SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1: I. FLOWS, FLUCTUATIONS AND COMPLEXITY

TRINITY TERM 2013

Tuesday, 11 June, 2.30 pm – 4.30 pm

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier-Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \ ,$$

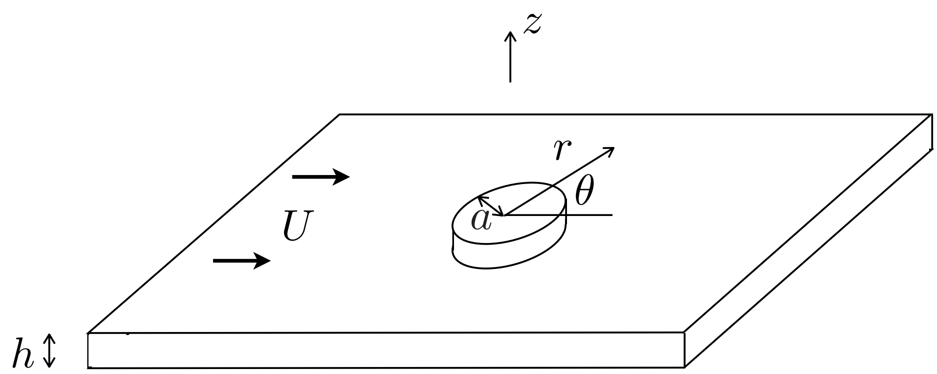
where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

1. Define the Reynolds number, Re. Show that in the limit Re $\ll 1$, the Navier-Stokes equation for fluid of constant density simplifies to

$$\nabla(p + \rho g z) = \rho\nu\nabla^2\mathbf{u}\ .$$ [5]

An experimental apparatus consists of a thin layer of such a fluid of uniform thickness h, occupying the region $0 \leq z \leq h$ and sandwiched between two solid plates, into which is placed a circular cylinder of radius a and height h. Far from the cylinder the fluid flows uniformly with velocity U at mid-depth (see diagram).



State the appropriate boundary conditions on the solid plates. Assuming Re $\ll 1$ and $\nabla_h p$ is independent of depth, where ∇_h is the horizontal gradient operator, and stating any further approximations, show that the horizontal velocity satisfies

$$\mathbf{u}_h = -\frac{1}{2\rho\nu} z(h-z)\nabla_h p\ .$$

Hence show that the horizontal pressure variations satisfy Laplace's equation,

$$\nabla_h^2 p = 0\ .$$

Using cylindrical coordinates and defining appropriate boundary conditions, verify the solution (for $r \geq a$)

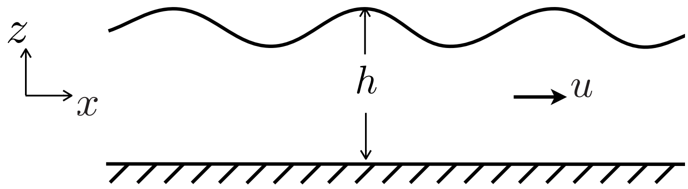
$$p = -\frac{8\rho\nu U}{h^2}\left(r + \frac{a^2}{r}\right)\cos\theta - \rho g z + p_0\ ,$$

where p_0 is a constant. [12]

Show that the horizontal flow obtained in this experiment at Re $\ll 1$ is equivalent, at each depth, to that found for two-dimensional ideal fluid at infinite Reynolds number, with the additional constraint that there can be no circulation around any closed obstacle. Sketch the streamlines for the flow.

Discuss how and why, in reality, two-dimensional flow past a cylinder at Re $\gg 1$ differs from that in an ideal fluid. [8]

2. Consider a shallow layer of water of thickness $h(x,t)$ as sketched in the diagram.



Assuming the dominant balance of forces in the vertical direction is between the pressure gradient and gravity, neglecting viscous forces, and assuming no variations or velocity in the y direction, show that the horizontal momentum equation can be written

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0 \; .$$

Starting from the equation of incompressibility and assuming appropriate boundary conditions for the vertical velocity, or otherwise, further show that

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0 \; .$$ [6]

By linearising these equations about a uniform mean flow of speed u_0 and uniform thickness h_0, derive expressions for the phase and group speeds of linear shallow water waves. Are these waves dispersive?

Estimate how long it takes a tsunami wave of amplitude 1 m and wavelength 200 km to cross an ocean basin of width 5000 km and depth 4 km. Confirm that background mean flows of magnitude $|u_0| \leq 1\,\mathrm{m\,s^{-1}}$ can be neglected. [8]

By defining a new variable P such that

$$dP = \sqrt{\frac{g}{h}}\,dh \; ,$$

show that the nonlinear equations can be rewritten:

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (u + \sqrt{gh}\,)\frac{\partial}{\partial x}\right](u+P) &= 0 \; , \\ \left[\frac{\partial}{\partial t} + (u - \sqrt{gh}\,)\frac{\partial}{\partial x}\right](u-P) &= 0 \; . \end{aligned}$$

Write down an implicit solution for u for a nonlinear shallow-water wave propagating in the positive x direction relative to the mean flow. Derive the corresponding solution for h. [8]

Sketch and physically interpret the evolution of h in a propagating finite-amplitude wave anomaly such as a tidal bore. Why must the solution eventually break down? [3]

3. The Lorenz system is defined by

$$\begin{aligned}\dot{x} &= \sigma(y-x)\ ,\\ \dot{y} &= rx - y - xz\ ,\\ \dot{z} &= xy - bz\ ,\end{aligned}$$

where $\sigma = 10$, $b = 8/3$ and $r \geq 0$.

Derive the fixed points and state the values of r for which each fixed point exists. Show that one of these fixed points is stable for $r < 1$. State the nature of the bifurcation that occurs at $r = 1$. [8]

Derive an equation for the rate of change of volume of an element moving around in phase space. Hence, or otherwise, deduce that all trajectories in the Lorenz system must converge on an attractor.

For the parameter value $r = 28$, it turns out that one of the local Lyapunov exponents is positive over a portion of each trajectory. Discuss what this implies for the geometry of the attractor and the predictability of the system. [9]

Now consider the modified Lorenz system

$$\begin{aligned}\dot{X} &= \sigma(Y-X)\ ,\\ \dot{Y} &= rx(t) - Y - x(t)Z\ ,\\ \dot{Z} &= x(t)Y - bZ\ ,\end{aligned}$$

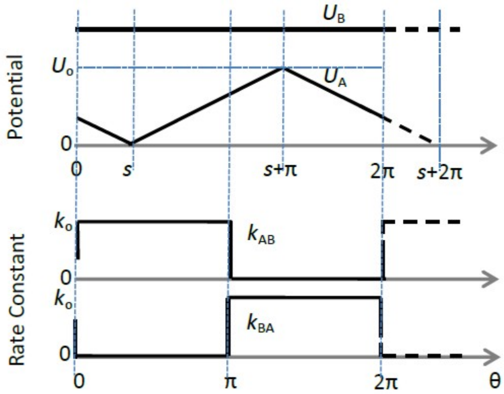
where $x(t)$ is a time series of x obtained by solving the original Lorenz system.

Write down dynamical equations for $e_x = X - x$, $e_y = Y - y$, $e_z = Z - z$, and for the quantity

$$E = \frac{e_x^2}{\sigma} + e_y^2 + e_z^2\ .$$

By deriving an equation for $\dot{E}$, or otherwise, show that it is possible, given $x(t)$, to recover the time series $y(t)$ and $z(t)$ from the original Lorenz system at long times, irrespective of the local Lyapunov exponents. [8]

4. A rotary molecular machine comprises a large number of protein motors, connected to form a ring, moving around a coaxial circular track. A motor is either in state A, bound to the track, with energy $U_{\mathrm{A}}(\theta)$ that depends on angular coordinate θ as shown, or in state B, detached from the track. The diagram also shows the angular dependence of rate constants for transitions between these states, $k_{\mathrm{AB}}(\theta)$, $k_{\mathrm{BA}}(\theta)$.



Probability densities $P_{\mathrm{A}}(\theta,t)$, $P_{\mathrm{B}}(\theta,t)$ satisfy the following equations:

$$\begin{aligned}
\frac{\partial P_{\mathrm{A}}(\theta,t)}{\partial t} &= -k_{\mathrm{AB}}(\theta)\, P_{\mathrm{A}}(\theta,t) + k_{\mathrm{BA}}(\theta)\, P_{\mathrm{B}}(\theta,t) - \ \omega\frac{\partial P_{\mathrm{A}}(\theta,t)}{\partial \theta};\\
\frac{\partial P_{\mathrm{B}}(\theta,t)}{\partial t} &= k_{\mathrm{AB}}(\theta)\, P_{\mathrm{A}}(\theta,t) - k_{\mathrm{BA}}(\theta)\, P_{\mathrm{B}}(\theta,t) - \ \omega\frac{\partial P_{\mathrm{B}}(\theta,t)}{\partial \theta};\\
P_{\mathrm{A}}(\theta) + P_{\mathrm{B}}(\theta) &= \frac{1}{2\pi}.
\end{aligned}$$

Brownian motion is neglected and each motor is assumed to rotate at the same, constant, angular velocity ω. Justify these assumptions. Show that the continuous, periodic, time-independent solution of these equations is

$$P_{\mathrm{A}}(\theta) = \begin{cases} \alpha \exp\left(-\beta\theta\right), & 0 \le \theta < \pi \\ 1/(2\pi) - P_{\mathrm{A}}(\theta-\pi), & \pi < \theta \le 2\pi \end{cases},$$

where α, β are constants to be determined, and sketch $P_{\mathrm{A}}(\theta)$, $P_{\mathrm{B}}(\theta)$ for $0 < \theta \le 2\pi$. (You may find it helpful to make the sketch first.) [12]

Show that the average torque exerted by a motor is given by

$$\bar{T}(s) = T_o\left(-\int_s^{s+\pi} P_{\mathrm{A}}(\theta)\, d\theta + \int_{s+\pi}^{s+2\pi} P_{\mathrm{A}}(\theta)\, d\theta\right)$$

(there is no need to evaluate these integrals) and determine T_o. Show that, for a given ω, the value of s that maximizes the average torque, $s_{\max}$, is a solution of the equation $P_{\mathrm{A}}(s) = P_{\mathrm{A}}(s+\pi) = 1/(4\pi)$. For each of the cases $\omega \ll k_o$ and $\omega \gg k_o$, sketch $P_{\mathrm{A}}(\theta)$ and the corresponding optimal potential $U_{\mathrm{A}}(\theta)$. Show that $\bar{T}(s_{\max}) \simeq T_o/2$ for $\omega \ll k_o$. [13]