A13480W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B1: FLOWS, FLUCTUATIONS AND COMPLEXITY

TRINITY TERM 2015

Tuesday, 16 June, 2.30 pm – 4.30 pm

10 minutes reading time

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The Navier-Stokes equation for viscous, incompressible, fluid flow under gravity is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = \nu \nabla^2 \mathbf{u} \ ,$$

where $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, g the acceleration due to gravity, $\mathbf{k}$ the vertical unit vector, ν the kinematic viscosity, and t is time.

The *Jacobian* $\mathbf{J}$ of a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where f_i and x_j are the i^{th} and j^{th} components of $\mathbf{f}(\mathbf{x})$ and $\mathbf{x}$ respectively. The trace τ of the Jacobian is equal to the sum of its eigenvalues, while the determinant Δ is equal to their product.

1. (a) Show through a Taylor expansion that the velocity field for any 2-dimensional flow can be described sufficiently close to the origin by

$$\begin{aligned} u &= u_0 + \frac{D}{2}x - \frac{\omega}{2}y + \frac{S_1}{2}x + \frac{S_2}{2}y + \cdots \ , \\ v &= v_0 + \frac{D}{2}y + \frac{\omega}{2}x - \frac{S_1}{2}y + \frac{S_2}{2}x + \cdots \ , \end{aligned}$$

where

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \ , \ \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ , \ S_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \ , \ S_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \ ,$$

and u_o and v_0 are constants. With the aid of suitable sketches, give brief physical interpretations of (u_0, v_0), D, ω, S_1 and S_2. [11]

(b) Using your answer to the previous question, sketch and physically interpret how a dyed fluid parcel evolves in a 2-dimensional, incompressible turbulent flow. Explain why molecular diffusion of the dye cannot be neglected in such flow. [6]

(c) Starting from the Navier-Stokes equation for 2-dimensional incompressible flow and ignoring gravity, show that

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\omega = \nu\nabla^2\omega \ ,$$

and write down the approximate form of this equation that holds in the high Reynolds number limit. [5]

(d) A patch of turbulence is introduced into a fluid in which the flow is constrained to be 2-dimensional. Without giving mathematical details, explain why the enstrophy

$$\frac{\partial}{\partial t}\iint \frac{\omega^2}{2}\,\mathrm{d}x\,\mathrm{d}y$$

is always dissipated. [3]

2. (a) The equations of motion for one-dimensional, adiabatic, compressible flow in an ideal gas are:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} \ ,$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \ ,$$

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma \ ,$$

where $\gamma = 7/5$, p_0 and ρ_0 are constants, and the other symbols have their usual meaning. Linearise these equations about a state of rest in the inviscid limit. Hence deduce that the non-Doppler shifted speed of linear sound waves is

$$c = \sqrt{\frac{\partial p}{\partial \rho}} \ .$$

Estimate the speed of sound in the Earth's atmosphere at sea level. [7]

(b) Show that the nonlinear equations for one-dimensional compressible, inviscid flow can be rewritten in the form

$$\begin{aligned} \left[\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right](u+P) &= 0 \ , \\ \left[\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right](u-P) &= 0 \ , \end{aligned}$$

where P is defined through $\mathrm{d}P = \mathrm{d}p/\rho c$. [8]

(c) Write down an implicit solution for u for a nonlinear sound wave propagating in the positive x direction relative to the mean flow.

Sketch the evolution of a wave anomaly as it propagates through the fluid and explain why the solution eventually forms a wave shock.

By considering the magnitude of the neglected viscous acceleration, or otherwise, show that the width of the shock wave is roughly

$$\delta \sim \frac{\nu}{c} \ .$$

Estimate δ for a sound wave in the Earth's atmosphere at sea level where the viscosity of air is 1.5×10^{-5} m^2 s^{-1}. [10]

3. (a) A dynamical system is defined by

$$\begin{aligned}\dot{x} &= yz \ , \\ \dot{y} &= x - y \ , \\ \dot{z} &= 1 - xy \ .\end{aligned}$$

Derive the fixed points and show that they are unstable. You may use the result that the cubic equation $r^3 + Ar^2 + Br + C = 0$ (for the variable r) where $A, B, C > 0$ has roots with positive real parts if and only if $C > AB$. [5]

(b) Derive from first principles the continuity equation for a volume element, δV, in the phase space of this system, and show that

$$\delta V = \delta V_0 \, \mathrm{e}^{-(t-t_0)} \ ,$$

where δV_0 is the volume at time t_0. Explain how this result indicates the existence of either an attracting limiting cycle or a strange attractor. [7]

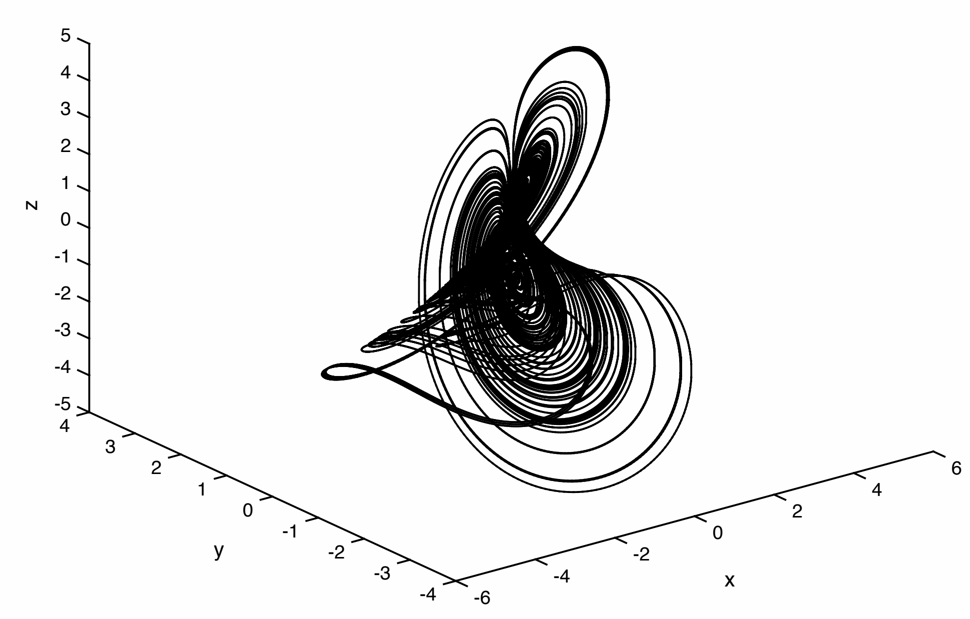
(c) Show that the length of a small displacement element, $\delta \mathbf{x}$, aligned with one of the eigenvectors of $(\mathbf{J} + \mathbf{J}^T)/2$, satisfies

$$||\delta \mathbf{x}|| = ||\delta \mathbf{x}||_0 \, \mathrm{e}^{\lambda(t-t_0)}$$

for sufficently short times, where λ is the corresponding eigenvalue of $(\mathbf{J} + \mathbf{J}^T)/2$ and $||\delta \mathbf{x}||_0$ is a constant vector.

Calculate the eigenvalues of $(\mathbf{J} + \mathbf{J}^T)/2$ at the origin. [8]

(d) The attractor in this system is plotted below and has a fractal dimension of approximately 2.174.



Discuss the geometry of the attractor and the extent to which it can be understood using the results obtained above. [5]

4. A freely jointed chain (FJC) in three dimensions is described by a random walk with N segments of Kuhn length b. If $N \gg 1$, the probability distribution function for the end to end vector $\mathbf{r} = (x, y, z)$ can be written as $P(\mathbf{r}) = P_x(x)P_y(y)P_z(z)$, where

$$P_x(x) = \left(\frac{3}{2\pi N b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{3x^2}{2Nb^2}\right) \ ,$$

and $P_y(y)$ and $P_z(z)$ have the same functional form of their respective arguments. Find an expression for the most probable distance $r = |\mathbf{r}|$ of the two chain ends. [3]

A FJC is tethered at one end to a surface at $z = 0$ and confined to $z > 0$. The probability distribution function can be written as $P(\mathbf{r}) = P_x(x)P_y(y)P_z(z)$, where $P_x(x)$ and $P_y(y)$ have the same functional form as for an untethered FJC, but

$$P_z(z) = \frac{z}{\Omega} \exp\left(-\frac{3z^2}{2Nb^2}\right) \quad \text{for } z > 0 \text{ and } P_z(z) = 0 \text{ for } z \leq 0 \ ,$$

where Ω is a normalisation factor that is independent of $\mathbf{r}$. Find an expression for the most probable distance r between the chain ends. Comment on your result in light of the untethered FJC model. The configurational entropy when the free end is confined to a specific z can be defined as $S(z) = k_B \ln[P(z)]$, where k_B is Boltzmann's constant. Find an expression for $S(z)$, sketch this as a function of z, and comment on your results. [8]

The probability that the ends of an untethered three-dimensional polymer are within a distance δ of each other is given by

$$\Pi(\delta) = \int_0^{\delta} 4\pi r^2 P(\mathbf{r}) \mathrm{d}r \ .$$

Evaluate this expression for a FJC in the case $\delta \ll b$. Explain why the loss of configurational entropy that would result if the ends were joined to form a loop can be written as $\Delta S = k_B \ln[\,\Pi(\delta)]$, and show that

$$\Delta S = B - \frac{3}{2} k_B \ln N \ ,$$

where B is a factor that does not depend on N. Give an expression for B. [7]

The elastic energy required to form a circular loop of radius R from double-stranded DNA with Kuhn length b at temperature T is

$$E_{\text{loop}} = \pi k_B T \, \frac{b}{2R} \ .$$

By using a FJC model to calculate the entropy of DNA, find an expression for the free energy change associated with the formation of such a loop. Sketch this as a function of the number of bases in the loop, and find the circumference in base-pairs of the DNA loop with minimum free energy. The Lac repressor binds two sites (one near the promotor) to form a DNA loop. Two possible pairs of sites are separated by 92 and 401 base pairs. Which pair is more likely to be linked by the repressor? [7]

[The Kuhn length of double stranded DNA is 100 nm and the separation between base-pairs is 0.34 nm.]