FLUIDS, FLOWS AND COMPLEXITY

PROBLEM SET 2 AND SOLUTIONS

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Changes for 2016: re-ordered to better reflect lectures; wave problem changed.
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1. **Rankine vortex**

   (a) A rotating impeller of radius a spins up a steady, two-dimensional vortex in a large container of water, in which the flow consists of a core of uniform angular velocity Ω for $r < a$ and irrotational flow with velocity given by $u_\theta = \Omega a^2/r$ for $r > a$. Determine the profile of vorticity $\omega(r)$ across the vortex and sketch the variations of $u_\theta(r)$ and $\omega(r)$. You may neglect viscous damping.

   (b) Use the Navier-Stokes equations in (cylindrical polars) to show that the difference in pressure between the centre of the vortex core and $r \to \infty$ is $-\rho\Omega^2 a^2$. Can you use Bernoulli's theorem for this calculation?

   (c) The water layer is bounded above by a free surface. Obtain an expression for the height $h(r)$ of the free surface across the vortex, and sketch the variation of h with r. Determine the change in the level of the free surface between the vortex core and large r for an impeller of radius 5 cm located at the bottom of the tank and rotating at 120 revolutions per minute. The container is 1.5 m deep. What is the maximum rotation rate of the impeller if the upper surface of the water is not to expose the impeller to the air?

Solution

(a) The vorticity and flow field are

$$u_\theta = \Omega r, \quad r < a; \qquad u_\theta = \Omega a^2/r \quad r > a;$$
$$\omega_z = 2\Omega \quad r < a; \qquad \omega_z = 0 \quad r > a;$$

(b) In cylindrical polars, with a velocity field $u_\theta(r)$ the Navier-Stokes equations are

$$\text{r-component: } \frac{u_\theta^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r}; \qquad \text{z-component: } 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Integrating the r-component for $r < a$ gives

$$p(a) - p(0) = \frac{\rho\Omega^2 a^2}{2}$$

Integrating for $r > a$ gives

$$p(\infty) - p(a) = \frac{\rho\Omega^2 a^2}{2}$$

so

$$p(\infty) - p(0) = \rho\Omega^2 a^2 \tag{1}$$

NB Bernoulli works for $r > a$ because the flow is irrotational, but not for $r < a$.

(c) We now need to include gravity. From the z-component of the Navier-Stokes equations we can find the z-dependence so

$$p(r,z) = -\rho g z + f(r).$$

Putting this together with the r-dependence

$$p(r) = p(\infty) - \rho g z - \frac{\rho \Omega^2 a^4}{2r^2}, \quad r > a; \qquad p(r) = p(0) - \rho g z + \frac{\rho \Omega^2 r^2}{2}, \quad r < a.$$

$p(a)$ must be continuous at $r = a$ and this is ensured by Eq. (1).
On the free surface $p = p(\infty)$ so

$$z = -\frac{\Omega^2 a^4}{2 g r^2}, \quad r > a; \qquad z = -\frac{\Omega^2 a^2}{g} + \frac{\Omega^2 r^2}{2g}, \quad r < a.$$

Height at $r = 0$ relative to height at infinity is $-\Omega^2 a^2/g = -4\text{cm}$
using $g = 10$, $a = 5 \times 10^{-2}$, $\Omega = 4\pi$

For $h = 1.5$, $\Omega \sim 80$ rad s^{-1}.

2. **The Bernoulli function**

(a) Starting from the Navier-Stokes equation, show that for incompressible flow

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla H = \mathbf{u} \times \omega + \nu \nabla^2 \mathbf{u}, \tag{1}$$

where $H = |\mathbf{u}|^2/2 + p/\rho$ is the Bernoulli function and $\omega = \nabla \times \mathbf{u}$ is the vorticity. Hence determine the conditions under which H is invariant along streamlines. Under what conditions is H constant everywhere?

(b) Form a vorticity equation from (1), and hence show that vorticity is conserved along streamlines if the flow is steady, two-dimensional, inviscid and incompressible.

(c) How high can water rise up one's arm hanging in the river from a punt?

[Assume the following vector identities:

$$(\mathbf{F}.\nabla)\mathbf{F} = \frac{1}{2}\nabla|\mathbf{F}|^2 - \mathbf{F} \times (\nabla \times \mathbf{F})$$

$$\nabla \times (\nabla^2 \mathbf{F}) = \nabla^2(\nabla \times \mathbf{F}) \quad]$$

Solution

(a) The Navier Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu \nabla^2 \mathbf{u}.$$

Using the given identity

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\nabla|\mathbf{u}|^2 - \mathbf{u} \times (\nabla \times \mathbf{u}) + \frac{1}{\rho}\nabla p = \nu \nabla^2 \mathbf{u}.$$

For incompressible flow ρ is constant so it immediately follows that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla H = \mathbf{u} \times \omega + \nu \nabla^2 \mathbf{u}. \tag{2}$$

Along streamlines H is constant if $\mathbf{u}.\nabla H = 0$. This is true for a steady, inviscid flow. If the flow is also irrotational $\nabla H = 0$ and H is constant throughout the flow.

(b) Taking the curl of Eq. (2), using the second given identity and $\nabla \times \nabla \equiv 0$,

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega = -(\mathbf{u} \cdot \nabla)\omega + (\omega \cdot \nabla)\mathbf{u} - \omega(\nabla \cdot \mathbf{u}) + \mathbf{u}(\nabla \cdot \omega) + \nu \nabla^2 \omega.$$

If flow is incompressible $(\nabla \cdot \mathbf{u}) = 0$
$\nabla \cdot \nabla \wedge \equiv 0$ so $(\nabla \cdot \omega) = 0$
If flow is inviscid $\nu = 0$
If flow is 2D $(\omega \cdot \nabla)\mathbf{u}) = 0$ because $\omega \perp \mathbf{u}$.

So Eq. (2) reduces to

$$\frac{D\omega}{Dt} = 0.$$

so the vorticity of each fluid element is conserved. If the flow is steady

$$(\mathbf{u} \cdot \nabla)\omega = 0$$

and the vorticity is constant along a streamline.

(c) This calls for a simple estimate using Bernoulli: $H = |\mathbf{u}|^2/2 + p/\rho + gz$ is constant along a streamline – take this to be the surface of the river at atmospheric pressure. Arm reduces velocity to zero. u is originally $\sim$1ms^{-1} so $z \sim u^2/2g \sim$ 5cm.

3. **Circulation and Lift**

Consider an irrotational, inviscid, incompressible two-dimensional flow with clockwise swirl around a cylinder of radius a. The flow is defined by the velocity potential,

$$\varphi = U\left(r + \frac{a^2}{r}\right)\cos\theta - A\theta, \qquad r > a.$$

(a) Find expressions for the velocity components perpendicular and parallel to the walls of the cylinder.
(b) Using Bernoulli's equation, show that the pressure on the cylinder is

$$p = \text{constant} - 2\rho U^2 \sin^2\theta - \frac{2\rho U A}{a}\sin\theta.$$

(c) Show that the lift force on the cylinder is

$$L = -\rho U \Gamma_C$$

where Γ_C is the circulation around a circle lying just outside the cylinder.

Solution

$$\mathbf{u} = \nabla\varphi \quad \rightarrow \quad u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{A}{r}$$

The flow is irrotational, inviscid, incompressible and two-dimensional so the Bernoulli function is constant throughout the flow:

$$|\mathbf{u}|^2/2 + p/\rho = \frac{1}{2}(-2U\sin\theta - A/a)^2 + p/\rho = \text{const}$$

$$\rightarrow \quad p = \text{const} - 2\rho U^2 \sin^2\theta - 2\rho UA\sin\theta/a.$$

Integrating $-p\sin\theta$ to get the lift force, the non-zero term is

$$F = \int_0^{2\pi} \frac{2\rho UA\sin^2\theta}{a}\, a d\theta = 2\rho UA\pi = -\rho U\Gamma_C$$

where the circulation is

$$\Gamma_C = \int_0^{2\pi} u_\theta \; a d\theta = -2\pi A.$$

4. **Surface waves**

(a) Small-amplitude waves are generated at the surface of a shallow channel filled with water to depth h. For two-dimensional, inviscid, irrotational flow in the water layer, the boundary conditions satisfied are

$$\frac{\partial\eta}{\partial t} = \frac{\partial\phi}{\partial y} \quad \text{and} \quad \frac{\partial\phi}{\partial t} + g\eta = 0 \quad \text{at the surface } y = h, \text{ and}$$

$$\frac{\partial\phi}{\partial y} = 0 \quad \text{at the bottom of the channel at } y = 0,$$

where η is the vertical displacement of the surface and ϕ the velocity potential in the water layer. Give a physical justification for these boundary conditions.

(b) Given that ϕ satisfies Laplace's equation in x and y, show that the waves satisfy the dispersion relation

$$\omega^2 = gk\tanh kh. \tag{3}$$

(c) Show that, for wavelengths for which kh is small but finite, this dispersion relation may be approximated by

$$\omega \simeq c_0\left(k - \frac{k^3h^2}{6}\right),$$

where $c_0 = (gh)^{1/2}$.

(d) The Korteweg–de Vries equation

$$\frac{\partial\eta}{\partial t} + c_0\frac{\partial\eta}{\partial x} + \left(\frac{3c_0}{2h}\right)\eta\frac{\partial\eta}{\partial x} + \frac{c_0h^2}{6}\frac{\partial^3\eta}{\partial x^3} = 0$$

is a model equation for weakly dispersive, nonlinear waves on the surface of a channel of water of depth h and surface elevation η. Without detailed mathematical derivation, give a brief physical justification of the origin of each term and its effect on the resultant dispersion relation and shape of the solution.

(e) A possible (soliton) solution to this equation is of the form

$$\eta = \frac{2hV}{c_0}\text{sech}^2\left[\frac{1}{2}\left\{\frac{V}{\sigma}\right\}^{1/2}\{x-(c_0+V)t\}\right],$$

where $\sigma = c_0h^2/6$ and V is a parameter. Sketch the form of this solution, clearly labelling its amplitude and extent in x, and indicating its propagation speed. How do the properties of this solution differ from those of small-amplitude linear waves which satisfy the dispersion relation (3)?

(f) Scott Russell first observed the formation of such a soliton wave on the surface of a canal in 1834 when a barge stopped abruptly, and determined its amplitude a and propagation speed C to be $a \approx 0.3\,\text{m}$ and $C \approx 3.5\,\text{m}\,\text{s}^{-1}$. Estimate the depth h of the canal.

Solution

If the flow is irrotational, curl $\vec{u} = 0$, and we can define a velocity potential ϕ by

$$\vec{u} = \text{grad}\ \phi. \tag{4}$$

If the flow is also incompressible, div $\vec{u} = 0$, and

$$\text{div}\ \vec{u} = \nabla^2\phi = 0. \tag{5}$$

(a) Assume small amplitude waves with the vertical displacement of the surface described by $\eta(x,t)$. The boundary conditions follow from

1. Bernoulli's equation, allowing the flow to be time-dependent, and neglecting term in the velocity squared because we are assuming that the velocity is small

$$\frac{\partial\phi}{\partial t} + \frac{p_0}{\rho} + g\eta = \text{constant} \qquad \text{at} \quad y = h.$$

(Taking $p = p_0$ (atmospheric pressure) means that we are ignoring surface tension which is a good approximation for waves with wavelength more than a few cm.)
2. Assume that the y component of the velocity is equal to the vertical velocity of the interface:

$$u_y = \frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \qquad \text{at} \quad y = h.$$

3. Fluid cannot penetrate into the bottom of the channel:

$$u_y = \frac{\partial\phi}{\partial y} = 0 \qquad \text{at} \quad y = 0.$$

(b) Assume a solution to Laplace's equation of the form

$$\phi = f(x-ct)g(y)$$

Separating variables gives

$$\frac{f''}{f} = -\frac{g''}{g} \equiv -k^2$$

so

$$\phi = A\exp i(kx-\omega t)\text{cosh}ky$$

where we have chosen the term in y to satisfy boundary condition (3).

Differentiating boundary condition (1) and substituting in (2) gives

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0 \qquad \text{at} \quad y = h.$$

Substituting in the solution for ϕ gives

$$-\omega^2 A \exp i(kx - \omega t)\cosh kh + gkA \exp i(kx - \omega t)\sinh kh = 0$$

leading to the dispersion relation

$$\omega^2 = gk\tanh kh.$$

(c) The Taylor series expansion for $\tanh x$ is

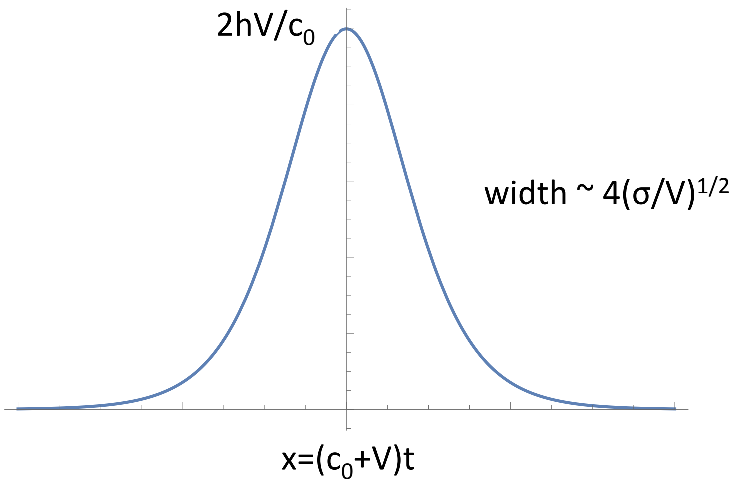
$$\tanh x = x - \frac{x^3}{3}.$$

Expanding the dispersion relation (3) for small kh

$$\omega \approx \sqrt{gk}\left(kh - \frac{(kh)^3}{3}\right)^{1/2} \approx \sqrt{gh}\,k\left(1 - \frac{(kh)^2}{6}\right) = c_0\left(k - \frac{(k^3h^2)}{6}\right).$$

(d) The first two terms describe a non-dispersive wave packet moving to the right with speed c_0.
The third term is a non-linear correction that will introduce interactions between different Fourier components of the wave packet. The wave steepens because the crest propagates more rapidly than the trough.
The fourth term gives dispersion (as in (c)) and the wave packet will change shape. Long wavelengths will move faster, and the wave packet will smooth out.

(e)



The shape of the wave packet stays the same over time (surprisingly) as there is a solution where the non-linearities and the dispersion balance.

(f)

$$C = c_0 + V = 3.5, \qquad a = \frac{2hV}{c_o} = 0.3.$$

Using $c_0 = \sqrt{gh}$ gives

$$0.3(gh)^{1/2} + 2h(gh)^{1/2} - 7h = 0$$

which is solved by $h \sim 1$m.

5. **Stokes drag**

The flow and pressure field for Stokes flow (Re=0) past a sphere of radius a, in spherical polar co-ordinates with z along the direction of the incoming flow, are

$$u_r = u_0 \cos\theta \left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right),$$

$$u_\theta = -u_0 \sin\theta \left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right),$$

$$p = p_0 - \frac{3\eta u_0 a}{2r^2}\cos\theta.$$

(a) Sketch the flow field.
(b) Confirm that the solution obeys (i) the Stokes equations (ii) the correct boundary conditions.
(c) The components of the viscous stress tensor in spherical polar co-ordinates are:

$$\sigma_{rr} = -p + 2\eta\frac{\partial u_r}{\partial r}$$

,

$$\sigma_{\theta\theta} = -p + 2\eta\left(\frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}\right),$$

$$\sigma_{\varphi\varphi} = -p + 2\eta\left(\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r}\right),$$

$$\sigma_{r\theta} = \eta\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right),$$

$$\sigma_{r\varphi} = \eta\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} + r\frac{\partial}{\partial r}\left(\frac{u_\varphi}{r}\right)\right),$$

$$\sigma_{\theta\varphi} = \eta\left(\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{u_\varphi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi}\right).$$

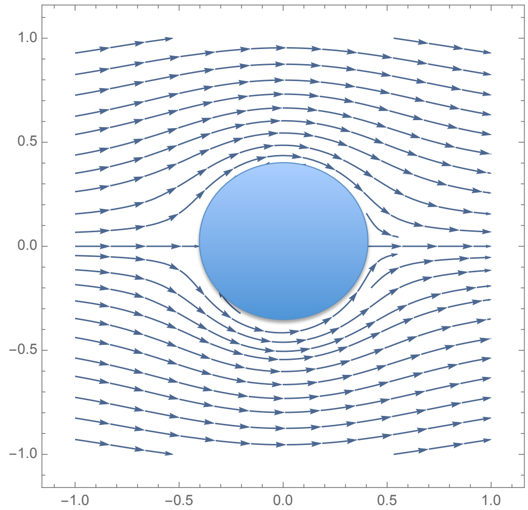
Calculate the viscous stress tensor at $r = a$ and hence identify the stress (force per unit area) acting on the sphere.
(d) By integrating over the sphere derive Stokes law for the drag D on the sphere:

$$D = 6\pi\eta a u_0.$$

(e) Assuming Stokes law, calculate the terminal velocity of a raindrop of radius 1mm falling in air. Discuss whether the use of Stokes law is in fact valid in this case.

Solution

(a)



(b) (i) The Stokes equations are

$$\nabla p = \eta \nabla^2 \mathbf{u}; \qquad \nabla \cdot \mathbf{u} = 0.$$

In spherical polars with no ϕ dependence these become

$$\partial_r p = \eta \left\{ \frac{1}{r^2}\partial_r(r^2 \partial_r u_r) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta \partial_\theta u_r) - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\partial_\theta(\sin\theta u_\theta) \right\},$$

$$\frac{1}{r}\partial_\theta p = \eta \left\{ \frac{1}{r^2}\partial_r(r^2 \partial_r u_\theta) + \frac{1}{r^2 \sin\theta}\partial_\theta(\sin\theta \partial_\theta u_\theta) - \frac{u_\theta}{r^2 \sin^2\theta} + \frac{2}{r^2}\partial_\theta(u_r) \right\},$$

$$\frac{1}{r^2}\partial_r(r^2 u_r) + \frac{1}{r\sin\theta}\partial_\theta(u_\theta \sin\theta) = 0.$$

Substitution and some careful algebra confirms that the given velocity and pressure fields satisfy these equations.

(ii) Check that boundary conditions on the sphere are no-slip:

$$u_r = 0, \qquad u_\theta = 0.$$

And that boundary conditions at infinity describe uniform flow of magnitude u_0 along z:

$$u_r \to u_0 \cos\theta, \qquad u_\theta \to -u_0 \sin\theta.$$

(c) At $r = a$ the stress tensor σ is

$$\begin{pmatrix} -p & \frac{-3\eta u_0 \sin\theta}{2a} & 0 \\ \frac{-3\eta u_0 \sin\theta}{2a} & -p & 0 \\ 0 & 0 & -p \end{pmatrix}$$

The normal to the surface is along $\hat{\mathbf{r}}$ so the components of the stress acting on the surface are

$$t_r = \sigma_{rr} = -p = -p_0 + \frac{3\eta u_0}{2a}\cos\theta, \qquad t_\theta = \sigma_{r\theta} = \frac{-3\eta u_0 \sin\theta}{2a}, \qquad t_\varphi = \sigma_{r\phi} = 0$$

By symmetry the net force on the sphere will be along z. The stress along z is

$$t_r \cos\theta - t_\theta \sin\theta = -p_0 \cos\theta + \frac{3\eta u_0}{2a}.$$

Integrating over the sphere to gives a force

$$\frac{3\eta u_0}{2a} \times (4\pi a^2) = 6\pi\eta u_0 a$$

which is the Stokes drag. (And the force along any direction perpendicular to z is zero as it must be by symmetry.)
(d) terminal velocity when

$$6\pi\eta u_0 a = mg \quad \rightarrow \quad u_0 = \frac{2a^2 g \rho_{\text{water}}}{9\nu\rho_{\text{air}}} = 125 \text{ ms}^{-1}$$

using $g = 10$, $\nu = 1.5 \times 10^{-5}$, $\rho_{\text{water}} = 10^3$, $\rho_{\text{air}} = 1.2$. The Reynolds number is $u_0 a/\nu \sim 10^4$ so Stokes drag is not a good approximation (but Re $\sim a^3$ so a=10 microns is OK).

6. **Thin film approximation**

A circular disk of radius a initially sticks to a flat ceiling at $z = 0$ by means of a thin film of viscous, incompressible liquid of dynamic viscosity η and thickness $h(t) \ll a$. Effects of surface tension may be neglected.

(a) Given that h and $W(t) = dh/dt$ are very small, explain briefly why the radial velocity u_r in the thin film obeys the approximate equation in cylindrical polar coordinates

$$\frac{\partial p}{\partial r} = \eta \frac{\partial^2 u_r}{\partial z^2}.$$

(b) Show that $p \approx p(r,t)$, and that u_r is given in this approximation by

$$u_r = \frac{1}{2\eta}\frac{\partial p}{\partial r} z(z+h).$$

(c) By integrating the incompressibility condition downwards in z from $z = 0$ to $z = -h$, show that

$$W = -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right).$$

(d) Hence show that the pressure distribution in the fluid film above the disk is given by

$$p - p_0 = \frac{3\eta W}{h^3}(a^2 - r^2),$$

where p_0 is atmospheric pressure.
(e) Show that the total downwards force on the disk needed to pull it away from the ceiling at speed W is

$$F = \frac{3\pi}{2}\frac{\eta a^4 W}{h^3}.$$

(f) Spiderman, of mass 50 kg, wishes to hang suspended from the ceiling by being attached to a circular disk of radius 10 cm on the underside of a thin film of a fluid of dynamic viscosity 300 kg m^{-1} s^{-1}. The film has an initial thickness 1 mm, and initially fills the entire space between the disk and the ceiling. The volume of the film V remains constant at $V = \pi a_0^2 h_0$, where a_0 and h_0 are the initial radius and thickness of the film. Estimate the length of time spiderman can remain attached to the ceiling. Comment on the likely accuracy of your estimate for a real fluid film (a real man?).

Solution

The Navier Stokes equations are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u}.$$

The flow is quasi-static so the time derivative can be neglected (the time dependence enters through the boundary conditions).
Slow flow, so assume $Re \ll 1$ and $(\mathbf{u}\cdot\nabla)\mathbf{u}$ can be neglected.
Symmetry $\rightarrow$ no θ dependence
$h \ll a$ and $u_r \gg u_z$ so $\frac{\partial u}{\partial z} \sim u_r/h \gg \frac{\partial u}{\partial r} \sim u_r/a$ so the rhs of N-S is approximately equal to $\nu\frac{\partial^2 \mathbf{u}}{\partial z^2}$.
So the components of the Navier-Stokes equation reduce to:

$$\frac{\partial p}{\partial r} = \eta\frac{\partial^2 u_r}{\partial z^2}, \tag{6}$$

$$\frac{\partial p}{\partial z} = \eta\frac{\partial^2 u_z}{\partial z^2} \approx 0 \rightarrow p \equiv p(r,t).$$

Integrating Eq. (6) twice with respect to z gives

$$u_r = \frac{1}{2\eta}\frac{\partial p}{\partial r}\,(z^2 + Cz + D).$$

The constants of integration follow from $u_r = 0$ at $z = 0$ and $z = -h$:

$$u_r = \frac{1}{2\eta}\frac{\partial p}{\partial r}\,z(z+h).$$

Incompressibility implies

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0$$

so

$$u_z \equiv W = \int_0^{-h} -\frac{1}{2\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) z(z+h)dz = -\frac{h^3}{12\eta}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right). \tag{7}$$

Integrating Eq. (7) gives the pressure distribution:

$$\left(r\frac{\partial p}{\partial r}\right) = -\frac{6\eta W r^2}{h^3} + C \quad\rightarrow\quad p = -\frac{3\eta W r^2}{h^3} + C\ln r + D.$$

The boundary conditions are p finite at the origin and $p = p_0$ at $R = a$ so

$$p = p_0 + \frac{3\eta W}{h^3}(a^2 - r^2).$$

The downwards force is

$$F = \int_0^a \frac{3\eta W}{h^3}(a^2 - r^2)2\pi r dr = \frac{3\pi\eta W a^4}{2h^3}.$$

Downwards force is provided by gravity, and writing everything in terms of h

$$mg = \frac{3\eta}{2\pi h^3}\frac{dh}{dt}\left(\frac{V}{h}\right)^2 \to \frac{dh}{dt} = \frac{2mg\pi h^5}{3\eta V^2} \to \frac{1}{4h_0^4} - \frac{1}{4h^4} = \frac{2mg\pi}{3\eta V^2}t$$

To estimate take time at which $h = \infty$

$$t = \frac{3\eta V^2}{8\pi mgh_0^4} = \frac{3\pi\eta a_0^4}{8mgh_0^2} \sim 70\text{s}$$

using $g = 10$, $\eta = 300$, $m = 50$, $a_0 = 10^{-1}$, $h_0 = 10^{-3}$.

The thin film approximation will fail as the film gets thicker and moves faster, but a real film will fail because adhesion between the liquid and ceiling is too weak and the film will peel off the ceiling.