FLUIDS, FLOWS AND COMPLEXITY

PROBLEM SET 1 AND SOLUTIONS

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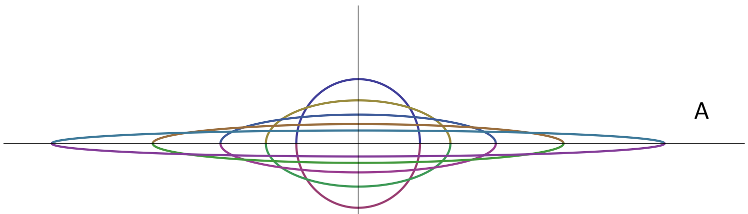
Changes for 2016: rearranged to more closely follow the lectures.

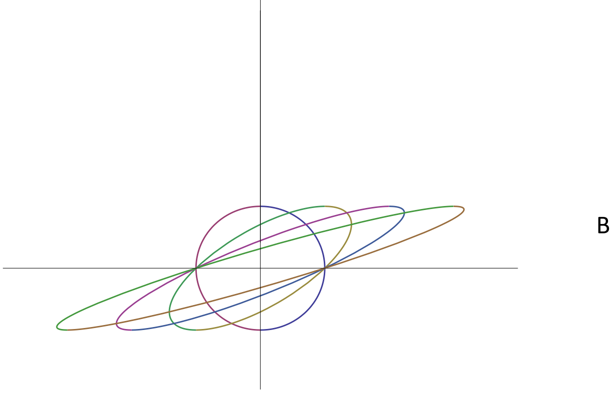
Thank you to Professor David Marshall who gave this course in previous years. I have used some of his problems and lecture notes. A few problems are from the web sites of Oxford Maths and Cambridge Maths.

1. **Streamlines and flows**

For (A) a 2D straining flow $\mathbf{u} = (\alpha x, -\alpha y)$ and (B) a simple shear flow $\mathbf{u} = (\gamma y, 0)$ where $\alpha$ and $\gamma$ are constants:
(a) Find the equation for a general streamline of the flow.
(b) At $t = 0$ dye is introduced to mark the curve $x^2 + y^2 = a^2$. Find the equation for this material fluid curve for $t > 0$ and sketch how the curve evolves with time.
(c) Does the area within the curve change in time, and why?
(d) Which of the two flows stretches the curve faster at long times?

**Solution**




A:
(a) Streamlines are tangent to the flow at every point. Therefore, for this 2D flow,

$$\frac{dx}{u_x} = \frac{dy}{u_y} \Rightarrow \frac{dx}{x} = -\frac{dy}{y} \Rightarrow \text{streamlines are } xy = \text{constant.}$$

(b)

$$u_x = \frac{dx}{dt} = \alpha x \text{ and } x = x(0) \text{ at } t = 0 \Rightarrow x(t) = x(0)e^{\alpha t}. \text{ Similarly } y(t) = y(0)e^{-\alpha t}.$$

Using $x^2(0) + y^2(0) = a^2$ this gives

$$x^2(t)e^{-2\alpha t} + y^2(t)e^{2\alpha t} = a^2$$

which is an ellipse with semi-axes of length $ae^{\alpha t}$ along $x$ and $ae^{-\alpha t}$ along $y$. So the flow stretches along $x$ and contracts along $y$ but
(c) the area within the curve does not change with time because the flow is incompressible, div $\mathbf{u} = 0$.
B:
(a) for shear flow the streamlines are $y =$ constant.
(b) $x(t) = x(0) + \gamma y(0)t, y(t) = y(0)$ so the circle of radius $a$ evolves to an ellipse with major axis at an angle to the $x$-axis which decreases with time:

$$(x(t) - \gamma y(t)t)^2 + y^2(t) = a^2.$$

(c) the flow is again incompressible.

(d) A - the stretching is exponential in time.

2. **Stream function and velocity potential**

(a) Is the motion incompressible for the flows given by the following velocity potentials:
(i) $\phi = C(x^2 + y^2)$ (ii) $\phi = C(x^2 - y^2)$?
If so, determine the corresponding stream functions.

(b) Is the motion irrotational for the flows given by the following stream functions:
(iii) $\psi = C(x^2 + y^2)$ (iv) $\psi = C(x^2 - y^2)$?
If so, determine the corresponding velocity potentials.

(iii) Sketch the streamlines for all cases (a)– (d) and the lines of constant $\phi$ where possible.

**Solution**

(a) (i) $\mathbf{u} = \nabla\phi = (2Cx, 2Cy)$, $\quad \nabla \cdot \mathbf{u} \neq 0$ so compressible
(ii) $\mathbf{u} = \nabla\phi = (2Cx, -2Cy)$, $\quad \nabla \cdot \mathbf{u} = 0$, incompressible, $\psi = 2Cxy$ (cf Q1A)
(b) (iii) $\mathbf{u} = \left(\frac{d\psi}{dy}, -\frac{d\psi}{dx}\right) = (2Cy, -2Cx)$, $\quad \nabla \wedge \mathbf{u} \neq 0$ so not irrotational
(iv) $\mathbf{u} = \left(\frac{d\psi}{dy}, -\frac{d\psi}{dx}\right) = (-2Cy, -2Cx)$, $\quad \nabla \wedge \mathbf{u} = 0$, irrotational, $\phi = -2Cxy$

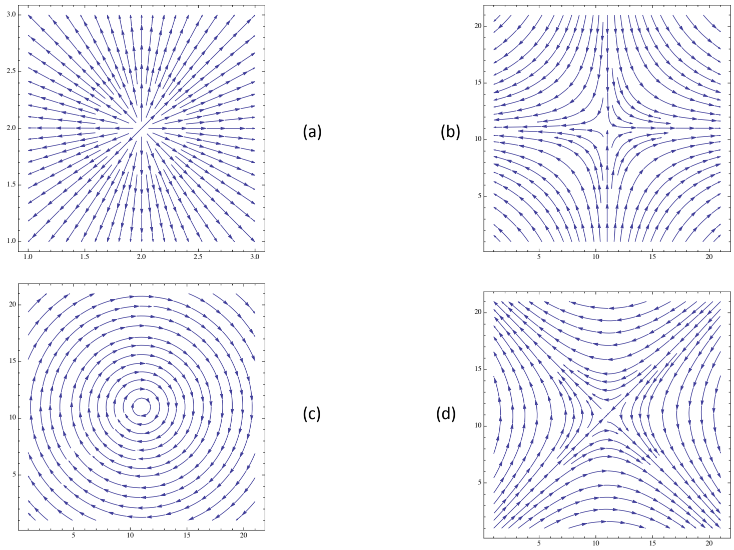


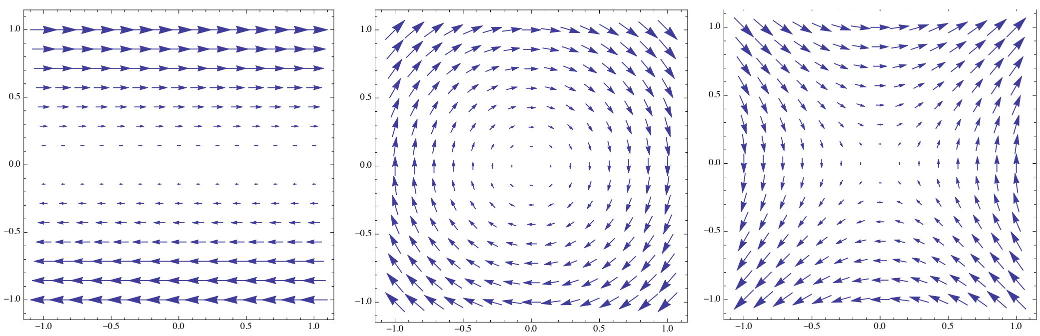

Figure 1: Lines of constant $\phi$ are perpendicular to the streamlines for (a), (b), (d).

3. **Velocity gradient tensor**

   Show that a simple shear flow $\mathbf{u} = (\alpha y, 0, 0)$ can be decomposed into a sum of a rotation and a straining flow (i) pictorially (ii) in terms of the velocity gradient tensor.

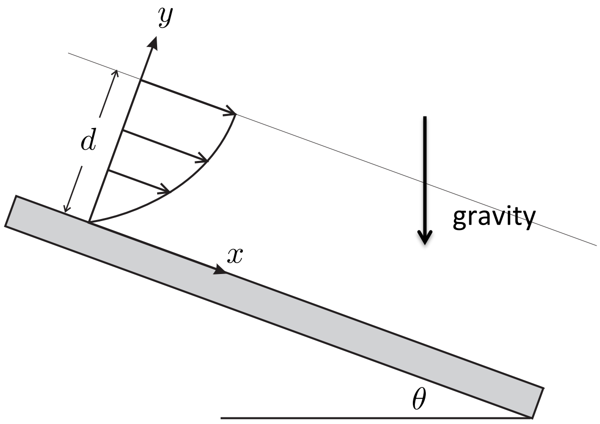
   **Solution**

| | shear | = | rotation | + | straining |
|---|---|---|---|---|---|
| $\mathbf{u} =$ | $(\alpha y, 0, 0)$ | | $(\frac{\alpha y}{2}, -\frac{\alpha x}{2}, 0)$ | | $(\frac{\alpha y}{2}, \frac{\alpha x}{2}, 0)$ |
| velocity gradient tensor | $\begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$ | | $\begin{pmatrix} 0 & \alpha/2 \\ -\alpha/2 & 0 \end{pmatrix}$ | | $\begin{pmatrix} 0 & \alpha/2 \\ \alpha/2 & 0 \end{pmatrix}$ |



4. **Solving Navier-Stokes: flow down an inclined plane**

Consider a steady, two-dimensional, incompressible, viscous flow down an inclined plane under the influence of gravity. Define the axes as shown in the diagram, and assume that the velocity $\mathbf{u}$ depends only on $y$.




(a) What are the boundary conditions for $\mathbf{u}$ at $y = 0$? Using incompressibility show that the $y$-component of the velocity is zero throughout the flow.
(b) Write down the $x$- and $y$-components of the Navier-Stokes equation.
(c) From the $y$-component show that the pressure

$$p = p_0 + \rho g(d - y)\cos\theta$$

where $p_0$ is the pressure at the free surface $y = d$.
(d) From the $x$-component, using the appropriate boundary conditions at $y = 0$ and the zero tangential stress condition $\nu\, du_x/dy = 0$ at the free surface $y = d$ show that

$$u_x = \frac{g}{2\nu} y(2d - y)\sin\theta.$$

(e) Show that the volume flux per unit distance along $z$ is $gd^3 \sin\theta/(3\nu)$.

**Solution**

(a) At $y = 0$ the boundary condition is no-slip, $\mathbf{u} = \mathbf{0}$. By symmetry only $y$-derivatives are non-zero. Therefore, from the continuity equation, $\partial u_y/\partial y = 0$ so $u_y$ =constant$= 0$ to match the boundary condition.

(b) The Navier Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u} + \mathbf{g}.$$

The flow is time-independent. So the $x$ and $y$ components of the Navier-Sokes equation are

$$0 = \nu\frac{d^2 u_x}{dy^2} + g\sin\theta, \qquad (1)$$

$$0 = -\frac{dp}{dy} - \rho g\cos\theta. \qquad (2)$$

(c) Integrate equation (2) with $p = p_0$ at $y = d$.

(d) Integrate equation (1) with $u_x = 0$ at $y = 0$ and $\nu\, du_x/dy = 0$ at $y = d$.

(e) The volume flux per unit distance along $z$ is

$$\int_0^d u_x \, dy.$$

5. **Reynolds number**

Estimate the magnitude of the Reynolds number for:

(a) flow past the wing of a jumbo jet,
(b) a human swimmer,
(c) a thick layer of treacle draining off a spoon,
(d) a bacterium swimming in water.

Take the kinematic viscosity $\nu$ to be $10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$ for water, $1.5 \times 10^{-5}\mathrm{m}^2\mathrm{s}^{-1}$ for air and $10^{-1}\mathrm{m}^2\mathrm{s}^{-1}$ for treacle.

**Solution**

Re=$UL/\nu$

Order of magnitude estimates:

$$\begin{array}{rl}
\text{plane} & \mathrm{Re} \sim \dfrac{150 \times 10}{1.5 \times 10^{-5}} \sim 10^8 \\
\text{human swimming} & \mathrm{Re} \sim \dfrac{1 \times 1}{10^{-6}} \sim 10^6 \\
\text{treacle} & \mathrm{Re} \sim \dfrac{10^{-3} \times 10^{-2}}{10^{-1}} \sim 10^{-4} \\
\text{bacterium swimming} & \mathrm{Re} \sim \dfrac{10^{-5} \times 10^{-5}}{10^{-6}} \sim 10^{-4}
\end{array}$$

6. **Dynamical similarity and dimensionless variables**

Determine the conditions for the dynamical similarity of steady incompressible flow of an electrically conducting fluid in a magnetic field, governed by the equations

$$\nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

$$\mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho\mu}(\nabla \wedge \mathbf{B}) \wedge \mathbf{B} + \nu \nabla^2 \mathbf{u}, \tag{5}$$

$$\mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \frac{1}{\sigma\mu}\nabla^2 \mathbf{B}. \tag{6}$$

You will need to define a length scale $L$, a velocity scale $U$ and a magnetic field scale $B_0$. Notation: $\mathbf{u}$=velocity, $\mathbf{B}$=magnetic field, $p$=pressure, $\rho$=density, $\nu$=kinematic viscosity, $\mu$=magnetic permeability, $\sigma$=electrical conductivity.

Comment on the physical meaning of the dimensionless control parameters.

**Solution**

Define dimensionless variables

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \quad \tilde{\mathbf{B}} = \frac{\mathbf{B}}{B_0}, \quad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}.$$

Substitute into equations (3)–(6) to give

$$\begin{aligned}
\nabla \cdot \tilde{\mathbf{u}} &= 0, \\
\nabla \cdot \tilde{\mathbf{B}} &= 0, \\
\tilde{\mathbf{u}} \cdot \tilde{\nabla}\tilde{\mathbf{u}} &= -\frac{1}{U^2\rho}\tilde{\nabla}p + \frac{B_0^2}{\rho\mu U^2}(\tilde{\nabla} \wedge \tilde{\mathbf{B}}) \wedge \tilde{\mathbf{B}} + \frac{\nu}{LU}\tilde{\nabla}^2\tilde{\mathbf{u}}, \\
\tilde{\mathbf{u}} \cdot \tilde{\nabla}\tilde{\mathbf{B}} &= \tilde{\mathbf{B}} \cdot \tilde{\nabla}\tilde{\mathbf{u}} + \frac{1}{\sigma\mu LU}\tilde{\nabla}^2\tilde{\mathbf{B}}.
\end{aligned}$$

Dimensionless ratios are:
$UL/\nu$ Reynolds number; ratio of inertial to viscous terms in N-S;
$B_0^2/\mu\rho U^2$ ratio of magnetic energy per unit volume to kinetic energy per unit volume; measures the importance of magnetic effects in controlling the flow;
$\sigma\mu UL$ magnetic Reynolds number.

7. **More dynamical similarity**

(a) A flat plate of width $L$ is placed at a right angle to the flow in a wind tunnel, in which the upstream wind speed is $U$.
Show that the expected scaling of the pressure variations is

$$\text{(i) } \Delta p \sim \frac{\rho\nu U}{L} \text{ in the limit } Re \ll 1,$$

$$\text{(ii) } \Delta p \sim \rho U^2 \text{ in the limit } Re \gg 1.$$

(b) In the wind tunnel vortices are shed behind the plate at a frequency of 0.5 $\mathrm{s}^{-1}$. The same plate is now placed into a water channel. Calculate the flow rate required, as a multiple of that in the wind tunnel, to produce dynamically similar behaviour, and calculate the frequency of the vortex shedding.

**Solution**

(a) The Navier Stokes equation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u}.$$

For $Re \ll 1$ the pressure term is balanced by the viscous term so

$$\frac{\Delta p}{\rho L} \sim \frac{\nu U}{L^2} \quad \rightarrow \quad \Delta p \sim \frac{\rho\nu U}{L}.$$

For $Re \gg 1$ the pressure term is balanced by the inertia so

$$\frac{\Delta p}{\rho L} \sim \frac{U^2}{L} \quad \rightarrow \quad \Delta p \sim \rho U^2.$$

(b) The Reynolds number is $UL/\nu$. $L$ is unchanged so for similar flows we require

$$\frac{U_{water}}{U_{wind}} = \frac{\nu_{water}}{\nu_{wind}} = 0.067.$$

frequency $\sim U/L$ so

$$\text{freq}_{water} = \frac{U_{water}}{U_{wind}} \text{freq}_{air} = 0.034.$$

8. **Vorticity**

(a) What is meant by the vorticity of a fluid flow? Illustrate your answer by discussing:
(i) a rectilinear flow that has vorticity eg simple shear (again) $\mathbf{u} = (\alpha y, 0, 0)$.
(ii) a rotating flow that does not have vorticity eg $u_r = 0, u_\theta = A/r$ (in plane polar co-ordinates).
(b) What is the vorticity of a flow in rigid body motion with angular velocity $\mathbf{\Omega}$?

**Solution**
The vorticity, $\omega = \nabla \wedge \mathbf{v}$.
(a)(i) $\omega = -\alpha \hat{\mathbf{z}}$
(ii) $\omega = 0$ $(r \neq 0)$

(b) $\mathbf{u} = \mathbf{\Omega} \wedge \mathbf{r} \Rightarrow \omega = \nabla \wedge (\mathbf{\Omega} \wedge \mathbf{r}) = (\mathbf{r} \cdot \nabla)\mathbf{\Omega} - (\mathbf{\Omega} \cdot \nabla)\mathbf{r} + \mathbf{\Omega}(\nabla \cdot \mathbf{r}) - \mathbf{r}(\nabla \cdot \mathbf{\Omega}) = 2\mathbf{\Omega}$

using $\mathbf{\Omega}$ is constant, $(\nabla \cdot \mathbf{r}) = 3$ and $(\mathbf{\Omega} \cdot \nabla)\mathbf{r} = \mathbf{\Omega}$.