Problem Set 3

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Problem Three: First-Order Negations (definitely needs to be checked/edited)

i. Negating the statement, "For all x in the set of real numbers, and for all y in the set of real numbers, if x is less than y then there exists q in the set of rational numbers such that x is less than q, and q is less than y."

$$\neg(\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y))) \tag{1}$$
$$\exists x \in \mathbb{R}.\neg(\forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y))) \tag{2}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\neg(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{3}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{4}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y)) \tag{5}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q \land q < y)) \tag{6}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q) \lor \neg(q < y) \tag{7}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.(x \geq q) \lor (q \geq y) \tag{8}$$
$$\tag{9}$$

The negation is, "There exists x in the set of real numbers, and there exists y in the set of real numbers, where x is less than y and for all q in the set of rational numbers x is greater than or equal to q, and q is greater than or equal to y."

ii. Negating the statement, "For all x, y, and z, if (R(x, y) and R(y, z) then R(x, z)) then if for all x, y, and z R(y, z) and R(z, y) then R(z, x)."

$$\neg(\forall x.\forall y.\forall z.(R(x,y) \land R(y,z) \rightarrow R(x,z))) \rightarrow (\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \rightarrow R(z,x))) \tag{10}$$
$$\exists x.\exists y.\exists z.\neg((R(x,y) \land R(y,z) \rightarrow R(x,z))) \rightarrow (\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \rightarrow R(z,x))) \tag{11}$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \rightarrow R(x,z))) \land \neg(\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \rightarrow R(z,x))) \tag{12}$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \rightarrow R(x,z))) \land \exists x.\exists y.\exists z.\neg(R(y,x) \land R(z,y) \rightarrow R(z,x)) \tag{13}$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \rightarrow R(x,z))) \land \exists x.\exists y.\exists z.\neg R(y,x) \lor \neg(R(z,y) \rightarrow R(z,x)) \tag{14}$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \rightarrow R(x,z))) \land \exists x.\exists y.\exists z.\neg R(y,x) \lor R(z,y) \land \neg R(z,x)) \tag{15}$$
$$\tag{16}$$

The negation is, "There exists x, y, and z, where if R(x, y) and R(y, z) then R(x, z), and there exists x, y, and z such that not R(y, z) or R(z, y) and not R(z, x)."

iii. Negating the statement, "For all x, there exists S such that S is a set and for all z, if z is in set S then z=x, and if z=x then z is in set S."

$$\neg(\forall x.\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x))) \tag{17}$$
$$\exists x.\neg(\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x))) \tag{18}$$
$$\exists x.\forall S.\neg(Set(S) \land \forall z.(z \in S \leftrightarrow z = x)) \tag{19}$$
$$\exists x.\forall S.\neg Set(S) \lor \neg(\forall z.(z \in S \leftrightarrow z = x)) \tag{20}$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x) \tag{21}$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x) \tag{22}$$
$$\tag{23}$$

How to distribute negation over bidirectional arrow?

Problem 4: 'Cause I'm Happy

i) Statment 2. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true everyone who is a person is happy. If this is true, then it is impossible that there exists a person who is unhappy.

ii) Statement 1.

iii) Statement 2.

iv) Statement 2.

v) Statement 2.

vi) Statement 2.

Problem 5: Translating into Logic

i) $\forall n Natural(n).\ \exists k Natural(k).\ \exists j Natural(j).\ (n = (k+k) \cap (n*n) = (j+j))$

ii)

iii) $\forall n Integer(n).\ \exists k Natural(k).\ \exists j Natural(j).\ (n = (k+k) \cap (n*n) = (j+j))$

The key to this problem is that an irrational number can not be represented as p/q.

iv) $\forall Q Set(Q).\ \exists P Set(P).\ \forall S Set(S).\ (S \in P \cap \forall x.\ \forall y.\ (x \in S \to x \in Q \cap y \notin S \to y \notin Q))$

This statement translated into English says "Any set Q has a powerset P that contains every subset S of Q."

v) $\exists x.\ (Lady(x) \cap \forall y.\ (Glitters(y) \to IsSureIsGold(x,y)) \cap \exists z.\ (StairwayToHeaven(z) \cap Buying(x,z)))$

Problem 6: Raven Paradox

The Raven Paradox results from taking the seemingly uninteresting statement "All ravens are black" and finding its contrapositive "Everything that is not black is not a raven." An important thing to note is that these statements are logically equivalent. Again, so far this is not especially interesting. The trouble arises when we start looking for phenomena that provide evidence for the statement. For instance, if you see a raven and that raven is black, this can be seen as providing evidence for the statement "All ravens are black." However, because the contrapositive is logically equivalent to this statement, observing something that is not black and also not a raven must also be seen as consituting evidence. This is paradoxical because the two do not seem connected at all; looking at a red clown nose does not (intuitively) seem to give us any evidence about the blackness of ravens.

In our discussions, we noted that the statement "All ravens are black" has the form of a scientific hypothesis. Everyone in the group had taken a course where we studied Popper and remembered what he had to say about scientific hypotheses: they can only be falsified, never confirmed. We have seen something similar in our dealings with first-order logic in this course. The statement "All ravens are black" can be turned into a statment in first-order logic with the form $\forall x.\ Raven(x) -> Black(x)$. This statement is only true if it is true for each and every raven. But no matter how many ravens we see that are black, it only takes a single albino raven to falsify the entire statement.

This shows that one of the ways of resolving this paradox is to attack the idea of a phenomenon providing evidence for a statement. If induction is in fact a myth, then seeing a black raven provides the same amount of evidence as seeing a red clown nose: none.

Problem 7: Graph Coloring

Problem 8: Tournament Cycles

A tournament is a directed graph with n nodes where there is exactly one edge between any pair of distinct nodes and there are no self-loops. Prove that if a tournament graph contains a cycle of any length, then it contains a cycle of length three.

Lemma 1. *Given a directed graph with n nodes, if the directed graph contains a cycle of length $k \geq$ three, then connecting any unconnected nodes in the cycle will create a cycle of length $< k$.*

Proof. The definition of a cycle in a directed graph is a path from a node to itself. Pick an arbitrary cycle in a connected graph of length $k \geq$ three and pick any two arbtrary nodes in that cycle m, n that are not already connected (i.e., not adjacent in the cycle). When we connect m and n we can either place the base of the arrow at m and the head at n or the base at n and the head at m.

□

Theorem. *If a tournament graph contains a cycle of any length, then it contains a cycle of length three.*

Proof. By strong induction. Let $P(n)$ be the statement "if a tournament contains a cycle of length n, then it contains a cycle of length three." We will prove that $P(n)$ holds for $n \geq 3$.

For our base case, we show that $P(3)$ is true. $P(3)$ states that a tournament with a cycle of length 3 contains a cycle of length 3. This is a tautology.

For our inductive step, assume that for some $k \geq 3$ that $P(k)$ is true; that is, that a tournament containing a cycle of length k also contains a cycle of length three. We will prove that $P(k+1)$ is true, that if a tournament contains a cycle of length $k+1$, then it also contains a cycle of length 3.

Consider any tournament with a cycle of length $k+1$. Remove all the edges from this graph except for this cycle. A representation of this graph is shown in Figure 1. Because this is a tournament, we know that every node must be connected to every other node, and that every node in the cycle must be connected to every other node in the cycle. As soon as we draw an edge to connect two of the nodes we create a smaller cycle (see Figure 2).

□

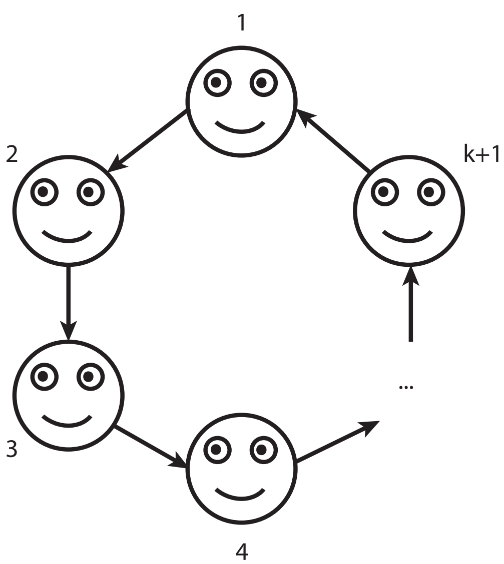

Figure 1: A cycle of length $k+1$

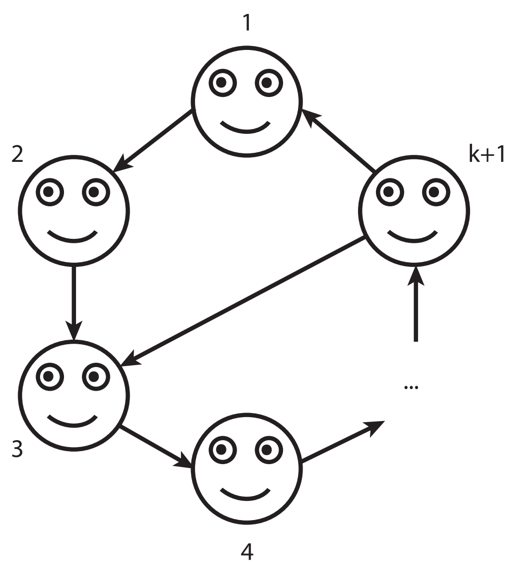

Figure 2: Connecting two nodes in the cycle