等价无穷小

$x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x \sim e^x-1 \sim \ln(1+x)$

$1-\cos x \sim \frac{1}{2}x^2$ $\quad 1-\cos^{\alpha}x \sim \frac{\alpha}{2}x^2$ $\quad \tan x - x \sim x - \arctan x \sim \frac{1}{3}x^3$

$(1+x)^a - 1 \sim ax$ $\quad x - \sin x \sim \arcsin x - x \sim \frac{1}{6}x^3$

$a^x - 1 \sim x\ln a \quad (a>0, a\neq 1)$

$x^m + x^k \sim x^m \quad (k>m>0)$

三角恒等式

$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha\pm\beta) = \dfrac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1-2\sin^2\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin\theta = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ $\quad \cos\theta = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$ $\quad \tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

导数

$(\tan x)' = \sec^2 x$ $\quad (\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x\tan x$ $\quad (\csc x)' = -\csc x\cot x$

$(a^x)' = a^x\ln a \quad (a>0, a\neq 1)$ $\quad (\log_a x)' = \dfrac{1}{x\ln a} \quad (a>0, a\neq 1)$

$(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$ $\quad (\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \dfrac{1}{1+x^2}$ $\quad (\operatorname{arccot} x)' = -\dfrac{1}{1+x^2}$

不定积分

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int x^\mu dx = \frac{x^{\mu+1}}{\mu+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \frac{1}{1+x^2} dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$$

$$\int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx = \tan x + C$$

$$\int \frac{1}{\sin^2 x} dx = \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

$$\int \tan x dx = -\ln|\cos x| + C$$

$$\int \cot x dx = \ln|\sin x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C$$

$$\int \csc dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \arcsin \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \sqrt{a^2-x^2} dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

$$\int \sqrt{x^2 \pm a^2} dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2} \ln|x + \sqrt{x^2 \pm a^2}| + C$$

$$\int u v^{(n+1)} dx = u v^{(n)} - u' v^{(n-1)} + u'' v^{(n-2)} - \cdots + (-1)^n u^{(n)} v + (-1)^{n+1} \int u^{(n+1)} v dx$$

定积分

$$\int_0^{\frac{\pi}{2}} f(\sin x) dx = \int_0^{\frac{\pi}{2}} f(\cos x) dx$$

$$\int_0^{\pi} x f(\sin x) dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x) dx$$

$$\int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0$$

$$\int_0^{\pi} \sin mx \sin nx dx = \int_0^{\pi} \cos mx \cos nx dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \end{cases}$$

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \int_0^{\frac{\pi}{2}} \cos^n x dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{2}{3} \cdot 1, & n \text{为大于1的奇数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \cdots \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{为正偶数} \\ 1, & n = 1 \end{cases}$$

(华里士公式)

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

Γ函数

$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx \quad (\alpha>0)$

$\Gamma(n+1)=n! \qquad \Gamma(\alpha+1)=\alpha\Gamma(\alpha) \quad (\alpha>0)$

令 $x=t^2$

$\Gamma(\alpha)=2\int_0^{+\infty}t^{2\alpha-1}e^{-t^2}dt$

$\Gamma(\frac{1}{2})=2\int_0^{+\infty}e^{-t^2}dt=2\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\pi}$ （通过广义二重积分求解，见18讲例11.22）

$\Gamma(\frac{7}{2})=\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma(\frac{1}{2})$

泰勒级数

△ ① $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ $(-1<x<1)$

② $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ $(-1<x<1)$ （① x 换成 $-x$）

③ $\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n$ $(-1<x\le 1)$ （①两边积分）

△ ④ $e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$ $(-\infty<x<+\infty)$

⑤ $a^x=\sum_{n=0}^{\infty}\frac{(\ln a)^n}{n!}x^n$ $(-\infty<x<+\infty)$ （④ x 换成 $x\ln a$）

△ ⑥ $\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ $(-\infty<x<+\infty)$

⑦ $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$ $(-\infty<x<+\infty)$ （⑥两边求导）

⑧ $\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$ $(-1<x<1)$ （① x 换成 x^2）

⑨ $\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$ $(-1\le x\le 1)$ （⑧两边积分）

级数常用

$\sin\frac{n\pi}{2}=\begin{cases}0, & n=2k\\(-1)^k, & n=2k+1\end{cases}$ $\qquad$ $\cos\frac{n\pi}{2}=\begin{cases}(-1)^k, & n=2k\\0, & n=2k+1\end{cases}$

不等式

$$\begin{cases} ab \le \frac{a^2+b^2}{2} \quad ((a-b)^2 = a^2+b^2-2ab > 0) \quad a+\frac{1}{a} \geq 2\sqrt{a\cdot\frac{1}{a}} = 2 \\ \frac{x_1+x_2+\cdots+x_n}{n} \geqslant \sqrt[n]{x_1x_2\cdots x_n} \quad (\text{算术平均} \geqslant \text{几何平均}) \\ |a+b| \le |a|+|b| \quad (\text{三角不等式}) \\ |\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}| \end{cases}$$

凹函数：$f\left(\frac{x_1+x_2}{2}\right) < \frac{f(x_1)+f(x_2)}{2}$

凸函数：$f\left(\frac{x_1+x_2}{2}\right) > \frac{f(x_1)+f(x_2)}{2}$

凹凸性的定义

凸函数：$f[(1-t)x_1+tx_2] \le (1-t)f(x_1)+tf(x_2)$，$0 \le t \le 1$ （琴生不等式）

$$\left\{ 0 \le \sin x \le x \quad (0 \le x \le \pi) \right.$$

例：$\int_0^1 (t-t^2)\sin^2 t\,dt \le \int_0^1 (t-t^2)t^2\,dt = \frac{1}{20}$

最大、最小值函数

$$\begin{cases} \max\{a,b\} = \frac{1}{2}[(a+b)+|a-b|] \\ \min\{a,b\} = \frac{1}{2}[(a+b)-|a-b|] \end{cases}$$

n阶导数

$$(e^{ax+b})^{(n)} = a^n e^{ax+b}$$

$$[\sin(ax+b)]^{(n)} = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$

$$[\cos(ax+b)]^{(n)} = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$

$$[(ax+b)^\beta]^{(n)} = a^n \beta(\beta-1)\cdots(\beta-n+1)(ax+b)^{\beta-n}$$

$$\left(\frac{1}{ax+b}\right)^{(n)} = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$$

$$[\ln(ax+b)]^{(n)} = (-1)^{n-1} a^n (n-1)! \frac{1}{(ax+b)^n}$$

$$\left\{ \frac{1}{x+\sqrt{1+x^2}} = -x+\sqrt{1+x^2} \Rightarrow x = \frac{1}{2}\left(x+\sqrt{1+x^2} - \frac{1}{x+\sqrt{1+x^2}}\right) \right.$$