On the Approximation of Optimal Portfolio Generating Functions in Stochastic Portfolio Theory

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The Basic Idea

We have stock data for times $\{0, ..., T'\}$ and a specific set of trading strategies we consider following. We choose one which maximizes wealth over time horizon $T < T'$.

Using the data for times $T+1, ..., T'$, we will test the performance of our chosen strategy relative to a benchmark strategy.

Overview

Functionally Generated Portfolios in Discrete Time

Introduction, Notation and Definitions

- We operate in a (frictionless) market of n stocks with market capitalizations $X_i(t)$ in discrete time, $i \in \{1, ..., n\}, t \in \mathbb{N}$. Assume $X_i(t) > 0$ a.s. for every t.
- A *portfolio* is any $\mathbb{R}^n$ valued process $\pi(t)$ such that

$$\mathbf{1}^T \pi(t) = 1 \quad \forall t \in \mathbb{N}$$

- Define

$$\mu_i(t) \equiv \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}$$

  The *market portfolio* is that which has ith element $\mu_i(t)$.
- We follow the *trading strategy* prescribed by a portfolio π if we have wealth $Z(t)$ at time t and hold a dollar value position of $\pi_i(t)Z(t)$ in the ith stock.
- $Z_\pi(t) \equiv$ wealth associated with portfolio π. We normalize to have $Z_\pi(0) \equiv 1$.

Introduction, Notation and Definitions

Let

$$V_\pi(t) \equiv \frac{Z_\pi(t)}{Z_\mu(t)}$$

This quantity is well defined since

$$Z_\pi(t+1) = Z_\pi(t) \sum_{i=1}^{n} \pi_i(t) \frac{X_i(t+1)}{X_i(t)}$$

$$\implies Z_\mu(t) = \frac{1}{\sum_j X_j(0)} \sum_i X_i(t) > 0 \quad \text{inductively.}$$

$$\implies V_\pi(t+1) = \frac{Z_\pi(t)}{Z_\mu(t)} \sum_i \pi_i(t) \frac{X_i(t+1)/Z_\mu(t+1)}{X_i(t)/Z_\mu(t)}$$

$$\implies V_\pi(t+1) = V_\pi(t) \sum_i \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)}$$

Introduction, Notation and Definitions

Inductively,

$$V_\pi(T) = \prod_{t=0}^{T-1} \left(\sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} \right) \tag{1.1}$$

Our goal is conceptually very simple: choose an "optimal" portfolio for long time horizon investments. Obviously, this is impossible, since we do not know the future. We will assume to be working in the following context:

Project Goal

We have stock data for times $\{0, ..., T'\}$. We will maximize $V_\pi(T)$ over some reasonable set of portfolios, where $T < T'$. Using the data for times $T+1, ..., T'$, we will test the performance of the maximizing portfolio relative to the market portfolio.

Introduction, Notation and Definitions

We define *functionally generated portfolios* as those of the form

$$\pi_i(t) = \mu_i(t)\left(D_i \log f(\mu(t)) + 1 - \sum_j \mu_j(t) D_j \log f(\mu(t))\right) \tag{1.2}$$

for $f \in C^k(\Delta^n \to (0,\infty))$, $k = 1$, or $k = 0$ and f concave/convex, where $\Delta^n \equiv \{x \in \mathbb{R}^n : \sum_i x_i = 1, x_i \geq 0\}$. More general definitions are reasonable of course; see [5], [6] for example.

Introduction, Notation and Definitions

We are also interested in the case where f is a function of the ranked market weights

$$\mu_{(1)}(t) \geq \ldots \geq \mu_{(n)}(t)$$

We let $\sigma_t : \{1, \ldots, n\} \to \{1, \ldots, n\}$ denote a permutation of $\{1, \ldots, n\}$ at time t which satisfies $\mu_{\sigma_t(k)}(t) = \mu_{(k)}(t)$ and define for any vector $v \in \mathbb{R}^n$ and permutation σ, $\sigma(v) \equiv (v_{\sigma(1)}, \ldots, v_{\sigma(n)})^T$. We further define

$$p(t) \equiv \sigma_t(\mu(t)) = (\mu_{(1)}(t), \ldots, \mu_{(n)}(t))^T,$$
$$q(t) \equiv \sigma_t(\mu(t+1))$$

We define *rank-based functionally generated portfolios* as those of the form

$$\pi_{\sigma_t(k))}(t) = p_k(t)\left(D_k \log f(p(t)) + 1 - \sum_{m=1}^{n} p_m(t) D_m \log f(p(t))\right) \tag{1.3}$$

Introduction, Notation and Definitions

We will refer to f in equations (1.2) and (1.3) as *(rank-based) portfolio generating functions* and say they *generate* the corresponding portfolio π. If f is concave and generates π, then for $g \equiv \log f$, we have

$$\sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} = 1 + \nabla g(\mu(t)) \cdot (\mu(t+1) - \mu(t))$$
$$\geq f(\mu(t+1))/f(\mu(t)) > 0$$

so $\log V_\pi(T)$ is well defined. The identical result holds in the ranked case with $\mu(t)$ replaced by p and $\mu(t+1)$ replaced by q.

We will optimize over the log generating function in all relevant scenarios, so from here on if e^g generates π we will write

$$V(g, T) \equiv V_\pi(T) \tag{1.4}$$

Introduction, Notation and Definitions

Some last definitions before we get started:

$$R(g, p, q) \equiv 1 + \nabla g(p) \cdot (q - p) \tag{1.5}$$

$$L(g, p, q) \equiv \log R(g, p, q) - (g(q) - g(p)) \tag{1.6}$$

We will abuse notation in future to write

$$R(g, t) = R(g, \mu(t), \mu(t+1)) \quad \text{or} \quad R(g, t) = R(g, p(t), q(t))$$

when the context is clear.

Motivation for Our "Reasonable Portfolios"

For *exponentially concave* g (i.e. e^g is concave), we have

$$\log V(g,T) = \sum_{t=0}^{T-1} \log R(g,\mu(t),\mu(t+1)) \qquad \text{(resp. } R(g,p(t),q(t)))$$

$$= \underbrace{g(\mu(T)) - g(\mu(0))}_{\text{bounded}} + \sum_{t=0}^{T-1} \underbrace{L(g,\mu(t),\mu(t+1))}_{\geq 0}$$

which is analogous to the continuous time decomposition of [2].

This decomposition is the motivation for our choice of constraints for the optimization problems in subsequent sections.

Optimizing Over a Subset of Concave Generating Functions

General Optimization Problem of Interest

$$\sup_{g \in \mathcal{G}} \frac{1}{T} \log V(g, T) \tag{2.1}$$

where $\mathcal{G} \subset C^1(\Delta^n)$, with the convention $\log(x) \equiv -\infty \quad \forall x \leq 0$.

Restricting $\mathcal{G}$

Our analysis will take g to be of the form

$$g(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \ell_j(x_i) \tag{2.2}$$

where the sets $\{C_j\}_{j=1}^J$ form a partition of $\{1, ..., n\}$, $(\lambda_j |C_j|)_{j=1}^J \in \Delta^J$ and $\ell_j \in C^1([0,1])$. The ℓ_j will be in one of the following two sets:

$$\mathcal{E}^1_\beta \equiv \left\{ \ell \in \mathcal{E}^1([0,1]) : \ell' \text{ is } \beta\text{-Lipschitz continuous}, \ell(1/2) = 0 \right\}$$

$$\mathcal{E}^2_{\alpha,\beta} \equiv \left\{ \ell \in \mathcal{E}^2([0,1]) : \ell'' \text{ is } \alpha\text{-Lipschitz continuous }, 0 \leq \ell' \leq \beta, \ell(0) = 0 \right\}$$

where $\mathcal{E}^k(S) = C^k(S) \cap \{\text{exponentially concave functions}\}$.

On the Spaces $\mathcal{E}^1_\beta$ and $\mathcal{E}^2_{\alpha,\beta}$

Define

$$d(f,g) \equiv \sup_{x\in[0,1]} |f'(x) - g'(x)|$$

for $f, g \in C^1([0,1])$.

Lemma 2.1

$(\mathcal{E}^1_\beta, d)$ and $(\mathcal{E}^2_{\alpha,\beta}, d)$ are compact metric spaces.

Proof:
(Sketch) One may show uniform boundedness of ℓ', ℓ'' in each case, respectively. Equicontinuity yields sequential compactness through an argument applying the Arzelà-Ascoli theorem. $\square$

Restricting $\mathcal{G}$

Theorem 2.2

Let $\mathcal{G}_1 \equiv \{g$ of the form (2.2) such that $\ell_j \in \mathcal{E}^1_\beta\}$ and $\mathcal{G}_2 \equiv \{g$ of the form (2.2) such that $\ell_j \in \mathcal{E}^2_{\alpha,\beta}\}$. Optimization problem (2.1) has a solution when $\mathcal{G} = \mathcal{G}_1$ or $\mathcal{G} = \mathcal{G}_2$.

Proof:
Note that the function $(\mathcal{E}^1_\beta)^{\otimes J} \ni \ell \mapsto \log V(g, T)$ is continuous, and likewise for $\mathcal{E}^2_{\alpha,\beta}$. □

Remark: The functions referenced in the proof are actually Lipschitz continuous.

Definition

We define

$$S_i \equiv \sup_{g \in \mathcal{G}_i} \frac{1}{T} \log V(g, T)$$

for $i \in \{1, 2\}$.

Approximating the Optimization Problems of Section 2

Approximating the Optimization Problems of Section 2

We discuss two frameworks for approximating the functions ℓ_j and their derivatives. For a partition $\mathcal{P} : 0 = x_1 < ... < x_d = 1$ of $[0, 1]$:

(i) Approximate $\ell \in \mathcal{E}^1_\beta$ using linear interpolations

$$\hat{\ell}(x) = \ell_i + \frac{\ell_{i+1} - \ell_i}{x_{i+1} - x_i}(x - x_i)$$

for $x \in [x_i, x_{i+1}]$ and maximize over the ℓ_i.

(ii) Approximate ℓ' for $\ell \in \mathcal{E}^2_{\alpha,\beta}$ by

$$\hat{\phi}(x) = \phi_i + \frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i}(x - x_i)$$

for $x \in [x_i, x_{i+1}]$ and maximize over the ϕ_i.

Approximating the Optimization Problems of Section 2

From here on, we denote the linear interpolation of $\boldsymbol{y} \in \mathbb{R}^d$ across the partition $\mathcal{P}$ by $\hat{y}(x)$, and define

$$\delta \equiv \max_{i \in [d-1]} x_{i+1} - x_i, \qquad \underline{\delta} \equiv \min_{i \in [d-1]} x_{i+1} - x_i$$

Remark: (i) is an altered version of the method of [1], (ii) is based on our own analysis.

(i) Discretizing the ℓ_j

Let $\hat{\mathcal{E}}^1_\beta$ denote the set of vectors $\boldsymbol{\ell} = (\ell_1, ..., \ell_d)^T \in \mathbb{R}^d$ which satisfy

(3.1) $$-\ell_i + \log\left(w_i e^{\ell_{i+1}} + (1-w_i)e^{\ell_{i-1}}\right) \leq 0 \quad , \quad w_i \equiv \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad , \quad \forall i \in \{2, ..., d-1\}$$

(3.2) $$\frac{\Delta_{i+1}\boldsymbol{\ell} - \Delta_i\boldsymbol{\ell}}{x_{i+2} - x_i} \geq -\beta/2 \quad , \quad \Delta_i\boldsymbol{\ell} \equiv \frac{\ell_{i+1} - \ell_i}{x_{i+1} - x_i} \quad , \quad \forall i \in \{1, ..., d-2\}$$

(3.3) $$|\Delta_1\boldsymbol{\ell}|, |\Delta_{d-1}\boldsymbol{\ell}| \leq \sqrt{\beta}$$

(3.4) $\ell_i = 0$ for the index i such that $x_i = 1/2$

(i) Discretizing the ℓ_j

We will use vectors $\{\ell_j\}_{j=1}^{J} \subseteq (\hat{\mathcal{E}}_{\beta}^{1})^{\otimes J}$ to construct functions of the form

$$\hat{g}(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \hat{\ell}_j(x_i) \tag{3.5}$$

to approximate $g \in \mathcal{G}_1$ and investigate their properties.

(3.1) approximates exponential concavity, (3.2) approximates the β Lipschitz requirement, (3.3) and (3.4) are analogous to conditions which holds for functions in $\mathcal{E}_{\beta}^{1}$.

(i) Discretizing the ℓ_j

We consider the following optimization problem to approximate S_1.

$$\hat{S}_1(\delta) \equiv \sup_{\{\ell_j\}_{j=1}^J \in (\hat{\mathcal{E}}_\beta^1)^{\otimes J}} \frac{1}{T} \log V\left(\sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \hat{\ell}_j(x_i), T\right) \tag{3.6}$$

This always has a solution due to the compactness of $\hat{\mathcal{E}}_\beta^1$. It is also a convex problem.

(i) Discretizing the ℓ_j

The following is due to [1].

Theorem 3.1

Assume that $\delta - \underline{\delta} \leq M\underline{\delta}^2$ for some $M \geq 0$. For every $\alpha \in (0,1), \beta \geq 0$, $\exists \delta^* > 0$ such that $\forall \delta < \delta^*$,

$$-4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\beta\delta \leq \hat{S}_1(\delta) - S_1 \leq 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta)$$

where

$$\epsilon_1(\delta) = \left(1 + \exp\left(\sqrt{\beta}\right)\delta^\alpha\right)\left(\frac{7}{2}e^{2\sqrt{\beta\delta}} + 1\right)\beta\delta + \exp\left(\sqrt{\beta}\right)\sqrt{\beta}\delta^\alpha$$

Large β in numerical examples would require computationally impossible choice of δ to guarantee convergence.

(i) Discretizing the ℓ_j

Can we do better? Yes, under the following assumptions on the stocks we trade in:

Market Assumption 1 (MA1)

$\exists\theta \in (0, 1]$ such that

$$q_k(t) \geq \theta p_k(t) \quad \forall k \in \{1, \ldots, n\}, t \in \{1, \ldots, T-1\}$$

Market Assumption 2 (MA2)

(Market Diversity) $\exists\gamma > 0$ such that

$$p_1(t) \leq 1 - \gamma \quad \forall t \in \{1, \ldots, T-1\}$$

Loosely speaking, (MA1) states that the stock with rank k at time t did not lose too much of its value relative to other stocks in the market over a one step time horizon. It is reasonable for large cap stocks. (MA2) is standard.

(i) Discretizing the ℓ_j

Theorem 3.2

Suppose (MA1) and (MA2) hold and that $n > \frac{(1-\theta)(1-\gamma)}{\theta\gamma}$. *If* $\boldsymbol{\lambda} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^J$, *then*

$$S_1 - C\delta \leq \hat{S}_1$$

where

$$C \equiv 4\left(1 - \frac{1}{T}\right)\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}\left(\frac{\beta(1-\theta)}{n}\right)$$

whenever

$$\frac{2(1-\theta)\beta\delta}{n} \leq \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)$$

We will only prove the rank-based case, as the proof for the non rank-based case is identical apart from notation.

(i) Discretizing the ℓ_j

Before we prove this, we need the following:

Lemma 3.3

For exponentially concave $\ell : [0,1] \to \mathbb{R}$ with right derivative ℓ', we have

$$-\frac{1}{1-x} \leq \ell'(x) \leq \frac{1}{x}$$

for every $x \in [0,1]$.

Proof:
By exponential concavity, ℓ' exists and satisfies

$$1 + \ell'(x)(y-x) \geq \exp\left(\ell(y) - \ell(x)\right) > 0 \qquad \forall x, y \in (0,1)$$

Letting $y \uparrow 1$ and $y \downarrow 0$ yields the result. □

Proof of Theorem 3.2

Proof of Theorem 3.2:
Let $\{\boldsymbol{\ell}_j^*\}_{j=1}^J \in (\hat{\mathcal{E}}_\beta^1)^{\otimes J}$ be a maximizer for problem 3.4 and $\{\ell_j^*\}_{j=1}^J \in (\mathcal{E}_\beta^1)^{\otimes J}$ be such that

$$g^*(y) \equiv \frac{1}{n} \sum_{j=1}^{J} \sum_{m \in C_j} \ell_j^*(y_m)$$

is a maximizer in problem 2.1 with $\mathcal{G} = \mathcal{G}_1$.
Define $\boldsymbol{\ell}_j \equiv \left(\ell_j^*(x_1), \dots, \ell_j^*(x_d)\right)^T \in \hat{\mathcal{E}}_\beta^1$ and let

$$\hat{g}(y) \equiv \frac{1}{n} \sum_{j=1}^{J} \sum_{m \in C_j} \hat{\ell}_j(y_m)$$

$$\hat{g}^*(y) \equiv \frac{1}{n} \sum_{j=1}^{J} \sum_{m \in C_j} \hat{\ell}_j^*(y_m)$$

Proof of Theorem 3.2 (continued)

By lemma 3.3 and market diversity,

$$-\gamma^{-1} \leq -\frac{1}{1-p_m(t)} \leq \ell_j^{*\prime}(p_m(t)) \leq \frac{1}{p_m(t)} \qquad \forall m \in \{1, \ldots, n\}, t \in \{1, \ldots T-1\}$$

Thus, dropping the t arguments and abbreviating summations, we have

$$\begin{aligned} R(g^*, t) &= 1 + \frac{1}{n} \sum_{\substack{j=1 \\ m \in C_j}}^{J} \ell_j^{*\prime}(p_m)(q_m - p_m) \\ &\geq 1 - \frac{1}{n} \sum_{q_m \geq p_m} \gamma^{-1}(q_m - p_m) + \frac{1}{n} \sum_{q_m < p_m} \left(\frac{q_m}{p_m} - 1 \right) \\ &= 1 + \frac{1}{n} \sum_{q_m < p_m} \left(\frac{q_m}{p_m} - 1 + \gamma^{-1}(q_m - p_m) \right) \end{aligned}$$

Proof of Theorem 3.2 (continued)

$$
\begin{aligned}
&\geq 1 + \frac{1}{n} \sum_{q_m < p_m} \left(\theta - 1 + \gamma^{-1}(\theta - 1) p_m\right) \\
&= 1 - \frac{1-\theta}{n} \sum_{q_m < p_m} \left(1 + \gamma^{-1} p_m\right) \\
&\geq 1 - \frac{(1-\theta)}{n} \left(n - 1 + \frac{1}{\gamma}\right) \\
&= \theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} > 0 \qquad (3.7)
\end{aligned}
$$

$\forall t \in \{1, ..., T-1\}$, where the first equality holds since $\sum_{m=1}^{n}(q_m - p_m) = 0$, the last inequality holds from the condition on n and we have applied market assumption 1.

Proof of Theorem 3.2 (continued)

For each $i \in \{1, \dots, d-1\}$, $j \in \{1, \dots, J\}$, $\exists c_{ij} \in [x_i, x_{i+1}]$ such that

$$\ell_j^{*\prime}(c_{ij}) = \Delta_i \ell$$

$$\implies \sup_{x \in [0,1]} \left| \ell_j^{*\prime}(x) - \hat{\ell}_j^{\prime}(x) \right| \leq \beta\delta$$

$$\begin{aligned}
\implies R(g^*, t) - R(\hat{g}, t) &\leq \frac{\beta\delta}{n} \left(\sum_{q_m > p_m} (q_m - p_m) + \sum_{p_m > q_m} (p_m - q_m) \right) \\
&= \frac{2\beta\delta}{n} \sum_{p_m > q_m} (p_m - q_m) \\
&\leq \frac{2\beta\delta(1-\theta)}{n} \leq \frac{1}{2} \left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} \right)
\end{aligned}$$

$$\implies R(\hat{g}, t) \geq \frac{1}{2} \left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} \right)$$

Proof of Theorem 3.2 (continued)

By the identical logic for $R(\hat{g}, t) - R(g^*, t)$, we have

$$|R(g^*, t) - R(\hat{g}, t)| \leq \frac{2\beta(1-\theta)\delta}{n}$$

Since log is Lipschitz continuous with constant $2\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}$ on the interval $\left[\frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right), \infty\right)$, we have

$$\begin{aligned} |S_1 - V(\hat{g}, T)| &\leq \frac{1}{T}\sum_{t=1}^{T-1} |\log(R(g^*, t)) - \log(R(\hat{g}, t))| \\ &\leq \frac{T-1}{T} \cdot 2\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1} \cdot \frac{2\beta(1-\theta)\delta}{n} = C\delta \end{aligned}$$

$$\implies S_1 - C\delta \leq V(\hat{g}, T) \leq V(\hat{g}^*, T) = \hat{S}_1$$

since one may verify that $\{\ell_j\}_{j=1}^J \in (\hat{\mathcal{E}}_\beta^1)^{\otimes J}$. $\square$

(ii) Discretizing the ℓ'_j

Let $\hat{\mathcal{E}}^2_{\alpha,\beta}$ be the set of vectors $\phi = (\phi_1, \ldots, \phi_d)^T \in \mathbb{R}^d$ such that

$$\text{(3.8)} \qquad \Delta_i\phi + \phi_{i+1}^2 \leq 0 \quad , \quad \Delta_i\phi \equiv \frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} \quad , \quad \forall i \in \{1, \ldots, d-1\}$$

$$\text{(3.9)} \qquad 0 \leq \phi_d \quad , \quad \phi_1 \leq \beta$$

$$\text{(3.10)} \qquad \left| \frac{\Delta_i\phi - \Delta_{i+1}\phi}{\frac{x_{i+2} - x_i}{2}} \right| \leq \alpha \qquad \forall i \in \{1, \ldots, d-2\}$$

(3.8) approximates exponential concavity, (3.9) is the same as the bound on ℓ' for $\ell \in \mathcal{E}^2_{\alpha,\beta}$, (3.10) approximates α Lipschitz continuity.

(ii) Discretizing the ℓ_j'

Defining $\ell_\phi(x) \equiv \int_0^x \hat{\phi}(u)du$ we consider the problem

$$\hat{S}_2(\delta) \equiv \sup_{\{\phi_j\}_{j=1}^J \in (\hat{\mathcal{E}}^2_{\alpha,\beta})^{\otimes J}} \frac{1}{T} \log V\left(\sum_{j=1}^J \sum_{i \in C_j} \lambda_j \ell_{\phi_j}(x_i), T\right) \tag{3.11}$$

This always has a solution due to the compactness of $\hat{\mathcal{E}}^2_{\alpha,\beta}$. It is also a convex problem.

(ii) Discretizing the ℓ_j'

Theorem 3.4

Suppose (MA1) and (MA2) hold and that $n > \frac{(1-\theta)(1-\gamma)}{\theta\gamma}$. *If* $\boldsymbol{\lambda} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^J$, *then*

$$S_2 - K\delta^2 \leq \hat{S}_2(\delta)$$

where

$$K \equiv 2\left(1 - \frac{1}{T}\right)\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}\left(\frac{\alpha(1-\theta)}{n}\right)$$

whenever

$$\frac{(1-\theta)\alpha\delta^2}{n} \leq \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)$$

Discretizing the ℓ_j'

Proof:
(Sketch). The identical strategy in proving theorem 3.2 applies here, since we only relied on exponential concavity to obtain (3.7). One may show that for $\ell \in \mathcal{E}^2_{\alpha,\beta}$, $\phi \equiv \{\ell'(x_i)\}_{i=1}^d \in \hat{\mathcal{E}}^2_{\alpha,\beta}$ and $d(\ell, \ell_\phi) \leq \frac{\alpha\delta^2}{2}$, so we may obtain the identical result to theorem 3.2 with $\beta\delta$ replaced by $\alpha\delta^2/2$. $\square$

Remark 1: We note that the bound does not depend on β, which may thus be taken arbitrarily large without penalty to the rate of convergence.

Remark 2: We note that the constraints corresponding to non-negative ϕ_i and non-negative ℓ' for $\ell \in \mathcal{E}^2_{\alpha,\beta}$ may be replaced by similar non-positive constraints (with alteration to (3.8) necessary as well) to obtain a similar result with identical bounds.

Empirical Results and Open Market Considerations

Open Market Considerations

- Previous sections' approach fixed sub-market of n stocks to trade in. Here we will consider an open market where we only trade in the top $n = 100$ stocks in the ranked based case.
- Benchmark portfolio is the weighted portfolio for the index consisting of the top n stocks. i.e. portfolio which invests the proportion

$$p_k(t) \equiv \mu_{(k)}(t) \equiv \frac{X_{(k)}(t)}{\sum_{m=1}^{n} X_{(m)}(t)}$$

of its wealth in the kth ranked stock. We abuse notation and denote its wealth by Z_μ.

Open Market Considerations

Let $\tilde{X}_k(t+1)$ be the market capitalization at time $t+1$ of the stock which had rank k at time t. It is easily seen that

$$Z_\mu(T) = \prod_{t=0}^{T-1} \frac{\sum_{k=1}^n \tilde{X}_k(t+1)}{\sum_{m=1}^n X_{(m)}(t)}$$

which implies that, for any rank-based portfolio π, we have

$$V_\pi(T) \equiv \frac{Z_\pi(T)}{Z_\mu(T)} = \prod_{t=0}^{T-1} \left(\sum_{k=1}^n \pi_{\sigma_t(k)}(t) \frac{q_k(t)}{p_k(t)} \right)$$

where

$$q_k(t) \equiv \frac{\tilde{X}_k(t+1)}{\sum_{m=1}^n \tilde{X}_m(t+1)}$$

Open Market Considerations

In particular,

$$V_\pi(T) = \prod_{t=0}^{T-1} \Big(1 + \nabla g(p(t)) \cdot (q(t) - p(t))\Big)$$

for π which is rank-based functionally generated by e^g, in similar fashion to the discussion of section 1.

The results of section 3 still hold in this context, provided market assumptions 1 and 2 still hold for the new definitions of p and q.

Empirical Results

- **Our Dataset:** Market capitalizations for the top $n = 100$ US stocks for each trading day between January 1st, 1957 and June 29th, 2022, along with their capitalizations on the next trading day[1].
- We solve[2] problems (3.6) and (3.11) under the open market setup over the time period January 1st, 1957 to December 31st, 2002, and test the performance of the resulting functions' portfolios relative to the benchmark portfolio over the time period January 1st, 2003 to June 30th, 2022.
- We fix $J = 3$, $\boldsymbol{\lambda} = \mathbf{1}/n$ for this example.

[1] Data collected by the Center for Research in Security Prices (CRSP) and distributed via Columbia University's subscription to the Wharton Research Data Services.

[2] The implementation is performed in MATLAB using the CVX software package; see [3], [4] for details. Full code may be found at `https://github.com/richardgroenewald/SPT-Project`. Much of the optimization code is motivated by [1] - see `https://github.com/stevenacampbell/FunctionalPortfolioOptimization` for details.

Empirical Results

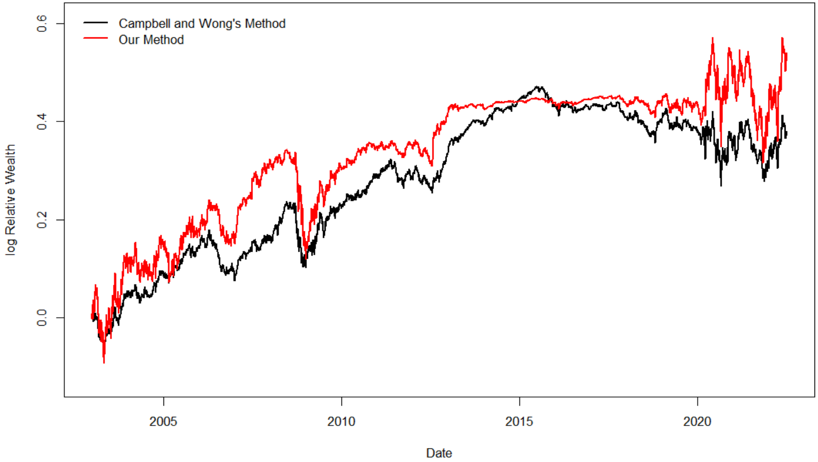

Figure #1: log of wealth relative to the weighted index portfolio of the top 100 US stocks, for the functionally generated portfolios created using our method and that of Campbell and Wong, [1]. The functionally generated portfolios are formed using the solution to problems (3.11) and (3.6), respectively, over the period between January 1st, 1957 and December 31st, 2002.

Potential Future Ideas

Potential Future Ideas

- Discover long run properties of these types of functionally generated portfolios; long run average growth rate, statistical properties/distributions of wealth (relative, or non-relative to the market/weighted index portfolio).
- Performance under transaction costs - partially assessed in [1], but not optimized (eg. optimal rebalancing times, incorporating costs into fitting procedure, etc.)
- Assumptions on market structure (perhaps statistical distributions on market capitalizations/formal model for them) which yield potential strategies to solve the infinite dimensional optimization problems.
- The addition of more structural conditions on the market may also produce better results in terms of the finite dimensional approximations.
- Generalizing inputs to generating functions (see [5], [7] for e.g.) - would require much more data.

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