Fluids PS2

Richard Fern

1 Rankine Vortex

a) A rotating impeller of radius a spins up a steady, two-dimensional vortex in a large container of water, in which the flow consists of a core of uniform angular velocity Ω for $r < a$ and irrotational flow with velocity given by $u_\theta = \Omega \frac{a^2}{r}$ for $r > a$. Determine the profile of vorticity $\omega(r)$ across the vortex and sketch the variations of $u_\theta(r)$ and $\omega(r)$. You may neglect viscous damping. [1]

Lets use the curl in cylindrical coordinates. So, for the outer vortex,

$$u_\theta = \Omega \frac{a^2}{r} \qquad \Longrightarrow \qquad \boldsymbol{\omega} = \frac{1}{r}\frac{\partial(r u_\theta)}{\partial r}\hat{\boldsymbol{z}} = 0.$$

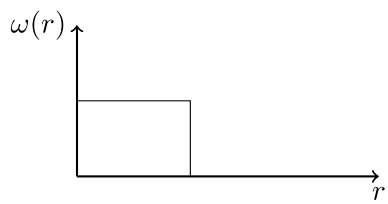
For the inner vortex however,

$$u_\theta = \Omega r \qquad \Longrightarrow \qquad \boldsymbol{\omega} = \frac{1}{r}\frac{\partial(r u_\theta)}{\partial r}\hat{\boldsymbol{z}} = 2\Omega\hat{\boldsymbol{z}}.$$

Therefore, the flow velocity has a profile that goes like

,

whereas the vorticity looks like



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b) Use Navier-Stokes equations (in cylindrical polars) to show that the difference in pressure between the centre of the vortex core and $r \to \infty$ is $-\rho\Omega^2 a^2$. Can you use Bernoulli's theorem for this calculation? [2]

Given that we have no variation in θ the Navier-Stokes equations (in cylindrical polars) are

$$\frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

Integrating the r-component for $r < a$ gives

$$p(a) - p(0) = \rho\Omega^2 \frac{a^2}{2}.$$

We then integrate to infinity to find

$$p(\infty) - p(a) = \frac{1}{2}\rho\Omega^2 a^2$$

and so

$$p(\infty) - p(0) = \rho\Omega^2 a^2.$$

Note that Bernoulli can only be used outside the centre where the flow is irrotational.

c) The water layer is bounded above by a free surface. Obtain an expression for the height $h(r)$ of the free surface across the vortex, and sketch the variation of h with r. Determine the change in the level of the free surface between the vortex core and large r for an impeller of radius 5cm located at the bottom of the tank and rotating at 120 revolutions per minute. The container is 1.5m deep. What is the maximum rotation rate of the impeller if the upper surface of the water is not to exposed the impeller to the air? [3]

Given the z-component of the Navier-Stokes equation we see that

$$p(r, z) = -\rho g z + f(r)$$

where we know from the previous part that

$$f(r) = p(\infty) - \frac{\rho\Omega^2 a^4}{2r^2}$$

outside the centre and

$$f(r) = p(0) + \frac{\rho\Omega^2 r^2}{2}$$

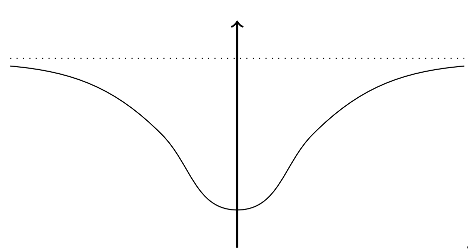
inside. Now, we know that the surface will be defined by the pressure $p = p(\infty)$. Therefore, we set $p(r, h) = p(\infty)$ to find $h(r)$. So outside the centre,

$$h - h_0 = -\frac{\Omega^2 a^4}{2gr^2}$$

and inside

$$h - h_0 = -\frac{\Omega^2 a^2}{g} + \frac{\Omega^2 r^2}{2g}.$$

This looks this the following diagram,



Therefore, we find the height drop in the first case to be

$$\Delta h = -\frac{(2 \times 2\pi)^2 \times (0.05)^2}{10} = -4\text{cm}.$$

Finally we set $h = 0$. Therefore, we're looking for when

$$h_0 = \frac{\Omega^2 a^2}{g}$$

which upon setting $h_0 = 1.5$ gives us

$$\Omega = 77\text{rad/s} = 730\text{rev/min}.$$

2 The Bernoulli Function

a) Starting from the Navier-Stokes equation, show that for incompressible flow

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla H = \boldsymbol{u} \times \boldsymbol{\omega} + \nu \nabla^2 \boldsymbol{u} \tag{2.1}$$

where $H = |\boldsymbol{u}|^2 + \frac{p}{\rho}$ is the Bernoulli function and $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ is the vorticity. Hence determine the conditions under which H is invariant along streamlines. Under what conditions is H constant everywhere? [2]

> To derive part one we just use the first vector identity (under part c)) and that $\nabla \rho = 0$ in an incompressible flow. Therefore,
>
> $$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \left(\frac{|\boldsymbol{u}|^2}{2} + \frac{p}{\rho} \right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u}$$
>
> as required. To be invariant along streamlines then we required $\boldsymbol{u} \cdot \nabla H = 0$, which holds for a steady ($\frac{\partial}{\partial t} = 0$) and inviscid ($\nu = 0$) flow. H is then constant everywhere when the flow is also irrotational, $\boldsymbol{\omega} = 0$.

b) Form a vorticity equation from (3.1), and hence show that vorticity is conserved along streamlines if the flow is steady, two-dimensional, inviscid and incompressible. [2]

> For the vorticity equation we take the curl of (2.1). Thus, using vector identity two and $\nabla \times (\nabla f) = 0$,
>
> $$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}.$$
>
> We then insert the form of curl of a cross product,
>
> $$\frac{\partial \boldsymbol{\omega}}{\partial t} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \cancel{\boldsymbol{\omega}(\nabla \cdot \boldsymbol{u})} + \cancel{\boldsymbol{u}(\nabla \cdot \boldsymbol{\omega})} + \nu \nabla^2 \boldsymbol{\omega}.$$
>
> The two divergence terms here are zero because we're incompressible (always, especially if we're using this form of the N-S equation) and $\nabla \cdot \boldsymbol{\omega} = 0$.
>
> Now, for 2D flow we have $\boldsymbol{\omega}$ is perpendicular to $\boldsymbol{u}$ and the $(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$ term goes. Secondly, we're incompressible so $\nabla \cdot \boldsymbol{u} = 0$. Then $\nabla \cdot \boldsymbol{\omega} = \nabla \cdot (\nabla \times \boldsymbol{u}) = 0$. Finally, $\nu = 0$. Therefore, we have
>
> $$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = 0$$
>
> and so the vorticity is constant along streamlines.

c) How high can water rise up one's arm hanging in the river from a punt? [1]
[Assume the following vector identities:

$$(\boldsymbol{F} \cdot \nabla)\boldsymbol{F} = \frac{1}{2}\nabla|\boldsymbol{F}|^2 - \boldsymbol{F} \times (\nabla \times \boldsymbol{F})$$

$$\nabla \times (\nabla^2 \boldsymbol{F}) = \nabla^2(\nabla \times \boldsymbol{F}) \quad]$$

Here we make the assumption that the flow is inviscid, which is ok as an estimation for the situation we're considering. Therefore, we can use that the Bernoulli function is constant along streamlines,

$$H = \frac{p}{\rho} + gz + \frac{|\boldsymbol{u}|^2}{2} = \text{const.}$$

On the surface far from the arm (in the arm's reference frame), $H = \frac{p_0}{\rho} + \frac{U^2}{2}$ where U is the punt speed. Therefore, taking the velocity at the top of the arm to be 0, we must solve

$$gh = \frac{U^2}{2} \tag{2.2}$$

So if $U \sim 1\text{ms}^{-1}$

$$h \sim 5\text{cm}.$$

Note that we didn't even need Bernoulli here. Equation (2.2) just says that the kinetic energy, $\frac{1}{2}\rho U^2$ is going to be equal to the gravitational potential, $\rho g h$.

3 Circulation and Lift

Consider an irrotational, inviscid, incompressible two-dimensional flow with clockwise swirl around a cylinder of radius a. The flow is defined by the velocity potential,

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - A\theta, \qquad r > a.$$

a) Find expressions for the velocity components perpendicular and parallel to the walls of the cylinder. [1]

This is a simple use of $\boldsymbol{u} = \nabla\phi$. So,

$$u_r = \frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{A}{r}.$$

b) Using Bernoulli's equation, show that the pressure on the cylinder is

$$p = \text{constant} - 2\rho U^2 \sin^2\theta - \frac{2\rho U A}{a}\sin\theta.$$

[1]

So we have

$$H = \frac{|\boldsymbol{u}|^2}{2} + \frac{p}{\rho} = \text{constant}.$$

Therefore, given the only component of $\boldsymbol{u}$ at a is from u_θ,

$$p = \text{constant} - \frac{\rho}{2}\left(-2U\sin\theta - \frac{A}{a}\right)^2$$

$$p = \text{constant}' - 2U^2\rho\sin^2\theta - \frac{2AU\rho\sin\theta}{a}.$$

c) Show that the lift force on the cylinder is

$$L = -\rho U \Gamma_c$$

where Γ_C is the circulation around a circle lying just outside the cylinder. [2]

The overall lift on the cylinder is given by the pressure applied,

$$F = \int_0^{2\pi} -p\sin\theta a d\theta$$

as p will push into the cylinder over the line element $a\mathrm{d}\theta$ and we want to resolve the vertical component of the force. Thus, we're left with

$$F = 2AU\rho \int_0^{2\pi} \sin^2\theta \mathrm{d}\theta$$

$$F = -\rho U(-2\pi A).$$

Therefore, we expect the circulation to be $-2\pi A$. Is it? The circulation is defined as

$$\Gamma_c = \int_0^{2\pi} a\mathrm{d}\theta u_\theta = -2\pi A.$$

Perfect. So under this scheme the cylinder feels lift due to the swirl. This can be undestood pretty directly from this Bernoulli picture as the fluid flows faster over the top, which causes a drop in pressure and therefore lift.

4 Surface Waves

a) Small-amplitude waves are generated at the surface of a shallow channel filled with water to depth h. For two-dimensional, inviscid, irrotational flow in the water layer, the boundary conditions satisfied are

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y} \qquad \text{and} \qquad \frac{\partial \phi}{\partial t} + g\eta = 0 \qquad \text{at the surface } y = h, \text{ and}$$

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{at the bottom of the channel at } y = 0,$$

where η is the vertical displacement of the surface and ϕ the velocity potential in the water layer. Give a physical justification for these boundary conditions. [2]

The first condition, $\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$, is that the height of the water should move at the velocity of the water at the surface.

The third condition, $\frac{\partial \phi}{\partial y} = 0$, is no-normal flow.

The second condition is the tricky one. Reconsider the Bernoulli problem in a time-dependent flow,

$$\frac{\partial}{\partial t}\nabla\phi + \nabla\left(\frac{|\nabla\phi|^2}{2} + \frac{p}{\rho}\right) = -g\nabla(\eta).$$

Now, the wave amplitude is small, allowing us to neglect the ϕ^2 term, and so

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{p_0}{\rho} + g\eta\right) = \nabla H(t) = 0$$

where p_0 is the surface pressure. Thus, $H(t)$ is some constant along streamlines which, far from the waves, should equal $\frac{p_0}{\rho}$ and so

$$\frac{\partial \phi}{\partial t} + g\eta = 0.$$

b) Given that ϕ satisfies Laplace's equation in x and y, show that the waves satisfy the dispersion relation

$$\omega^2 = gk\tanh(kh). \tag{4.1}$$

[2]

We assume that ϕ is some wave solution along x,

$$\phi = f(x - ct)g(y).$$

We can solve Laplace's equation by separating the variables,

$$\frac{f''}{f} = -\frac{g''}{g} = -k^2$$

$$\implies \qquad \phi = e^{i(kx-\omega t)}\left(ae^{ky} + be^{-ky}\right).$$

To satisfy the no-normal flow condition then $a = b$,

$$\phi = Ae^{i(kx-\omega t)}\cosh(ky).$$

We then mix together the other two boundary conditions,

$$\eta = -\frac{1}{g}\frac{\partial\phi}{\partial t} \qquad \Longrightarrow \qquad \frac{\partial\eta}{\partial t} = -\frac{1}{g}\frac{\partial^2\phi}{\partial t^2}$$

$$\therefore \qquad \frac{\partial\phi}{\partial y} + \frac{1}{g}\frac{\partial^2\phi}{\partial t^2} = 0.$$

We substitute our general solution into this,

$$k\sinh(ky) - \frac{\omega^2}{g}\cosh(ky) = 0$$

$$\Longrightarrow \qquad \omega^2 = gk\tanh(ky).$$

c) Show that, for wavelengths for which kh is small but finite, this dispersion relation may be approximated by

$$\omega \simeq c_0\left(k - \frac{k^3h^2}{6}\right),$$

where $c_0 = (gh)^{\frac{1}{2}}$. [1]

We simply Taylor expand at the surface, using

$$\tanh(x) = x - \frac{x^3}{3} + \ldots$$

to find that the dispersion relation for the waves at the surface is

$$\begin{aligned}\omega &\simeq \sqrt{gk}\left(kh - \frac{(kh)^3}{3}\right)^{\frac{1}{2}}\\ &= \sqrt{gh}\,k\left(1 - \frac{(kh)^2}{6}\right)\end{aligned}$$

as required.

d) The Korteweg-de Vries equation

$$\frac{\partial\eta}{\partial t} + c_0\frac{\partial\eta}{\partial x} + \left(\frac{3c_0}{2h}\right)\eta\frac{\partial\eta}{\partial x} + \frac{c_0h^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0$$

is a model equation for weakly dispersive, nonlinear waves on the surface of a channel of water of depth h and surface elevation η. Without detailed mathematical derivation, give a brief physical justification of the origin of each term and its effect on the resultant dispersion relation and shape of the solution. [2]

This question really wants physical explanations of the terms, as opposed to some justification that they should exist. The justification of the equation is then something something Occam's razor something something.

To think about this equation it is easiest to posit a wave solution, $\eta \sim e^{i(kx-\omega t)}$ and see what the dispersion relation must be. This gives

$$-\omega + c_0 \left(k + \frac{3}{2h} k\eta - \frac{k^3 h^2}{6} \right) = 0.$$

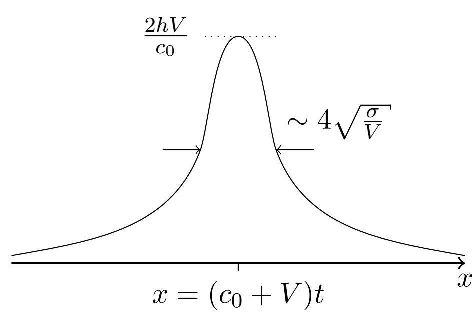
The first two terms are therefore our simple linear wave equation. The second term introduces nonlinearities, with the wave speed depending on its amplitude. This term also introduces complex interactions between wavepackets of different wavenumbers. Finally, we have the strange ∂^3 term which gives the strange dispersion we have in equation (4.1), making the long-wavelength waves move faster than their shorter wavelength counterparts.

e) A possible (soliton) solution to this equation is of the form

$$\eta = \frac{2hV}{c_0} \operatorname{sech}^2 \left[\frac{1}{2} \left(\frac{V}{\sigma} \right)^{\frac{1}{2}} (x - (c_0 + V)t) \right],$$

where $\sigma = \frac{c_0 h^2}{6}$ and V is a parameter. Sketch the form of this solution, clearly labelling its amplitude and extent in x, and indicating its propagation speed. How do the properties of this solution differ from those of small-amplitude linear waves which satisfy the dispersion relation (4.1)? [2]

The width of this wave is defined roughly by the argument of the sech being 1, so $\Delta x \sim 2\sqrt{\frac{\sigma}{V}}$. Thus, the wavepacket looks like this,



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These waves do not change shape over time as the nonlinear effects cancel the dispersive effects.

f) Scott Russell first observed the formation of such a soliton wave on the surface of a canal in 1834 when a barge stopped abruptly, and determined its amplitude a and propagation speed C to be $a \simeq 0.3$m and $C \simeq 3.5\text{ms}^{-1}$. Estimate the depth h of the canal. [2]

We are given two equations,

$$\frac{2hV}{c_0} = a, \qquad \text{and} \qquad c_0 + V = C.$$

We then rearrange and solve,

$$h^{\frac{1}{2}}\left(-2\sqrt{g}h + 2Ch^{\frac{1}{2}} - a\sqrt{g}\right) = 0.$$

We find three solutions from this,

$$h = 0\text{m}, 0.16\text{m and } 0.95\text{m}.$$

The first two must be wrong has $a > h$ in both cases so we must have

$$h \simeq 1\text{m}.$$

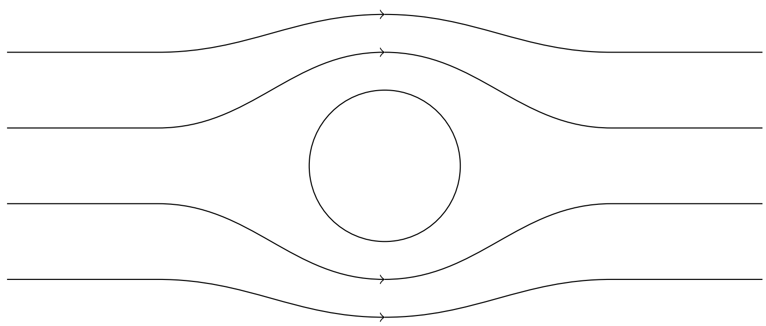
5 Stokes Drag

The flow and pressure field for Stokes flow (Re $= 0$) past a sphere of radius a, in spherical polar co-ordinates with z along the direction of the incoming flow, are

$$
\begin{aligned}
u_r &= u_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right), \\
u_\theta &= -u_0 \sin\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right), \\
p &= p_0 - \frac{3\eta u_0 a}{2r^2}\cos\theta.
\end{aligned}
$$

a) Sketch the flow field. [1]

This is a simple case of drawing what's shown here. It's actually pretty obvious. The result is shown in this figure.



But a note should be made on where these solutions come from as it turns out the Navier-Stokes equations in spherical co-ordinates is not as one would expect by just inserting the expressions for the gradient operator. The reason for this is that we're differentiating vectors, so when performing the differentiation we need to be worried about $\frac{\partial}{\partial r_i}\hat{\boldsymbol{e}}_j$, which might be non-zero in a non-Cartesian co-ordinate system. In this notation r_i is a co-ordinate in direction i and $\hat{\boldsymbol{e}}_j$ is the unit vector along co-ordinate direction j.

Let's consider an example. For simplicity I'll consider a cylindrical geometry (because the maths so much easier here). Along the radial direction of motion the Navier-Stokes equation reads (where $\boldsymbol{u} = (v_r, v_\theta, v_z)$),

$$\rho\left(\frac{\partial v_r}{\partial t} + \boldsymbol{u}\cdot\nabla v_r - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left(\nabla^2 v_r - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

which is as we expect but with a couple of anomalous terms from v_θ. This arises because in this geometry we have $\frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \theta} = -\hat{\boldsymbol{e}}_r$ and comes from the term

$$\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \left(v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}\right)\left(v_r\hat{\boldsymbol{e}}_r + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{e}}_z\right).$$

Therefore, considering only the part along the $\hat{\boldsymbol{e}}_r$ direction,

$$
\begin{aligned}
\boldsymbol{u}\cdot\nabla\boldsymbol{u} &= \hat{\boldsymbol{e}}_r(\boldsymbol{u}\cdot\nabla)v_r + v_\theta\left(\frac{v_\theta}{r}\frac{\partial}{\partial\theta}\right)\hat{\boldsymbol{e}}_\theta \\
&= \hat{\boldsymbol{e}}_r\left[(\boldsymbol{u}\cdot\nabla)v_r - \frac{v_\theta^2}{r}\right].
\end{aligned}
$$

b) Confirm that the solution obeys (i) the Stokes equations and (ii) the correct boundary conditions. [4]

In these very viscous (remember what Stokes flow describes) and incompressible limits the Navier-Stokes and continuity equations become

$$\nabla p = \eta \nabla^2 \boldsymbol{u} \qquad\qquad \nabla \cdot \boldsymbol{u} = 0.$$

These are the Stokes equations. We should note that these can be solved *very* easily as we can take a divergence to see that $\nabla^2 p = 0$ or a curl to see that $\nabla^2 \boldsymbol{\omega} = 0$.

To satisfy the Stokes equations as given we require

$$\frac{\partial p}{\partial r} = \eta \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_r}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u_r}{\partial \theta} \right) - \frac{2u_r}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial (\sin\theta u_\theta)}{\partial \theta} \right)$$

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \eta \left(\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u_\theta}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial u_\theta}{\partial \theta} \right) - \frac{u_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right)$$

$$\frac{1}{r^2} \frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r \sin\theta} \frac{\partial (\sin\theta u_\theta)}{\partial \theta} = 0.$$

An alternative method that circumvents these awful equations is to use the identity

$$\nabla^2 \boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{u})$$

in the incompressible case. This is a little cleaner.

The second part of the problem requires one to check that $\boldsymbol{u}(a) = 0$ and that

$$u_r \to u_0 \cos\theta \qquad\qquad u_\theta \to -u_0 \sin\theta$$

in the large-$\boldsymbol{r}$ limit. These are obvious by inspection.

c) The components of the viscous stress tensor in the spherical polar co-ordinates are:

$$\sigma_{rr} = -p + 2\eta \frac{\partial u_r}{\partial r},$$

$$\sigma_{\theta\theta} = -p + 2\eta \left(\frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right),$$

$$\sigma_{\phi\phi} = -p + 2\eta \left(\frac{1}{r \sin\theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} \right),$$

$$\sigma_{r\theta} = \eta \left(r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right),$$

$$\sigma_{r\phi} = \eta \left(\frac{1}{r \sin\theta} \frac{\partial u_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{u_\phi}{r} \right) \right),$$

$$\sigma_{\theta\phi} = \eta \left(\frac{\sin\theta}{r} \frac{\partial}{\partial \theta} \left(\frac{u_\phi}{\sin\theta} \right) + \frac{1}{r \sin\theta} \frac{\partial u_\theta}{\partial \phi} \right).$$

Calculate the viscous stress tensor at $r = a$ and hence identify the stress (force per unit area) acting on the sphere. [3]

Consider element by element at $r = a$. So,

$$\sigma_{rr} = -p$$

which we find by differentiating and just finding that everything cancels.

$$\sigma_{\theta\theta} = -p$$

becasue the differentiate of u_θ by θ leaves the r-dependent parts unchanged and, given $\boldsymbol{u}(a) = 0$, everything except p is zero.

$$\sigma_{\phi\phi} = -p$$

purely because $\boldsymbol{u}(a) = 0$.

$$\sigma_{r\theta} = -\frac{3\eta u_0 \sin\theta}{2a}$$

by simply differentiating the first term here (the second term is zero). Finally, the last two are zero because there's no ϕ-dependence anywhere. So overall,

$$\sigma(a) = \begin{pmatrix} -p & -\frac{3\eta u_0 \sin\theta}{2a} & 0 \\ -\frac{3\eta u_0 \sin\theta}{2a} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}.$$

The stress is defined as

$$\frac{\mathrm{d}F_i}{\mathrm{d}A_j} = \sigma_{ij}.$$

So, given the surface of the sphere has a normal $\hat{\boldsymbol{e}}_r$, we find

$$\frac{\mathrm{d}F_r}{\mathrm{d}A_j} = \sigma_{rj} = \begin{pmatrix} -p \\ -\frac{3\eta u_0 \sin\theta}{2a} \\ 0 \end{pmatrix}.$$

d) By integrating over the sphere derive Stokes law for the drag D on the sphere:

$$D = 6\pi\eta a u_0.$$

[2]

By symmetry we notice that the only non-zero component of force will be along the z-direction with the total stress,

$$\begin{aligned} t_z &= \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta \\ &= -p_0\cos\theta + \frac{3\eta u_0}{2a}\left(\cos^2\theta + \sin^2\theta\right). \end{aligned}$$

We integrate this over the surface of the sphere and the first term cancels, leaving

$$D = 4\pi a^2 \frac{3\eta u_0}{2a} = 6\pi\eta u_0 a.$$

e) Assuming Stokes law, calculate the terminal velocity of a raindrop of radius 1mm falling in air. Discuss whether the use of Stokes law is in fact valid in this case. [2]

We resolve the upward and downward forces,

$$mg = \frac{4}{3}\pi a^3 \rho g = 6\pi\eta u_0 a$$

$$\implies u_0 = \frac{2a^2\rho g}{9\eta}$$

and put in the numbers to find

$$u_0 = \frac{2 \times (10^{-3})^2 \times 10^3 \times 10}{9 \times (2 \times 10^{-5})} \simeq 110\text{m/s}.$$

To see if this is valid we should consider Reynolds number,

$$\text{Re} \sim \frac{10^2 \times 10^{-3}}{10^{-5}} \sim 10^4.$$

So viscous flow is an awful description for a raindrop, as might be expected. In this limit the drag is entirely inertial with

$$D \sim u_0^2.$$

6 Thin Film Approximation

A circular disk of radius a initially sticks to a flat ceiling at $z = 0$ by means of a thin film of viscous, incompressible liquid of dynamic viscosity η and thickness $h(t) \ll a$. Effects of surface tension may be neglected.

a) Given that h and $W(t) = \frac{\mathrm{d}h}{\mathrm{d}t}$ are very small, explain briefly why the radial velocity u_r in the thin film obeys the approximate equation in cylindrical polar coordinates

$$\frac{\partial p}{\partial r} = \eta \frac{\partial^2 u_r}{\partial z^2}. \tag{6.1}$$

[1]

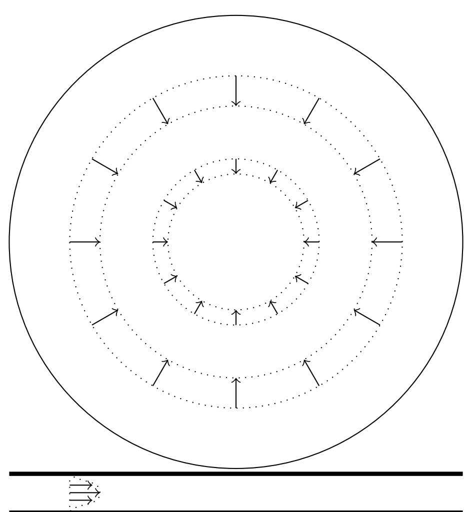
The liquid is incompressible and viscous so we have

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = \eta \nabla^2 \boldsymbol{u} \qquad \nabla \cdot \boldsymbol{u} = 0.$$

By symmetry we can ignore anything in the θ direction. We can then consider the size of $\frac{\partial}{\partial r} \sim \frac{1}{a}$ and $\frac{\partial}{\partial z} \sim \frac{1}{h}$. These are of very different sizes so we expect $\nabla^2 \sim \frac{\partial^2}{\partial z^2} + \text{small}$. Therefore, the r-component of NS is, ignoring also the small rate of change of the flow as W is small,

$$\frac{\partial p}{\partial r} = \eta \frac{\partial^2 u_r}{\partial z^2}.$$

We can think of this physically too. Whilst there must be variation in both directions, the variation of u_r along r will be constrained by the geometry as water must flow in to increase the height. On the other hand, variation in the z direction is driven by no-slip boundary conditions and so, given h is so small, this variation is significant. This is summed up in this figure.



Therefore, as h becomes very small, the z-variation is far more significant.

b) Show that $p \simeq p(r,t)$, and that u_r is given in this approximation by

$$u_r = \frac{1}{2\eta} \frac{\partial p}{\partial r} z(z+h).$$

[2]

For this we want the other half of the NS equations,

$$\frac{\partial p}{\partial z} = \eta \frac{\partial^2 u_z}{\partial z^2}.$$

Now, we've been told that $u_z = W$ is tiny. Therefore, we can ignore it and find

$$\frac{\partial p}{\partial z} \simeq 0 \qquad \Longrightarrow \qquad p = p(r,t).$$

For the second part we then simply integrate the radial equation, (6.1), using $u_r = 0$ at $z = 0, -h$ so

$$u_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z(z+h).$$

c) By integrating the incompressibility condition downwards in z from $z = 0$ to $z = -h$ show that

$$W = -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right).$$

[1]

Incompressibility tells us that

$$\frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r).$$

Inserting our expression for u_r above we therefore see

$$\frac{\partial u_z}{\partial z} = -\frac{1}{2\eta} z(z+h)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)$$

$$u_z(-h) - u_z(0) = -\frac{1}{2\eta}\left(\frac{-h^3}{3} + h\frac{h^2}{2}\right)\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)$$

$$W = -u_z(-h) = \frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)$$

(and so there's a sign error here that can be fixed by redefining $W = -\frac{\mathrm{d}h}{\mathrm{d}t}$, which is the speed at which the disk ascends instead of descends).

d) Hence show that the pressure distribution in the fluid film above the disk is given by

$$p - p_0 = \frac{3\eta W}{h^3}(a^2 - r^2),$$

where p_0 is atmospheric pressure. [1]

Another simple integration problem. We take what's above given that W doesn't depend on r to find

$$r\frac{\partial p}{\partial r} = \frac{6\eta W}{h^3} r^2 + c$$

$$p - p_0 = \frac{3\eta W}{h^3}(r^2 - a^2)$$

given that $p(a) = p_0$. We also know used that the pressure must be finite for all r so $c = 0$.

e) Show that the total downwards for on the disk needed to pull it away from the ceiling as speed W is

$$F = \frac{3\pi}{2}\frac{\eta a^4 W}{h^3}.$$

[1]

We now need some force which will cancel this upwards pressure and keep the disk moving at speed W. So we integrate the pressure difference over the surface of the disk so find

$$F = \frac{3\eta W}{h^3}\int_0^a 2\pi r \mathrm{d}r(a^2 - r^2)$$

$$F = \frac{3\pi\eta W a^4}{2h^3}.$$

f) Spiderman, of mass 50kg, wishes to hang suspended from the ceiling by being attached to a circular disk of radius 10cm on the underside of a thin film of a fluid of dynamic viscosity $300\mathrm{kgm}^{-1}\mathrm{s}^{-1}$. The film has an initial thickness 1mm, and initially fills the entire space between the disk and the ceiling. The volume of the film V remains constant at $V = \pi a_0^2 h_0$, where a_0 and h_0 are the initial radius and thickness of the film. Estimate the length of time spiderman can remain attached to the ceiling. Comment on the likely accuracy of your estimate for a real fluid film. [3]

Spiderman's mass gives us the force we're pulling with to be mg so

$$mg = \frac{3\pi\eta}{2}\frac{a^4}{h^3}\frac{\mathrm{d}h}{\mathrm{d}t}.$$

We can solve this by using that the volume is constant so $a(t) = \sqrt{\frac{V}{\pi h(t)}}$, so

$$mg = \frac{3\eta}{2\pi}\frac{V^2}{h^5}\frac{\mathrm{d}h}{\mathrm{d}t}.$$

Therefore, we can simply solve the equation to find

$$\frac{1}{4h_0^4} - \frac{1}{4h^4} = \frac{2\pi mg}{3\eta V^2}t$$

and so, taking $h \to \infty$,

$$t = \frac{3\eta V^2}{8\pi mg h_0^4} = \frac{3\eta\pi a_0^4}{8mg h_0^2} \simeq 71\mathrm{s}.$$

Obviously taking $h \to \infty$ seems a bit odd here as the thin film approximation has long broken by that point but the time at which h is large is actually a very small part of this 71s (plot $h(t)$ to convince yourself).

As for the thin film approximation, it's good whilst the film is thin, the speeds are slow and the forces are very small. However, for Spiderman that last one is the issue as the disk is going to peel away very easily from the fluid and he will die (unless he's just in his bedroom or something). The bad approximation that causes this is incompressibility.