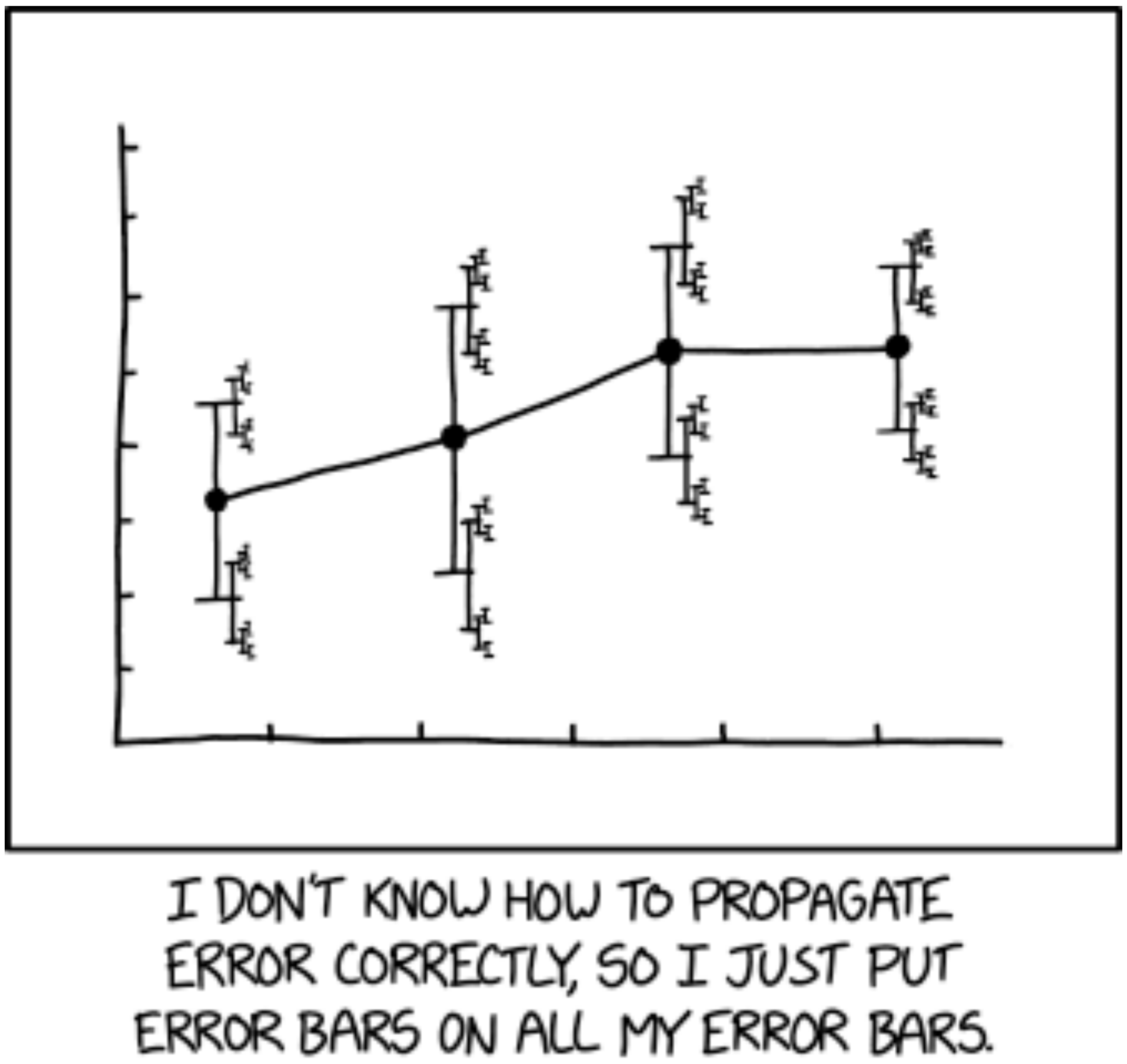
Intro to Uncertainty and Curve Fitting



Every measurement has an **uncertainty** (often called 'error' but that's a misleading name).

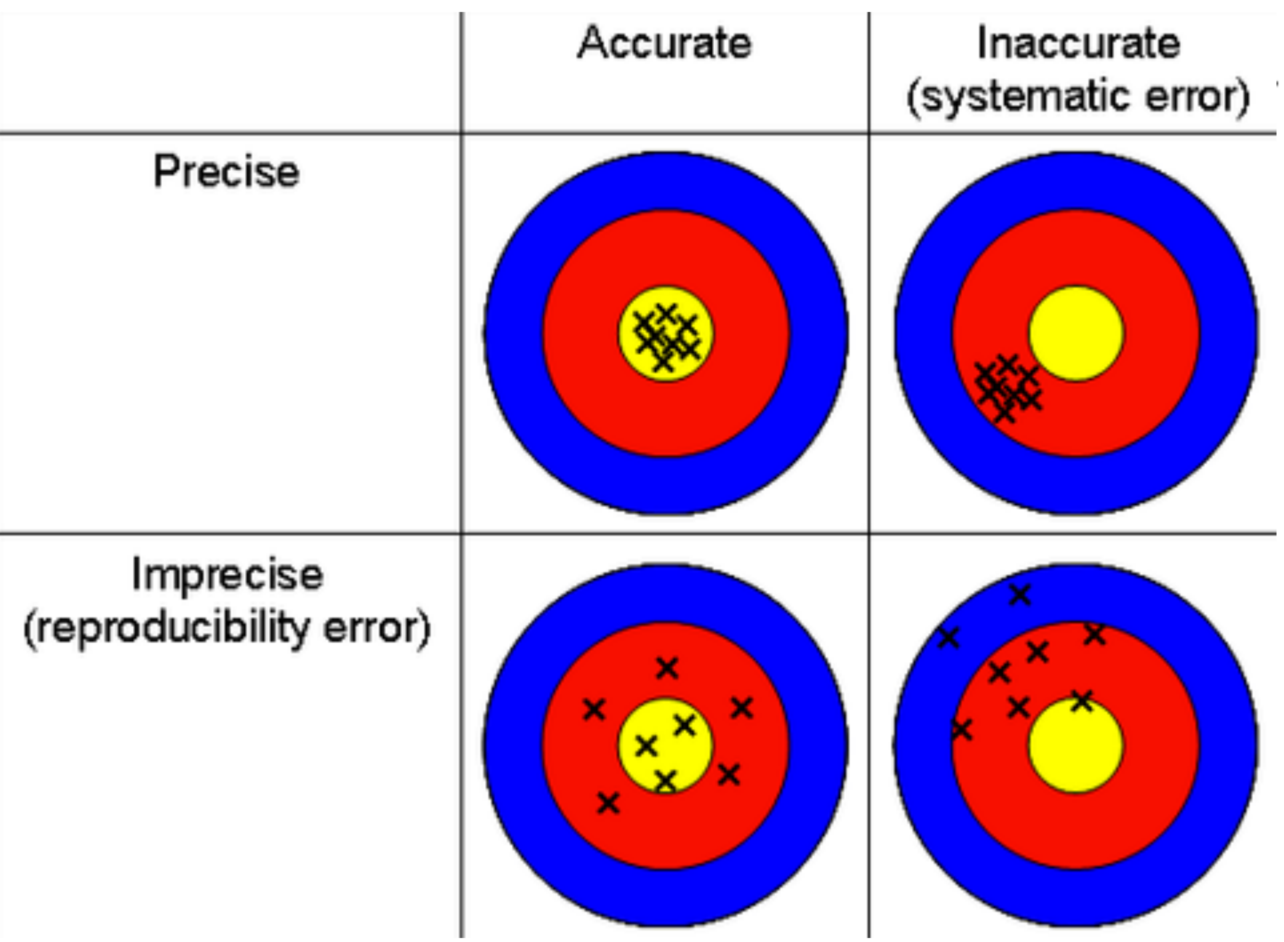
Uncertainty is an experimentally measured/calculated quantity.
Uncertainty is **never** found by comparing your result to an expected result.

***Accuracy** describes* how close the result of the experiment is to the 'true' value.

Precision describes the process, how well the result was determined, measure of reproducibility

(a) *Absolute precision of measurement x is* Δx (error bars on the graph).

(b) *The fractional precision of measurement x* is Δ*x*/*x* (in decimals or in percent).



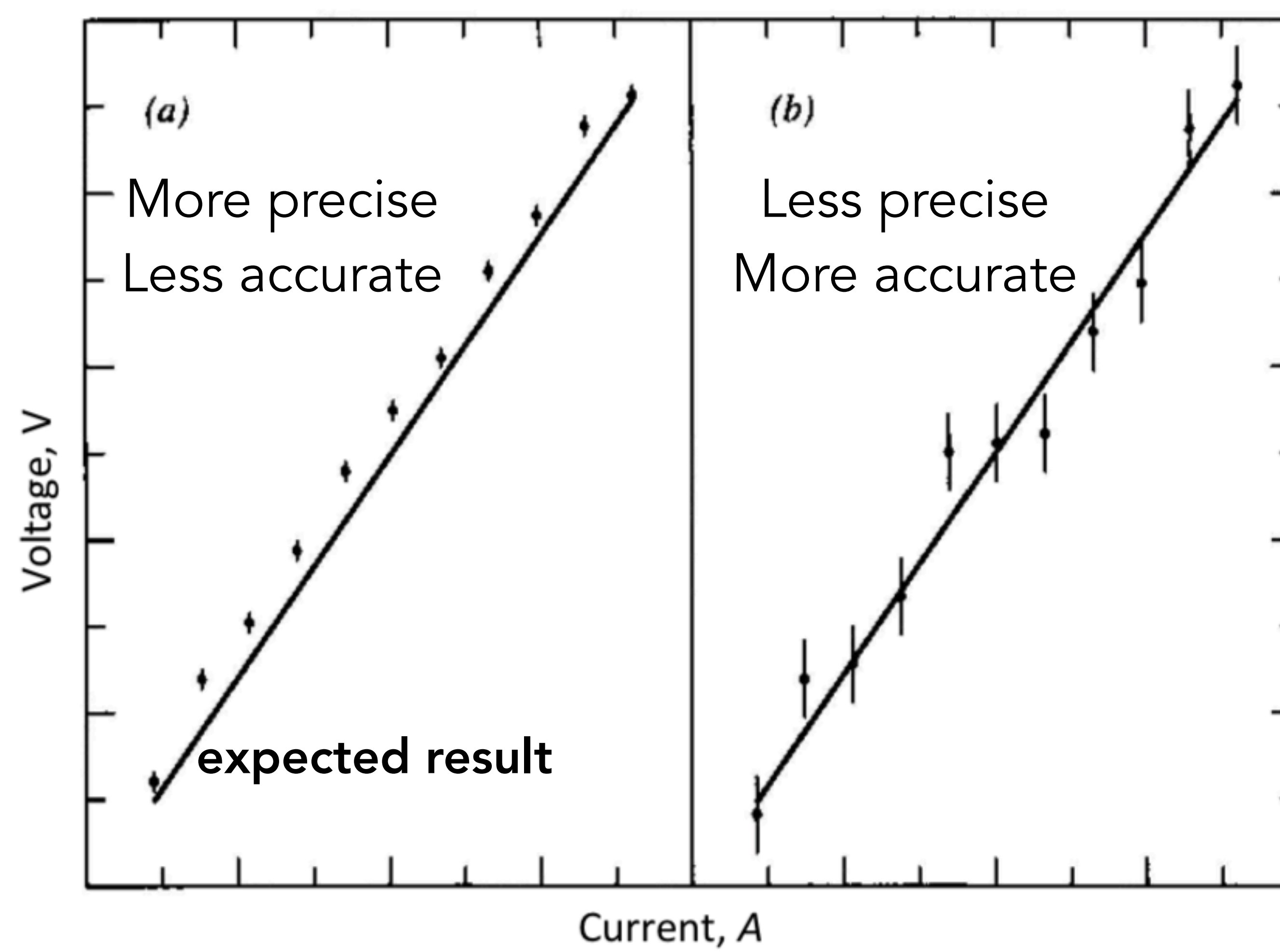


Fig. I. Precision and accuracy for measurements with different voltmeters.

Reading uncertainty depends on the scale of the measuring device (it is the *resolution*)

Analog:

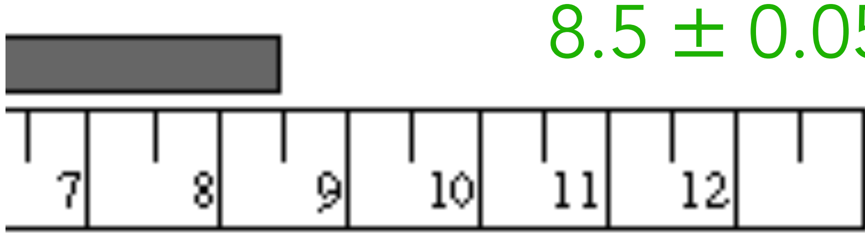
Reading uncertainty

= ± a fraction of least division

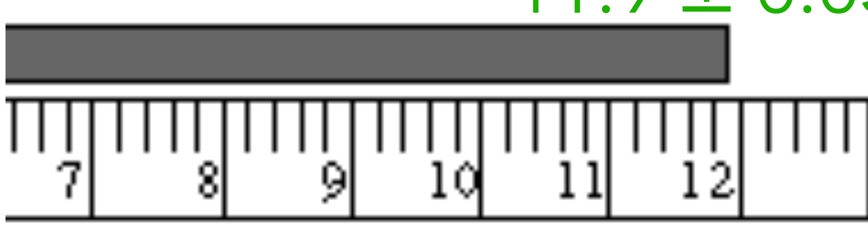
(judgement call)

9.5 ± 0.1

8.5 ± 0.05



11.9 ± 0.03



Digital:

Reading uncertainty

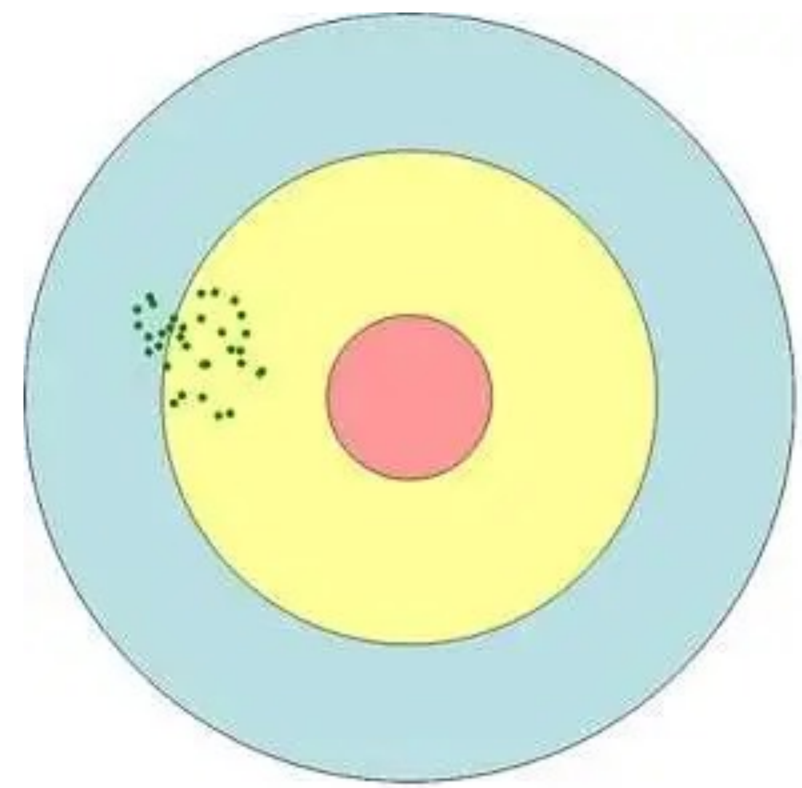
= ± 1/2 of last digit

(Some sources recommend ± the last digit)

12.8±0.05 °C

Systematic uncertainty (bias):

- A predictable (reproducible) discrepancy between measured and true value
- Corrections/calibrations that are made to eliminate the systematic error improve the *accuracy* (repeating trials won't help!)



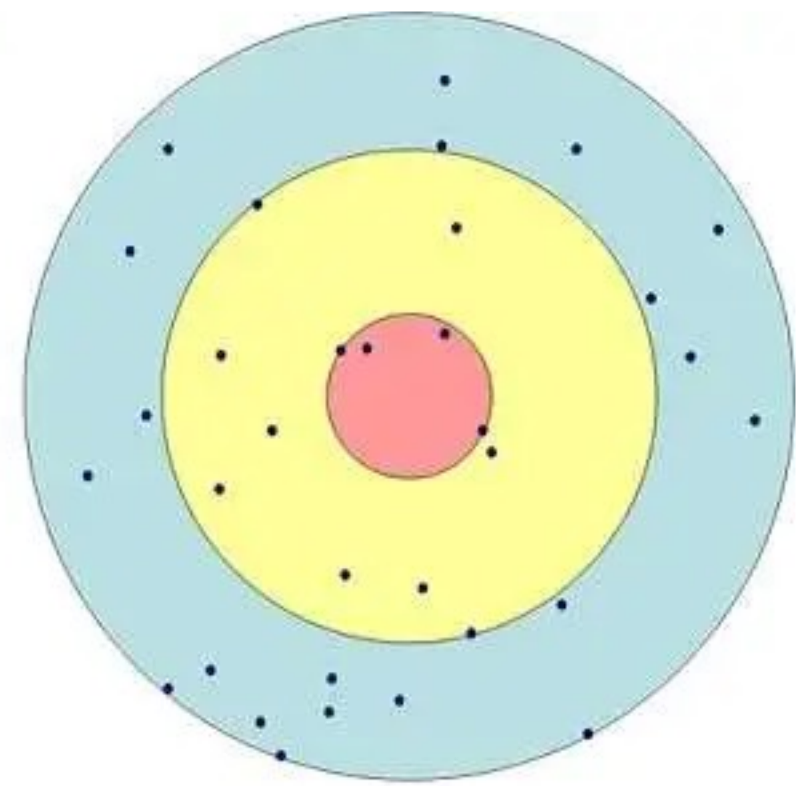
Systematic Uncertainty

Ex: poor alignment of an optical system

Ex. a mass scale that hasn't been zeroed

***Random (statistical)* uncertainty:**

- Fluctuations of the measured value when the measurements are repeated.
- The random uncertainty affects the *precision*
- To reduce random uncertainty, use better equipment and/or make more measurements
- The lower limit is the reading uncertainty.



Statistical Uncertainty

To estimate the random uncertainty, need to study the distribution of results of the repeated trials about the *mean value* (find the standard deviation)

The Mean

The “true” mean μ (of the parent distribution) is unknown, but we can estimate it by finding the mean of finite samples x_i (the sample distribution).

Δx_i = uncertainty of each measured data point

N = number of measurements

If the uncertainty of all points is equal ($\Delta x_i = \Delta x$),

the mean is: $$\bar{x} = \frac{1}{N}\sum x_i$$

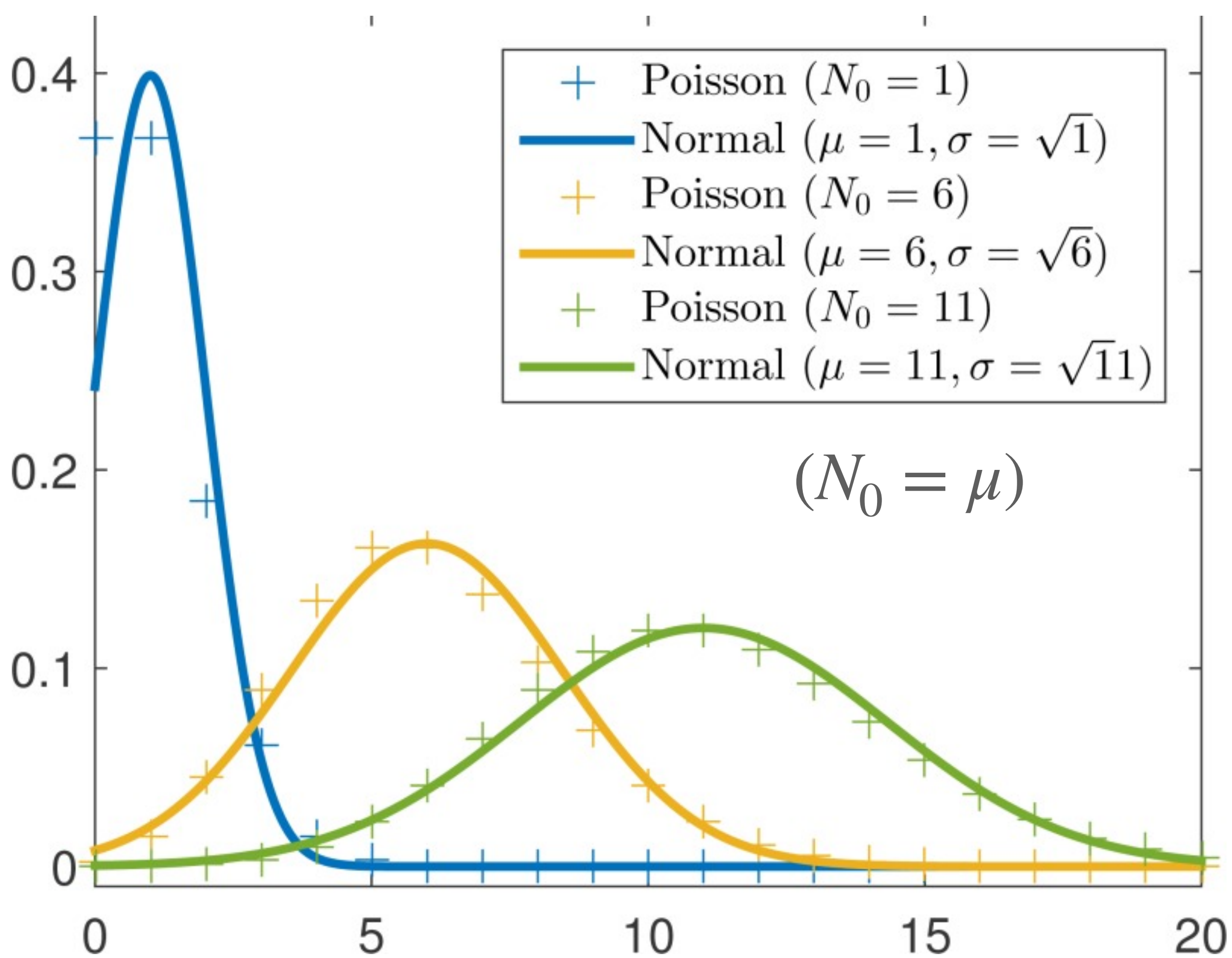
Otherwise: $$\boxed{\bar{x} = \frac{\sum x_i/(\Delta x_i)^2}{\sum 1/(\Delta x_i)^2}}$$

This is a weighted mean, measurements with small uncertainties have higher weights

The Standard Deviation

This tells you how much the values vary **within** your sample. It is an estimate of the standard deviation of the population.

$$\boxed{\sigma = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2}}$$



For Poisson distributions: $\sigma \xrightarrow[N>>1]{} \sqrt{\mu}$

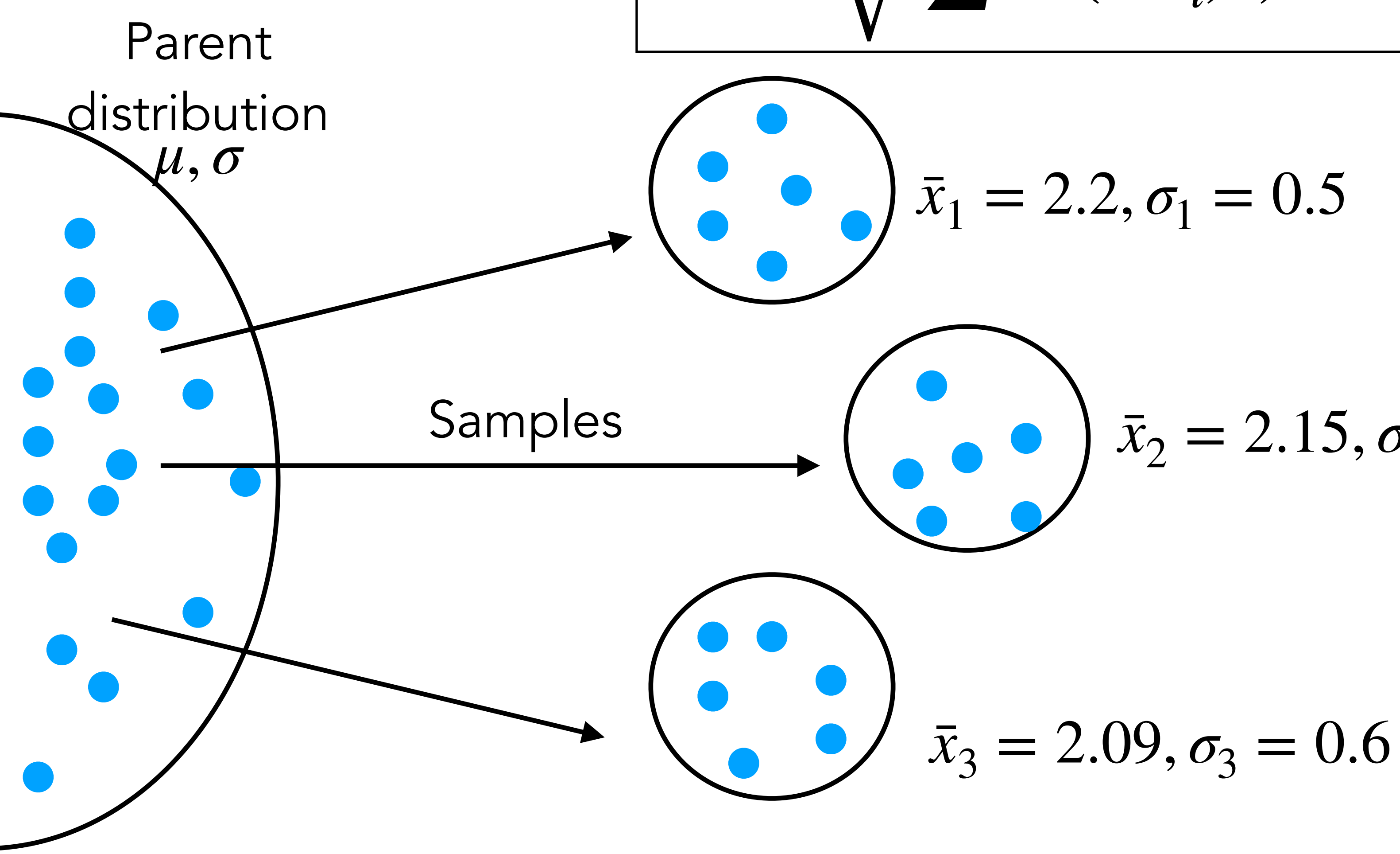
(Ex. counting events in time intervals, μ is the expected number of events in a given time period)

The Poisson distribution approaches a Gaussian for large μ

The ~~Uncertainty~~ Standard Deviation of the Mean (Standard Error)

This tells you how much your estimate of the mean would vary across different samples.

$$\boxed{\sigma_{\bar{x}} = \frac{1}{\sqrt{\sum 1/(\Delta x_i)^2)}} \xrightarrow[\Delta x_i = \Delta x]{} \frac{\Delta x}{\sqrt{N}}}$$



In this case, $\sigma_{\bar{x}} \approx 0.1$

Notice:

- the uncertainty of the mean < uncertainty of individual data points
- The uncertainty decreases with N

See Ch. 4 of Data Reduction…

Ex. The mean of a set of measurements is 2.6437 and the standard deviation is 0.2432. The measurement should be expressed as:

i. 2.6437 ± 0.2432

ii. 2.6 ± 0.2

iii. 2.6437 ± 0.2

iv. 3 ± 0.2

Sig Figs

"The Sun has a Radius of 432,288 miles or a Circumference of 14341248821.16985804821504 Feet"

- actual email I got RE: secret frequencies of the great pyramids

The uncertainty defines the reasonable number of significant figures in the result.

Ex. If you calculate $\bar{x} = 5.4321$ *and* $\sigma = 0.7654$, *you should report your result as* $x = 5.4 \pm 0.8$

The reasonable number of significant figures in the uncertainty itself is usually **one** (or sometimes two, such as when a manufacturer lists the reading error as a percentage).

The number of sig figs in the uncertainty itself could also be estimated using the uncertainty of the standard deviation: $\Delta\sigma = \dfrac{\sigma}{\sqrt{2N-2}}$

Ex. The manual of a digital thermometer says that its tolerance is 1.2%. If the reading says 201.5 °C, the measurement should be expressed as:

I. 202 ± 2 °

II. 201.5 ± 2.4 °

III. 201.5 ± 2.418 °

IV. 201.500 ± 2.418 °

Putting it Together

- For a unique measurement without repeated trials: uncertainty is the reading uncertainty or tolerance given by a manufacturer

- For repeated measurements that differ, random uncertainty (the standard deviation) can be calculated

- More often, we directly measure several quantities – independent variables with their uncertainties, then we calculate the quantity of interest, calculating the uncertainty of this indirect measurement is called ***propagation of uncertainty.***

If the fluctuations of the measured variables are not correlated, the formulas for the propagated uncertainty are:

$$z = x \pm y \qquad \Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$z = xy \qquad z = \frac{x}{y} \qquad \frac{\Delta z}{z} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2}$$

In general:

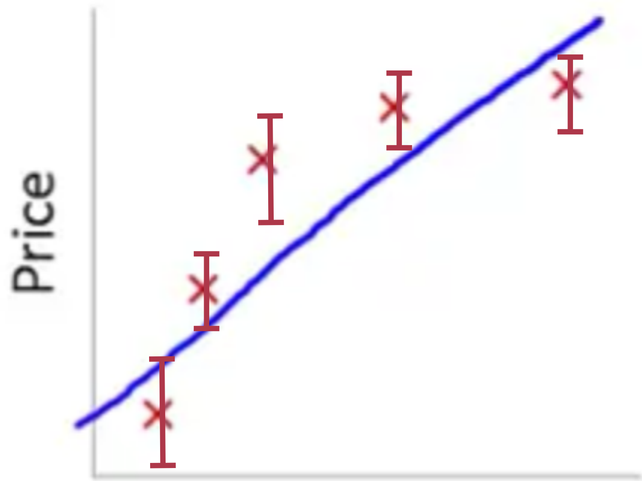
$$\boxed{z = f(x, y, \ldots) \qquad (\Delta z)^2 = \left(\frac{\partial f}{\partial x}\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\Delta y\right)^2 + \ldots}$$

If one of these terms is < 10% of the largest term, you can neglect it.

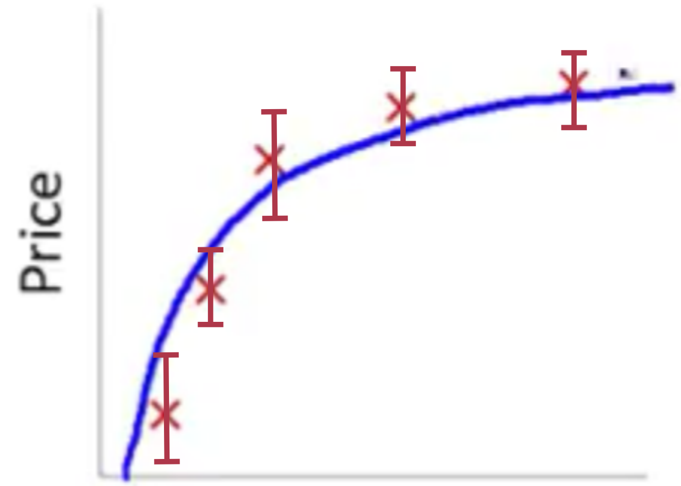
In some cases (involved in curve fitting), the fluctuations are correlated and there are additional 'covariance' terms (see Ch. 3 of Bevington)

If the experiment involves changing an independent variable (rather than a set of repeated trials), the goal is to know which function better fits a set of data you measured.

This procedure is called ***curve fitting***.

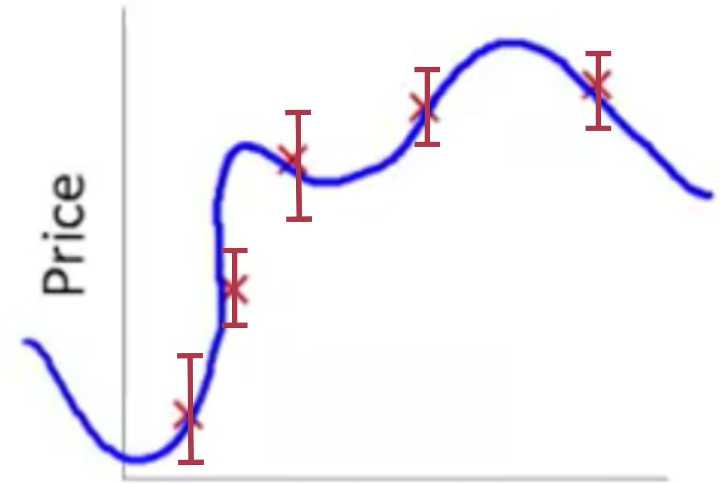


High bias
(underfit)



"Just right"

This one looks good but we need to apply more rigorous criteria to check



High variance
(overfit)

The criterion of how close the line is to all data points is called the ***goodness of fit criterion***.

You need to include 2 goodness of fit criteria in each of your reports (Chi-squared, residuals, or other)

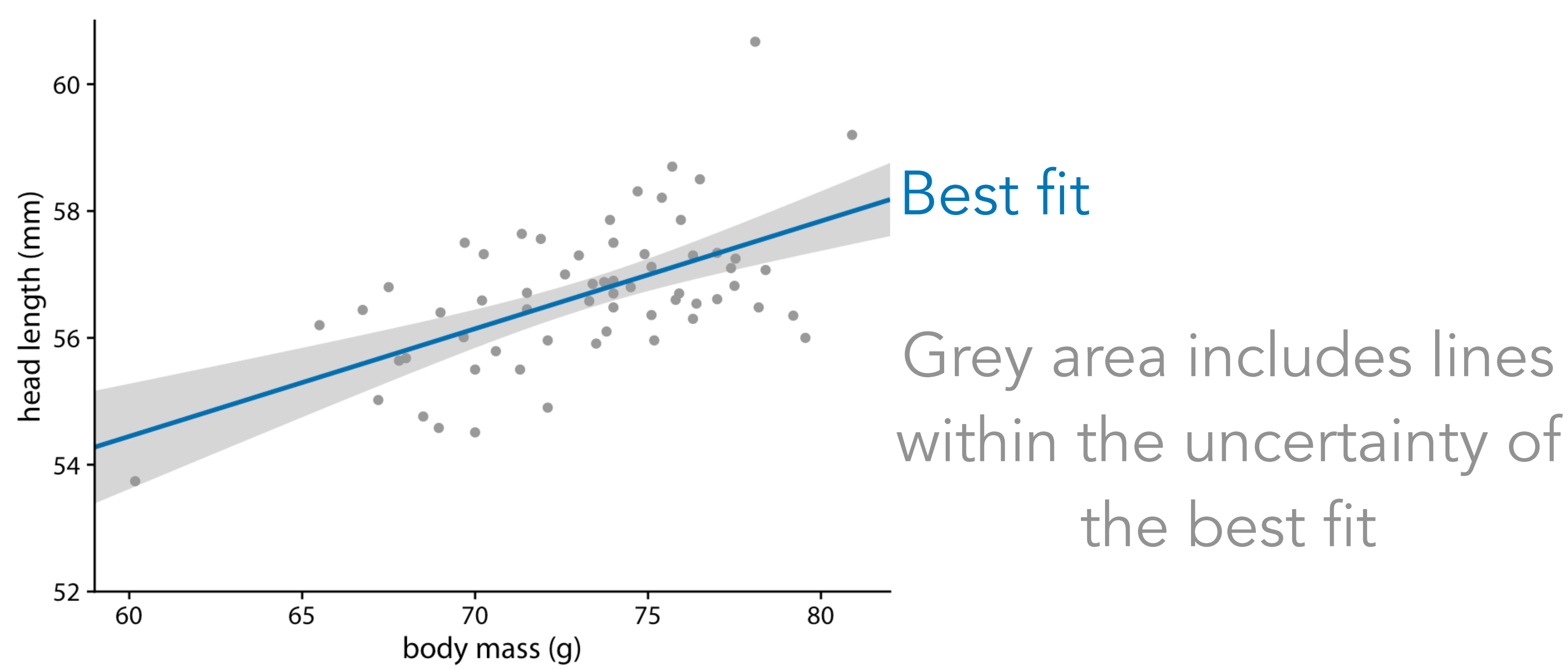
Uncertainty of Linear Fit

- To find a line of best fit you can use the "least-squares" method (write your own Python code, use Matlab's linear regression, or Excel)

- For a linear fit $f(x) = mx + b$ you need to find m and b **and** their uncertainties s_m and s_b. These quantities are not directly measured and must be calculated using your data.

- See *Uncertainties for Linear Fits* on Quercus for formulae to calculate these uncertainties. (or Ch. 6 of Bevington, watch out for different notation!)

- You can also use more the advanced Excel function LINEST



The Chi-Squared Goodness of Fit Criterion (Sec. 4.4 in Bevington)

- Uses the *Least Squares Method* to find unknown parameters of the fitting function f which minimize the value of **chi-squared**,

$$\chi^2 = \sum_{i=1}^{N} \frac{[y_i - f(x_i)]^2}{\sigma_{y_i}^2}$$

$$y_i = \text{data}$$
$$f(x_i) = \text{fitting function}$$
$$\sigma_{y_i} = \text{uncertainty of } \mathrm{y_i}$$

- To be a good fit, χ^2 **must be close to the number of degrees of freedom** of the system, ν.

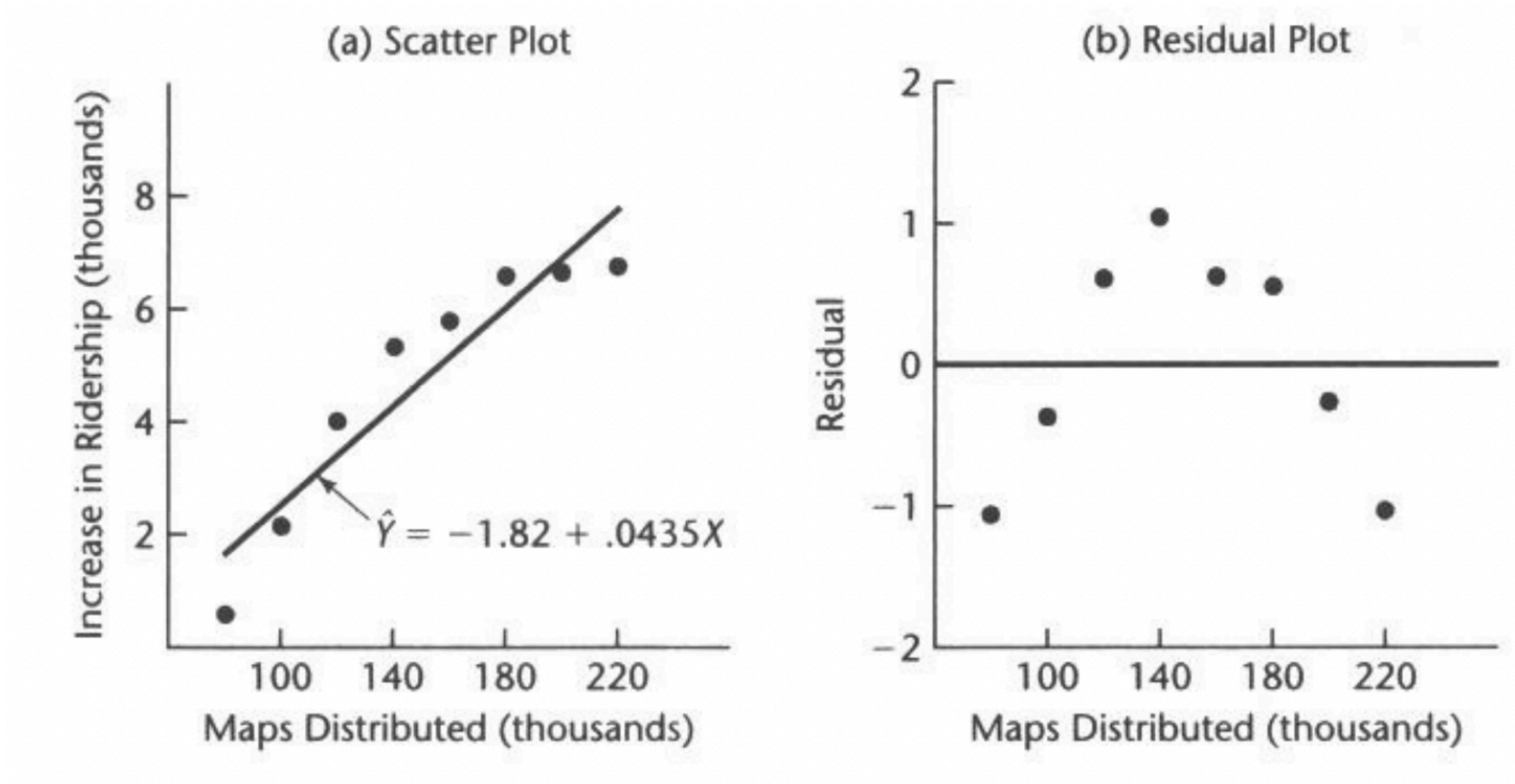
- ν = number of data points (x_i, y_i) minus the number of unknown parameters of $f(x)$.

- For a linear fit, $f(x) = mx + b$, m and b are unknown so $\nu = N - 2$.

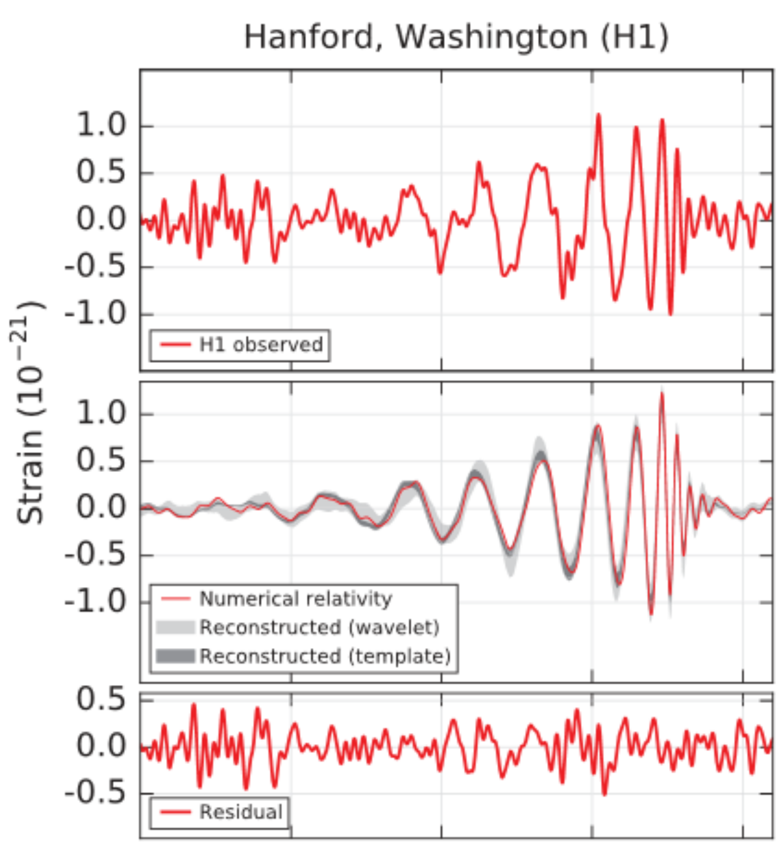
- The reduced chi-squared is $\chi_\nu^2 = \dfrac{\chi^2}{\nu}$ and must be ≈ 1 for a good fit

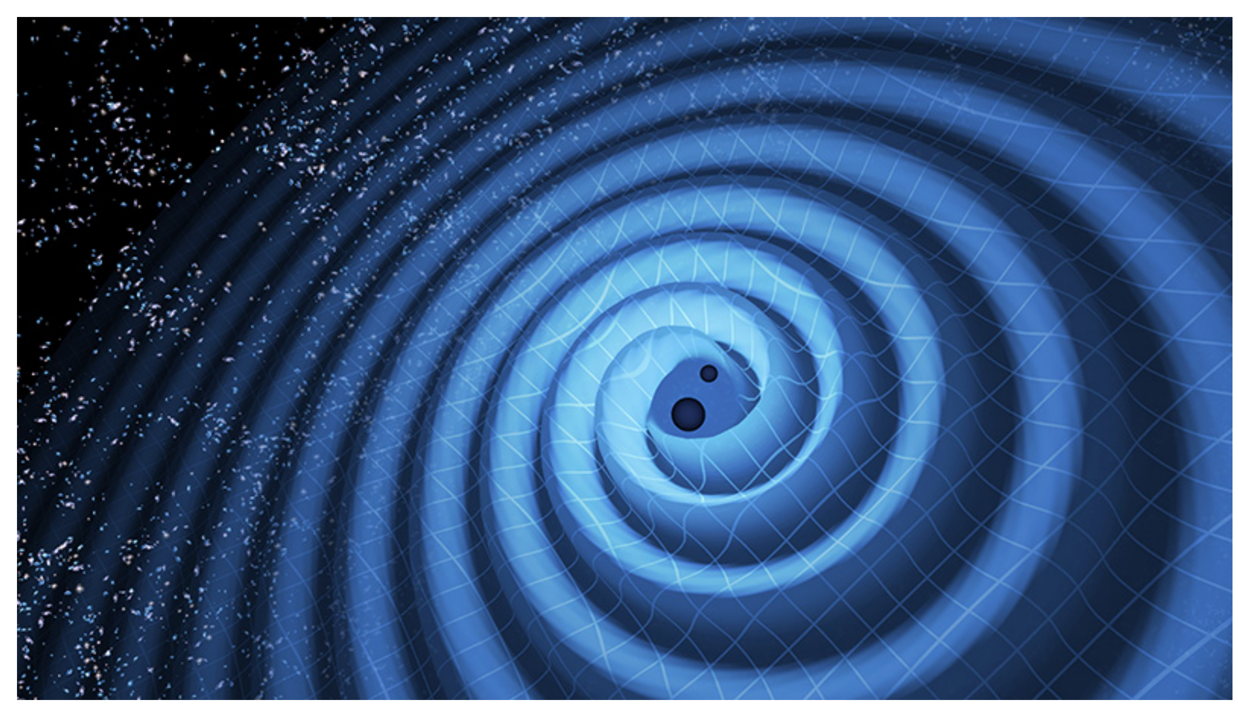
Residuals Graph Goodness of Fit Criterion

Residuals = dependent variable data points - fit function



If a pattern is discernible in the residuals plot, the fit function is chosen poorly





TO DO

- Read *Error Analysis In Experimental Physical Science (1st Year Labs)* (if you'd like a review)
- Read *Error Analysis for 2nd Year Experimental Physics* ("Notes on Error Analysis")
- *Data Reduction and Error Analysis…* by Bevington & Robinson is a great text and reference (download from Quercus)
- Ask questions on Ed Discussion Board