A13480W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1. FLOWS, FLUCTUATIONS AND COMPLEXITY

TRINITY TERM 2017

Monday, 5 June, 2.30 pm – 4.30 pm

Candidates are strongly advised to use the first 10 minutes to read the whole paper before starting writing.

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

The use of approved calculators is permitted.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier–Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \ ,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector and ν the kinematic viscosity.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as

$$J_{ij} = \frac{\partial f_i}{\partial x_j},$$

where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

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1. Show that an incompressible and irrotational flow with velocity $\mathbf{u}$ satisfies Laplace's equation

$$\nabla^2 \phi = 0 \ ,$$

where $\phi(x, y, z, t)$ is the velocity potential, x, y, z are spatial coordinates and t is time. [2]

Consider small amplitude, two-dimensional inviscid ocean surface waves in the $x - z$ plane. The ocean has infinite depth and there is no y-dependence. The vertical displacement of the free-surface is given by $z = \eta(x, t)$. The ocean mean surface is at $z = 0$ and the fluid occupies $z < \eta$. The flow is incompressible and irrotational with velocity potential $\phi(x, z, t)$. At large depths ($z \to -\infty$), the fluid is at rest.

The boundary conditions at the mean surface $z = 0$ are given by

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad \frac{\partial \phi}{\partial t} = -g\eta \ ,$$

to within the linear approximation, where g is the acceleration due to gravity. Give a brief physical explanation of how each of these boundary conditions arises (without detailed calculation). Write down the boundary condition for $z \to -\infty$ in terms of ϕ. [4]

Assume that the solution for the free-surface takes the form $\eta(x, t) = A \ \sin{(kx - \omega t)}$, where A, k and ω are constants. Find the corresponding form of the velocity potential everywhere and hence find the dispersion relation for deep water waves. [6]

Assume that displacements of fluid particles about their mean positions are small. Show that the paths of fluid particles are circular in shape and sketch or describe their variation with depth below the surface. [7]

If the effects of surface tension are included, the pressure at the surface is reduced by an amount proportional to the curvature of the surface, i.e.

$$p(\text{surface}) = p_{\text{atm}} - \sigma \frac{\partial^2 \eta}{\partial x^2} \ ,$$

where p_{atm} is the atmospheric pressure and σ is a constant. The new boundary conditions for ϕ at $z = 0$ are therefore

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{and} \quad \frac{\partial \phi}{\partial t} = -g\eta + \frac{\sigma}{\rho} \frac{\partial^2 \eta}{\partial x^2} \ .$$

Show that the dispersion relation is given by $\omega^2 = c_1 k + c_2 k^3$ and find c_1 and c_2. Sketch the wave speed $c = \omega / k$ as a function of wavenumber k or wavelength λ for $\sigma = 0$ and for a small value of σ. Which waves are most affected by the surface tension and why? [6]

2. By writing the Navier-Stokes equations for a 2D (in the x-y plane) steady incompressible flow in terms of dimensionless variables, derive an expression for the Reynolds number in terms of a characteristic length scale, L, and a characteristic velocity scale, U. Describe typical behaviour of flows for different Reynolds numbers.

[7]

Consider an inviscid, incompressible and irrotational fluid. The velocity potential for a flow around a cylinder, placed in a uniform stream, may be obtained by adding the velocity potentials for a uniform stream, a dipole and a source. The potential can then take the form

$$\phi = Ur\cos\theta + \frac{\mu\cos\theta}{r} + m\ \log\ r.$$

Assume that the uniform stream has speed U, the normal (outward) velocity at the surface of the cylinder is V $(\neq 0)$, the radius of the cylinder is a and the circulation around the cylinder is zero. μ and m are constants.

Find expressions for μ and m in terms of U, V and a. Write down expressions for the velocity potential and the velocity components (u_r and u_θ).

[4]

Show that if V equals $4U$, then there is a stagnation point at a distance $\left(2+\sqrt{5}\right)a$ upstream of the cylinder. Sketch streamlines for the flow.

[7]

Given $(\mathbf{u}\cdot\nabla)\,\mathbf{u} = \nabla\left(\frac{1}{2}\mathbf{u}^2\right) + \omega\times\mathbf{u}$ and $\omega = \nabla\times\mathbf{u}$, show that $B = p + \frac{1}{2}\rho\mathbf{u}^2 + \rho g z$ is constant along streamlines, for a steady, incompressible and inviscid flow.

[3]

A large tank of uniform surface cross section A_0 has a small hole in the side of the tank, with area A_e. The hole is located at a depth H below the top of the tank. The tank is filled with water. Find the velocity and volume flux at which the flow exits the tank. Clearly describe any assumptions made. Sketch the streamlines.

[4]

3. Consider the differential equation

$$\dot{x} = rx - 2x^2 + x^3,$$

where r is a constant. Show that $x^* = 0$ is a fixed point for any value of the parameter r. Determine its stability. Identify a value, r_1, at which there is a bifurcation point associated with the fixed point $x^* = 0$.

[4]

Determine the additional fixed points and the values of r for which these fixed points exist. Determine their stability and identify a value, r_2, at which there is a further bifurcation point.

[6]

Sketch the bifurcation diagram for all values of r and x, showing all the fixed points and the flow. Using a Taylor expansion of $\dot{x} = rx - 2x^2 + x^3$, or otherwise, determine the normal (canonical) form of the bifurcation at r_1 and classify it. Do the same for the bifurcation at r_2.

[6]

Consider the dynamical system in the x-y plane

$$\dot{x} = x^2 - y - 1,$$

$$\dot{y} = (x - 2)y.$$

Find the fixed points and determine their stability. Draw a full phase portrait.

[9]

4. A biological polymer is modeled as a three-dimensional freely jointed chain (FJC) of N rigid segments of length b. Using a sketch, show that for a FJC with non-interacting segments, the mean square end-to-end distance is $< \mathbf{r}^2 >= Nb^2$. Show that if the polymer is modeled as a wormlike chain with persistence length P, contour length L and a tangent vector correlation function $< \mathbf{r}_s \cdot \mathbf{r}_t >= \mathrm{e}^{-|s-t|/P}$, the corresponding distance becomes

$$< \mathbf{r}^2 >= 2P^2(\mathrm{e}^{-L/P} - 1 + \frac{L}{P}).$$

Explore the limits $L>>P$ and $L<<P$ and discuss your findings. [7]

The polymer conformations for a FJC have similarities to three-dimensional particle diffusion. The probability distribution for finding a diffusing particle along a single axis is

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(\frac{-x^2}{4Dt}\right),$$

where D is the diffusion coefficient, and t the diffusion time. Find an expression for the end-to-end distance distribution $P(\mathbf{r}, N)$ for a FJC with N segments and plot the distribution. How does the distribution width relate to $< \mathbf{r}^2 >$?

Show that the entropic energy term for extending a FJC to length r is given by

$$A(\mathbf{r}) = a + c\mathbf{r}^2,$$

where a and c are constants. Provide an expression for c and explain why a FJC can act as an entropic spring. Derive what the force-extension curve would look like in the high-force limit for the FJC. [10]

A bacterial chromosome can be treated as a single open chain 5×10^6 basepairs (bp) long. If the DNA is double-stranded and is modelled as a wormlike chain with P =150 bp and each bp step is 0.34 nm, what is its mean end-to-end distance? What would be the mean end-to-end distance if the DNA strand had the same length, but was single-stranded, thus having a persistence length equivalent to 5 bp steps?

For both chains above, calculate the ratio of the probability that $R = 0$ compared to the probability that $R = 50$ μm, and comment on your results. Sketch the force-extension curves at the low-force regime for both single-stranded DNA and double-stranded DNA, and comment on your results. [8]