A13480W1

SECOND PUBLIC EXAMINATION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1. FLOWS, FLUCTUATIONS AND COMPLEXITY

TRINITY TERM 2016

Monday, 6 June, 2.30 pm – 4.30 pm

Answer **five** *questions with at least* **one** *from each section:*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier–Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \,,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as

$$J_{ij} = \frac{\partial f_i}{\partial x_j},$$

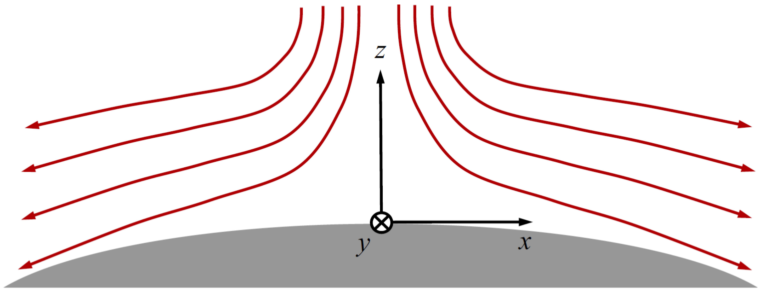
where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

1. Show that an irrotational, incompressible flow can be described by a velocity potential, ϕ, where

$$\nabla^2\phi = 0.$$

[2]

Consider such a flow heading towards the surface of a stationary and impenetrable object, thus creating a *stagnation point* as shown in the following figure.



Explain why the velocity potential in the vicinity of the stagnation point has the form

$$\phi(x, y, z) = W(x^2 + y^2 - 2z^2).$$

Determine the shape of the streamlines. [6]

A sphere of radius R and mass density ρ_0 is moving with constant velocity $\mathbf{v}$ (taken to be along the z-axis) in an inviscid, incompressible fluid of constant density, ρ, creating a steady flow around it. Determine the velocity potential, ϕ, and the fluid velocity profile $\mathbf{u}(\mathbf{r})$, in the rest frame of the sphere, using the standard axially symmetric solution to Laplace equation in spherical coordinates:

$$\phi(r, \theta) = A_0 + \frac{B_0}{r} + \left(A_1 r + \frac{B_1}{r^2}\right)\cos\theta + \cdots.$$

[7]

Identify both stagnation points in the flow, and using a Taylor expansion or otherwise, show that your result agrees with the expected form of flow near a stagnation point. [5]

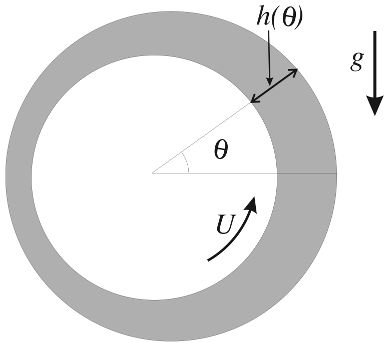
Using the Bernoulli equation (ignoring gravity)

$$\frac{\mathbf{u}^2}{2} + \frac{p}{\rho} = \text{constant},$$

calculate the pressure on the surface of the sphere. [5]

2. Write down the equations governing slow viscous flow, giving clear physical interpretation of their meaning and the criteria for them to be valid. [3]

A thin layer of viscous fluid of thickness $h(\theta)$ (shown shaded, and with exaggerated thickness, in the diagram) covers the outer surface of an infinitely long cylinder whose circumference rotates at a speed U about a horizontal axis through the centre of the cylinder, in the presence of a gravitational acceleration g.



Show that under the lubrication approximation, the fluid velocity satisfies the equation

$$\nu \frac{\partial^2 u}{\partial z^2} = g \cos\theta \, ,$$

where z is the distance from the surface of the cylinder in the normal direction, $u(\theta, z)$ is the velocity of the fluid in the θ direction, and ν is the kinematic viscosity. Write down the boundary conditions at $z = 0$ and $z = h(\theta)$ and show that

$$u(\theta, z) = U + \left(\frac{z^2}{2} - zh \right) \frac{g}{\nu} \cos\theta.$$

[7]

Obtain an expression for the volume flux

$$Q(\theta) = \int_0^{h(\theta)} u \, \mathrm{d}z$$

across any section (with $\theta =$ constant). Hence show that

$$\frac{1}{H^2} - \frac{1}{H^3} = \alpha \cos\theta \, ,$$

where

$$H(\theta) = Uh(\theta)/Q \quad \text{and} \quad \alpha = \frac{gQ^2}{3\nu U^3} \, .$$

Explain why Q is constant. [5]

Find a perturbative solution for $h(\theta)$ assuming $\alpha \ll 1$, to first order in α. [3]

Show that a solution $H(\theta)$ can exist for all values of θ only if $\alpha \le 4/27$. Show that the maximum film thickness the system can stably accommodate is

$$h_{\text{max}} = \frac{3}{2} \frac{Q}{U}.$$

Explain the physical mechanism behind this phenomenon. [7]

3. Consider the dynamical system

$$\dot{x} = rx - \sin x,$$

on the real x-axis, where r is a positive and real tuning parameter.

For $r = 0$, find and classify all the fixed points and summarize your results by showing the flows along the x-axis. [3]

Discuss what happens as r is changed from 0 to 2, and classify all the fixed points. For $0 < r \ll 1$, find an approximate expression for values of r at which bifurcations occur. Estimate the number of fixed points as a function of r. [8]

What happens when the (same) dynamical system is defined on the imaginary x-axis (i.e. $x \mapsto \mathrm{i}x$ in the dynamical equation)? Plot the bifurcation diagram. [5]

For the dynamical system

$$\begin{aligned} \dot{x} &= y^3 - y \\ \dot{y} &= x - y^2 \end{aligned}$$

find all the fixed points, classify them, and produce the full phase portrait. [9]

4. During gene expression of a single gene, the RNA polymerase molecule transcribes DNA into RNA with a rate k_R, and RNA is subsequently translated to protein with a rate k_P. The RNA and protein molecules are also degraded with rates γ_R and γ_P, respectively. In the absence of noise, the changes in the RNA concentration R and in the protein concentration P are described by equations:

$$\begin{aligned} \frac{\mathrm{d}R}{\mathrm{d}t} &= k_R - \gamma_R R \\ \frac{\mathrm{d}P}{\mathrm{d}t} &= k_P R - \gamma_P P. \end{aligned}$$

Derive expressions for R as a function of time, as well as for R and P when a steady state is reached. If the single gene is transcribed in 1 min, and its RNA lifetime is 10 min, what would be the average number of this RNA molecule found in a single cell? [5]

Assume that the rate of RNA and protein concentration change contains additional noise terms $n_R(t)$ and $n_P(t)$ respectively, characterised by the following statistics:

$$\langle \eta_i(t) \rangle = 0 \qquad \text{and} \qquad \langle \eta_i(t)\eta_j(t-t') \rangle = q_i \delta(t')\delta_{ij},$$

where q_i is the noise strength and δ is the Dirac δ-function. Using Fourier transforms and the standard integral

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{2\pi(x^2+a^2)} = \frac{1}{2a},$$

show that the mean square fluctuations in R equal $q_R/(2\gamma_R)$. [10]

Using similar arguments, the steady-state fluctuations in P can be described by

$$\left\langle \delta P^2 \right\rangle = \langle P \rangle \left(1 + \frac{k_P}{\gamma_R + \gamma_P} \right).$$

Rewrite the Fano factor $F = \langle \delta P^2 \rangle / \langle P \rangle$ as $F = 1 + b/(1+\phi)$ and explain what the quantities b and ϕ represent. What is the value of F for gene expression with Poissonian noise? Sketch and discuss the dependence of F with increasing translation rates (considering that the RNA of a gene is much more unstable than its protein). [6]

Explain what the terms *intrinsic* and *extrinsic noise* mean for gene expression, and discuss how *intrinsic noise* can be beneficial or detrimental for cells. [4]