# Fluids Collection

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# 1 Poiseuille Flow

Explain why a fluid may be considered as a continuous medium when viewed on a large enough length scale. Derive an estimate of the length scale involved, in the case of air at standard temperature and pressure. [5]

On a large enough length scale the volume properties we measure are averages over a very large number of particles and so any statistical fluctuations can be ignored and the system appears as a continuum. Large enough is therefore length scales that involve a 'large' number of particles. [2]

For air we could consider this length scale to be the mean free path of the gas. So we take

$$\lambda \sim \frac{1}{n\sigma}$$

where $\sigma$ is our cross sectional area and $n = \frac{p}{kT}$ is the number density ($p$ and $T$ are then the pressure and temperature). So,

$$\lambda \sim \frac{kT}{p\sigma}.$$

Now we are given that $p \sim 10^5$Pa and $T \sim 300$K. We can then estimate $\sigma$ using some multiple of the Bohr radius (let's call that multiple 8 because the molecules are diatomic so probably have an end to end width of about $4a_0$) such that $\sigma \sim \pi(5 \times 10^{-10})^2$. Therefore,

$$\lambda \sim 80\text{nm}.$$

[3]

Derive the acceleration of a moving fluid element in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u},$$

where $\boldsymbol{u}$ is the fluid velocity at a fixed point $\boldsymbol{r}$ and time $t$. [3]

This is a problem about total derivatives. The acceleration is

$$\boldsymbol{a} = \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}\left(\{x^i\}, t\right).$$

So we have

$$\mathrm{d}\boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial x^i}\mathrm{d}x^i + \frac{\partial \boldsymbol{u}}{\partial t}\mathrm{d}t$$
$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\partial \boldsymbol{u}}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial \boldsymbol{u}}{\partial t}.$$

Now, by definition, $u^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$ so

$$\boldsymbol{a} = u^i \partial_i \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}$$
$$\boldsymbol{a} = (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \frac{\partial \boldsymbol{u}}{\partial t}.$$

[3]

A viscous fluid of uniform density $\rho$ and uniform kinematic viscosity $\nu$ flows steadily in the $x$-direction along a tube of circular cross-section with radius $a$. It is subject to a constant pressure gradient $-G$ in the $x$-direction. The flow speed $u$ depends only on $r$, the distance from the axis of the tube. Given that the $x$-component of $\nabla^2 \boldsymbol{u}$ in these circumstances is

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right)$$

show that

$$u(r) = A(a^2 - r^2),$$

and find $A$ in terms of $G$, $\rho$ and $\nu$. Find also the mass of fluid crossing the surface $x = 0$ in unit time. [10]

This question is just about solving the Navier-Stokes equation. So, given we have a steady flow we have

$$(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} + \frac{\nabla p}{\rho} = \nu\nabla^2\boldsymbol{u}.$$

Now we use the rotational symmetry about the axis of the tube and conservation arguments to claim that $\boldsymbol{u}$ is only along $x$. We then use translational symmetry along the tube to claim that $u$ depends only on $r$. Therefore,

$$(\boldsymbol{u}\cdot\nabla) = u(r)\frac{\partial}{\partial x}u(r) = 0.$$

We then insert $\nabla p = -G$ and we're left with

$$-\frac{G}{\rho\nu} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right).$$

We integrate once

$$r\frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{G}{2\rho\nu}r^2 + c_1$$

and then again

$$u(r) = -\frac{G}{4\rho\nu}r^2 + c_1\ln r + c_2.$$

At this point we say that $u(r)$ is necessarily finite at $r = 0$ and also zero at the boundary. Therefore, $c_1 = 0$ and $c_2$ is chosen such that

$$u(r) = \frac{G}{4\rho\nu}(a^2 - r^2).$$

[6]

For the second part we want to integrate the mass flux, $\rho u(r)$, over the cross-section of the tube. So

$$\Phi = \int 2\pi r\mathrm{d}r\rho u(r)$$

$$\Phi = \frac{\pi G}{2\nu}\int_0^a \mathrm{d}r(a^2 r - r^3)$$

$$\Phi = \frac{\pi G}{2\nu}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$$\Phi = \frac{\pi a^4 G}{8\nu}.$$

[4]

Explain what is meant by the *Reynolds Number*, and derive an expression for it for this flow. What is found experimentally as the imposed gradient is increased from zero? [7]

The Reynolds number is the ratio of the inertial accelerations in the flow to the viscous forcing terms. A scale analysis shows

$$\mathrm{Re} \sim \frac{u\nabla u}{\nu\nabla^2 u} \sim \frac{LU}{\nu}$$

where $L$, $U$ and $\nu$ are the length scale, velocity scale and viscosity respectively. [2]

For this flow we can choose

$$L \sim a \qquad\qquad U \sim \frac{Ga^2}{4\rho\nu}.$$

Therefore,

$$\mathrm{Re} \sim \frac{Ga^3}{4\rho\nu^2}.$$

[2]

As we increase the pressure gradient we take Re from 0 to $\infty$. Initially the flow is laminar. As we increase the pressure gradient the flow becomes slightly turbulent up to a fully turbulent and chaotic mess. The crossover is at around Re $\sim$ 1000 but can persist higher depending on the channel geometry, roughness of walls and sizes of external perturbations. [3]

# 2 Lift

Explain what is meant by the *vorticity* $\omega$ of a fluid flow, and show how it is related to the *circulation* of the velocity field (as suitably defined). Obtain expressions for the vorticity and circulation of (a) a flow in rigid-body rotation with angular velocity $\Omega$, and (b) a two-dimensional line vortex whose velocity is given by $\boldsymbol{u} = (-Ay/r^2, Ax/r^2, 0)$, where $r^2 = x^2 + y^2$ and $A$ is a constant. [6]

The vorticity is the local rotation of a fluid element. It is defined as

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{u}.$$

The circulation is the total flow around some curve, $C$, defined as

$$\Gamma_C = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{r}.$$

Using Stokes' theorem then

$$\Gamma_C = \int_S (\nabla \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S} = \int_S \boldsymbol{\omega} \cdot \mathrm{d}\boldsymbol{S}.$$

[2]

For example (a) the flow field can be described by $\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$. Therefore,

$$\begin{aligned}
\omega_i &= \epsilon_{ijk}\partial_j(\epsilon_{klm}\Omega_l r_m) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\Omega_l \partial_j r_m \\
&= \Omega_i \partial_j r_j - \Omega_j \partial_j r_i \\
&= 3\Omega_i - \Omega_i.
\end{aligned}$$

Thus, $\boldsymbol{\omega} = 2\boldsymbol{\Omega}$. The circulation is then the total flux of $2\boldsymbol{\Omega}$ through our curve,

$$\Gamma_C = 2\Omega A_C,$$

where $A_C$ is the area enclosed by curve $C$ as projected onto the 2D plane perpendicular to $\boldsymbol{\Omega}$. [2]

For example (b) we plug it in to find

$$\begin{aligned}
\boldsymbol{\omega} &= \left[\partial_x\left(\frac{Ax}{r^2}\right) - \partial_y\left(-\frac{Ay}{r^2}\right)\right]\hat{\boldsymbol{z}} \\
&= \left(\frac{A}{r^2} - \frac{2Ax^2}{r^4} + \frac{A}{r^2} - \frac{2Ay^2}{r^4}\right)\hat{\boldsymbol{z}} \\
&= \left(\frac{2A}{r^2} - \frac{2A(x^2+y^2)}{r^4}\right)\hat{\boldsymbol{z}} = 0.
\end{aligned}$$

So the vorticity is zero for all $r > 0$. Therefore, any path $C$ that does not enclose the origin has zero circulation. However, if the path does enclose the origin we know it must have non-zero circulation. Therefore, consider that

$$\boldsymbol{u} = \frac{A}{r}\hat{\boldsymbol{\phi}}.$$

Therefore, the circulation around the origin is going to be

$$\Gamma_C = \frac{A}{r} \times 2\pi r = 2\pi A.$$

[2]

Consider an inviscid, incompressible flow, parallel to the $xy$-plane with no variation in the $z$-direction, whose velocity field is given by $\boldsymbol{u} = (u_x, u_y, 0) = \nabla\phi$, where $\phi$ is a scalar function of $x$ and $y$. Show that this flow has zero vorticity everywhere, and that $\phi$ satisfies Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0.$$

A steady, two-dimensional, invisicid, incompressible flow around a long circular cylinder, aligned in the $z$-direction and of radius $a$, is given by the velocity potential

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - G\theta,$$

where $x = r\cos\theta$ and $y = r\sin\theta$, and $U$ and $G$ are constants. Verify that $\phi$ and the radial and azimuthal velocity components of this field satisfy Laplace's equation and appropriate boundary conditions at the surface of the cylinder. Why does $u_\theta$ not need to be zero at the surface of the clinder? Obtain expressions for the vorticity of the flow and circulation $\Gamma$ around a path enclosing the cylinder. [8]

Given that $\nabla \times \nabla\phi = 0$ for any $\phi$, this flow must necessarily have zero vorticity as $\boldsymbol{u} = \nabla\phi$. We are also told that the flow is incompressible, so $\nabla \cdot \boldsymbol{u} = 0$, and hence

$$\nabla^2\phi = 0.$$

The form given then assumed no variation in $z$, as we are told. [2]

Using cylindrical polars we have

$$\begin{aligned}
\nabla^2\phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r - \frac{a^2}{r}\right)U\cos\theta - \frac{1}{r^2}U\left(r + \frac{a^2}{r}\right)\cos\theta \\
&= U\left(\frac{1}{r} + \frac{a^2}{r^3}\right)\cos\theta - U\left(\frac{1}{r} + \frac{a^2}{r^3}\right)\cos\theta = 0.
\end{aligned}$$

So it satisfies Laplace's equation. The velocity is then

$$\boldsymbol{u} = \left(U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{G}{r}, 0\right).$$

The boundary conditions are

- No normal flow into the sphere. This is satsfied as $u_r(r = a) = 0$.
- Planar flow at infinity. This is satisfied as $\boldsymbol{u}(r \to \infty) = (U\cos\theta, -U\sin\theta, 0)$, which is the cylindrical form for a flow $U\hat{\boldsymbol{x}}$. (Note that because we're not told the physical situation, it's only implied, missing this one lost you no marks).

The flow along the surface is not zero in this simple flow as we are considering the fluid to be inviscid. Technically there will be some term, important very close to the boundary, that does ensure no flow along the boundary. The vorticity is zero (because $\boldsymbol{u} = \nabla\phi$, as you argued above). We find the circulation by ignoring the sinusoidal terms (they will give zero) so

$$\begin{aligned}\Gamma_C &= \oint_C -\frac{G}{r}\hat{\boldsymbol{\theta}} \cdot \mathrm{d}\boldsymbol{l} \\ &= \oint_C -\frac{G}{r} r\mathrm{d}\theta = -2\pi G.\end{aligned}$$

[6]

By applying Bernulli's equation,

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + gz = 0,$$

where $\rho$ is the density and $g$ is the acceleration due to gravity, to this situation, obtain an expression for the pressure $p$ at the surface of the cylinder. Hence, show that the drag force $F_x$ on the cylinder in the $x$-direction is zero and there is a net lift force in the $y$ direction of the form

$$F_y = -\rho U \Gamma$$

per unit length of the cylinder in the $z$-direction. [7]

The velocity potential is independent of time. Therefore,

$$p = -\rho g z - \frac{1}{2}(u_r^2 + u_\theta^2).$$

Now at the surface we know that $u_r = 0$ so we just need $u_\theta$, and so

$$p = -\rho g z - \frac{\rho}{2}\left(2U\sin\theta + \frac{G}{a}\right)^2.$$

Ignoring this $z$ dependance (we consider it at a particular height) we consider this pressure acting inwards along $-\hat{\boldsymbol{r}}$. Therefore,

$$\begin{aligned}F_x &= -\int_0^{2\pi} p\cos\theta a\mathrm{d}\theta \\ &= \frac{\rho a}{2}\int_0^{2\pi}\left(4U^2\sin^2\theta + \frac{4GU}{a}\sin\theta + \frac{G^2}{a^2}\right)\cos\theta\mathrm{d}\theta\end{aligned}$$

and all of these sinusoidal terms evaluate to zero between 0 and $2\pi$. For the lift we have

$$\begin{aligned}F_y &= -\int_0^{2\pi} p\sin\theta a\mathrm{d}\theta \\ &= \frac{\rho a}{2}\int_0^{2\pi}\left(4U^2\sin^2\theta + \frac{4GU}{a}\sin\theta + \frac{G^2}{a^2}\right)\sin\theta\mathrm{d}\theta.\end{aligned}$$

The second term is non-zero and so

$$F_y = \frac{\rho a}{2}\frac{4GU}{a}\pi = -\rho U(-2\pi G)$$

as required. [7]

Give a brief qualitative discussion of how this result can be extended to explain the physics of aircraft flight. [4]

You've got four marks to get here so we need

- When a plane starts to move a vortex is formed near the trailing edge of the wing.
- This vortex forms due to viscous effects in the boundary layer.
- The aerofoil is left with a net circulation around the wing, causing lift.
- We can apply this result more directly by applying a conformal transformation to the sphere.

[4]

# 3 Dynamical Systems

Two competing antibodies in an immune system have populations $x(t)$ and $y(t)$ at time $t$. The evolution of the twospecies is described by the following coupled time-dependent equations:

$$\dot{x} = x(3 - \beta x - y),$$

$$\dot{y} = y(2 - x - y).$$

The parameter $\beta$ is determined by the environment. Consider $\beta = 1$. Find the fixed points and determine their stabilities. Hence sketch the phase portrait in the $(x, y)$ plane for the system. [13]

So we want to set $\dot{x} = \dot{y} = 0$. The first solution is the obvious $(0, 0)$ solution. By inspection we can also have $(0, 2)$ or $(3, 0)$. The Jacobian for the system is

$$J = \begin{pmatrix} 3 - 2\beta x - y & -x \\ -y & 2 - x - 2y \end{pmatrix}.$$

So consider each point in turn.

$$J_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix},$$

so this point is unstable.

$$J_{(0,2)} = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}.$$

The eigenvalues of this are 1 and $-2$, making a saddle. The eigenvalues of the form $(a, b)$ then satisfy

$$a = \lambda a \qquad\qquad - 2(a + b) = \lambda b.$$

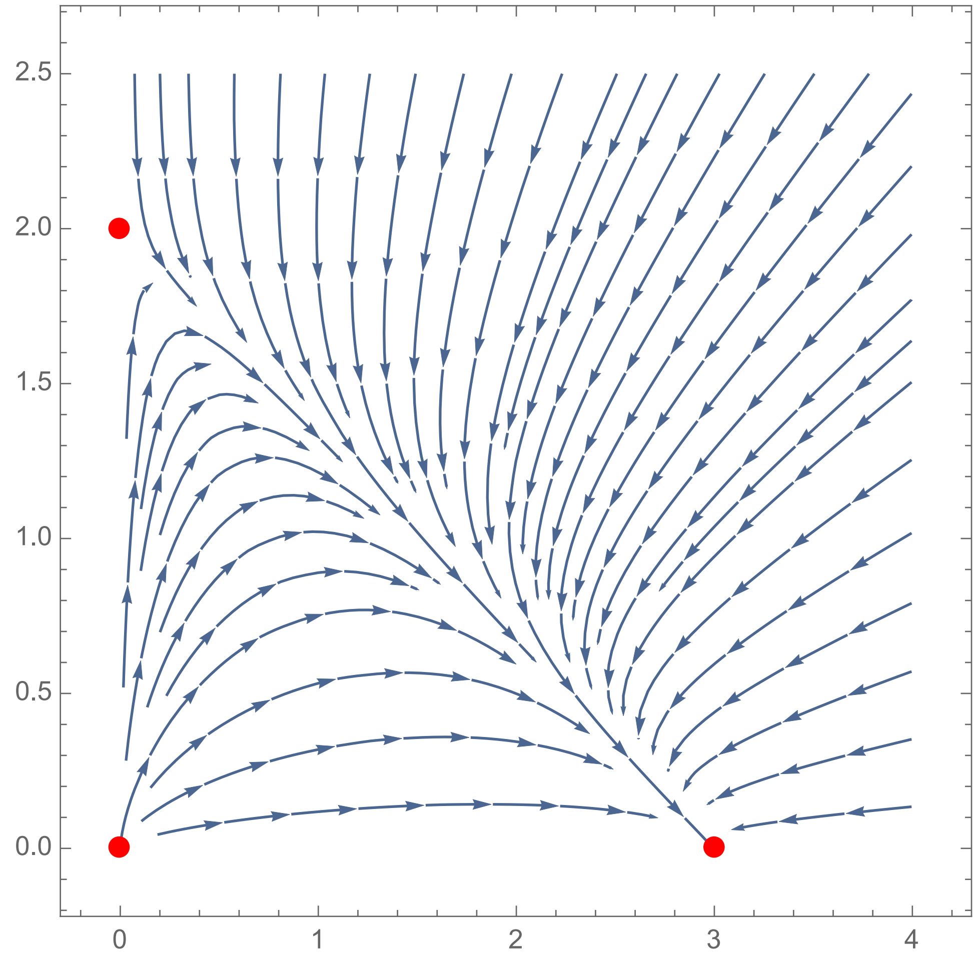
Therefore we have

$$\boldsymbol{v}_1 = \left(1, -\frac{2}{3}\right) \qquad\qquad \boldsymbol{v}_{-2} = (0, 1).$$

Finally we have the point

$$J_{(3,0)} = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix},$$

so this point is stable.




[13]

Now consider $\beta = 2$. Find the corresponding fixed points and determine their stabilities. Sketch the phase portrait. [8]

So now we can have an extra fixed point. We have $(0,0)$, $(0,2)$, $(3/2,0)$ and $(1,1)$. The Jacobian for the first two fixed points in unchanged. For the latter two we have

$$J_{3/2,0} = \begin{pmatrix} -3 & -3/2 \\ 0 & 1/2 \end{pmatrix}.$$

Therefore we have a saddle with eigenvectors

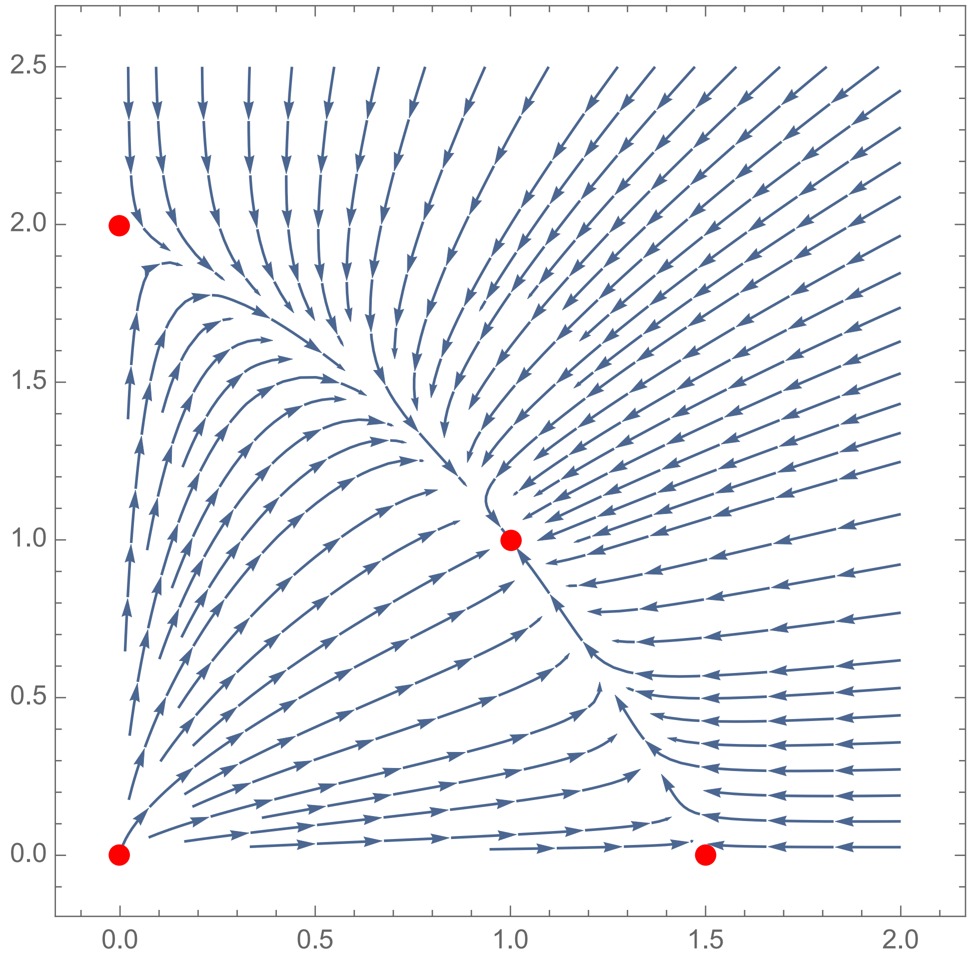
$$-3(a + b/2) = \lambda a \qquad\qquad b/2 = \lambda b$$

$$\implies \qquad \boldsymbol{v}_{-3} = (1,0) \qquad\qquad \boldsymbol{v}_{1/2} = (1, -7/3).$$

The new fixed point then has

$$J_{(1,1)} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}.$$

The eigevalues here are $\frac{-3\pm\sqrt{5}}{2}$, which are both negative. So this is stable.




[8]

Comment on the significance of the particular forms of the phase portrait for $\beta = 1$ and $\beta = 2$. [4]

For $\beta = 1$ the stable fixed point is the one in which $x$ wins is any of it is present at the start. This kills $y$. However, for the $\beta = 2$ case the system evovles to a final state where there is an equal amount of $x$ and $y$. This is a pitchfork bifurcation, but a pretty weird one because we're restricted to $x$ and $y$ being greater than zero, so one of our stable prongs isn't seen. [4]

# 4 Biophysics

Derive an expression for $\langle z^2 \rangle$, the mean square end-to-end distance for an ideal freely-jointed chain (FJC, often also called a Gaussian chain) consisting of $N$ rigid segments of length $b$, freely hinged where they join. You may neglect possible consequences of interference between different parts of the chain. Write down an expression for the partition function of the FJC in the case that a force $f$ is applied between the ends of the chain to separate them. Show that $\langle z \rangle$, the mean end-to-end distance of the chain is related to the applied force by

$$\langle z \rangle = Nb(\coth\alpha - 1/\alpha)$$

where $\alpha = fb/kT$. [14]

This is an easy 14 marks. Let's call $\boldsymbol{n}_i$ the unit vector along the $i^{\text{th}}$ segment of the chain. The length is then $z = |\sum_i b\boldsymbol{n}_i|$. Therefore,

$$\langle z^2 \rangle = b^2 \left\langle \sum_{ij} \boldsymbol{n}_i \cdot \boldsymbol{n}_j \right\rangle.$$

Here we use that the segments are uncorrelated so

$$\langle z^2 \rangle = b^2 \sum_{ij} \boldsymbol{n}_i \cdot \boldsymbol{n}_j \delta_{ij} = Nb^2.$$

[3]

For the partition function we need the energy. This is

$$U(\{\boldsymbol{n}_i\}) = -fz$$

where $z$ is defined above. Therefore,

$$\mathcal{Z} = \int \mathrm{d}^2\boldsymbol{n}_1 \dots \mathrm{d}^2\boldsymbol{n}_N e^{fz/kT}.$$

[5]

Here we define $z$ in terms of angular variables, $z = \sum_i b\cos\theta_i$. Then we use $\mathrm{d}^2\boldsymbol{n}_i = \mathrm{d}\phi_i \mathrm{d}(\cos\theta_i)$. Therefore,

$$\begin{aligned}
\mathcal{Z} &= \int \prod_i 2\pi \mathrm{d}(\cos\theta_i) e^{fb/kT \sum_i \cos\theta_i} \\
&= \prod_i \int_{-1}^{1} 2\pi \mathrm{d}x_i e^{\alpha x_i} \\
&= \left(4\pi \frac{\sinh\alpha}{\alpha}\right)^N.
\end{aligned}$$

For the average position we differentiate with respect to $f$. So

$$\langle z \rangle = kT \frac{\partial \mathrm{Ln}\mathcal{Z}}{\partial f} = Nb(\coth\alpha - 1/\alpha).$$

[6]

The figure shows a force-extension curve for $\lambda$-phage double-stranded DNA of length $L$ ($\sim 10^{-5}$m) with fits to Hooke's law (at small extensions), and to the FJC and Worm-Like Chain (WLC) models. Discuss the physical origin of the observed behaviour of double-stranded DNA in the low and high force regimes. Find an expression for the spring constant (i.e., the constant of proportionality between force and extension) for a FJC for small applied forces. (You may assume that, for $\alpha \ll 1$, $\coth(\alpha) - 1/\alpha \simeq \alpha/3$.) Hence, using the information provided in the figure, estimate the effective segment length for double stranded DNA to 1 s.f. To 1 s.f., what is the FJC prediction for the force required to stretch this piece of DNA to 0.999 of its contour length? (At the temperature of the measurement, $kT = 4$pm nm.) [11]

Another remarkably achievable 11 marks. We start by explaining that elasticity arises due to entropic reasons. The number of possible configurations with a large end-to-end distance is significantly smaller than the number of configurations with a very small end-to-end distance. Therefore, entropy wants the chain to be short. In the low-force regime this explains why the chain is short, as entropy is winning the battle and energy is inconsequential. In the high-force regime the chain begins to lengthen with applied force as the energy becomes an important factor. However, the force needed diverges as we approach full extension as there will only be one configuration left at that extension. [4]

For small applied forces we expand in small $\alpha$. Therefore,

$$\langle z \rangle \simeq \frac{Nb\alpha}{3} = \frac{Nb^2}{3kT} f.$$

We can invert this relation to see $f = \kappa \langle z \rangle$ where this spring constant is

$$\kappa = \frac{3kT}{Nb^2}.$$

[1]

Given the Hooke's law approximation we have that $f(1) = 0.1$pN. Therefore,

$$0.1\text{pN} = \frac{3 \times 4\text{pN nm}}{Nb^2} Nb$$

$$\implies b = 120\text{nm}.$$

[3]

Finally we find the force for a 0.999 extension. We assume the force is large so take $\coth \alpha \simeq 1$. Therefore,

$$\langle z \rangle \simeq Nb(1 - 1/\alpha)$$

$$\implies f = \frac{kT}{b(1 - \langle z \rangle / Nb)}.$$

Therefore, setting $\langle z \rangle / Nb = 0.001$ we find

$$f \simeq 33\text{pN}.$$

Finally we must check that $\alpha$ is indeed large. We find it to be

$$\alpha = \frac{33 * 120}{4} = 1000.$$

This is large. [3]