COLLECTION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

FLUIDS, FLOWS AND COMPLEXITY

HILARY TERM

Two hours including 10 minutes reading time
Answer **two** *questions.*

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier–Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \, ,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

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1. Explain why a fluid may be considered as a continuous medium when viewed on a large enough length scale. Derive an estimate of the length scale involved, in the case of air at standard temperature and pressure. [5]

Derive the acceleration of a moving fluid element in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \, ,$$

where $\mathbf{u}$ is the fluid velocity at a fixed point $\mathbf{r}$ and time t. [3]

A viscous fluid of uniform density ρ and uniform kinematic viscosity ν flows steadily in the x-direction along a tube of circular cross-section with radius a. It is subject to a constant pressure gradient $-G$ in the x-direction. The flow speed u depends only on r, the distance from the axis of the tube. Given that the x-component of $\nabla^2 \mathbf{u}$ in these circumstances is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) ,$$

show that

$$u(r) = A(a^2 - r^2) \, ,$$

and find A in terms of G, ρ and ν. Find also the mass of fluid crossing the surface $x = 0$ in unit time. [10]

Explain what is meant by the *Reynolds Number*, and derive an expression for it for this flow. What is found experimentally as the imposed pressure gradient is increased from zero? [7]

2. Explain what is meant by the *vorticity* ω of a fluid flow, and show how it is related to the *circulation* of the velocity field (as suitably defined). Obtain expressions for the vorticity and circulation of (a) a flow in rigid-body rotation with angular velocity Ω, and (b) a two-dimensional line vortex whose velocity is given by $\mathbf{u} = (-Ay/r^2, Ax/r^2, 0)$, where $r^2 = x^2 + y^2$ and A is a constant. [6]

Consider an inviscid, incompressible flow, parallel to the xy-plane with no variation in the z-direction, whose velocity field is given by $\mathbf{u} = (u_x, u_y, 0) = \nabla\phi$, where ϕ is a scalar function of x and y. Show that this flow has zero vorticity everywhere, and that ϕ satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

A steady, two-dimensional, inviscid, incompressible flow around a long circular cylinder, aligned in the z-direction and of radius a, is given by the velocity potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - G\theta,$$

where $x = r\cos\theta$ and $y = r\sin\theta$, and U and G are constants. Verify that ϕ and the radial and azimuthal velocity components of this velocity field satisfy Laplace's equation and appropriate boundary conditions at the surface of the cylinder. Why does u_θ not need to be zero at the surface of the cylinder? Obtain expressions for the vorticity of the flow and circulation Γ around a path enclosing the cylinder. [8]

By applying Bernoulli's equation,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\mathbf{u}|^2 + \frac{p}{\rho} + gz = 0,$$

where ρ is the density and g is the acceleration due to gravity, to this situation, obtain an expression for the pressure p at the surface of the cylinder. Hence, show that the drag force F_x on the cylinder in the x-direction is zero and there is a net lift force in the y direction of the form

$$F_y = -\rho U\Gamma$$

per unit length of the cylinder in the z-direction. [7]

Give a brief qualitative discussion of how this result can be extended to explain the physics of aircraft flight. [4]

3. Two competing antibodies in an immune system have populations $x(t)$ and $y(t)$ at time t. The evolution of the two species is described by the following coupled time-dependent equations:

$$\begin{aligned}\dot{x} &= x(3 - \beta x - y),\\ \dot{y} &= y(2 - x - y).\end{aligned}$$

The parameter β is determined by the environment. Consider $\beta = 1$. Find the fixed points and determine their stabilities. Hence sketch the phase portrait in the (x, y) plane for the system. [13]

Now consider $\beta = 2$. Find the corresponding fixed points and determine their stabilities. Sketch the phase portrait. [8]

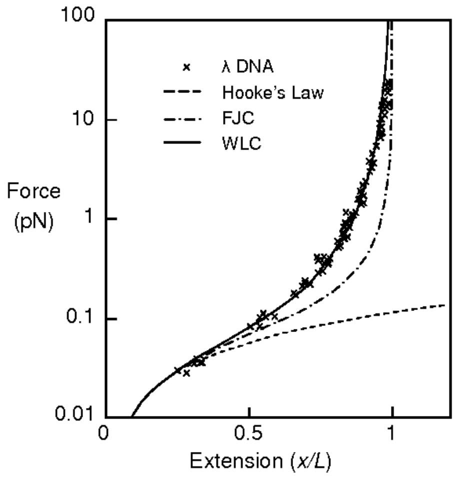
Comment on the significance of the particular forms of the phase portrait for $\beta = 1$ and $\beta = 2$. [4]

4. please see next page

4. Derive an expression for $\langle z^2 \rangle$, the mean square end-to-end distance for an ideal freely-jointed chain (FJC, often also called a Gaussian chain) consisting of N rigid segments of length b, freely hinged where they join. You may neglect possible consequences of interference between different parts of the chain. Write down an expression for the partition function of the FJC in the case that a force f is applied between the ends of the chain to separate them. Show that $\langle z \rangle$, the mean end-to-end distance of the chain, is related to the applied force by

$$\langle z \rangle = Nb\,(\coth\alpha - 1/\alpha)$$

where $\alpha = fb/k_B T$. [14]



The figure shows a force-extension curve for λ-phage double-stranded DNA of length L ($\sim 10^{-5}$m) with fits to Hooke's law (at small extensions), and to the FJC and Worm-Like Chain (WLC) models. Discuss the physical origin of the observed behaviour of double-stranded DNA in the low and high force regimes. Find an expression for the spring constant (i.e., the constant of proportionality between force and extension) for a FJC for small applied forces. (You may assume that, for $\alpha \ll 1$, $\coth(\alpha) - 1/\alpha \simeq \alpha/3$.) Hence, using the information provided in the figure, estimate the effective segment length for double-stranded DNA to 1 s.f. To 1 s.f.,what is the FJC prediction for the force required to stretch this piece of DNA to 0.999 of its contour length? (At the temperature of the measurement, $k_B T = 4$ pN nm.) [11]