2701

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1. FLOWS, FLUCTUATIONS AND COMPLEXITY

TRINITY TERM 2014

Tuesday, 17 June, 2.30 pm – 4.30 pm

10 minutes reading time

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier–Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \ ,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

---

1. Show that an irrotational, incompressible flow can be described by a velocity potential, ϕ, where

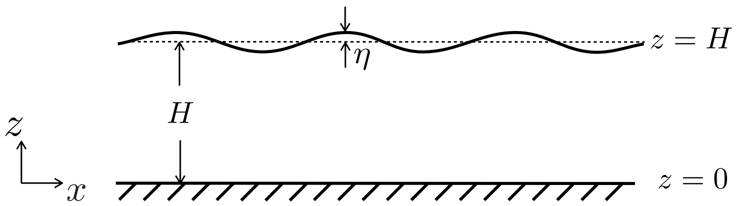
$$\nabla^2 \phi = 0.$$

Starting with the momentum equation for incompressible, irrotational flow in an inviscid fluid of constant density, show that

$$\frac{\partial \phi}{\partial t} + \frac{\nabla\phi \cdot \nabla\phi}{2} + \frac{p}{\rho} + gz = \text{constant},$$

where the symbols have their conventional meanings. You may wish to use the identity $\mathbf{A} \cdot \nabla \mathbf{A} = (\nabla \times \mathbf{A}) \times \mathbf{A} + \nabla(\mathbf{A} \cdot \mathbf{A}/2)$. [6]

Consider linear surface waves in a layer of water of depth at rest H, as sketched in the following figure.



Assuming the free surface elevation is $\eta = \eta_0 \sin(kx - \omega t)$, write down suitable linear boundary conditions for the vertical velocity at the upper and lower boundaries.

Assuming the flow is irrotational and seeking a solution of the form $\phi = X(x,t)Z(z)$, show that:

$$\phi = -\frac{\omega \eta_0}{k} \cos(kx - \omega t) \frac{\cosh kz}{\sinh kH},$$

subject to the dispersion relation

$$\omega^2 = gk \tanh kH.$$

[13]

Write down approximate dispersion relations that hold in the limits $kH \ll 1$ and $kH \gg 1$ and obtain expressions for the group velocity of the waves in each limit.

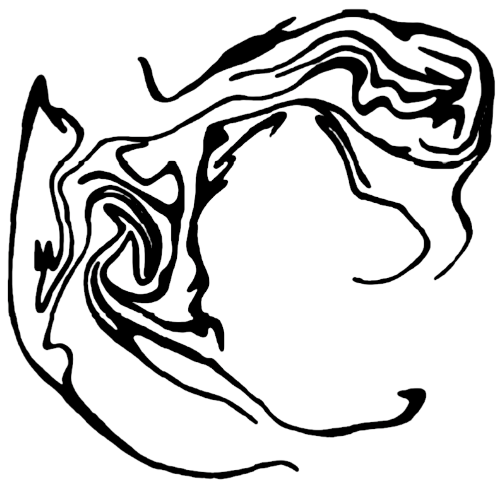
A motor boat passes by a jetty, causing the free surface to move up and down. Discuss how the frequency of the movement of the free surface at the jetty varies with time after the boat has passed. [6]

2. Define what is meant by *dynamical similarity* and the *Reynolds number*.

Starting with the Navier–Stokes equation with constant density and ignoring the gravitational accelation, show that two fluid flows require the same Reynolds number in order to be dynamically similar. [5]

Sketch the evolution of an initially rectangular fluid element, $|x| \leq 1$, $|y| \leq 1$ $(t = 0)$, in a deformation flow, $u = ax$, $v = -ay$, where a is a constant, and derive a formula for the evolution of the aspect ratio of the fluid element.

Using this result, or otherwise, explain the filamentary structure observed in the fluid flow shown in the following figure.



Explain why molecular diffusion becomes significant within a finite time for the evolution of a passive trace substance injected into such a flow. [6]

Describe what is meant by a *turbulent cascade*.

Deduce, using dimensional arguments, that the energy spectrum in a three-dimensional turbulent cascade should scale as

$$E(k) \propto k^{-5/3}\epsilon^{2/3},$$

where $E(k)$ is the energy per unit wavenumber, k is the wavenumber and ϵ is the energy dissipation rate.

Roughly what value do you expect for the Reynolds number at the Kolmogorov scale at which the turbulent cascade is arrested?

Using dimensional arguments, deduce that the Kolmogorov scale is

$$l \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4},$$

where ν is the kinematic viscosity. [10]

The Gulf Stream is an ocean current with a typical velocity of magnitude $0.2\,\mathrm{m\,s^{-1}}$ and width 50 km. In a numerical ocean-atmosphere climate model, computational constraints require that the viscosity be $10^4\,\mathrm{m^2\,s^{-1}}$, compared with $10^{-6}\,\mathrm{m^2\,s^{-1}}$ in water. Discuss how you expect the modelled Gulf Stream to differ from reality. [4]

3. Consider a dynamical system, $\dot{x}_i = f_i(x_i)$, $(i, j = 1, ..., n)$. Derive the equation for the rate of change of volume, ΔV, of an element moving in phase space:

$$\dot{\Delta V} = \sum_{i=1}^{n} \frac{\partial \dot{x}_i}{\partial x_i} \Delta V.$$

[3]

A damped pendulum is described by the dynamical system:

$$\begin{aligned} \dot{\omega} &= -ga \sin\theta - r\omega, \\ \dot{\theta} &= \omega, \end{aligned}$$

where ω is the angular velocity of the pendulum, θ is the angular displacement of the pendulum relative to the downward position, g is the gravitational acceleration, a is the length of the pendulum and r is the linear damping coefficient.

Derive an expression for the time-evolution of volume elements in phase space of this system and use this result to prove that all trajectories converge on an attractor if $r > 0$.

Solve for the fixed points in the case $r > 0$ and determine their stability properties.

Sketch a phase portrait for the path of trajectories as a function of θ and ω in the limit $r < 2\sqrt{ga}$. [11]

Now consider the addition of a constant torque to the pendulum:

$$\begin{aligned} \dot{\omega} &= -ga \sin\theta - r\omega + T, \\ \dot{\theta} &= \omega, \end{aligned}$$

where T is the applied torque (of either sign).

Solve for the fixed points and their stability, and sketch a bifurcation diagram for the behaviour of θ at the fixed points as the parameter T is varied.

Show that stable limit cycles must exist for $|T| > ga$. Describe in physical terms why the attracting fixed point transitions to an attracting limit cycle as $|T|$ is increased.

State *two* reasons why these limit cycles cannot emerge through Hopf bifurcations.

[11]

4. Derive an expression for $\langle z^2 \rangle$, the mean square end-to-end distance for an ideal freely-jointed chain consisting of N rigid segments of length b, freely hinged where they join. You may neglect possible consequences of interference between different parts of the chain.

The corresponding result for a wormlike chain, which models a polymer as a continuous filament with a non-zero bending modulus, is

$$\langle z^2 \rangle = 2A^2 \left(\mathrm{e}^{-L/A} - 1 + \frac{L}{A} \right),$$

where L is the contour length and A the persistence length. Evaluate this expression in the limits $L \ll A$ and $L \gg A$ (show your working) and comment on both results. [8]

Write down an expression for the partition function of the freely-jointed chain in the case that a force f is applied between the ends of the chain to separate them. Show that $\langle z \rangle$, the mean end-to-end distance of the chain, is related to the force f by

$$\langle z \rangle = Nb \left(\coth \alpha - 1/\alpha \right)$$

where $\alpha = fb/k_B T$. [11]

Show that the work done in stretching the freely-jointed chain close to its contour length is approximately given by

$$\Delta W \simeq c - W_0 \ln(1 - z/z_0),$$

where c is a constant that you need not evaluate, and provide expressions for constants W_0 and z_0. [6]