigma}_b^2 = 1.775 = 1.332^2)$

$\varepsilon_{ij} \sim N(0, \hat{\sigma}^2 = 1.211)$

$\hat{\beta}_1 = 8.556 \quad \text{stderror}(\hat{\beta}_1) = \underline{0.576}$

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$\hat{\beta}_1 = \frac{1}{9} \sum Y_{i1}$ (Exercise: check this numerically)

$$Var(\hat{\beta}_1) = \frac{1}{81} Var\left(\sum_{i=1}^{9} Y_{i1}\right)$$

$$= \frac{1}{81} Var\left(\sum_i (\beta_1 + b_i + \varepsilon_{i1})\right)$$

$$= \frac{1}{81} \sum \left(Var(b_i) + Var(\varepsilon_{i1})\right)$$ as $b_i$ & $\varepsilon_{i1}$ are indep.

$$= \frac{1}{81}\left(9\sigma_b^2 + 9\sigma^2\right)$$

$$= \frac{1}{9}\left(\sigma_b^2 + \sigma^2\right)$$

In comparison, a fixed effect model $Y_{ij} = \beta_j + \alpha_i + \varepsilon_{ij}$, $\sum \alpha_i = 0$

we'd still find $\hat{\beta}_1 = \frac{1}{9} \sum Y_{1j}$

but now $Var(\hat{\beta}_1) = \frac{\sigma^2}{9} < \frac{1}{9}(\sigma_b^2 + \sigma^2)$

This is because $\beta_j$ in the fixed effect model is the effect averaged over only 9 subjects

For random effects model $\beta_j$ is the effect for the entire population.

Please start on Question I.

$$I = \int_0^{10} \frac{1}{1+x^2}\, dx$$

Choose a distribution $g(x)$

& then $I = \int \frac{1}{(1+x^2)} \frac{1}{g(x)}\, g(x)\, dx = \mathbb{E}_{X \sim g}\left[\frac{1}{(1+x^2)\, g(x)}\right]$

$$\approx \frac{1}{n} \sum \frac{1}{(1+X_i^2)\, g(X_i)} \quad \text{where } X_i \sim g(\cdot)$$

<u>Example</u> $g(x) = U[0,10]$ $\quad g(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$

$g(x) = \text{Exp}(\lambda)$

Cauchy.