MATH3027: Optimization 2022

Week 11: Computer lab 8

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Projected Gradient Descent

Consider the minimization problem

$$\begin{aligned}\min_{\mathbf{x}} \; & 2x_1^2 + 3x_2^2 + 4x_3^2 + 2x_1x_2 - 2x_1x_3 - 8x_1 - 4x_2 - 2x_3 \\ \text{subject to} \quad & x_1, x_2, x_3 \geq 0\,.\end{aligned}$$

- Show that the vector $\left(\frac{17}{7}, 0, \frac{6}{7}\right)^T$ is an optimal solution.
- Construct a gradient projection method with constant stepsize $\frac{1}{L}$(L being the Lipschitz constant of the gradient of the objective function). Show the function values of the first 100 iterations and the produced solution. Try different initial guesses.

Constrained Least Squares

Consider the minimization problem

$$\begin{aligned}\min_{\mathbf{x}} \; & ||\mathbf{A}\mathbf{x} - \mathbf{b}||_2^2 \\ \text{subject to} \quad & ||\mathbf{x}||_2^2 \leq \alpha\,,\end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is assumed to be of full column rank, $\mathbf{b} \in \mathbb{R}^m$, and $\alpha > 0$. We call this a *constrained least squares* (CLS) problem.

- Derive the Lagrangian associated with this problem and the KKT conditions.
- Are the KKT conditions necessary and sufficient here?
- Find a solution to the KKT system. It will help to first consider the case $\lambda = 0$, and to think about when the resulting $\mathbf{x}$ satisfies the inequality constraints. In the case where $\lambda \neq 0$ we must have $||\mathbf{x}||_2^2 = \alpha$ by the complimentary slackness constraint. Explain why there must exist a λ for which this is true.

- Write code that inputs $\mathbf{A}$, $\mathbf{b}$, and α, and outputs the solution of the constrained least squares problem.
- Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

  and compare your results against the linear least squares problem.
- Compare constrained least squares to regularized least squares?
- Use CVXR to check your answer.