Problem Set 3

Katherine Cheng, Richard Davis, Marty Keil

April 24, 2015

Problem 1: Simplifying Propositional Formulas

i) $\neg p$. In this formula whenever p is false the entire statement is true. This is because p implies q in both sub-formulas, and anytime an antecedent is false, the entire statement is true. Conversely, whenever p is true, the entire statement is false. For instance, when p is true and q is true, the negation of q creates a false second statement. However, when p is true and q is false, the first statement is now false. Either way one is left with one false statement and since the "and" conjuction requires that both statements be true, the entire statement is always false in this scenario.

ii) $\top$. In any case the result is true. For p not to imply q, p must true and q must be false. If we assume this then q must imply p because anytime q is false, the implication statement is true. Therefore the overall statement is true since it uses an "or" conjunction, which requires one true statement.

iii) p $\leftrightarrow$ q. Whenever p and q match the overall statement is true, otherwise the statement is false. When one variable is true, but the other is false, the first statement is true due to an "or" conjunction. Under these same conditions the second statement is false due to an "and" conjunction. With a true antecedent and false consequent, the entire formula is always false. However, when one considers identical truth values for p and q, the "or" and "and" conjunctions are no longer different, and the statement will always be true in this case. The bi-conditional conjunction satisfies this same restriction, requiring matching statement outcomes for a true result.

Problem 2: Ternary Conditionals

i) Truth Table

p	q	r	p ? q : r
F	F	F	F
F	T	F	F
T	F	F	F
T	F	T	F
F	T	T	T
F	F	T	T
T	T	F	T
T	T	T	T

ii) $(p \wedge q) \vee (\neg p \wedge r)$

Truth Table

p	q	r	$(p \wedge q) \vee (\neg p \wedge r)$
F	F	F	F
F	T	F	F
T	F	F	F
T	F	T	F
F	T	T	T
F	F	T	T
T	T	F	T
T	T	T	T

iii) p ? $\perp$: $\top$

iv) p ? q : $\top$

Problem Three: First-Order Negations (definitely needs to be checked/edited)

i. Negating the statement, "For all x in the set of real numbers, and for all y in the set of real numbers, if x is less than y then there exists q in the set of rational numbers such that x is less than q, and q is less than y."

$$\neg(\forall x \in \mathbb{R}.\ \forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \wedge q < y))) \tag{1}$$
$$\exists x \in \mathbb{R}.\neg(\forall y \in \mathbb{R}.(x < y \to \exists q \in \mathbb{Q}.(x < q \wedge q < y))) \tag{2}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.\neg(x < y \to \exists q \in \mathbb{Q}.(x < q \wedge q < y)) \tag{3}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \wedge \neg(\exists q \in \mathbb{Q}.(x < q \wedge q < y)) \tag{4}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \wedge \neg(\exists q \in \mathbb{Q}.(x < q \wedge q < y)) \tag{5}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \wedge \forall q \in \mathbb{Q}.\neg(x < q \wedge q < y)) \tag{6}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \wedge \forall q \in \mathbb{Q}.\neg(x < q) \vee \neg(q < y) \tag{7}$$
$$\exists x \in \mathbb{R}.\ \exists y \in \mathbb{R}.(x < y) \wedge \forall q \in \mathbb{Q}.(x \geq q) \vee (q \geq y) \tag{8}$$
$$\tag{9}$$

The negation is, "There exists x in the set of real numbers, and there exists y in the set of real numbers, where x is less than y and for all q in the set of rational numbers x is greater than or equal to q, and q is greater than or equal to y."

ii. Negating the statement, "For all x, y, and z, if (R(x, y) and R(y, z) then R(x, z)) then if for all x, y, and z R(y, z) and R(z, y) then R(z, x)."

$$\neg(\forall x.\forall y.\forall z.(R(x,y) \wedge R(y,z) \to R(x,z))) \to (\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \to R(z,x))) \tag{10}$$
$$\exists x.\exists y.\exists z.\neg((R(x,y) \wedge R(y,z) \to R(x,z))) \to (\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \to R(z,x))) \tag{11}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \to R(x,z))) \wedge \neg(\forall x.\forall y.\forall z.(R(y,x) \wedge R(z,y) \to R(z,x))) \tag{12}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \to R(x,z))) \wedge \exists x.\exists y.\exists z.\neg(R(y,x) \wedge R(z,y) \to R(z,x)) \tag{13}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \to R(x,z))) \wedge \exists x.\exists y.\exists z.\neg R(y,x) \vee \neg(R(z,y) \to R(z,x)) \tag{14}$$
$$\exists x.\exists y.\exists z.((R(x,y) \wedge R(y,z) \to R(x,z))) \wedge \exists x.\exists y.\exists z.\neg R(y,x) \vee R(z,y) \wedge \neg R(z,x)) \tag{15}$$
$$\tag{16}$$

The negation is, "There exists x, y, and z, where if R(x, y) and R(y, z) then R(x, z), and there exists x, y, and z such that not R(y, z) or R(z, y) and not R(z, x)."

iii. Negating the statement, "For all x, there exists S such that S is a set and for all z, if z is in set S then z=x, and if z=x then z is in set S."

$$\neg(\forall x.\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x))) \tag{17}$$

$$\exists x.\neg(\exists S.(Set(S) \land \forall z.(z \in S \leftrightarrow z = x))) \tag{18}$$

$$\exists x.\forall S.\neg(Set(S) \land \forall z.(z \in S \leftrightarrow z = x)) \tag{19}$$

$$\exists x.\forall S.\neg Set(S) \lor \neg(\forall z.(z \in S \leftrightarrow z = x)) \tag{20}$$

$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x) \tag{21}$$

$$\exists x.\forall S.\neg Set(S) \lor \exists z.\neg(z \in S \leftrightarrow z = x) \tag{22}$$

$$\tag{23}$$

How to distribute negation over bidirectional arrow?

Problem 4: 'Cause I'm Happy

i) Statment 2. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true everyone who is a person is happy. If this is true, then it is impossible that there exists a person who is unhappy.

ii) Statement 1. The only way to make a conditional statement false is if the antecedent is true and the consequent is false. In this case, when the antecedent is true there is at least one person in the world who is happy. To make the consequent false, we say that not everyone in the world is happy.

iii) Statement 2. If we negate this statement we get the formula...

iv) Statement 2. Take the negation and show that it is always false.

v) Statement 2. The only way to make an implication false is...

vi) Statement 2.

Problem 5: Translating into Logic

i)

ii)

iii)

iv) $\forall Q Set(Q).\ \exists P Set(P).\ \forall S Set(S).\ (S \in P \cap \forall x.\ \forall y.\ (x \in S \rightarrow x \in Q \cap y \notin S \rightarrow y \notin Q))$

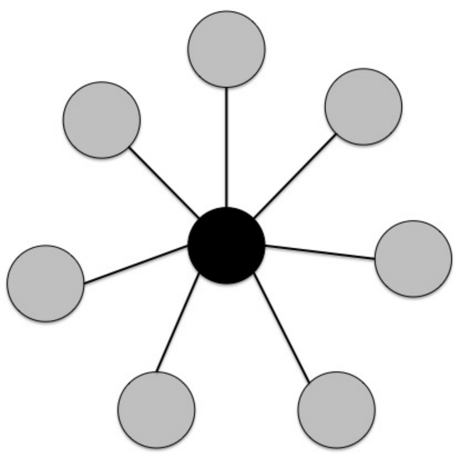
v)

Problem 6: Raven Paradox

The Raven Paradox involves a scientific statement. As we know from Popper, it is only possible to falsify scientific statements. Induction is a myth.

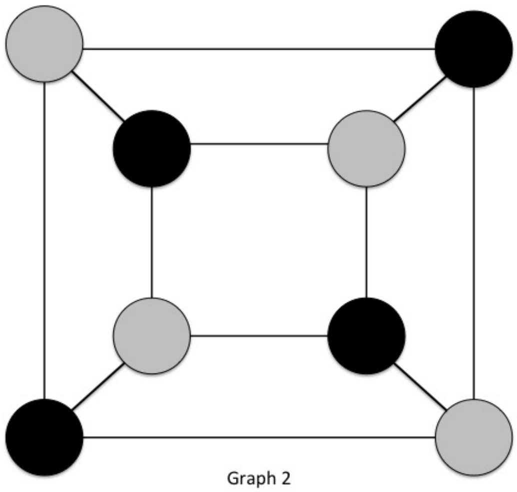
Problem 7: Graph Coloring

i) In this graph the inner-node is connected to all other nodes, giving it a degree of seven. All other nodes are only connected to the inner node, which allow these nodes to all be of the same color and the inner node to be of a different color. This creates a two-colorable graph since no two nodes of the same color are joined by an edge.



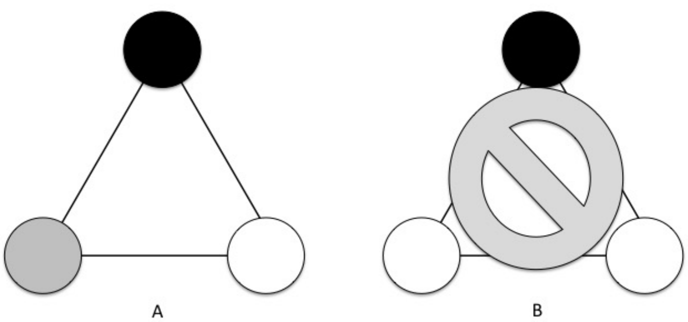
Graph 1

ii) This graph represents a cube, in which each corner shares an edge with three other corners. This means there is a degree of three for every node. It is possible for each corner of the cube to be connected to a corner with only one different color, createing a two-colorable graph.



Graph 2

iii) In a triangle each node is connected to the other two nodes, meaning each node has a degree of 2. This graph is only 3-colorable because each node shares an edge with one another. If one tried to just use two colors, it is guaranteed that these two nodes would share an edge, violating the rule of a 2-colorable graph.



Graph 3

Problem 8: Tournament Cycles

A tournament is a directed graph with n nodes where there is exactly one edge between any pair of distinct nodes and there are no self-loops. Prove that if a tournament graph contains a cycle of any length, then it contains a cycle of length three.

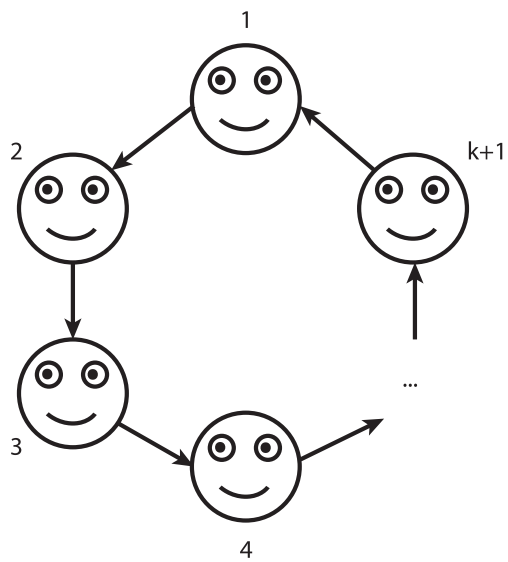


Figure 1: A figure

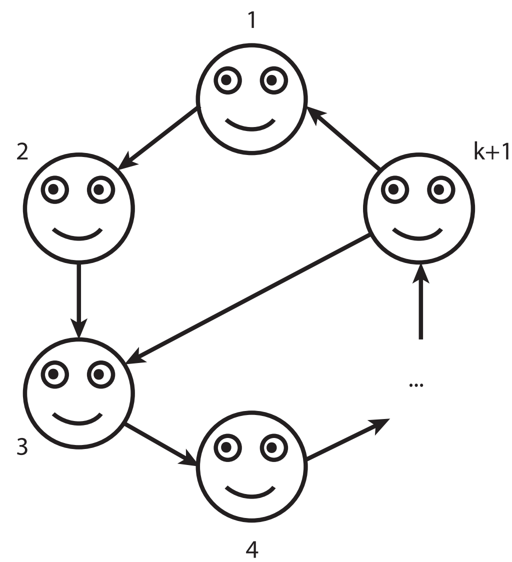


Figure 2: Another figure

Theorem. *If a tournament graph contains a cycle of any length, then it contains a cycle of length three.*

Proof. By strong induction. Let $P(n)$ be the statement "if a tournament contains a cycle of length n, then it contains a cycle of length three." We will prove that $P(n)$ holds for $n \geq 3$.

For our base case, we show that $P(3)$ is true. $P(3)$ states that a tournament with a cycle of length 3 contains a cycle of length 3. This is a tautology.

For our inductive step, assume that for some $k \geq 3$ that $P(k)$ is true; that is, that a tournament containing a cycle of length k also contains a cycle of length three. We will prove that $P(k+1)$ is true, that if a tournament contains a cycle of length $k+1$, then it also contains a cycle of length 3.

Consider any tournament with a cycle of length $k+1$. Remove all the edges from this graph except for this cycle. A representation of this graph is shown in Figure 1. Because this is a tournament, we know that every node must be connected to every other node.

$\square$