Homework 4

See materials and assigned reading from Lectures 6 and 7.

Problem 1. Superposition, and the interpretation of "s" as a derivative.

This problem is meant to emphasize: (1) the use/interpretation of the "s" operator as a derivative, and (2) superposition. There is much preamble/set-up; actual problem (next page) is pretty short.

(1) Compare the two following differential equations between input, *u* or *v*, and output, *y*:

$$a\ddot{y} + b\dot{y} + cy = du \qquad \text{vs} \qquad a\ddot{y} + b\dot{y} + cy = f\dot{v}$$

Let's write those equations using Laplace (i.e., "s") notation:

$$(as^2 + bs + c) \cdot Y(s) = d \cdot U(s) \qquad \text{and} \qquad (as^2 + bs + c) \cdot Y(s) = fs \cdot V(s)$$

The two corresponding transfer functions (relating input and output) are:

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{d}{as^2+bs+c} \qquad \text{and} \qquad G_2(s) = \frac{Y(s)}{V(s)} = \frac{fs}{as^2+bs+c}$$, respectively.

The "s" operator in the numerator and the scaling factor (d vs f) are the only differences.

Now, assume we have the differential relationship below:

$$a\ddot{y} + b\dot{y} + cy = f\dot{w} + dw \, .$$

The new transfer function is: $$G_3(s) = \frac{Y(s)}{W(s)} = \frac{fs+d}{as^2+bs+c} \, .$$

If we compare a unit step response for Y(s)/U(s) vs for Y(s)/V(s), the second one will be (f/d) times the derivative of the first one. We have a 2nd-order system, meaning there will be two poles, s_1 and s_2. The response of Y(s)/U(s) to a unit step will be: $y(t) = K_{fv} + C_1 e^{s_1 t} + C_2 e^{s_2 t}$. The derivative of this is easy to calculate! It is: $\dot{y}(t) = 0 + s_1 C_1 e^{s_1 t} + s_2 C_2 e^{s_2 t}$.

Because Y(s)/V(s) is (f/d)*s times Y(s)/U(s), a unit step response for Y(s)/V(s) will just be (f/d) times the expression above for the *derivative* of y(t), "y dot".

Again: that "s" operator represents a derivative in a transfer function.

(2) Now for superposition. For linear, constant-coefficient different equations (like the ones represented by all the 2nd-order transfer functions above), linearity and superposition mean that:

- If $G_3(s) = G_1(s) + G_2(s)$, the unit step response to $G_3(s)$ will be exactly the sum of the unit step responses to $G_1(s)$ plus the unit step response to $G_2(s)$.

Use MATLAB to create unit step responses for two transfer functions, G1 and G2, as follows:

```
G1 = tf([1],[1 4 25])
G2 = tf([1 0],[1 4 25])
figure(1); clf
step(G1); hold on
step(G2);
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(a) What are the initial position, velocity and acceleration of a unit step response for G1, and what are the final position, velocity and acceleration? (Hint: Final Value Theorem, Initial Value Theorem, and multiplication of G1 by "s" for "derivative".) If there is a discontinuity at $t=0$, determine the value at $t=0+$ (just after this discontinuity).

(b) Repeat part (a) to find these same six values for G2.

In MATLAB, create 5 unit steps responses for $P(s) = \frac{1+as}{s^2+4s+25}$, using a = 1, .5, 0, -.5 and -1. Notice they all have the same "shape", because they ALL HAVE THE SAME POLES. Zeros are solutions for s to set the numerator of the transfer function to zero, analogous to poles on the s-plane. The effect of the zero is to change the phase and amplitude of the decaying sinusoidal output solution, but "zeta" and "omega n" are set only by the POLES.

Because the zeros change the phase of the decaying sinusoidal solution, the step response may "overshoot" more or less than for the case without a zero. If a zero is "in the righthand plane", meaning it has a positive real part on the complex plane, the response actually "goes the wrong way" first! This just means it has negative slope at t=0+, although its final value is positive.

(c) Show work to solve for the exact time at which all 5 step responses above first intersect one another at some t>0. (These would obviously also be the "zero crossings" for "step(G2)", which you plotted earlier… The time will be directly related to the "damped natural frequency", which is the magnitude of the imaginary parts of the poles. It is one half-cycle, or pi radians.)

(d) Find a set of values of b, c, and d in $P(s) = \frac{bs^2+cs+d}{s^2+4s+25}$ such that:
- The initial value of a unit step response is 10
- The final value of a unit step response is 5

(No, your choice of "c" does not matter here…)

Setting initial velocity for the unit step response above is trickier. Here is one trick to do so. Break P(s) into two pieces, so that $P(s) = K + \frac{f}{s^2+4s+25} = \frac{Ks^2+4Ks+25K+f}{s^2+4s+25}$. Through superposition, the total unit step response will be the unit step response to $P_1(s)=K$, which is just a constant output value of "K", plus the unit step response to $P_2(s) = f/(s^2+4s+25)$, which has zero as both initial value and initial slope.

(e) Pick the value of c needed for your solution in part (d) to set the initial slope of a unit step response to zero. (Please check your solution on your own, using MATLAB.)

Problem 2. Wheeled-vehicle kinematics

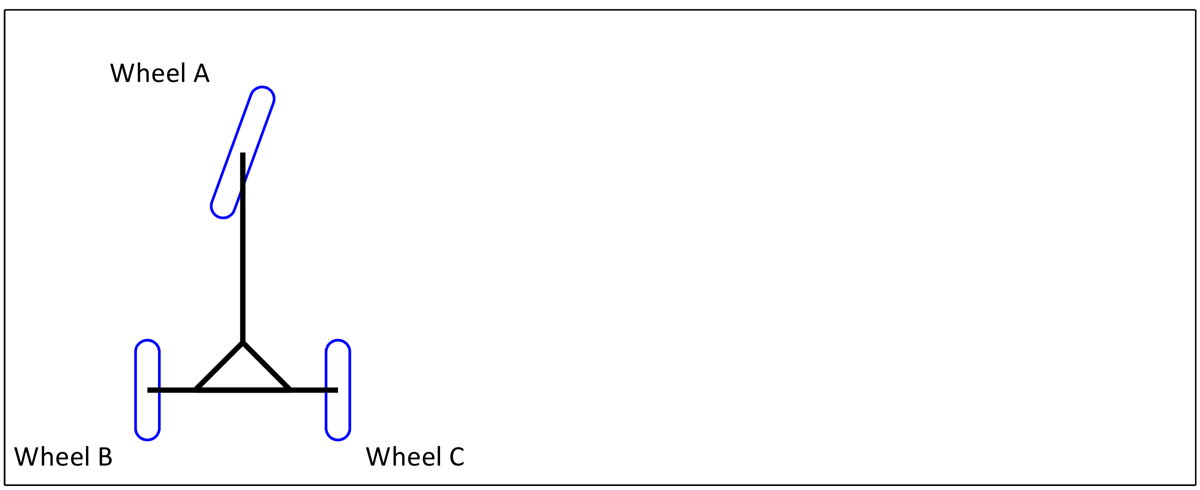
Below is an overhead sketch of a tricycle, moving upward on the page.

a) Sketch the instantaneous center of rotation (ICR), given the wheel orientations shown.

b) Label the sketch as needed (e.g., with wheel radii and distances from ICR to wheel contacts at the ground) to relate the rotational velocities of each the wheels and rotational velocity of the tricycle frame itself (about the ICR at this instance).

c) Finally, use a ruler to estimate and relate any lengths (radii, etc.) as needed, and estimate the rotational velocities of wheels B and C, as well as the rotational rate of the frame about the ICR, given that wheel A is rotating at 1 rad/sec.

d) Assume this is a scaled drawing, and that wheel A has a radius of 0.3 meters. How fast is wheel A traveling with respect to the ground at the moment, in meters/sec?

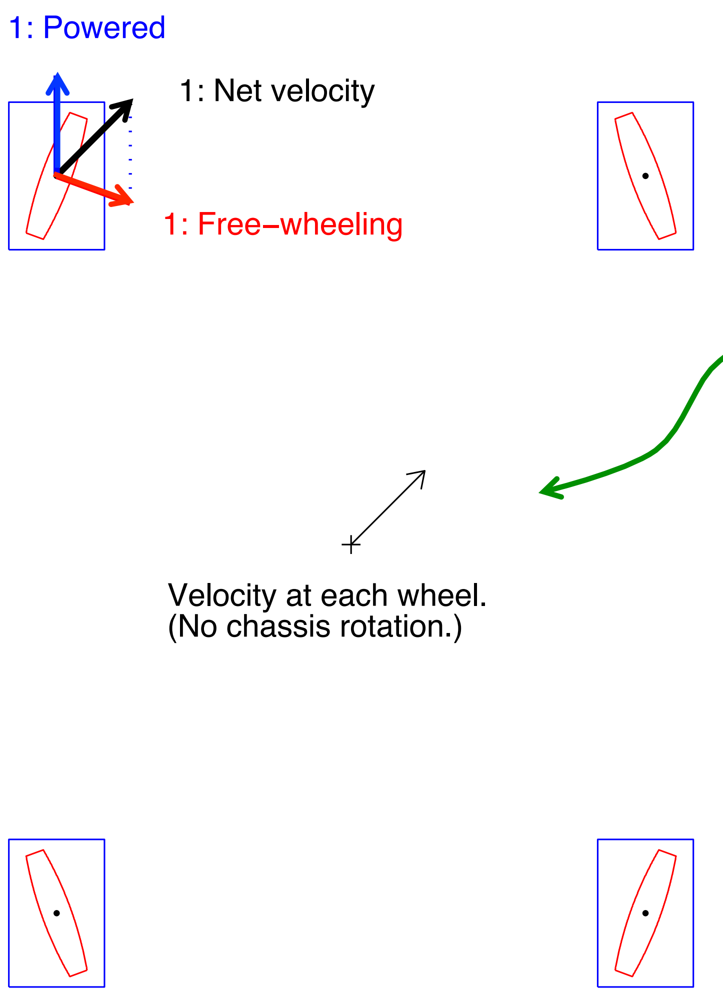


Problem 3. Omni-directional wheels.

At right is an omni-directional wheel. Unlike Lab 2, the roller axes are not at 90 degrees with respect to the "powered" (main) axis of the wheel.

Below is a sketch showing 4 such omni-wheels as mounted on a robot chassis. Assume the drawing shows only the free roller that is currently touching the ground. The key idea in this problem is that the total velocity for each wheel comes from adding the vector velocities in the "powered" and "free-wheeling" directions.

(a) In order for the vehicle to move 45 degrees ("up and to the right"), as illustrated, sketch the vector contributions from the powered and free directions of motions at each wheel-ground contact. This is already done at wheel 1, to get you started.

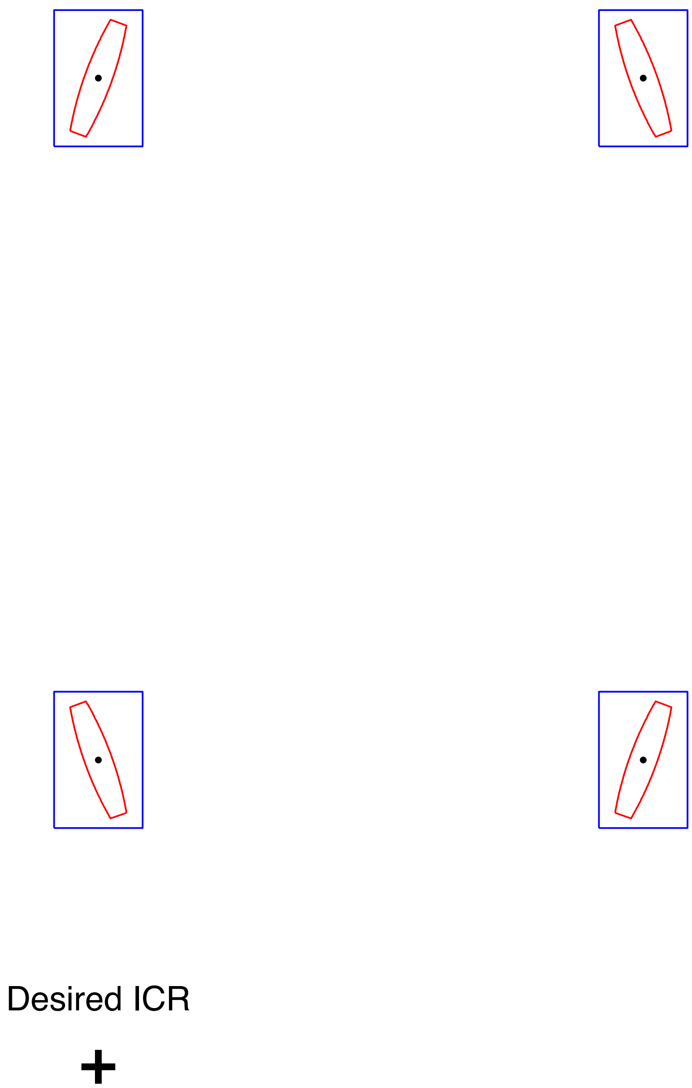


WHAT TO DO: Make each "net velocity" line up to be parallel to the overall chassis velocity, going **45 degrees up and to the right. It should also be the same length as that 45-degree vector, to represent magnitude of velocity.** Note that the required powered velocities for each wheel will be proportional to the length of each blue line. The free-wheeling direction must always be perpendicular to the roller axis, as shown in the Wheel #1 case.

Wheel numbering goes CLOCKWISE, starting with Wheel 1 (upper left), as shown. Red shows the free-wheeling velocity contribution. Blue shows the velocity due to the powered wheel.

(b) Now, assume you wish to plan motions for the powered wheels so that the vehicle chassis rotates as shown about the desired ICR point on the plot below. At each wheel-ground contact point, sketch the contributions from the powered and free-wheeling directions of motion required. Wheel numbering is the same as in part a), starting with wheel 1 at top left. Use ruler measurements to make the MAGNITUDE of each velocity as accurate as possible. That is, the velocity of each wheel will be proportional to the distance from the desired ICR… For example, if you start by drawing the vectors at Wheel 1, the "net velocity" at each other wheel will be either or larger. (Plan ahead, to be able to draw all vectors "at a reasonable scale" for the paper.)

Assume the robot chassis is to rotation counter-clockwise (CCW) about the "Desired ICR" shown above.



(c) For parts (a) and (b), assume the powered axis of Wheel #1 turns at +1 (rad/sec), and estimate the velocities of wheels 2, 3, and 4, to achieve the specified motion. (i.e., estimate relative lengths of the blue lines…)