Lecture II: The Savings Problem

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ECS506: PhD Macroeconomics I

Fall 2021

This lecture: The Savings Problem

Dynamic Programming: Some Theory

The Savings Problem

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The Savings Problem

Same simple example from before

- Let's start again from the Bellman equation for the Stochastic Growth Model

$$V(k,z) = \max_{k'} u(zf(k) - k') + \beta \mathbb{E}_{z'|z} V(k', z).$$

- There was a bit of a leap of faith in what we have done so far: Does $V(.)$ actually exist? Is it unique?

Objective: Introduce some of the mathematical tools required to show that we do have existence and uniqueness

Fixed point: A detour by the Solow model

- Recall the basic Solow model where we define $\mathcal{T}$:

$$k_{n+1} = \mathcal{T}(k_n) = f(k_n) + (1-\delta)k_n$$

The steady-state is a fixed point $k^* = \mathcal{T}(k^*)$.

- If we start from some $k_0 > 0$ and iterate, we get

$$k_1 = \mathcal{T}(k_0), \quad k_2 = \mathcal{T}(k_1) = \mathcal{T}^2(k_0), \quad k_3 = \dots$$

- Under some conditions on $f(.)$, we have

$$\lim_{n\to\infty} ||k_n - k_{n-1}|| = \lim_{n\to\infty} ||\mathcal{T}^n(k_0) - \mathcal{T}^{n-1}(k_0)|| = 0,$$

so that $\lim_{n\to\infty} \mathcal{T}^n(k_0) = k^*$

Fixed point: Applied to the Bellman equation

- Now instead we have $\mathcal{T}$: functions $\rightarrow$ functions

  $$V_{n+1} = \mathcal{T}(V_n) := \max_{k'} u(zf(k) - k') + \beta \mathbb{E}_{z'|z} V_n(k', z).$$

  The value function is a fixed point $V^* = \mathcal{T}(V^*)$.

- We are looking for conditions on $\mathcal{T}$ such that we can start from some V_0, apply $\mathcal{T}$ recursively,

  $$V_1 = \mathcal{T}(V_0), \quad V_2 = \mathcal{T}(V_1) = \mathcal{T}^2(V_0), \quad V_3 = \dots$$

  and these conditions imply

  $$\lim_{n\to\infty} ||V_n - V_{n-1}|| = \lim_{n\to\infty} ||\mathcal{T}^n(V_0) - \mathcal{T}^{n-1}(V_0)|| = 0,$$

  so that $\lim_{n\to\infty} \mathcal{T}^n(V_0) = V^*$

Contraction Mapping Theorem

- These conditions are given by the **Contraction Mapping Theorem**

- A Contraction Mapping $\mathcal{T}$ is defined as

$$\sup_{x \in X} |\mathcal{T}f(x) - \mathcal{T}g(x)| \leq \beta \sup_{x \in X} |f(x) - g(x)|$$

  for $\beta \in (0, 1)$

- We require $f, g \in B(X)$, the space of bounded functions of X and $\mathcal{T} : B(X) \to B(X)$

- Intuition: the definition doesn't make much sense if $f, g, \mathcal{T}f$, or $\mathcal{T}g$ go to infinity

In words: Applying the mapping to two functions "shrinks" the distance between them by at least β

Contraction Mapping Theorem

Theorem: If $\mathcal{T} : B(X) \to B(X)$ is a contraction mapping with modulus β:

1. $\mathcal{T}$ has a unique fixed point $V^* \in B(X)$ such that $\mathcal{T}V^* = V^*$
2. For any $V_0 \in B(X)$,

$$\sup_{x \in X} |\mathcal{T}^n V_0(x) - \mathcal{T}V^*(x)| \leq \beta^n \sup_{x \in X} |V_0(x) - V^*(x)|$$

   $n = 0, 1, 2, \ldots$

In words: we can get arbitrarily close to V^* by repeatedly applying $\mathcal{T}$ to any V_0

Remarks

1. $\lim_{n\to\infty} ||\mathcal{T}^n(V_0) - \mathcal{T}^{n-1}(V_0)||$ is exactly what we're doing with Value Function Iteration

2. In "practice", it is easier to work with **Blackwell's conditions**, which imply that the functional equation is a contraction mapping.

Blackwell's conditions

A mapping $\mathcal{T} : B(X) \to B(X)$ is a Contraction Mapping if:

1. *Monotonicity*: For $f, g \in B(X)$, we have:

$$f(x) > g(x) \implies \mathcal{T}f(x) > \mathcal{T}g(x) \quad \forall x$$

2. *Discounting*: There is some $\beta \in (0, 1)$ such that

$$\mathcal{T}(f + a)(x) \leq \mathcal{T}f(x) + \beta a \quad \forall x,\ a \in \mathbb{R}$$

Sidenote: David Blackwell was the first African American tenured at Berkeley

Blackwell's conditions applied to the Growth model

- This is actually easier than it might seem
- Let's show these conditions hold in the (deterministic) Growth model from Lecture I

$$\mathcal{T}(V) := \max_{k'} u(f(k) - k') + \beta V(k')$$

Note: Showing that $\mathcal{T} : B(X) \to B(X)$ is the difficult part. We just assume it here.

Blackwell's conditions: Monotonicity

Let V, W be two functions such that $V(k) \geq W(k)$, $\forall k$, then

$$\begin{aligned}
\mathcal{T}(V) &= \max_{k'} u(f(k) - k') + \beta V(k') \\
&\geq \max_{k'} u(f(k) - k') + \beta W(k') \\
&= \mathcal{T}(W)
\end{aligned}$$

So we have Monotonicity: $V \geq W \implies \mathcal{T}(V) \geq \mathcal{T}(W)$

Blackwell's conditions: Discounting

Let $\alpha \in \mathbb{R}$ be a constant, then

$$\begin{aligned}
\mathcal{T}(V + \alpha) &= \max_{k'} u(f(k) - k') + \beta(V + \alpha)(k') \\
&= \max_{k'} u(f(k) - k') + \beta V(k') + \beta\alpha \\
&= \mathcal{T}(V) + \beta\alpha
\end{aligned}$$

So we have discounting.

A general framework

The framework we introduced is way more general than the Stochastic growth model we have studied so far.

The same reasoning applies for the more general problem

$$\max_{y_t, x_{t+1}} \mathbb{E}_0 \sum_t^{\infty} \beta^t r(y_t, x_t) \quad \text{s.t.} \quad x_{t+1} \leq g(x_t, y_t, \varepsilon_{t+1}).$$

with

- $y_t,\ x_t,\ \varepsilon_{t+1}$ are (potentially large)vectors of choices, states, and shocks
- $r(.)$ some "reward" function (utility, profits)
- $g(.)$ a law of motion for the states next period (constraint)

A general framework

The same reasoning applies for the more general problem

$$\max_{y_t, x_{t+1}} \mathbb{E}_0 \sum_t^{\infty} \beta^t r(y_t, x_t) \quad \text{s.t.} \quad x_{t+1} \leq g(x_t, y_t, \varepsilon_{t+1}).$$

Specifically:

1. The agent's problem admits a recursive formulation

$$V(x) = \max_y r(x, y) + \beta \mathbb{E}_\varepsilon V(x') \quad \text{s.t.} \quad x' \leq g(x, y, \varepsilon')$$

2. Under some conditions on the function $r(.)$ and the set defined by $g(.)$, we can apply the Contraction Mapping Theorem

3. Many, many econ problems satisfy these conditions

Example 1: Rust (1987)

The function $V_\theta(x_t)$ defined in (3.2) is the *value function* and is the unique solution to *Bellman's equation* given by[5]

(3.5) $$V_\theta(x_t) = \max_{i_t \in C(x_t)} [u(x_t, i_t, \theta_1) + \beta E V_\theta(x_t, i_t)]$$

where $C(x_t) = \{0, 1\}$ and where the function $EV_\theta(x_t, i_t)$ is defined by

(3.6) $$EV_\theta(x_t, i_t) \equiv \int_0^\infty V_\theta(y) p(dy \mid x_t, i_t, \theta_2).$$

Using Bellman's equation, I have shown elsewhere (Rust, (1986a)) that there is an optimal stationary, Markovian replacement policy $\Pi = (f, f \ldots)$ where f is given by

(3.7) $$i_t = f(x_t, \theta) = \begin{cases} 1 & \text{if} \quad x_t > \gamma(\theta_1, \theta_2), \\ 0 & \text{if} \quad x_t \leqslant \gamma(\theta_1, \theta_2), \end{cases}$$

Rust, John. "Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher." Econometrica (1987)

Example 2: Hopenhayn and Rogerson (1993)

III. Equilibrium

A. Notation

We begin by examining the decision problem of a firm in more detail. In anticipation of a stationary equilibrium, a constant output price of p is assumed. Specifically, consider a firm that employed n workers last period, decided to remain in the industry for the current period, and has received a new value for its shock equal to s. The Bellman equation corresponding to the firm's decision problem at this point is

$$W(s, n; p) = \max_{n' \geq 0} \{pf(n', s) - n' - pc_f - g(n', n) + \beta \max[E_s W(s', n'; p), -g(0, n')]\},$$

where E_s denotes expectations conditional on the current value of s, and s' denotes next period's (random) value of s. For reasons that

Hopenhayn, Hugo, and Richard Rogerson. "Job turnover and policy evaluation: A general equilibrium analysis." Journal of political Economy (1993)

Summary: Theory of Dynamic Programming

- Wanted to justify that the solution to the Bellman Equation exists and is unique, which was implicit until now
- The Bellman equation is a functional equation, **mapping functions into functions**
- The **Contraction Mapping Theorem** gives conditions for existence and uniqueness of a solution to such an equation
- Many, many problems in econ have a "discounted infinite sum" form, which satisfies these conditions

This lecture: The Savings Problem

Dynamic Programming: Some Theory

The Savings Problem

Introduction

The savings problem is:

1. A twist on the growth model we have introduced so far
2. A key microfoundation to many macro models: it describes the trade-off between consumption and savings

Environment

Let's again start from the stochastic growth model

$$\max_{\{c_t,k_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + k_{t+1} \leq z_t f(k_t), \;\; t = 0, 1, 2, \ldots$$

$$c_t \geq 0$$

but with the following changes

$$\max_{\{c_t,a_{t+1}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t.} \quad c_t + a_{t+1} \leq (1+r)a_t + y_t, \;\; t = 0, 1, 2, \ldots$$

$$c_t \geq 0$$

$$a_{t+1} \geq \underline{a}$$

The savings problem

An agent chooses consumption and assets to

$$\begin{aligned}
\max_{\{c_t, a_{t+1}\}} \; & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & c_t + a_{t+1} \leq (1+r)a_t + y_t, \;\; t = 0, 1, 2, \dots \\
& c_t \geq 0 \\
& a_{t+1} \geq \underline{a}
\end{aligned}$$

- r is interest rate (sometime $R := 1 + r$)
- y_t is income—the source of uncertainty
- $\underline{a} \leq 0$ is a borrowing limit

Self-insurance

The ingredients

- r is interest rate (sometime $R := 1 + r$)
- y_t is income—fluctuates stochastically
- $\underline{a} \leq 0$ is a borrowing limit

induce a **self-insurance motive**

There is a trade-off between consumption today and savings in anticipation of a rainy day (“rainy day” means a bad day *outside* Bergen)

The borrowing limit implies the person cannot borrow as desired when y_t is low

Remarks on borrowing limit

- We assume $a \geq \underline{a}$ with $\underline{a} \leq 0$ ($\underline{a} = 0$ is the no-borrowing case)
- This is sometimes referred to as a **ad-hoc** borrowing constraint (imperfection in financial markets, etc.)
- Alternative is **natural** borrowing limit: the max amount one can hope to repay with $c_{t+j} = 0$ onwards and the worst realization of income $\min y_t := \underline{y}$:

$$\underline{a} = -\sum_{j=0}^{\infty} \frac{\underline{y}}{(1+r)^{j+1}} = -\underline{y}/r$$

  (Homework: Derive that expression from the budget constraint.)
- If $\underline{y} = 0$ we're back to no-borrowing case

Recursive formulation

We know how to write this problem recursively by now

$$\begin{aligned} V(a,y) = & \max_{\{c,a'\}} u(c) + \beta \mathbb{E}_{y'|y}\left[V(a',y')\right] \\ \text{s.t.} \quad & c + a' \leq (1+r)a + y \qquad (1) \\ & c \geq 0, \qquad (2) \\ & a' \geq \underline{a} \qquad (3) \end{aligned}$$

(1) always binds (why?)

(2) never binds if u satisfies the Inada condition:
$\lim_{c \to 0} u(c) = -\infty$

(3) needs to be dealt with!

A modified Euler equation

With $\mu \geq 0$ the lagrange multiplier associated with the borrowing constraint (3), we obtain

$$\begin{aligned} u'(c) &= \beta(1+r)\mathbb{E}\big[u'(c')\big] + \mu \\ \Leftrightarrow u'(c) &\geq \beta(1+r)\mathbb{E}\big[u'(c')\big] \qquad (= \text{if } a' > \underline{a}) \end{aligned}$$

Make sure you can derive this expression!

Question: Why didn't we worry about this constraint in the Stochastic Growth Model?

Intuition in a two-period model

Focus on a simple two-period model for intuition

$$\max_{c_1, c_2, a_2} \; u(c_1) + \beta \mathbb{E}\left[u(c_2)\right]$$

$$\text{s.t.} \quad c_1 + a_2 = y_1$$

$$c_2 = a_2(1 + r) + y_2$$

$$a_2 \geq \underline{a} \quad (\mu \geq 0)$$

Note: Implicitly $a_3 \geq 0$, so no debt left behind.

Same Euler equation

We again have the same Euler equation

$$
\begin{aligned}
u'(c_1) &= \mathbb{E}\big[u'(c_2)\big] + \mu \\
u'(c_1) &\geq \mathbb{E}\big[u'(c_2)\big] && (= \text{if } a_2 > \underline{a})
\end{aligned}
$$

where we assume $\beta(1+r) = 1$ for simplicity.

Three special cases to gain intuition on the impact of borrowing limit and income shocks:

1. No uncertainty, no borrowing limit
2. No uncertainty, with borrowing limit
3. Uncertainty, no borrowing limit

Special case 1: no uncertainty, no borrowing limit

In this scenario,

$$u'(c_1) = \mathbb{E}\big[u'(c_2)\big] + \mu$$

now becomes

$$u'(c_1) = u'(c_2) \quad \Rightarrow \quad c_1 = c_2 \qquad\qquad (u'' < 0)$$

We get perfect consumption smoothing without any restriction on y_2

$$c_1 = c_2 = \frac{y_1 + y_2}{2}.$$

Special case 2: no uncertainty, borrowing limit

The Euler equation

$$u'(c_1) = \mathbb{E}[u'(c_2)] + \mu$$

is now

$$u'(c_1) \geq u'(c_2).$$

There are two cases:

1. The constraint doesn't bind, then we again get $c_1 = c_2$.
2. If the constraint binds, then $c_1 < c_2$ (since $u'' < 0$). This situation only arises if the income path is increasing (why?).

So borrowing constraint may prevent agent from reaching its target consumption level: **first motive to save**

Special case 3: uncertainty, no borrowing limit

Again starting from the Euler equation,

$$u'(c_1) = \mathbb{E}\big[u'(c_2)\big] + \mu$$

this case can be written

$$u'(-a_2 + y_1) = \mathbb{E}\big[u'(y_2 + a_2)\big].$$

Define $\overline{y} := \mathbb{E}\big[y_2\big]$, then it can be shown that savings are higher with uncertainty than with a deterministic income $\overline{y}$.

Uncertainty is a **second motive to save**.

Special case 3: uncertainty, no borrowing limit

Proof: Denote $\tilde{a}$ savings in the economy without uncertainty. Assume $\tilde{a} > a_2$, then $u'(-\tilde{a} + y_1) > u'(-a_2 + y_1)$, but

$$u'(-a_2 + y_1) = \underbrace{\mathbb{E}[u'(y_2 + a_2)] \geq u'(\overline{y} + a_2)}_{\text{Jenssens' inequality}}$$
$$> u'(\overline{y} + \tilde{a}) = u'(-\tilde{a} - y_1),$$

a contradiction. So, $\tilde{a} <= a'$.

In words: marginal utility is very high in case of a bad y, so the optimum requires to save for this event

Note: Implicitly, we assumed that u' is convex ($u''' > 0$).

Back to general case

- Borrowing limit and income uncertainty represent two motives to save

$$
\begin{aligned}
V(a,y) = \max_{\{c,a'\}} &\; u(c) + \beta \mathbb{E}_{y'|y}\left[V(a',y')\right] \\
\text{s.t.} \quad & c + a' \leq (1+r)a + y \\
& c \geq 0, \\
& a' \geq \underline{a}
\end{aligned}
$$

- The agent's consumption-saving behavior is encoded in the policy functions $c(a,y)$ and $a'(a,y)$

- No closed form solutions in general (some special cases, such as quadratic preferences)

- But we know how to get a numerical solution using VFI!

The Marginal Propensity to Consume (MPC)

- Euler equation informs about how consumption varies over time

- The Marginal Propensity to Consume (MPC) tells us how consumption would respond to a small, **unanticipated**, change in income or wealth

- It is defined as

$$\text{MPC}(a) = \frac{\partial c(a, y)}{\partial a}$$
$$\text{MPC}(y) = \frac{\partial c(a, y)}{\partial y}$$

respectively for the MPC wrt income and wealth

- Requires to compute the consumption function!

Why is the MPC such an object of interest?

- Key object for macro policy. How does consumption respond to changes in income? In undergrad macro, you have seen

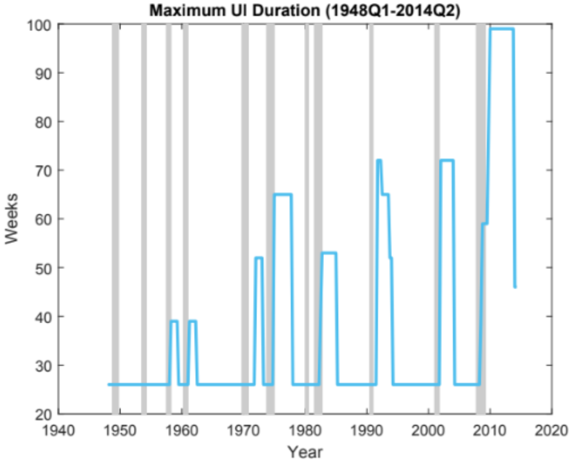
$$C_t = \alpha + \beta Y_t \implies \text{MPC}(Y_t) = \beta$$

- The savings problem provides a microfoundation for the MPC, as an outcome of an intertemporal decision problem.

- Also makes clear that it shouldn't be the same at all wealth levels/income levels

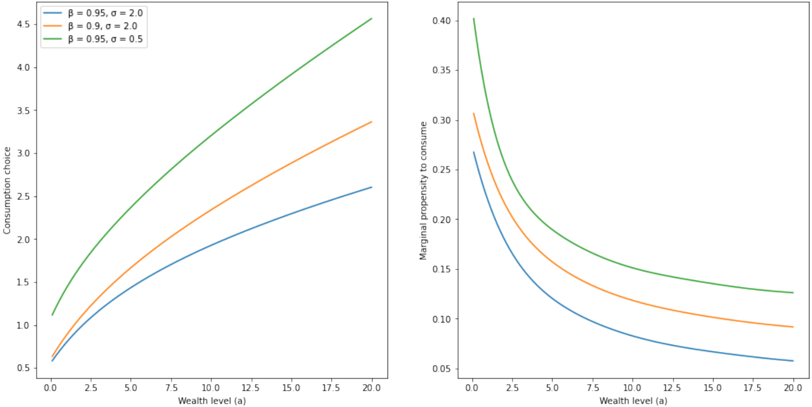
- Opportunity to target transfers to people with higher MPCs to support demand during economic contractions

Example of targeted transfers: Cyclicality of UI duration



Source: Rujiwattanapong (2020)

MPC in the savings problem



MPC by wealth level

- The savings model suggests that MPCs should be stronger for households with low wealth, sometimes referred to as **“hand-to-mouth” households**

- Important to distinguish between liquid and illiquid wealth—Kaplan et al. (2013) point to a large fraction of wealthy hand-to-mouth. These are households with a large fraction of wealth tied up in illiquid assets, such as housing

Consistent with Survey Evidence

What fraction would you consume if you unexpectedly receive one month of income?

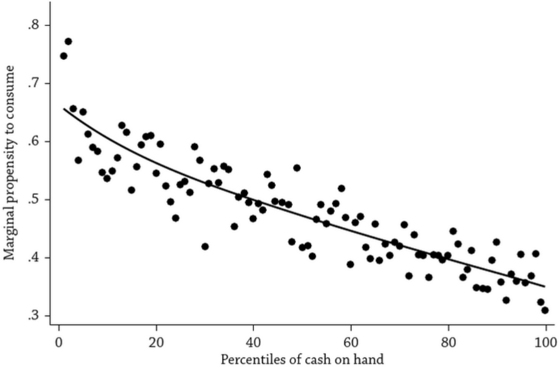


Figure 9.1: The sample distribution of the MPC by percentiles of cash on hand
Note: The figure is reproduced with permission from Jappelli and Pistaferri (2014).

Cash on hand $:= y_t + (1 + r)a_t$

MPC: Empirical strategies

- Very brief intro to a vast empirical literature
- Starting point

$$\ln c_{it} = z_{it}'\lambda + \underbrace{\alpha E_{t-1}\Delta \ln y_{it}}_{\text{anticipated change}} + \underbrace{\sum_{k=1}^{K} \phi_k \pi_{it}^k}_{\text{unanticipated shocks}} + \xi_{it}$$

- Example of empirical strategies
  1. Quasi-experiments
  2. Covariance restrictions

Quasi-experiments: Find some credible exogenous variation

1. Payouts from government: tax rebates, etc.
2. Illness, disability, unemployment
3. Weather shocks and crop losses (developing country context)
4. Lottery winners in Norway (Fagereng et al. 2020)
5. ...

Not suprisingly, estimates vary a lot with context, size and sign of shocks

Good to keep in mind if you “stumble” on some interesting variation

Covariance restrictions on income and consumption

- Specify a model for income process with transitory shocks (u_{it}) and permanent shocks (ζ_{it})

$$\ln y_{it} = p_{it} + u_{it} + \nu_{it}$$
$$p_{it} = p_{it-1} + \zeta_{it}$$

- Gives pair of equation for consumption and income changes

$$\Delta \ln c_{it} = z_{it}'\lambda_c + \phi_1\zeta_{it} + \phi_2 u_{it} + \phi_3\nu_{it} + \Delta\xi_{it}$$
$$\Delta \ln y_{it} = z_{it}'\lambda_y + \zeta_{it} + \Delta u_{it} + \Delta\nu_{it}$$

- Retrieve estimates of ϕ_1, ϕ_2 by expressing the variance and covariance between consumption changes and income changes $\rightarrow$ Method of Moments

- Blundell et al. (2008): $\phi_1 \approxeq .65$ and $\phi_2 \approxeq .05$

This lecture: Wrapping up

Key concepts:

- Functional equation, Contraction Mapping Theorem
- Borrowing constraint (natural vs ad-hoc), Euler equation
- Marginal Propensity to Consume

Next lecture: Relax infinite-horizon assumption to introduce a notion of age in the savings problem

References

Theory of Dynamic Programming

RMT - Chapter 3 & Appendix A is a good place to start to learn more

“Recursive Methods in Economic Dynamics” by Nancy L. Stokey, Robert E. Lucas, Jr. (with Edward C. Prescott) is the classic reference, but it is a very, very formal treatment

References

Savings Problem

RMT - Chapter 17

EoC - Chapters 5, 6, and 7

Measuring MPCs

EoC - Chapter 9

Blundell, Richard, Luigi Pistaferri, and Ian Preston. "Consumption inequality and partial insurance." American Economic Review 98.5 (2008): 1887-1921