Quantum PS3

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1 2.7) Time Dependence

A particle is confined in a potential well such that its allowed energies are $E_n = n^2\mathcal{E}$, where $n = 1, 2, \ldots$ is an integer and $\mathcal{E}$ a positive constant. The corresponding energy eigenstates are $|1\rangle, |2\rangle, \ldots, |n\rangle, \ldots$. At $t = 0$ the particle is in the state

$$|\psi(0)\rangle = 0.2\,|1\rangle + 0.3\,|2\rangle + 0.4\,|3\rangle + 0.843\,|4\rangle\,.$$

(a) What is the probability, if the energy is measured at $t = 0$, of finding a number smaller than $6\mathcal{E}$? [1]

The energies of these states are $\mathcal{E}, 4\mathcal{E}, 9\mathcal{E}$ and $16\mathcal{E}$ so the probability is the mod square of the first two (divided by the normalisation),

$$P(E < 6\mathcal{E}) = 0.1299.$$

(b) What is the mean value and what is the rms deviation of the energy of the particle in the state $|\psi(0)\rangle$? [2]

The average energy is

$$\langle E\rangle = \langle\psi(0)|\,H\,|\psi(0)\rangle\,.$$

We are told that $H\,|n\rangle = n^2\mathcal{E}\,|n\rangle$ so

$$\langle E\rangle = \sum_n |c_n|^2 E_n$$

(assuming $|\psi(0)\rangle$ is normalised) where $c_n = \langle n|\psi(0)\rangle$. Thus,

$$\langle E\rangle = 13.20\mathcal{E}.$$

The expectation of E^2 is then

$$\left\langle E^2\right\rangle = \langle\psi(0)|\,H^2\,|\psi(0)\rangle = \sum_n |c_n|^2 E_n^2 = 196.1\mathcal{E}^2.$$

Therefore the standard deviation is

$$\sigma = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = 4.7\mathcal{E}.$$

(c) Calculate the state vector $|\psi\rangle$ at time t. Do the results found in (a) and (b) for time t remain valid for arbitrary time t? [2]

Each individual state picks up an $e^{-i\frac{E}{\hbar}t}$. Thus,

$$|\psi(t)\rangle = 0.2e^{-i\frac{\mathcal{E}}{\hbar}t}\,|1\rangle + 0.3e^{-i\frac{4\mathcal{E}}{\hbar}t}\,|2\rangle + 0.4e^{-i9\frac{\mathcal{E}}{\hbar}t}\,|3\rangle + 0.843e^{-i16\frac{\mathcal{E}}{\hbar}t}\,|4\rangle\,.$$

This doesn't change the results in (a) and (b) as they only required the mod square of the amplitude. This time dependence only adds a phase so nothing changes.

(d) When the energy is measured it turns out to be $16\mathcal{E}$. After the measurement what is the state of the system? What result is obtained if the energy is measured again? [1]

After the measurement,

$$|\psi(t)\rangle = |4\rangle \,.$$

The wavefunction has been collapsed so this is the wavefunction for all time.

2 2.8) Well-defined Observables

A particle moves in the potential $V(\boldsymbol{x})$ and is known to have energy E_n. (a) Can it have well-defined momentum for some particular $V(\boldsymbol{x})$? (b) Can the particle simultaneously have well-defined energy and position? [4]

Given that the energy of the system is know the wavefunction must be some eigenstate of the Hamiltonian,

$$H\left|\psi\right\rangle = E_n\left|\psi\right\rangle.$$

Can this state have well defined momentum? This would demand that

$$\hat{\boldsymbol{p}}\left|\psi\right\rangle = \boldsymbol{p}_n\left|\psi\right\rangle.$$

So consider some $\left|\psi\right\rangle$ which is an eigenstate of both H and $\hat{\boldsymbol{p}}$. Then,

$$H\hat{\boldsymbol{p}}\left|\psi\right\rangle = E_n\boldsymbol{p}\left|\psi\right\rangle = \hat{\boldsymbol{p}}H\left|\psi\right\rangle.$$

So for any eigenstate of H to be a state of well-defined momentum we required

$$[H, \hat{\boldsymbol{p}}] = 0$$

which is not true for generic $V(\boldsymbol{x})$, only if $V =$ constant.

Part (b) is exactly the same but considering $\hat{\boldsymbol{x}}$ as the operator. Thus, we would need

$$[H, \hat{\boldsymbol{x}}] = 0$$

which for any Hamiltonian of the form

$$H = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\boldsymbol{x}})$$

cannot be true. So no, the particle can never have well-defined energy and position.

3 2.9) Hermiticity

Which of the following operators are Hermitian, given that $\hat{A}$ and $\hat{B}$ are Hermitian:

$$\hat{A}+\hat{B}; \qquad c\hat{A}; \qquad \hat{A}\hat{B}; \qquad \hat{A}\hat{B}+\hat{B}\hat{A}.$$

Show that in one dimension, for functions which tend to zero as $|x| \to \infty$, the operator $\frac{\partial}{\partial x}$ is not Hermitian, but $-i\hbar\frac{\partial}{\partial x}$ is. Is $\frac{\partial^2}{\partial x^2}$ Hermitian? [5]

By inspection,

$$\begin{aligned} \hat{A}+\hat{B} \quad & \text{is Hermitian,} \\ c\hat{A} \quad & \text{is not necessarily Hermitian (unless } c^* = c\text{),} \\ \hat{A}\hat{B} \quad & \text{is not necessarily Hermitian,} \\ \hat{A}\hat{B}+\hat{B}\hat{A} \quad & \text{is Hermitian.} \end{aligned}$$

For the second part consider what Hermiticity means,

$$\begin{aligned} \langle\psi| A |\phi\rangle = \langle\psi|A\phi\rangle &= \int \mathrm{d}x \psi^*(x) A\phi(x) \\ &= \langle A\psi|\phi\rangle = \int \mathrm{d}x \left(A\psi(x)\right)^* \phi(x). \end{aligned}$$

However, if we take $A = \frac{\partial}{\partial x}$ then

$$\langle\psi|A\phi\rangle = \cancel{\left[\psi(x)\frac{\partial\phi}{\partial x}\right]_{-\infty}^{\infty}} - \int \mathrm{d}x \frac{\partial\psi^*}{\partial x}\phi(x)$$

but

$$\langle A\psi|\phi\rangle = \int \mathrm{d}x \left(\frac{\partial\psi}{\partial x}\right)^* \phi(x).$$

Therefore, A cannot be Hermitian.

However, appending an i to it does make it Hermitian as, upon complex conjugation of $A\psi(x)$, we pick up the required minus sign.

Finally, $\frac{\partial^2}{\partial x^2}$ is Hermitian. To see this we integrate by parts twice, which gives two minus signs, which therefore cancel.

4 2.10) Hermitian Commutators

Given that $\hat{A}$ and $\hat{B}$ are Hermitian operators, show that $i[\hat{A}, \hat{B}]$ is a Hermitian operator. [2]

This is very simple.

$$\begin{aligned}
(i[A,B])^\dagger &= -i\,(AB - BA)^\dagger \\
&= -i\left[(AB)^\dagger - (BA)^\dagger\right] \\
&= -i\left(B^\dagger A^\dagger - A^\dagger B^\dagger\right) \\
&= i\,(AB - BA) \\
&= i[A,B].
\end{aligned}$$

So it's Hermitian.

5 2.11) Hermitian Products

Given that for any two operators $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger \hat{A}^\dagger$, show that

$$(\hat{A}\hat{B}\hat{C}\hat{D})^\dagger = \hat{D}^\dagger \hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger.$$

[1]

We just plug this in

$$\begin{aligned}
[ABCD]^\dagger &= [(AB)(CD)]^\dagger \\
&= (CD)^\dagger (AB)^\dagger \\
&= D^\dagger C^\dagger B^\dagger A^\dagger.
\end{aligned}$$

Simple.

6 2.12) Simultaneous Eigenstates

Show that if there is a complete set of mutual eigenkets of the Hermitian operators $\hat{A}$ and $\hat{B}$, then $[\hat{A}, \hat{B}] = 0$. Explain the physical significance of this result. [3]

If $|n\rangle$ are mutual eigenkets of A and B then

$$A|n\rangle = a_n |n\rangle,$$

$$B|n\rangle = b_n |n\rangle.$$

Therefore, consider some general state

$$|\psi\rangle = \sum_n \psi_n |n\rangle.$$

Then consider the action of the commutator on this state,

$$\begin{aligned}[A, B]|\psi\rangle &= \sum_n \psi_n (AB - BA)|n\rangle \\ &= \sum_n \psi_n (a_n b_n - b_n a_n)|n\rangle = 0.\end{aligned}$$

Therefore, $[A, B] = 0$.

Physically this means that, if two operators A and B commute, then the states of the system can be states of well-defined A and B.

7 2.13) Mutual Eigenstates

Does it always follow that if a system is an eigenstate of $\hat{A}$ and $[\hat{A}, \hat{B}] = 0$ then the system will be in an eigenstate of $\hat{B}$? If not, give a counterexample. [4]

Consider two operators which commute, $[A, B] = 0$. Now consider an eigenstate of A,

$$A|n\rangle = a_n |n\rangle .$$

Then consider applying B to this. This will a priori generate some new state, $|\phi\rangle$,

$$B|n\rangle = |\phi\rangle .$$

Thus, let's consider the following. Given that the operators commute

$$\begin{aligned} AB|n\rangle &= BA|n\rangle \\ A|\phi\rangle &= Ba_n|n\rangle \\ A|\phi\rangle &= a_n|\phi\rangle . \end{aligned}$$

Therefore, this new state $|\phi\rangle$ is also an eigenstate of A. How could this have happened?

Consider what $B|n\rangle$ could be in general. B is some operator which a priori will mix in a lot of eigenstates

$$B|n\rangle = |\phi\rangle = \sum_{n'} b_{n'} |n'\rangle$$

where each $|n'\rangle$ is an eigenstate of A with eigenvalue $a_{n'}$. Thus, if we know that $A|\phi\rangle = a_n|n\rangle$ then it must be the case that all $b_{n'} = 0$ except for b_n. So we conclude that

$$B|n\rangle = b_n|n\rangle$$

and the eigenstates of A are simultaneously eigenstates of B.

So this is the reverse of the previous proof. There we saw that $|\psi\rangle$ was the eigenstate of two operators A and B and we saw they had to commute for this to be true. Here we have two operators which commute, $[A, B] = 0$, and we see that the eigenstates of A must also be the eigenstates of B. This is automatic. Therefore consider our previous problems where we assumed that eigenstates have definite parity. We knew that $[H, P] = 0$ and so eigenstates of H must also be eigenstates of parity.

However, this is slightly deceptive. The above proof only works if the eigenstates of A are not degenerate. A very similar proof works in the degenerate case but the final conclusion is somewhat different. We just showed that "If A and B commute then the eigenstates of A must be eigenstates of B". In the degenerate case you should find that "If A and B commute then the eigenstates of A can always be made to coincide with the eigenstates of B". This is a statement about the non-uniqueness of degenerate eigenstates (the linear combination of any two is always an eigenstate with the same eigenvalue).

As such, we finally reach the answer to the question; No. The issue is that if A has any degenerate eigenstates then a generic eigenstate of A will not necessarily be an eigenstate of B. For example, consider the free particle Hamiltonian,

$$H = \frac{p^2}{2m},$$

which commutes with p. The eigenstates of H are

$$\psi = Ae^{ikx} + Be^{-ikx}.$$

However,

$$p\psi = \hbar k \left(Ae^{ikx} - Be^{-ikx} \right) \neq \hbar k \psi.$$

8 2.14) Commutators

Show that
(a) $[\hat{A}\hat{B}, \hat{C}] = A[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$. [2]

We just write this out,

$$\begin{aligned}[AB, C] &= ABC - CAB \\ &= ABC - CAB + ACB - ACB \\ &= A(BC - CB) + (AC - CA)B \\ &= A[B, C] + [A, C]B.\end{aligned}$$

(b) $[\hat{A}\hat{B}\hat{C}, \hat{D}] = \hat{A}\hat{B}[\hat{C}, \hat{D}] + \hat{A}[\hat{B}, \hat{D}]\hat{C} + [\hat{A}, \hat{D}]\hat{B}\hat{C}$. Explain the similarity with the rule for differentiating a product. [2]

We use the rule we just derived,

$$\begin{aligned}[ABC, D] &= AB[C, D] + [AB, D]C \\ &= AB[C, D] + A[B, D]C + [A, D]BC.\end{aligned}$$

Much like differentiation therefore, we have a product rule,

$$\frac{\partial}{\partial x}(ABC) = A'BC + AB'C + ABC'.$$

(c) $[\hat{x}^n, \hat{p}] = i\hbar n\hat{x}^{n-1}$. [2]

Consider this commutator, using the rule from (a),

$$[x^n, p] = [x, p]x^{n-1} + x[x^{n-1}, p] = i\hbar x^{n-1} + x[x^{n-1}, p].$$

Thus, we have a rule which we can apply to the x^{n-1} commutator,

$$[x^n, p] = i\hbar x^{n-1} + x\left(i\hbar x^{n-2} + x[x^{n-2}, p]\right) = 2i\hbar x^{n-1} + x^2[x^{n-2}, p].$$

This shows a clear pattern resulting in

$$[x^n, p] = i\hbar n x^{n-1}.$$

(d) $[f(\hat{x}), \hat{p}] = i\hbar\frac{\mathrm{d}f}{\mathrm{d}x}$ for any function $f(x)$. [2]

Here we expand the function,

$$f(x) = \sum_n f_n x^n.$$

Thus,

$$\begin{aligned}
[f(x), p] &= \left[\sum_n f_n x^n, p\right] \\
&= \sum_n f_n [x^n, p] \\
&= i\hbar \sum_n f_n n x^{n-1}.
\end{aligned}$$

By inspection this is clearly just the derivative,

$$[f(x), p] = i\hbar \frac{\mathrm{d}f}{\mathrm{d}x}.$$