Fluids PS1

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1 Streamlines and Flows

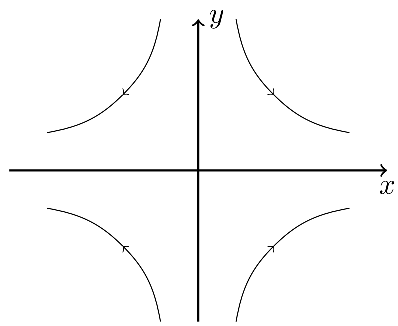
For (A) a 2D straining flow $\boldsymbol{u} = (\alpha x, -\alpha y)$ and (B) a simple shear flow $\boldsymbol{u} = (\gamma y, 0)$ where α and γ are constants:
a) Find the equation for a general streamline of the flow. [1]

The question is looking for you to use $\frac{\mathrm{d}x}{u_x} = \frac{\mathrm{d}y}{u_y}$. We see this by using $u_x = \frac{\mathrm{d}x}{\mathrm{d}t}$ and $u_y = \frac{\mathrm{d}y}{\mathrm{d}t}$ and then dividing. For flow (A) this yields

$$\frac{\mathrm{d}x}{\alpha x} = \frac{\mathrm{d}y}{-\alpha y}$$

$$\implies \qquad \mathrm{dLn}x + \mathrm{dLn}y = 0$$

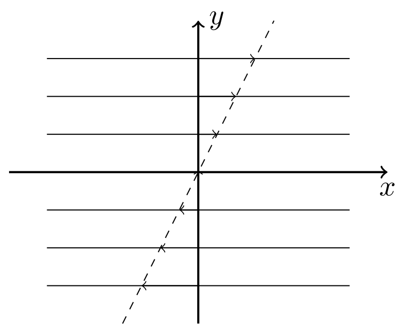
and so $xy =$ constant along a streamline. This flattens the flow along the x-axis, as shown in this figure.



For the second flow we have

$$\frac{\mathrm{d}x}{\gamma y} = \frac{\mathrm{d}y}{0}$$

so $\mathrm{d}y = 0$ and the streamlines are $y =$ const. This is shown in this figure.



Alternatively, we can just solve the equations. In case (A),

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha x \implies x = x_0 e^{\alpha t}$$

$$y = y_0 e^{-\alpha t}$$

and for flow (B)

$$y = y_0$$
$$\implies \frac{\mathrm{d}x}{\mathrm{d}t} = \gamma y_0$$
$$\implies x = x_0 + \gamma y_0 t.$$

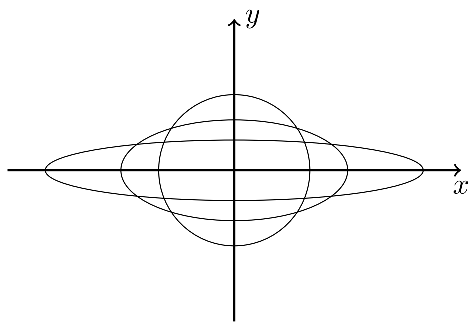
These give the same results as the solutions above.

b) At $t = 0$ dye is introduced to mark the curve $x^2 + y^2 = a^2$. Find the equation for this material fluid curve for $t > 0$ and sketch how the curve evolves with time. [2]

We solved for the flow in the previous section and now just insert that in here. So we begin with $x_0^2 + y_0^2 = a^2$. Therefore, putting these initial values in terms of the general values, $x = x(t)$ and $y = y(t)$,

$$x^2 e^{-2\alpha t} + y^2 e^{2\alpha t} = a^2$$

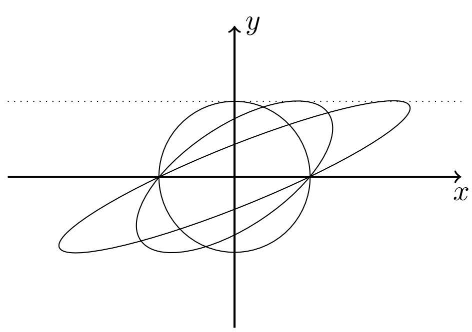
which traces out an ellipse with major axis $ae^{\alpha t}$ and minor axis $ae^{-\alpha t}$ as shown in the figure.



For flow (B) we have

$$(x - \gamma y t)^2 + y^2 = a^2.$$

What this does to the flow is less instantly clear but we can use out intution from part a) which tells us that the circle gets pulled into an ellipse as shown in this figure.



c) Does the area within the curve change in time, and why? [1]

This could be made exceedingly difficult but is remarkably easy when we notice that both flows are incompressible, i.e. $\nabla \cdot \boldsymbol{u} = 0$. Therefore, the density is constant by the continuity equation and so the area enclosed by the curve cannot have changed.

d) Which of these two flows stretches the curve faster at long times? [1]

Flow (A) stretches the flow exponentially in time at long times whereas the shear flow in (B) is only linear. Therefore, flow (A).

2 Stream Function and Velocity Potential

a) Is the motion incompressible for the flows given by the following velocity potentials:
(i) $\phi = C(x^2 + y^2)$ (ii) $\phi = C(x^2 - y^2)$?
If so, determine the corresponding stream functions. [2]

The velocity potential is defined by $\boldsymbol{u} = \nabla\phi$. Therefore, for incompressibility, we require $\nabla^2\phi = 0$. So is this the case?

$$\nabla^2\phi_1 = 4C$$

$$\nabla^2\phi_2 = 0.$$

So the latter velocity field is incompressible with

$$\boldsymbol{u} = \begin{pmatrix} 2Cx \\ -2Cy \end{pmatrix}.$$

To find the streamfunction we use

$$\boldsymbol{u} = \nabla \times (\psi\hat{\boldsymbol{z}})$$

to see that

$$\psi = 2Cxy.$$

b) Is the motion irrotational for the flows given by the following stream functions:
(iii) $\psi = C(x^2 + y^2)$ (iv) $\psi = C(x^2 - y^2)$?
If so, determine the corresponding velocity potentials. [2]

To check this one we should convert straight to $\boldsymbol{u}$. So

$$\boldsymbol{u}_1 = \begin{pmatrix} 2Cy \\ -2Cx \end{pmatrix}$$

which has a curl. Therefore, this is not irrotational. The second has

$$\boldsymbol{u}_2 = \begin{pmatrix} -2Cy \\ -2Cx \end{pmatrix}$$

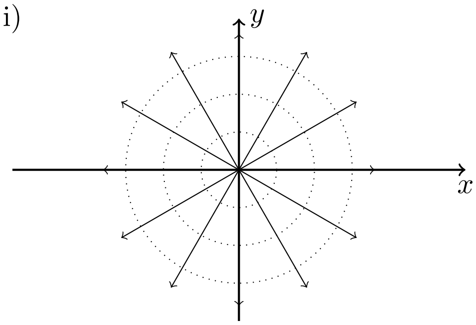
for which $\nabla \times \boldsymbol{u}_2 = 0$. This flow is therfore irrotational with a velocity potential

$$\phi = -2Cxy.$$

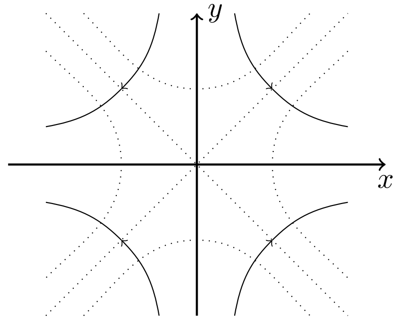
c) Sketch the streamlines for all cases (i)-(iv) and the lines of constant ϕ where possible. [2]

Solid arrows denote streamlines and dotted lines denote lines of constant velocity potential in these figures.

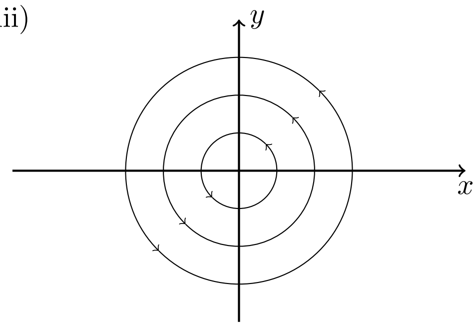
i)



ii)

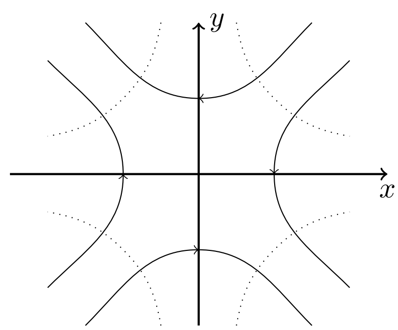


iii)



(there is no velocity potential describing this flow)

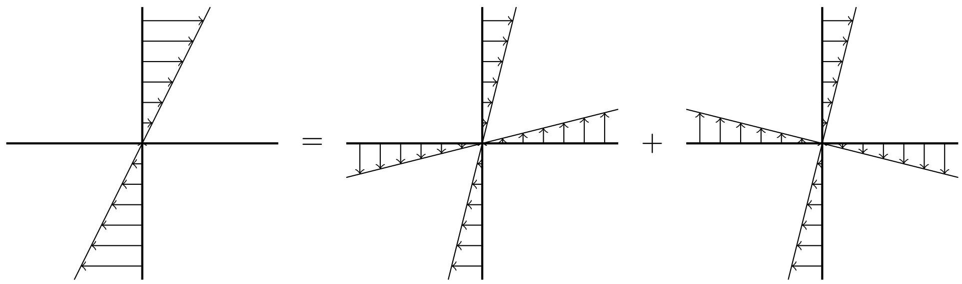
iv)



3 Velocity Gradient Tensor

Show that a simple shear flow $\boldsymbol{u} = (\alpha y, 0, 0)$ can be decomposed into a sum of a rotation and a straining flow (i) pictorially (ii) in terms of the velocity gradient tensor. [3]

Pictorially one can consider the following figure.



In terms of the velocity gradient tensor $L_{ij} = \partial_i u_j$, we can consider each flow

$$\boldsymbol{u}^{\text{shear}} = \begin{pmatrix} \alpha y \\ 0 \end{pmatrix} \qquad \Longrightarrow \qquad L^{\text{shear}} = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

$$\boldsymbol{u}^{\text{rotation}} = \begin{pmatrix} \alpha y \\ -\alpha x \end{pmatrix} \qquad \Longrightarrow \qquad L^{\text{rotation}} = \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}$$

$$\boldsymbol{u}^{\text{strain}} = \begin{pmatrix} \alpha y \\ \alpha x \end{pmatrix} \qquad \Longrightarrow \qquad L^{\text{strain}} = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

and so

$$L^{\text{shear}} = \frac{1}{2}\left(L^{\text{rotation}} + L^{\text{strain}}\right).$$

4 Solving Navier-Stokes

Consider a steady, two-dimensional, incompressible, viscous flow down an inclined plane under the influence of gravity. Define the axes as shown in the diagram, and assume that the velocity $\boldsymbol{u}$ depends only on y.

a) What are the boundary conditions for $\boldsymbol{u}$ at $y = 0$? Using incompressibility, show that the y-component of the velocity is zero throughout the flow. [2]

At $y = 0$ we have the no-slip and no normal flow boundary conditions, $\boldsymbol{u} = \mathbf{0}$. We then consider that translational symmetry demands $\frac{\partial}{\partial x}\boldsymbol{u} = 0$ and so the continuity equation (for the incompressible fluid) becomes

$$\nabla \cdot \boldsymbol{u} = \frac{\partial u_y}{\partial y} = 0.$$

Therefore, there is no velocity in the y direction.

b) Write fown the x- and y-components of the Navier-Stokes equation. [2]

The NS equation is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{\nabla p}{\rho} = \nu \nabla^2 \boldsymbol{u} + \boldsymbol{g}.$$

We have steady flow ($\frac{\partial}{\partial t} = 0$) and translational symmetry along x ($\frac{\partial}{\partial x} = 0$ and $\boldsymbol{u} \cdot \nabla = 0$ as $\boldsymbol{u}$ is purely along x). Therefore,

$$0 = \nu \frac{\partial^2 u_x}{\partial y^2} + g \sin\theta$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\cos\theta.$$

c) From the y-component show that the pressure

$$p = p_0 + \rho g(d - y)\cos\theta$$

where p_0 is the pressure at the free surface $y = d$. [1]

We simply integrate and apply our boundary conditions,

$$p - p_0 = -\rho g \cos\theta (y - d)$$

$$p = p_0 + \rho g(d - y)\cos\theta.$$

d) From the x-component, using the appropriate boundary conditions at $y = 0$ and the zero tangential stress condition $\nu \frac{\mathrm{d}u_x}{\mathrm{d}y} = 0$ at the free surface $y = d$ show that

$$u_x = \frac{g}{2\nu} y(2d - y)\sin\theta.$$

[2]

This is another simple integration problem.

$$\frac{\partial u_x}{\partial y} - 0 = -\frac{g}{\nu}(y-d)\sin\theta$$

$$u_x - 0 = -\frac{g}{\nu}\left(\frac{y^2}{2} - dy\right)\sin\theta$$

$$u_x = \frac{g}{2\nu}y(2d-y)\sin\theta.$$

e) Show that the volume flux per unit distance along z is $\frac{gd^3\sin\theta}{3\nu}$. [2]

Here we integrate the velocity profile over the height,

$$\Phi = \int_0^d u_x \mathrm{d}y,$$

another integration problem.

$$\Phi = \frac{g}{2\nu}\sin\theta\int_0^d (2dy - y^2)\mathrm{d}y$$

$$\Phi = \frac{g}{2\nu}\sin\theta\frac{2d^3}{3}$$

$$\Phi = \frac{gd^3}{3\nu}\sin\theta.$$

5 Reynolds Number

Estimate the magnitude of the Reynolds number for:
a) flow past the wing of a jumbo jet,
b) a human swimmer,
c) a thick layer of treacle draining off a spoon,
d) a bacterium swimming in water.
Take the kinematic viscosity ν to be $10^{-6}\text{m}^2\text{s}^{-1}$ for water, $1.5\times10^{-5}\text{m}^2\text{s}^{-1}$ for air and $10^{-1}\text{m}^2\text{s}^{-1}$ for treacle. [2]

The Reynolds number is $\frac{UL}{\nu}$. So we have

$$a) \qquad \text{Re} \sim \frac{150 \times 10}{1.5 \times 10^{-5}} \sim 10^8$$

$$b) \qquad \text{Re} \sim \frac{1 \times 1}{10^{-6}} \sim 10^6$$

$$c) \qquad \text{Re} \sim \frac{10^{-3} \times 10^{-2}}{10^{-1}} \sim 10^{-4}$$

$$d) \qquad \text{Re} \sim \frac{10^{-5} \times 10^{-5}}{10^{-6}} \sim 10^{-4}.$$

So at scales we're interested in, things are often turbulent. At smaller scales, bacteria really do swim through what looks like treacle to them.

6 Dynamical Similarity

Determine the conditions for the dynamical similarity of steady incompressible flow of an electrically conducting fluid in a magnetic field, governed by the equations

$$\nabla \cdot \boldsymbol{u} = 0, \tag{6.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{6.2}$$

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho\mu}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u}, \tag{6.3}$$

$$\boldsymbol{u} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{u} + \frac{1}{\sigma\mu}\nabla^2 \boldsymbol{B}. \tag{6.4}$$

You will need to define a length scale L, a velocity scale U and a magnetic field scale B_0.
Notation: $\boldsymbol{u}$=velocity, $\boldsymbol{B}$=magnetic field, p=pressure, ρ=density, ν=kinematic viscosity, μ=magnetic permeability, σ=electrical conductivity.
Comment on the physical meaning of the dimensionless control parameters. [6]

What we're looking for is all the dimensionless parameters of the flow. To do this we could simply use dimensional analysis on our parameters,

$$p \sim \rho U^2$$

$$\rho \sim \rho$$
$$\boldsymbol{u} \sim U$$
$$\boldsymbol{x} \sim L$$
$$\nu \sim LU$$
$$\mu \sim B^2 \rho^{-1} U^{-2}$$
$$\sigma \sim \rho U L^{-1} B^{-2}$$
$$B \sim B$$

where I've chosen the independent set of units ρ, U, L and B. So we're expecting four dimensionless parameters defining the flow.

The nicer way to do things is to start with the equations and put them in terms of dimensionless variables. The first two are already dimensionless so we needn't worry about them. For the others we define dimensionless variables

$$\boldsymbol{u} = U\tilde{\boldsymbol{u}}$$

$$\boldsymbol{B} = B_0 \tilde{\boldsymbol{B}}$$

$$\boldsymbol{x} = L\tilde{\boldsymbol{x}}.$$

These can be inserted into our equations above,

$$\frac{U^2}{L}\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\frac{1}{L}\frac{\tilde{\nabla}p}{\rho} + \frac{B_0^2}{L\rho\mu}(\tilde{\nabla} \times \tilde{\boldsymbol{B}}) \times \tilde{\boldsymbol{B}} + \frac{U\nu}{L^2}\tilde{\nabla}^2\tilde{\boldsymbol{u}}$$

$$\tilde{\boldsymbol{u}} \cdot \tilde{\nabla}\tilde{\boldsymbol{u}} = -\frac{\tilde{\nabla}p}{U^2\rho} + \frac{B_0^2}{U^2\rho\mu}(\tilde{\nabla} \times \tilde{\boldsymbol{B}}) \times \tilde{\boldsymbol{B}} + \frac{\nu}{UL}\tilde{\nabla}^2\tilde{\boldsymbol{u}}$$

and for the second one,

$$\frac{UB_0}{L}\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}\tilde{\boldsymbol{B}} = \frac{UB_0}{L}\tilde{\boldsymbol{B}}\cdot\tilde{\nabla}\tilde{\boldsymbol{u}} + \frac{B_0}{L^2\sigma\mu}\tilde{\nabla}^2\tilde{\boldsymbol{B}}$$

$$\tilde{\boldsymbol{u}}\cdot\tilde{\nabla}\tilde{\boldsymbol{B}} = \tilde{\boldsymbol{B}}\cdot\tilde{\nabla}\tilde{\boldsymbol{u}} + \frac{1}{UL\sigma\mu}\tilde{\nabla}^2\tilde{\boldsymbol{B}}.$$

Notice I haven't said anything about p or ρ yet. That term is dimensionless though and we'll deal with it at the end.

So what dimensionless parameters do we have? Firstly, we have

$$\frac{B_0^2}{U^2\rho\mu}.$$

Physically this is a ratio of energy scales with $\frac{B_0^2}{\mu}$ the energy density of the magnetic field and ρU^2 the kinetic energy. Secondly we have the Reynolds number

$$\frac{UL}{\nu}$$

which determines the size of inertial terms versus the size of viscous terms in the Navier-Stokes equations. Finally, we have

$$UL\sigma\mu.$$

This one is a 'magnetic Reynolds number' so named because it tells us about the size of inertial terms relative to 'magnetic friction', which is some force with dimensions $\frac{L^2\sigma\mu}{U}$.

Finally, and despite what the official answers say, we have the $\frac{\tilde{\nabla}p}{\rho U^2}$ term. If we call $p = P\tilde{p}$ then the dimensionless parameter out front here is

$$\frac{P}{\rho U^2}.$$

This is usually the Mach number of the fluid, as $\frac{p}{\rho}$ is the speed of sound squared. However, here the speed of sound is infinite (your fluid is incompressible) so it actually just encodes how your plasma is forced. This perhaps excuses it's presence in the official answers as a dynamical similarity experiment would assume the forcing is the same in both cases.

7 More Dynamical Similarity

a) A flat plate of width L is placed at a right angle to the flow in a wind tunnel, in which the upstream wind speed is U. Show that the expected scaling of the pressure variations is
(i) $\Delta p \sim \frac{\rho\nu U}{L}$ in the limit Re $\ll 1$,
(ii) $\Delta p \sim \rho U^2$ in the limit Re $\gg 1$. [2]

We start by quoting the Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{\nabla p}{\rho} = \nu \nabla^2 \boldsymbol{u}.$$

Therefore, in the first limit. Where the Reynolds number is small viscosity dominates so we can ignore the inertial term and

$$\frac{\Delta p}{\rho L} \sim \frac{\nu U}{L^2}$$

$$\Delta p \sim \frac{\rho \nu U}{L}.$$

In the other limit we ignore viscosity and

$$\frac{\Delta p}{L\rho} \sim \frac{U^2}{L}$$

$$\Delta p \sim \rho U^2.$$

b) In the wind tunnel vortices are shed behind the plate at a frequency of $0.5s^{-1}$. The same plate is now placed into a water channel. Calculate the flow rate required, as a multiple of that in the wind tunnel, to produce dynamically similar behaviour, and calculate the frequency of the vortex shedding. [2]

We need to preserve the dimensionless parameter that defines the entire flow, namely

$$\text{Re} = \frac{UL}{\nu}$$

so the difference must be

$$\frac{U_{\text{water}}}{\nu_{\text{water}}} = \frac{U_{\text{air}}}{\nu_{\text{air}}}$$

$$\implies U_{\text{water}} = 0.067 U_{\text{air}}.$$

The frequency can then be scaled according to $f \sim \frac{U}{L}$ so

$$f_{\text{water}} = 0.067 f_{\text{air}} = 0.033 s^{-1}.$$

8 Vorticity

a) What is meant by the vorticity of a fluid flow? Illustrate your answer by discussing:
(i) A rectilinear flow that has vorticity, e.g. simple shear (again) $\boldsymbol{u} = (\alpha y, 0, 0)$.
(ii) a rotating flow that does not have vorticity, e.g. $u_r = 0, u_\theta = \frac{A}{r}$ (in plane polar coordinates).
[3]

Vorticity is defined only as the local rotation that a fluid element has. So given the definition, $\omega = \nabla \times \boldsymbol{u}$ lets consider the examples given. In example (i) we have

$$\boldsymbol{u} = \begin{pmatrix} \alpha y \\ 0 \\ 0 \end{pmatrix}$$

$$\implies \boldsymbol{\omega} = -\alpha \hat{\boldsymbol{z}}.$$

This makes sense because we have a fluid element which becomes stretched, almost like it is rotating along that shear direction.

In the second case, which seems like it is indeed rotating, we have

$$\boldsymbol{u} = A \begin{pmatrix} -\frac{y}{r^2} \\ \frac{x}{r^2} \\ 0 \end{pmatrix}$$

where $r^2 = x^2 + y^2$ so

$$\boldsymbol{\omega} = \left(\frac{2}{r^2} - \frac{2x^2}{r^4} - \frac{2y^2}{r^4} \right) \hat{\boldsymbol{z}} = 0$$

where we've subtly assumed that $r \neq 0$ (see first year maths). Therefore, for this fluid which is clearly undergoing global rotation, the vorticity is zero. Any fluid element just travels around the origin, without rotating. Only a packet of fluid at the centre, over $r = 0$ would rotate.

b) What is the vorticity of a flow in rigid body motion with angular velocity Ω? [1]

We begin by quoting the velocity field

$$\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}.$$

For this we find the vorticity,

$$\begin{aligned} \boldsymbol{\omega} &= \nabla \times (\boldsymbol{\Omega} \times \boldsymbol{r}) \\ \omega_i &= \partial_j r_j \Omega_i - \partial_j \Omega_j r_i \\ &= 3\Omega_i - \Omega_j \delta_{ij} \\ \boldsymbol{\omega} &= 2\boldsymbol{\Omega}. \end{aligned}$$