Problem Set 3

Katherine Cheng, Richard Davis, Marty Keil

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Problem 1: Simplifying Propositional Formulas

i) $\neg p$. In this formula whenever p is false the entire statement is true. This is because p implies q in both sub-formulas, and anytime an antecedent is false, the entire statement is true. Conversely, whenever p is true, the entire statement is false. For instance, when p is true and q is true, the negation of q creates a false second statement. However, when p is true and q is false, the first statement is now false. Either way one is left with one false statement and since the "and" conjuction requires that both statements be true, the entire statement is always false in this scenario.

ii) $\top$. In any case the result is true. For p not to imply q, p must true and q must be false. If we assume this then q must imply p because anytime q is false, the implication statement is true. Therefore the overall statement is true since it uses an "or" conjunction, which requires one true statement.

iii) p $\leftrightarrow$ q. Whenever p and q match the overall statement is true, otherwise the statement is false. When one variable is true, but the other is false, the first statement is true due to an "or" conjunction. Under these same conditions the second statement is false due to an "and" conjunction. With a true antecedent and false consequent, the entire formula is always false. However, when one considers identical truth values for p and q, the "or" and "and" conjunctions are no longer different, and the statement will always be true in this case. The bi-conditional conjunction satisfies this same restriction, requiring matching statement outcomes for a true result.

Problem 2: Ternary Conditionals

i) Truth Table

p	q	r	p ? q : r
F	F	F	F
F	T	F	F
T	F	F	F
T	F	T	F
F	T	T	T
F	F	T	T
T	T	F	T
T	T	T	T

ii) $(p \land q) \lor (\neg p \land r)$

Truth Table

p	q	r	$(p \land q) \lor (\neg p \land r)$
F	F	F	F
F	T	F	F
T	F	F	F
T	F	T	F
F	T	T	T
F	F	T	T
T	T	F	T
T	T	T	T

iii) p ? $\bot$: $\top$

iv) p ? q : $\top$

Problem Three: First-Order Negations

i. Negating the statement, "For all x in the set of real numbers, and for all y in the set of real numbers, if x is less than y then there exists q in the set of rational numbers such that x is less than q, and q is less than y.”

$$\neg(\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y)))$$
$$\exists x \in \mathbb{R}.\neg(\forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y)))$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\neg(x < y \to \exists q \in \mathbb{Q}.(x < q \land q < y))$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y))$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \neg(\exists q \in \mathbb{Q}.(x < q \land q < y))$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q \land q < y))$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.\neg(x < q) \lor \neg(q < y)$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \land \forall q \in \mathbb{Q}.(x \geq q) \lor (q \geq y)$$

The negation is, “There exists x in the set of real numbers, and there exists y in the set of real numbers, where x is less than y and for all q in the set of rational numbers x is greater than or equal to q, and q is greater than or equal to y.”

ii. Negating the statement, “For all x, y, and z, if (R(x, y) and R(y, z) then R(x, z)) then if for all x, y, and z R(y, z) and R(z, y) then R(z, x).”

$$\neg(\forall x.\forall y.\forall z.(R(x,y) \land R(y,z) \to R(x,z))) \to (\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \to R(z,x)))$$
$$\exists x.\exists y.\exists z.\neg((R(x,y) \land R(y,z) \to R(x,z))) \to (\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \to R(z,x)))$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \to R(x,z))) \land \neg(\forall x.\forall y.\forall z.(R(y,x) \land R(z,y) \to R(z,x)))$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \to R(x,z))) \land \exists x.\exists y.\exists z.\neg(R(y,x) \land R(z,y) \to R(z,x))$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \to R(x,z))) \land \exists x.\exists y.\exists z.\neg R(y,x) \lor \neg(R(z,y) \to R(z,x))$$
$$\exists x.\exists y.\exists z.((R(x,y) \land R(y,z) \to R(x,z))) \land \exists x.\exists y.\exists z.\neg R(y,x) \lor R(z,y) \land \neg R(z,x))$$

The negation is, “There exists x, y, and z, where if R(x, y) and R(y, z) then R(x, z), and there exists x, y, and z such that not R(y, z) or R(z, y) and not R(z, x).”

iii. Negating the statement, “For all x, there exists S such that S is a set and for all z, if z is in set S then z=x, and if z=x then z is in set S.”

$$\neg(\forall x.\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x)))$$
$$\exists x.\neg(\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x)))$$
$$\exists x.\forall S.\neg(Set(S) \land \forall z.(z \in S \leftrightarrow z = x))$$
$$\exists x.\forall S.\neg Set(S) \lor \neg(\forall z.(z \in S \leftrightarrow z = x))$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x)$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x)$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg((z \in S \to z = x) \land (z = x \to z \in S))$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg((z \in S \to z = x) \lor \neg(z = x \to z \in S))$$
$$\exists x.\forall S.\neg Set(S) \lor \exists z.((z \in S \land z \neq x) \lor (z = x \land z \notin S))$$

Problem 4: 'Cause I'm Happy

i) Type 2: The statement is always true, regardless of which people in P are happy. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true everyone who is a person is happy. If this is true, then it is impossible for there to exist a person who is unhappy. Thus, in all cases, the statement holds true.

ii) Type 1: The statement is true only when either everyone in P is happy or no one in P is happy. In the case that everyone in P is happy, the antecedent is true (there exists a person in P who is happy) and the consequent is true (all people in P are happy). In the case that no one in P is happy, the antecedent is false (there does not exist a person in P who is happy), making the statement vacuously true. However, in the case where at least one person in P is unhappy, the antecedent can be true (there exists a person in P who is happy) while the consequent is false (not all people in P are happy). Thus, the statement is only true when everyone or no one in P is happy.

iii) Type 2: The statement is always true, regardless of which people in P are happy. In the case that everyone in P is happy, it's true that for all people in P, if the person is happy then there exists a person in P who is happy. In the case that no on in P is happy, for all people in P, the antecedent is always false, so the statement is vacuously true. In the case that at least one person in P is unhappy, for all people in P, if the person is happy then the statement holds, and if the person is unhappy, then the statement is vacuously true. Thus, in all cases, the statement holds true.

iv) Type 2: The statement is always true, regardless of which people in P are happy. In the case that everyone in P is happy, it's true that there exists a person in P where if the person is happy, then everyone in P is happy. In the case that no one in P is happy, it's true that there exists a person in P where the statement is true (because the antecedent is false, so the statement is vacuously true). Likewise, in the case that at least one person is unhappy, it's true that there exists a person in P where the statement is true (because for the one unhappy person, the antecedent is false, so the statement is vacuously true). Thus, in all cases, the statement holds true.

v) Type 2: The statement is always true, regardless of which people in P are happy. In the case that everyone is happy, it's true that for all people in P there exists a person in P where the first person's happiness implies the second person's happiness. In the case that no one is happy, the statement is still true, because for all people in P, the falsification of the antecedent makes the statement vacuously true. In the case that at least one person is unhappy, it's true that for all people in P there exists a person in P where the first person's happiness implies the second person's happiness, or, for the one unhappy person, the antecedent is false so the statement is vacuously true). Thus, in all cases, the statement holds true.

vi) Type 2: The statement is always true, regardless of which people in P are happy. In the case that everyone is happy, it's true that there exists a person where for all people, everyone's happiness implies that person's happiness. In the case that no one is happy, it's true that there exists a person where for all people, the antecedent is false making the statement vacuously true. And finally, in the case that at least one person is unhappy, it's true that there exists a person where for all people, the antecedent is false making the statement vacuously true (namely the one unhappy person). Thus, in all cases, the statement holds true.

Problem 5: Translating into Logic

i) $\forall n.\ (Natural(n) \to \exists k.\ \exists j.\ (Natural(k) \land Natural(j) \land ((n = (k + k)) \land ((n * n) = (j + j)))$

ii) ∃p. ∃k. ∃j.(Person(p) ∧ k $\neq j$ ∧ Kitten(k) ∧ Kitten(j) ∧ HasPet(p, k) ∧ HasPet(p, j) ∧ ∀b.((b ≠ k ∧ b ≠ j) → (¬ HasPet(p, b))))

iii) The key to this problem is that an irrational number can not be represented as p/q, or $\sqrt{2} \neq p/q$. We can simplify this to $2 * q^2 \neq p^2$, then use the same method as in i). This results in the following formula

$\forall p.\ \forall q.\ ((Integer(p) \land Integer(q)) \to (((q \cdot q) + (q \cdot q) \neq (p \cdot p)) \land (p + q \neq p))$

iv) $\forall Q.\ (Set(Q) \to \exists P.\ (Set(P) \land \forall S.\ (Set(S) \to (S \in P \land \forall x.\ \forall y.\ (x \in S \to x \in Q \land y \notin Q \to y \notin S)))))$

This statement translated into English says "Any set Q has a powerset P that contains every subset S of Q."

v) $\exists x.\,(Lady(x) \land \forall y.\,(Glitters(y) \rightarrow IsSureIsGold(x, y)) \land \exists z.\,(StairwayToHeaven(z) \land Buying(x, z)))$

Problem 6: Raven Paradox

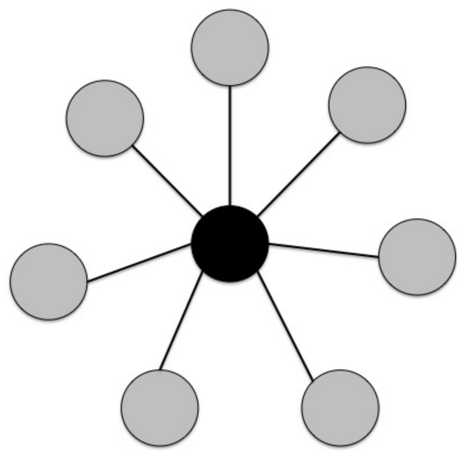
The Raven Paradox results from taking the seemingly uninteresting statement "All ravens are black" and finding its contrapositive "Everything that is not black is not a raven." An important thing to note is that these statements are logically equivalent. Again, so far this is not especially interesting. The trouble arises when we start looking for phenomena that provide evidence for the statement. For instance, if you see a raven and that raven is black, this can be seen as providing evidence for the statement "All ravens are black." However, because the contrapositive is logically equivalent to this statement, observing something that is not black and also not a raven must also be seen as consituting evidence. This is paradoxical because the two do not seem connected at all; looking at a red clown nose does not (intuitively) seem to give us any evidence about the blackness of ravens.

In our discussions, we noted that the statement "All ravens are black" has the form of a scientific hypothesis. Everyone in the group had taken a course where we studied Popper and remembered what he had to say about scientific hypotheses: they can only be falsified, never confirmed. We have seen something similar in our dealings with first-order logic in this course. The statement "All ravens are black" can be turned into a statment in first-order logic with the form $\forall x.\ Raven(x) - > Black(x)$. This statement is only true if it is true for each and every raven. But no matter how many ravens we see that are black, it only takes a single albino raven to falsify the entire statement.

This shows that one of the ways of resolving this paradox is to attack the idea of a phenomenon providing evidence for a statement. If induction is in fact a myth, then seeing a black raven provides the same amount of evidence as seeing a red clown nose: none.

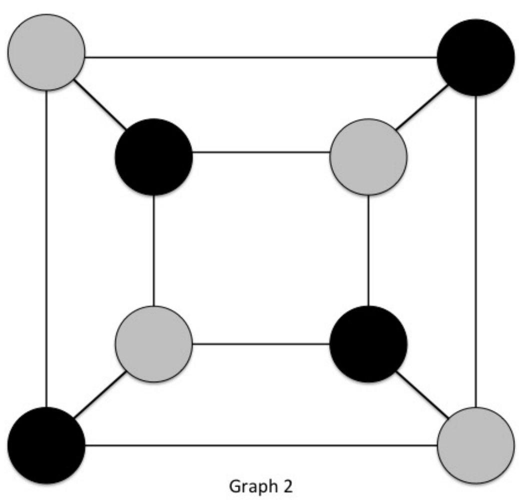
Problem 7: Graph Coloring

i) In this graph the inner-node is connected to all other nodes, giving it a degree of seven. All other nodes are only connected to the inner node, which allow these nodes to all be of the same color and the inner node to be of a different color. This creates a two-colorable graph since no two nodes of the same color are joined by an edge.



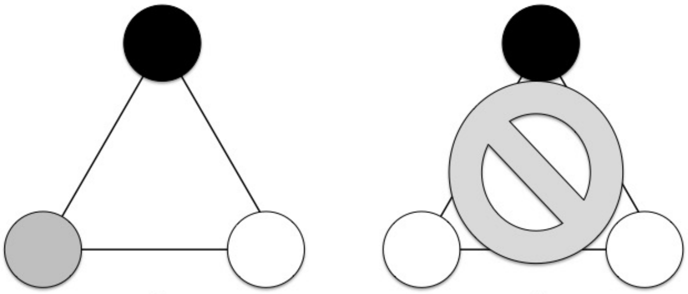
Graph 1

ii) This graph represents a cube, in which each corner shares an edge with three other corners. This means there is a degree of three for every node. It is possible for each corner of the cube to be connected to a corner with only one different color, createing a two-colorable graph.



Graph 2

iii) In a triangle each node is connected to the other two nodes, meaning each node has a degree of 2. This graph is only 3-colorable because each node shares an edge with one another. If one tried to just use two colors, it is guaranteed that these two nodes would share an edge, violating the rule of a 2-colorable graph.



A B

Graph 3

Problem 8: Tournament Cycles

A tournament is a directed graph with n nodes where there is exactly one edge between any pair of distinct nodes and there are no self-loops. Prove that if a tournament graph contains a cycle of any length, then it contains a cycle of length three.

Lemma 1. *Given a directed graph with n nodes, if the directed graph contains a cycle of length $k > 3$, then connecting any unconnected nodes in the cycle will create a cycle of length $< k$.*

Proof. The definition of a cycle in a directed graph is a path from a node to itself. Pick an arbitrary cycle in a connected graph of length k and pick any two arbtrary nodes in that cycle m, n that are not already connected (i.e., not adjacent in the cycle). To connect m and n we can either place the base of the arrow at m and the head at n or the base at n and the head at m. Assume we place the base at m and the head at n. We know there already exists a cycle that starts at m, passes through at least one node that is not n, passes through n, and then passes through at least one node that is not n before returning to m. By connecting m and n we create a new path that skips all the nodes between m and n. Because we know there is at least one node between them, we create a new cycle that cuts out at least one node in the cycle and thus must be $< k$. We know that this is true no matter how we place the base and head by symmetry. □

Theorem. *If a tournament graph contains a cycle of any length, then it contains a cycle of length three.*

Proof. By strong induction. Let $P(n)$ be the statement "if a tournament contains a cycle of length n, then it contains a cycle of length three." We will prove that $P(n)$ holds for $n \geq 3$.

For our base case, we show that $P(3)$ is true. $P(3)$ states that a tournament with a cycle of length 3 contains a cycle of length 3. This is a tautology.

For our inductive step, assume that for some $n \geq 3$, that for any $k \leq n$, that $P(k)$ is true; that is, that a tournament containing a cycle of length k also contains a cycle of length three. We will prove that $P(k+1)$ is true, that if a tournament contains a cycle of length $k+1$, then it also contains a cycle of length three.

Consider any tournament with a cycle of length $k+1$. Remove all the edges from this graph except for this cycle. A representation of this graph is shown in Figure 1. Because this is a tournament, we know that every node in the cycle must be connected to every other node in the cycle. As soon as we draw an edge to reconnect two of the nodes we know by Lemma 1 that we create a cycle of length $< k+1$ (see Figure 2 for illustration). By our inductive hypothesis we know that any cycle of length $\leq k$ contains a cycle of length three.

□

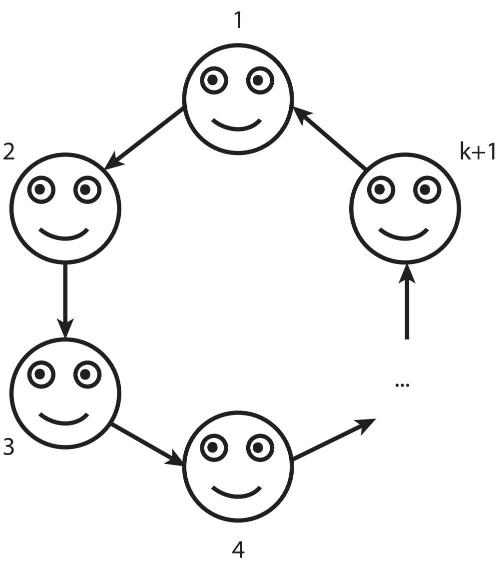


Figure 1: A cycle of length $k+1$

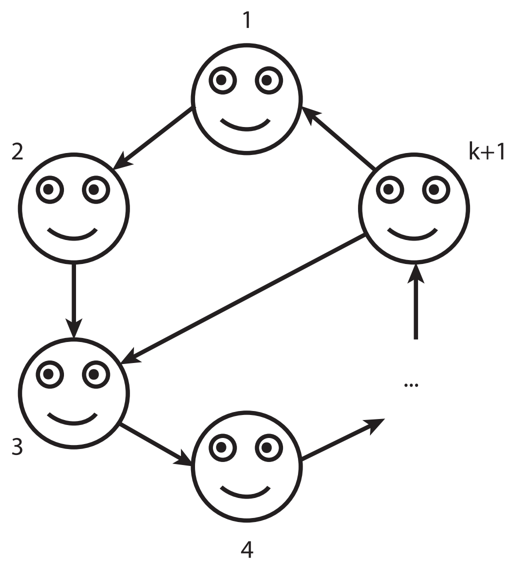


Figure 2: Connecting two nodes in the cycle

Problem Nine: Bipartite Graphs

Suppose G is an undirected graph $G = (V, E)$ where V is the set of nodes and E is the set of edges, where G has no cycles of odd length. Assume G has just one connected component.

Let v be any node $v \in V$.
Let V_1 be the set of all nodes that are connected to v by a path of odd length.
Let V_2 be the set of all nodes connected to v by a path of even length.

i.

Theorem. *V_1 and V_2 have no nodes in common.*

Proof. Assume for the sake of contradiction that there exists a node n that is common between V_1 and V_2. By our definitions of sets V_1 and V_2, if n is common between both sets, there must be at least one path from v to n that is of odd length, and at least one path from v to n that is of even length. Since a path cannot be both odd and even in length, we may conclude that there are at least two paths from v to n.

Since there are multiple paths from v to n, we know by **Lemma 2** that the paths must traverse part of a cycle. If the path traverses part of a cycle, it must start on/enter the cycle at some node, and it must end on/exit the cycle at some other node. We know by **Lemma 3** that there are two paths from the start/entry node on the cycle to the end/exit node on the cycle. Furthermore, because we have stipulated that our graph G only has cycles of even path length, **Lemma 4** indicates that the two paths must both be of even length, or both of odd length.

To take **Lemma 4** even further, no matter how many cycles are traversed between v and n, the path from v to n will either always be even in length, or always be odd in length. This is because each time you enter a cycle, the two possible paths to enter and exit the cycle are both even, or both odd.

This contradicts our earlier assumption that there is at least one path from v to n that is of odd length, and one path from v to n that is of even length. Thus, by contradiction, we have shown that it cannot be true that there exists a node that is common between V_1 and V_2. □

Lemma 2. *If there are multiple paths from v to n, the paths must traverse part of a cycle.*

Proof. The only way for there to be multiple paths from v to n is if the path traverses part of a cycle. We know this to be true from the following line of reasoning: If we start at node v, and we know there are at least two paths from v to n, then there must be a point at which the paths diverge. From node v, the paths either diverge immediately, or they diverge at a later node in the path. Since the path must end at node n, the paths must converge again, either at or before reaching node n. These diverging and converging paths form a cycle: if we take one path from the point of divergence to the point of convergence, and take a different path from that point back to the point of divergence, we have traversed a path from a node to itself, which is the definition of a cycle, as stated in Lecture 8. □

Lemma 3. *For any given cycle, there are two paths from any node on the cycle to another.*

Proof. A cycle in a graph is a path from a node to itself. Choose an arbitrary node on the cycle n, and another arbitrary node on the cycle m. From n, we can traverse the cycle starting either to the left or to the right. If we move from n to the left until we reach m, we have found one path. If we move from n to the right until we reach m, we have found another path. Thus, for any given cycle, there are two paths from any node on the cycle to another. □

Lemma 4. *On a cycle of even length, if the cycle is divided into two subpaths, the two paths will both be of even length, or both of odd length.*

Proof. Take an arbitrary subpath of the cycle.

Case 1: Assume this subpath has an odd path length. Let us denote the path length of this subpath as $2k + 1$, where $k \in \mathbb{N}$. Since the cycle is of even length, denoted as $2j$ where $j \in \mathbb{N}$, we know the remainder of the cycle must be of length $2j - 2k + 1 = 2(j - k) + 1$. Thus, by the definition of odd numbers in Lecture 1, both the arbitrary subpath and the remainder are of odd path length.

Case 2: Assume this subpath has an even path length. Let us denote the path length of this subpath as $2k$, where $k \in \mathbb{N}$. Since the cycle is of even length, denoted as $2j$ where $j \in \mathbb{N}$, we know the remainder of the cycle must be of length $2j - 2k = 2(j - k)$. Thus, by the definition of even numbers in Lecture 1, both the arbitrary subpath and the remainder are of even path length.

Therefore, for a cycle of even length, if the cycle is divided into two subpaths, both subpaths will be of even length, or both will be of odd length. □

ii.

Theorem. *If $G = (V, E)$ has no cycles of odd length, then G is bipartite*

Proof. Take any two adjacent nodes, n and m, where $n \in V$ and $m \in V$. Since n and m are adjacent, you traverse one edge to get from one to the other. We know from Lecture 8 that the length of a path is the number of edges it contains, so the path from n to m, a path with one edge, has a length of one. Let us examine two cases:

Case 1: The path from v to n is of even length

From our proof in Part i. we know that in our graph G, every possible path from v to n must be even in length. We will denote this length as $2k$, where $k \in \mathbb{N}$. Since n and m are adjacent, we know that the path from v to m must be either $2k+1$ or $2k-1 = 2k-2+1 = 2(k-1)+1$. In either case, by the definition of odd numbers, we know that the path length from v to m is odd. Thus, the path from v to n is even, and the path from v to m is odd, which means n and m are in sets V_2 and V_1 respectively.

Case 2: The path from v to n is of odd length

From our proof in Part i. we know that in our graph G, every possible path from v to n must be odd in length. We will denote this length as $2k+1$, where $k \in \mathbb{N}$. Since n and m are adjacent, we know that the path from v to m must be either $2k+1-1 = 2k$ or $2k+1+1 = 2k+2 = 2(k+1)$. In either case, by the definition of even numbers, we know that the path length from v to m is even. Thus, the path from v to n is odd, and the path from v to m is even, which means n and m are in sets V_1 and V_2 respectively.

In both cases, when n and m are adjacent nodes, that is, when there is an edge from one node to the other, n is in one set, and m must be in the other.

The definition of a bipartite graph states that an undirected graph $G = (V, E)$ is called bipartite if there is a way to partition the nodes V into two sets V_1 and V_2 so that every edge in E has one endpoint in V_1 and the other in V_2. So, in accordance with the definition of a bipartite graph, we have found two sets, V_1 and V_2, where for every pair of adjacent nodes n and m, n is in one set, and m must be in the other. From this we may conclude that if $G = (V, E)$ has no cycles of odd length, then G is bipartite

$\square$