On the Approximation of Optimal Portfolio Generating Functions in Stochastic Portfolio Theory

Richard Groenewald

Abstract

Please note that this document still needs a formal introduction and conclusion, and is of course a rough draft. In this document, we develop methodology to approximate several portfolio optimization problems in the discrete time setup of stochastic portfolio theory. We study the properties of portfolios which are particular functions of the market weights and propose two infinite dimensional optimization problems whose goal is to maximize investor wealth over a finite time horizon. We approximate each problem with a finite dimensional one by partitioning the unit interval and discretizing the evaluation of either the functions or their first derivatives. We show that the maximum wealth associated with the finite dimensional problems is bounded below by the maximum wealth associated with the infinite dimensional problems minus a term that tends to zero as the mesh size of the partition tends to zero.

1 Functionally Generated Portfolios in Discrete Time

We presume to be operating on a filtered probability space in discrete time $(\Omega, \mathcal{F}, P, \mathbb{F})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is a nested sequence of sigma fields on Ω. We will operate with trades being executed in discrete time (days, in all numerical examples) in the absence of trading costs on a finite time horizon $\{0, ..., T\}$ for some $T \in \mathbb{N}$. We will trade in a total of n stocks, with the market capitalization of the ith stock being an $\mathbb{F}$-adapted process denoted by $X_i(t)$ at each time $t \in \{1, ..., T\}, i \in \{1, ..., n\}$ and assume that

$$P\Big(X_i(t) > 0 \quad \forall t \in \mathbb{N}, i \in \{1, ..., n\}\Big) = 1 \tag{1.1}$$

Notation: We let $\mathbf{1} \in \mathbb{R}^n$ denote the vector of ones, $e_i \in \mathbb{R}^n$ denote the ith standard basis vector - i.e. the vector containing a one as its ith entry and zeroes elsewhere and Δ^n denote the standard unit simplex in $\mathbb{R}^n$:

$$\Delta^n \equiv \Big\{x \in \mathbb{R}^n : \mathbf{1}^T x = 1, e_i^T x \geq 0 \quad \forall i \in \{1, ..., n\}\Big\}$$

For any vector $x \in \mathbb{R}^n$, we denote its ith element by x_i and its order statistics by $x_{(1)} \geq ... \geq x_{(n)}$.

For a set $U \subseteq \mathbb{R}^n$, let $C^k(U)$ denote the set of real valued functions with domain U for which all partial derivatives of order k exist and are continuous on U, with $C^0(U)$ the set of continuous functions on U.

Given a function f, we let $D_{i_1...i_m} f$ denote the mth mixed partial derivative of a function f with respect to its i_1st to i_mth arguments, and ∇f denote the gradient of f, when the appropriate derivatives exist.

Lastly, if f is concave, in all relevant contexts we will give a specific supergradient, and denote it by ∇f with ith element $D_i f$ as well.

Definition 1.1. We list several important terms:

- The ith *market weight* is

$$\mu_i(t) \equiv \frac{X_i(t)}{\sum_{j=1}^n X_j(t)}$$

- A vector of *portfolio weights* is any $\mathbb{F}$-adapted $\mathbb{R}^n$ valued process $\pi(t)$ such that

$$\mathbf{1}^T \pi(t) = 1 \quad \forall t \in \mathbb{N} \text{ a.s.}$$

  We will frequently refer to a vector of portfolio weights simply as a *portfolio*.

- The *market portfolio* $\mu(t) \equiv (\mu_1(t), ...\mu_n(t))^T$ is a specific example (and perhaps the most important one) of a portfolio.

- We will say we follow the trading strategy prescribed by a vector of portfolio weights π if we hold wealth $Z(t)$ at time t and $\pi_i(t)Z(t)$ is the dollar value we hold (or are short) in the ith stock. We will denote by $Z_\pi(t)$ the wealth process associated with portfolio weights π. Without loss of generality, we will presume all wealth processes are normalized to have $Z_\pi(0) = 1$.

- Finally, we define the *wealth relative to the market portfolio* of a portfolio π by

$$V_\pi(t) \equiv \frac{Z_\pi(t)}{Z_\mu(t)}$$

Lemma 1.2. *For any portfolio π, $T \in \mathbb{N}$, we have*

$$V_\pi(T) = \prod_{t=0}^{T-1} \left(\sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} \right)$$

Proof:
Note that

$$Z_\pi(t+1) = Z_\pi(t) \sum_{i=1}^{n} \pi_i(t) \frac{X_i(t+1)}{X_i(t)}$$

by definition of the total return on the ith stock. From the definition of the μ_i and the fact that $Z_\mu(0) = 1$ we obtain by induction

$$Z_\mu(t) = \frac{1}{\sum_{j=1}^{n} X_j(0)} \sum_{i=1}^{n} X_i(t) \neq 0$$

so that V_π is well defined in the first place.

$$\begin{aligned} \implies V_\pi(t+1) = \frac{Z_\pi(t+1)}{Z_\mu(t+1)} &= \frac{Z_\pi(t)}{Z_\mu(t)} \sum_{i=1}^{n} \pi_i(t) \frac{X_i(t+1)/Z_\mu(t+1)}{X_i(t)/Z_\mu(t)} \\ &= V_\pi(t) \sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} \end{aligned}$$

The claim follows by induction on T. □

Definition 1.3. A portfolio π is *functionally generated* if

$$\pi_i(t) = \mu_i(t) \left(D_i \log f(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log f(\mu(t)) \right)$$

for some function $f : \Delta^n \to (0, \infty)$ such that $f \in C^1(\Delta^n)$ or $\log f$ is concave. In this case, f is known as a *portfolio generating function*, and we will say it *generates* π. We note that several authors specify more general definitions of functionally generated portfolios; see [6], [7], for example.

We will primarily concern ourselves with concave portfolio generating functions, as the portfolios they have characteristics which are excellent over long time horizons and provide opportunities to outperform the market portfolio under very few assumptions on the market itself. As is suggested by the form of functionally generated portfolios, we will be concerned with approximating the derivative of the logarithm of the generating function, which motivates the following definition.

Definition 1.4. Let U be a convex subset of $\mathbb{R}^n$. $g : U \to \mathbb{R}$ is *(strictly) exponentially concave* if e^g is (strictly) concave.

Since we are concerned with outperforming the market portfolio, we define this notion explicitly.

Definition 1.5. A *strong relative arbitrage* opportunity is a portfolio π for which $\exists T \in \mathbb{N}$ such that

$$V_\pi(T) > 1 \quad \text{a.s.}$$

If $g \in C^1([0,1]^n)$ is exponentially concave, then for the portfolio π functionally generated by e^g we have

$$\begin{aligned}\sum_{i=1}^{n} \pi_i(t) \frac{\mu_i(t+1)}{\mu_i(t)} &= 1 + \nabla g(\mu(t)) \Big(\mu(t+1) - \mu(t) \Big) \\ &\geq \exp \Big(g(\mu(t+1)) - g(\mu(t)) \Big) > 0 \end{aligned} \tag{1.2}$$

where the first inequality holds by exponential concavity. Hence, $V_\pi(T) > 0$ for all $T \in \mathbb{N}$ and we may write

$$\boxed{\log V_\pi(T) = g(\mu(T)) - g(\mu(0)) + \sum_{t=0}^{T-1} \left\{ \log \Big(1 + \nabla g(\mu(t)) \Big(\mu(t+1) - \mu(t) \Big) \Big) - \Big(g(\mu(t+1)) - g(\mu(t)) \Big) \right\}}$$

Specifically, $\log V_\pi(T)$ is equal to the sum of two terms; the first, $g(\mu(t)) - g(\mu(0))$, is bounded by the continuity of g and the second summation term is nondecreasing in T by the exponential concavity of g, regardless of the market weights. In terms of economic interpretation, the first term, when g is symmetric in its n arguments, is a measure of so called "market diversity", as it increases if one shifts market capitalization from larger stocks to smaller stocks. This may be shown using the concavity of g: suppose $\mu = (\mu_1, ..., \mu_n)^T \in \Delta^n$ is such that $\mu_i > \mu_j$. Let $h(\lambda) \equiv g(\mu + \lambda(\mu_i - \mu_j)(e_j - e_i))$ and note that evaluating h at $\lambda \in [0, 1]$ represents a shift of $\lambda(\mu_i - \mu_j)$ from the ith weight to the jth weight. It is easily seen that h is concave in λ with $h(0) = h(1)$ by symmetry.

$$\implies g(\mu + \lambda(\mu_i - \mu_j)(e_j - e_i)) \geq g(\mu)$$

where the inequality is strict if g is strictly exponentially concave. The second term is representative of the contribution of market volatility. If $\mu_i(t + 1)$ and $\mu_i(t)$ differ often enough and by a large enough quantity for strictly exponentially concave g, we may obtain strong relative arbitrage over a long enough time horizon.

For ease in exposition going forward, we introduce the following notation for some relevant quantities:

$$L(g, p, q) \equiv \log\Big(1 + \nabla g(p)\,(q - p)\Big) - \Big(g(q) - g(p)\Big) \tag{1.3}$$

$$R(g, p, q) \equiv 1 + \nabla g(p)\,(q - p) \tag{1.4}$$

$$\implies \boxed{\log V_\pi(T) = g(\mu(T)) - g(\mu(0)) + \sum_{t=0}^{T-1} L(g, \mu(t), \mu(t+1)) = \sum_{t=0}^{T-1} \log R(g, \mu(t), \mu(t+1))} \tag{1.5}$$

We will also be interested in portfolios which are based on the order statistics of the market weights:

$$\mu_{(1)}(t) \geq ... \geq \mu_{(n)}(t)$$

based on the observed stability of the distribution of capital in the global stock markets [3] [2].

Definition 1.6. Let $\sigma_t : \{1, ..., n\} \to \{1, ..., n\}$ denote a permutation of $\{1, ..., n\}$ at time t which satisfies $\mu_{\sigma_t(j)}(t) = \mu_{(j)}(t)$ and define for any vector $v \in \mathbb{R}^n$ and permutation σ, $\sigma(v) \equiv (v_{\sigma(1)}, ..., v_{\sigma(n)})^T$. We further define

$$\begin{aligned} p(t) &\equiv \sigma_t(\mu(t)) = (\mu_{(1)}(t), ..., \mu_{(n)}(t))^T, \\ q(t) &\equiv \sigma_t(\mu(t+1)) \end{aligned}$$

Of course, we will use $p_j(t), q_j(t)$ to refer to the jth elements of $p(t), q(t)$, respectively.

Definition 1.7. A portfolio π is *rank based functionally generated* if

$$\pi_{\sigma_t(j))} = p_j(t)\left(D_j \log f(p(t)) + 1 - \sum_{k=1}^{n} p_j(t) D_k \log f(p(t))\right)$$

for some function $f : \Delta^n \to (0, \infty)$ such that $f \in C^1(\Delta^n)$ or $\log f$ is concave. In this case, we say f is a *rank based portfolio generating function*, and we will say it *generates* π.

By lemma 1.1, we see that for the portfolio π that is rank based functionally generated by e^g, $g \in C^1([0,1]^n)$,

$$\begin{aligned} V_\pi(T) &= \prod_{t=0}^{T-1}\left(\sum_{j=1}^{n} \pi_{\sigma_t(j)} \frac{q_j(t)}{p_j(t)}\right) \\ &= \prod_{t=0}^{T-1}\Big(1 + \nabla g(p(t)) \cdot \Big(q(t) - p(t)\Big)\Big) \end{aligned}$$

If g is exponentially concave, each term in the product is strictly positive, so in similar fashion to the non-rank based case, we have

$$\boxed{\log V_\pi(T) = \sum_{t=0}^{T-1} \log R(g, p(t), q(t))} \tag{1.6}$$

To conclude this section, we will introduce one final definition: in an abuse of notation, we will define

$$V(g, T) \equiv V_\pi(T) \tag{1.7}$$

if the function e^g (rank based) functionally generates the portfolio π.

2 Optimization Over a Subset of Concave Generating Functions

We desire to compute or approximate

Problem 2.1.

$$\sup_{g\in\mathcal{G}} \frac{1}{T} \log V(g, T)$$

where $\mathcal{G} \subseteq C^1(\Delta^n)$, and we use the convention $\log x \equiv -\infty \ \forall x \leq 0$.

Choosing $\mathcal{G}$ will be heavily motivated by the desirable long run properties of concave and exponentially concave functions established in section 1. We prove several results concerning such functions, some of which will only be used later sections, before further discussing some technical requirements on $\mathcal{G}$ which will imply the existence of a maximizer.

Definition 2.2. Let $U \subseteq \mathbb{R}^n$ be convex and define

$$\mathcal{E}^k(U) \equiv \left\{g \in C^k(U) : g \text{ is exponentially concave}\right\}$$

Lemma 2.3. *$\mathcal{E}^k(U)$ is a convex set for all convex $U \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$.*

Proof:
Let $g_1, g_2 \in \mathcal{E}^k(U)$, $x_1, x_2 \in U$, $\lambda, \rho \in (0, 1)$ and define $g_\lambda \equiv \lambda g_1 + (1-\lambda)g_2 \in C^k(U)$, $x_\rho \equiv \rho x_1 + (1-\rho)x_2$.

Further, define the functions $f_1 \equiv e^{g_1}, f_2 \equiv e^{g_2}$ and the measure $Q \equiv \rho\delta_{x_1} + (1-\rho)\delta_{x_2}$ on U, where δ_x is the Dirac measure centered at x.

$$\begin{aligned}
\exp(g_\lambda(x_\rho)) &= \left(f_1\left(\int x dQ(x)\right)\right)^\lambda \left(f_2\left(\int x dQ(x)\right)\right)^{1-\lambda} \\
&\geq \left(\int f_1 dQ\right)^\lambda \left(\int f_2 dQ\right)^{1-\lambda} \\
&\geq \int f_1^\lambda f_2^{1-\lambda} dQ \\
&= \int g_\lambda dQ = \rho \exp(g_\lambda(x_1)) + (1-\rho)\exp(g_\lambda(x_2))
\end{aligned}$$

where the first inequality follows by applying Jensen's inequality to the concave functions f_1, f_2 and the second inequality follows by applying Hölder's inequality. Thus, $\lambda g_1 + (1-\lambda)g_2$ is exponentially concave by definition, concluding the proof. $\square$

Lemma 2.4. *If $g : [0,1] \to \mathbb{R}$ is exponentially concave, then its right derivative g' satisfies*

$$-\frac{1}{1-x} \leq g'(x) \leq \frac{1}{x} \qquad \forall x \in (0,1)$$

Proof:
By exponential concavity, g' exists and satisfies

$$1 + g'(x)(y-x) \geq \exp\left(g(y) - g(x)\right) > 0 \qquad \forall x, y \in (0,1)$$

Letting $y \uparrow 1$ and $y \downarrow 0$ yields the result. $\square$

Throughout this paper, we will restrict our attention to optimizing over *additive* log portfolio generating functions $g : \Delta^n \to \mathbb{R}$ of the form

$$g(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \ell_j(x_i)$$

in the rank based case, where the sets $\{C_j\}_{j=1}^J$ form a partition of $\{1, ..., n\}$, $\ell_j \in C^1([0,1])$ $\forall j \in \{1, ..., J\}$ are concave and $\lambda \in \mathbb{R}^J$ satisfies

$$\sum_{j=1}^{J} |C_j| \lambda_j = 1 \tag{2.1}$$

and is fixed. Such g are of course also exponentially concave by lemma 2.3. The choice of this additive form is made to reduce computational complexity when approximating the gradient of g, as will be discussed in section 3.

Definition 2.5. We define the following operator, d, on $C^1([0,1]) \times C^1([0,1])$ by

$$d(f,g) \equiv \sup_{x \in [0,1]} |f'(x) - g'(x)|$$

Definition 2.6. Let $\alpha, \beta > 0$ and define

$$\mathcal{E}^1_\beta \equiv \left\{\ell \in \mathcal{E}^1([0,1]) : \ell' \text{ is } \beta\text{-Lipschitz continuous}, \ell(1/2) = 0\right\}$$
$$\mathcal{E}^2_{\alpha,\beta} \equiv \left\{\ell \in \mathcal{E}^2([0,1]) : \ell'' \text{ is } \alpha\text{-Lipschitz continuous }, 0 \leq \ell' \leq \beta, \ell(0) = 0\right\}$$

We consider two approaches to further limit the space of functions we are considering to ensure that the optimization problem 2.1 has a solution. The first is a small modification of the methodology of [1]: the restriction that $\ell_i \in \mathcal{E}^1_\beta$. The second is of course that $\ell_i \in \mathcal{E}^2_{\alpha,\beta}$. Explicitly, given market data $\mu(1), ..., \mu(T)$, we will show that the optimization problem 2.1 has a maximizer in both the non rank based and rank based cases whenever

$$\mathcal{G} = \mathcal{G}_1 \equiv \left\{ g \in C^1([0,1]^n) : g(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \ell_j(x_i), \{\ell_j\}_{j=1}^J \subseteq \mathcal{E}^1_\beta \right\} \tag{2.2}$$

or

$$
\mathcal{G} = \mathcal{G}_2 \equiv \left\{ g \in C^2([0,1]^n) : g(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \ell_j(x_i), \{\ell_j\}_{j=1}^{J} \subseteq \mathcal{E}^2_{\alpha,\beta} \right\} \tag{2.3}
$$

In section 3, we will discuss the differences between these choices of $\mathcal{G}$, particularly with respect to their computational approximations. First, we prove some auxiliary results concerning $\mathcal{E}^1_\beta$ and $\mathcal{E}^2_{\alpha,\beta}$.

Lemma 2.7. *If $\ell \in \mathcal{E}^1_\beta$, then*

$$
|\ell'(x)| \leq \sqrt{\beta} \quad \forall x \in [0,1]
$$

Proof:
By the $\beta-$Lipschitz property of ℓ', $0 \leq -\ell''(x) \leq \beta$ for all x at which ℓ'' exists. By exponential concavity,

$$
\begin{aligned}
\left(e^{\ell(x)}\right)'' &= e^{\ell(x)} \left((\ell'(x))^2 + \ell''(x) \right) \leq 0 \\
&\implies (\ell'(x))^2 \leq -\ell''(x) \leq \beta
\end{aligned}
$$

for all x at which ℓ'' exists. Now fix any $x \in (0,1)$. By the concavity of ℓ, ℓ' is monotone decreasing $\implies \ell''$ exists Lebesgue almost everywhere on $(0,1) \implies \exists\, c_1, c_2$ with $0 < c_1 \leq x \leq c_2 < 1$ such that $\ell''(c_1)$ and $\ell''(c_2)$ exist. Thus,

$$
-\sqrt{\beta} \leq \ell'(c_2) \leq \ell'(x) \leq \ell'(c_1) \leq \sqrt{\beta}
$$

Continuity of ℓ' yields the bound for $x \in [0,1]$. □

Lemma 2.8. *Both (i) $\left(\mathcal{E}^1_\beta, d\right)$ and (ii) $\left(\mathcal{E}^2_{\alpha,\beta}, d\right)$ are compact metric spaces.*

Proof:
The fact that the *(i)* and *(ii)* are metric spaces follows trivially by checking the definition, noting that each of $\mathcal{E}^1_\beta$ and $\mathcal{E}^2_{\alpha,\beta}$ specify the value of their functions at a single point. It suffices to show sequential compactness:

- *(i)*: Let $\{\ell_i\}_{i\in\mathbb{N}} \subseteq \mathcal{E}^1_\beta$ be an arbitrary sequence. As $\{\ell_i'\}_{i\in\mathbb{N}}$ is uniformly bounded by lemma 2.6 and equicontinuous by the assumption of β-Lipschitz continuity, the Arzelà-Ascoli theorem implies the existence of a subsequence $\{\ell_{i_j}'\}_{j\in\mathbb{N}}$ which converges uniformly on $[0,1]$ to a limit f. It is trivial that f is β-Lipschitz continuous and thus integrable, so letting

$$
\ell(x) \equiv \int_0^x f(u)du - \int_0^{1/2} f(u)du \in \mathcal{E}^1_\beta
$$

  we have

$$
\ell_{i_j} \to \ell
$$

  under the metric d, which shows sequential compactness by definition.

- *(ii)*: Again let $\{\ell_i\}_{i\in\mathbb{N}} \subseteq \mathcal{E}^2_{\alpha,\beta}$ be an arbitrary sequence. By the mean value theorem, $\exists c_i \in (0,1)$ such that

$$
\begin{aligned}
&|\ell_i''(c_i)| = |\ell_i'(1) - \ell_i'(0)| \leq 2\beta \\
\implies \forall x \in [0,1], \qquad &|\ell_i''(x)| \leq \alpha|x - c_i| + 2\beta \leq \alpha + 2\beta
\end{aligned}
$$

so that $\{\ell_i''\}_{i\in\mathbb{N}}$ is uniformly bounded and equicontinuous. Thus, we may apply the Arzelà-Ascoli theorem again to obtain a subsequence whose second derivatives converge to an α-Lipschitz continuous function f. By the uniform bound on the ℓ_i', there exists a further subsequence $\{\ell_{i_j}\}_{j\in\mathbb{N}}$ such that $\ell_{i_j}'' \xrightarrow{k\to\infty} f$ uniformly on $[0,1]$ and $\ell_{i_j}'(1/2)$ converges. This implies (see [9], exercise 3.7.2, for example) that ℓ_{i_j}' converges uniformly on $[0,1]$ to some differentiable function g with $g' = f$.

Clearly g is integrable and inherits the bounds on the ℓ_{i_j}', so putting $\ell(x) \equiv \int_0^x g(u)du$ implies that $\ell \in \mathcal{E}^2_{\alpha,\beta}$ and that $\ell_{i_j} \to \ell$ under the metric d, again showing sequential compactness by definition. □

We are now ready to prove the main theorem in this section:

Theorem 2.9. *The optimization problem 2.1 is convex and has a solution for (i) $\mathcal{G} = \mathcal{G}_1$ or (ii) $\mathcal{G} = \mathcal{G}_2$, given by equations* (2.2) *and* (2.3)*, in both the non rank based and rank based cases.*

Proof:
We prove the result for non rank based case only, since the proof for the rank based generating functions is identical apart from notation. We define the following operators for $f, g \in \left(C^1([0,1])\right)^{\otimes J}$:

$$d_J(f,g) \equiv \max_{j\in\{1,\ldots,J\}} d(f_j, g_j)$$
$$g(f) \equiv \sum_{j=1}^{J}\sum_{i\in C_j} \lambda_j f_j$$

By lemma 2.8, both $\left((\mathcal{E}^1_\beta)^{\otimes J}, d_J\right)$ and $\left((\mathcal{E}^2_{\alpha,\beta})^{\otimes J}, d_J\right)$ are compact metric spaces. Note that

$$\sup_{g\in\mathcal{G}_1} \frac{1}{T}\log V(g,T) = \sup_{\ell\in(\mathcal{E}^1_\beta)^{\otimes J}} \frac{1}{T}\log V\left(g(\ell), T\right)$$

Lastly, we let $R(g,t)$ denote $R(g,\mu(t),\mu(t+1))$ for ease in notation.

- *(i)*: Note that $\log V(g,T)$ is well defined $\forall g \in \mathcal{G}_1$ by (1.2). Fix $\ell \in (\mathcal{E}^1_\beta)^{\otimes J}$. We have

$$\begin{aligned}
\log R(g(\ell),t) &= \log\left(1 + \sum_{j=1}^{J}\sum_{i\in C_j} \lambda_j \ell_j'(\mu_i(t))\Big(\mu_i(t+1) - \mu_i(t)\Big)\right) \\
&\geq \sum_{j=1}^{J}\sum_{i\in C_j} \lambda_j \Big(\ell_j(\mu_i(t+1)) - \ell_j(\mu_i(t))\Big) \\
&\geq \sum_{j=1}^{J}\sum_{i\in C_j} \lambda_j (-\sqrt{\beta})\,|\mu_i(t+1) - \mu_i(t)| \\
&\geq -2\sqrt{\beta}
\end{aligned}$$

Since log is Lipschitz continuous with constant $e^{2\sqrt{\beta}}$ on $\left[e^{-2\sqrt{\beta}}, \infty\right)$, we see $\forall \ell, \tilde{\ell} \in (\mathcal{E}^1_\beta)^{\otimes J}$,

$$\left|\log V(g(\ell),T) - \log V(g(\tilde{\ell}),T)\right| \leq \sum_{t=0}^{T-1}\left|\log R(g(\ell),t) - \log R(g(\tilde{\ell}),t)\right|$$

$$
\begin{aligned}
&\leq \sum_{t=0}^{T-1} e^{2\sqrt{\beta}} \left| \sum_{j=1}^{J} \sum_{i\in C_j} \lambda_j \Big(\ell_j'(\mu_i(t)) - \tilde{\ell}_j'(\mu_i(t))\Big)\Big(\mu_i(t+1)-\mu_i(t)\Big)\right| \\
&\leq \sum_{t=0}^{T-1}\sum_{j=1}^{J}\sum_{i\in C_j} e^{2\sqrt{\beta}}\lambda_j \Big|\mu_i(t+1)-\mu_i(t)\Big| d_J(\ell,\tilde{\ell}) \\
&\leq 2Te^{2\sqrt{\beta}} d_J(\ell,\tilde{\ell})
\end{aligned}
$$

so that the function $\ell \mapsto \frac{1}{T}\log V(g(\ell),T)$ is continuous on $(\mathcal{E}_\beta^1)^{\otimes J}$. By the compactness of the space,

$$
\ell^* = \underset{\ell\in(\mathcal{E}_\beta^1)^{\otimes J}}{\operatorname{argmax}} \frac{1}{T}\log V(g(\ell),T)
$$

exists and $g^* = g(\ell^*)$ solves problem 2.1 for $\mathcal{G} = \mathcal{G}_1$.

- *(ii)*: As in lemma 2.8, we note that $\ell \in \mathcal{E}_{\alpha,\beta}^2 \implies |\ell''| \leq \alpha + 2\beta$. Therefore, the continuity of the function $(\mathcal{E}_{\alpha,\beta}^2)^{\otimes J} \ni \ell \mapsto \frac{1}{T}\log V(g(\ell),T)$ is established exactly the same way for $\mathcal{G}_2$, but with β replaced with $\alpha+2\beta$. The proof is concluded by noting compactness of the space $(\mathcal{E}_{\alpha,\beta}^2)^{\otimes J}$.

$\square$

In light of theorem 2.9, we conclude this section with the following definition:

Definition 2.10.

$$
S_i \equiv \max_{g\in\mathcal{G}_i} \frac{1}{T}\log V(g,T)
$$

for $i \in \{1,2\}$.

3 Approximating the Optimization Problems

Since $\mathcal{E}_\beta^1$ and $\mathcal{E}_{\alpha,\beta}^2$ are both infinite dimensional, the problem of finding $\operatorname{argmax}_{g\in\mathcal{G}} V(g,T)$ for $\mathcal{G}=\mathcal{G}_1$ or $\mathcal{G}=\mathcal{G}_2$ is incredibly complex, particularly when the structure of our market has very few assumptions placed on it. In this section, we will discretize both problems, relating each to a maximization problem in $(\mathbb{R}^d)^{\otimes J}$, $d\in\mathbb{N}$, and show that the optimal values can be made to be arbitrarily close to that of the solutions to problem 2.1 in either case. This section is split into three subsections; the first addresses and makes some slight extensions to the approach of [1], while the second and third address the optimization problems involving $\mathcal{G}_1$ and $\mathcal{G}_2$, respectively, in the context of several additional assumptions on the market of stocks.

3.1 Optimizing Over $\mathcal{G}_1$ in the Market of (1.1)

In this subsection, we follow the approach of [1], where the only assumption we make on the market is that specified by (1.1).

Definition 3.1. Let $d \geq 3$ and fix a partition $\mathcal{P} = (x_1, ..., x_d)^T$ of $[0,1]$ with $0 = x_1 < x_2 < ... < x_d = 1$ and $1/2 \in \{x_i\}_{i=1}^d$. Let

$$
\underline{\delta} \equiv \min_{i\in\{1,\ldots,d-1\}} |x_{i+1}-x_i| \qquad \text{and} \qquad \delta \equiv \max_{i\in\{1,\ldots,d-1\}} |x_{i+1}-x_i|
$$

and let $\hat{\mathcal{E}}^1_\beta$ denote the set of vectors $\boldsymbol{\ell} = (\ell_1, ..., \ell_d)^T \in \mathbb{R}^d$ which satisfy

$$- \ell_i + \log\left(w_i e^{\ell_{i+1}} + (1 - w_i)e^{\ell_{i-1}}\right) \leq 0 \quad , \quad w_i \equiv \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}} \quad , \quad \forall i \in \{2, ..., d-1\} \tag{3.1}$$

$$\frac{\Delta_{i+1}\boldsymbol{\ell} - \Delta_i\boldsymbol{\ell}}{x_{i+2} - x_i} \geq -\beta/2 \quad , \quad \Delta_i\boldsymbol{\ell} \equiv \frac{\ell_{i+1} - \ell_i}{x_{i+1} - x_i} \quad , \quad \forall i \in \{1, ..., d-2\} \tag{3.2}$$

$$|\Delta_1\boldsymbol{\ell}|, |\Delta_{d-1}\boldsymbol{\ell}| \leq \sqrt{\beta} \tag{3.3}$$

$$\ell_i = 0 \text{ for the index } i \text{ such that } x_i = 1/2 \tag{3.4}$$

Definition 3.2. For any vector $\boldsymbol{\ell} \in \mathbb{R}^d$, define $\hat{\ell}$ to be the linear interpolation of $\boldsymbol{\ell}$ across the partition $\mathcal{P}$. Specifically,

$$\hat{\ell}(x) = \ell_i + \Delta_i\boldsymbol{\ell}(x - x_i) \quad \forall x \in [x_i, x_{i+1}]$$

We will use the interpolations $\{\hat{\ell}_j\}_{j=1}^J$ of vectors $\{\boldsymbol{\ell}_j\}_{j=1}^J \subseteq (\hat{\mathcal{E}}^1_\beta)^{\otimes J}$ to construct functions of the form

$$\hat{g}(x) = \sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \hat{\ell}_j(x_i) \tag{3.5}$$

to approximate $g \in \mathcal{G}_1$ and investigate their properties. (3.1) approximates exponential concavity, (3.2) approximates the β Lipschitz requirement (only one inequality direction is required since the concavity of $\log(x)$ and definition of w_i imply the difference quotient is non-positive), (3.3) is analogous to lemma 2.7 and (3.4) is the specification that $\hat{\ell}(1/2) = 0$, the same as functions in $\mathcal{E}^1_\beta$.

The following is a very basic lemma we will apply to interpolations of vectors in $\hat{\mathcal{E}}^1_\beta$. Its proof is included for completeness but may be skipped.

Lemma 3.3. *If $f : [x_0, x_1] \to \mathbb{R}$ is continuous and has a monotone decreasing right derivative, f'_+, on $[x_0, x_1)$, then f is concave.*

Proof:
Let $h : [x_0, x_1] \to \mathbb{R}$ be any continuous, right differentiable function with $h'_+(x) \leq 0 \ \forall x \in [x_0, x_1)$. We prove that $h_\epsilon(x) \equiv h(x) - \epsilon x$ is monotone decreasing $\forall \epsilon > 0$:

Suppose not. Then $\exists x_0 \leq a < b \leq x_1$ such that $h_\epsilon(a) < h_\epsilon(b)$ for some $\epsilon > 0$. Letting $c = \operatorname{argmin}_{w \in [a,b]} h_\epsilon(w)$, which exists by compactness, we must have $c < b$. Thus, $(h_\epsilon)'_+(c)$ exists and satisfies

$$\begin{aligned}
&(h_\epsilon)'_+(c) \leq -\epsilon < 0 \\
&\implies \exists \delta > 0 \text{ such that } c + \delta < b \text{ and } \frac{h_\epsilon(c+\delta) - h_\epsilon(c)}{\delta} < 0 \\
&\implies c \neq \operatorname*{argmin}_{w \in [a,b]} h(w) \implies \text{contradiction}
\end{aligned}$$

Since ϵ is arbitrary, we see that h is thus monotone decreasing as well. Now, let $\lambda \in (0, 1)$, $x_0 \leq x < y \leq x_1$ and $x_\lambda \equiv \lambda x + (1 - \lambda)y$ and consider the functions

$$\begin{aligned}
g(u) &\equiv f'_+(x_\lambda)(u - x_\lambda) - f(u) \\
h(u) &\equiv f(u) - f'_+(x_\lambda)(u - x_\lambda)
\end{aligned}$$

By the monotonicity of f'_+, $g'_+(u) \leq 0$ on $[x, x_\lambda)$ and $h'_+(u) \leq 0$ on $[x_\lambda, y)$, so that

$$\begin{aligned} g(x_\lambda) &= -f(x_\lambda) \leq f'_+(x_\lambda)(1-\lambda)(x-y) - f(x) = g(x) \\ h(x_\lambda) &= f(x_\lambda) \geq f(y) - f'_+(x_\lambda)\lambda(y-x) = h(y) \end{aligned}$$

Multiplying both sides of the first inequality by λ and both sides of the second by $1-\lambda$ and adding them together yields

$$\begin{aligned} f(x_\lambda) &\geq \lambda f(x) - \lambda(1-\lambda)f'_+(x_\lambda)(x-y) - \lambda(1-\lambda)f'_+(x_\lambda)(y-x) + (1-\lambda)f(y) \\ &= \lambda f(x) + (1-\lambda)f(y) \end{aligned}$$

upon rearranging terms, which is concavity by definition. □

Corollary 3.3.1. *If $\boldsymbol{\ell} \in \mathbb{R}^d$ satisfies* (3.1)*, then the linear interpolations of $\boldsymbol{\ell}$ and $\boldsymbol{f} \equiv (e^{\ell_1}, ..., e^{\ell_d})^T$ across the partition $\mathcal{P}$ are both concave.*

Proof:
We see that $\Delta_{i+1}\boldsymbol{f} \leq \Delta_i \boldsymbol{f}$ and $\Delta_{i+1}\boldsymbol{\ell} \leq \Delta_i \boldsymbol{\ell}$ $\forall i \in \{1, ..., d-2\}$, by exponentiating both sides of (3.1) and rearranging in the case of $\boldsymbol{f}$ and by noting the concavity of log and rearranging in the case of $\boldsymbol{\ell}$. Thus, $\hat{\ell}$ and $\hat{f}$ have monotone decreasing right derivatives on $[0, 1)$. □

Corollary 3.3.1 implies that functions $\hat{g}$ of the form (3.5) are concave and thus $e^{\hat{g}}$ is seen to be a portfolio generating function. To eliminate any ambiguity, we will use the right derivative of the functions $\hat{\ell}_j$ in our portfolio weights. That is, in the non rank based case, for $i \in C_j$, we will have portfolio weight

$$\pi_i(t) = \mu_i(t) \left(\lambda_j \ell'_j(x_i) + 1 - \sum_{k=1}^{J} \sum_{m \in C_k} \lambda_k \ell'_k(x_m)\mu_m(t) \right)$$

where $'$ denotes the right derivative. The portfolio weights are defined analogously in the rank based case using the right derivatives of the ℓ_j.

We introduce the following optimization problem, which will approximate problem 2.1 in the case $\mathcal{G} = \mathcal{G}_1$:

Problem 3.4.

$$\sup_{\{\boldsymbol{\ell}_j\}_{j=1}^{J} \in (\hat{\mathcal{E}}^1_\beta)^{\otimes J}} \frac{1}{T} \log V \left(\sum_{j=1}^{J} \sum_{i \in C_j} \lambda_j \hat{\ell}_j(x_i), T \right)$$

Lemma 3.5. *A maximizer exists for problem 3.4. We will define $\hat{S}_1(\delta)$ to be the associated optimal value.*

Proof:
First, note that (3.1), (3.3) and (3.4) imply that $\forall i \in \{1, ..., d-1\}$

$$\begin{aligned} -\sqrt{\beta} \leq \Delta_{d-1}\boldsymbol{\ell} \leq \Delta_i \boldsymbol{\ell} \leq \Delta_1 \boldsymbol{\ell} \leq \sqrt{\beta} \\ \implies |\ell_{i+1} - \ell_i| \leq \delta_i \sqrt{\beta} \end{aligned}$$

$$\implies |\ell_i| = \left| \sum_{j=i\wedge k}^{i\vee k-1} \ell_{j+1} - \ell_j \right| \le \sum_{j=i\wedge k}^{i\vee k-1} \sqrt{\beta}\delta_j \le \frac{\sqrt{\beta}}{2}$$

Thus, $\left((\hat{\mathcal{E}}_\beta^1)^{\otimes J}, \|\cdot\|\right)$, where $\|\{\boldsymbol{\ell}_j\}_{j=1}^J\| \equiv \sqrt{\sum_{j=1}^J \sum_{i=1}^d (\boldsymbol{\ell}_j)_i^2}$ is a compact normed space, and the function $\{\boldsymbol{\ell}_j\}_{j=1}^J \mapsto V\left(\sum_{j=1}^J \sum_{i\in C_j} \lambda_j \hat{\ell}_j(x_i), T\right)$ is continuous, which shows that a maximizer exists. $\square$

We note that problem 3.4 is a convex optimization problem in $(\hat{\mathcal{E}}_\beta^1)^{\otimes J}$ so it is easily solvable with software, provided J and d are chosen to be of reasonable size.

We now turn to analyzing the difference between S_1 and $\hat{S}_1$. We start by constructing a function using $\boldsymbol{\ell} \in \hat{\mathcal{E}}_\beta^1$ which we will show to be in the set $\mathcal{E}_\beta^1$. This is the content of lemmas 3.6, 3.7, 3.8 and 3.9, most of which are tedious calculations whose proofs may be skipped if the reader chooses. Lemma 3.10 and theorem 3.11 are the most important results from this section, as they provide From this point forward, unless otherwise mentioned, we will be considering a fixed vector $\boldsymbol{\ell} \in \mathbb{R}^d$, with $\boldsymbol{f} \equiv (e^{\ell_1}, ..., e^{\ell_d})^T$. We define the following functions:

$$g_i(x) \equiv \frac{\hat{f}(x_i) + \hat{f}(x_i - \underline{\delta})}{2} + \frac{\hat{f}(x_i) - \hat{f}(x_i - \underline{\delta})}{\underline{\delta}}\left(x - x_i + \frac{\underline{\delta}}{2}\right) + \frac{\hat{f}(x_i + \underline{\delta}) - 2\hat{f}(x_i) + \hat{f}(x_i - \underline{\delta})}{2\underline{\delta}^2}\left(x - x_i + \frac{\underline{\delta}}{2}\right)^2$$

and

$$\boxed{s(x) \equiv \begin{cases} g_i(x) & x \in \left[x_i - \frac{\underline{\delta}}{2}, x_i + \frac{\underline{\delta}}{2}\right), i \in \{2, ..., d-1\} \\ \hat{f}(x) & \text{otherwise} \end{cases}} \tag{3.6}$$

We note that s is linear on each of the intervals $\left[0, x_2 - \frac{\underline{\delta}}{2}\right), \left[x_i + \frac{\underline{\delta}}{2}, x_{i+1} - \frac{\underline{\delta}}{2}\right), \left[x_{d-1} + \frac{\underline{\delta}}{2}, 1\right], i \in \{2, ..., d-2\}$ whenever such intervals are nonempty.

Lemma 3.6. *If $\boldsymbol{\ell}$ satisfies* (3.1)*, then $s \in C^1([0,1])$ and is concave.*

Proof:
To show that $s \in C^1([0,1])$ it suffices to show that

$$\begin{aligned}
\hat{f}(x_i - \underline{\delta}/2) &= g_i(x_i - \underline{\delta}/2) && \text{(i)} \\
\hat{f}(x_i + \underline{\delta}/2) &= g_i(x_i + \underline{\delta}/2) && \text{(ii)} \\
\hat{f}'_-(x_i - \underline{\delta}/2) &= g'_i(x_i - \underline{\delta}/2) && \text{(iii)} \\
\hat{f}'_+(x_i + \underline{\delta}/2) &= g'_i(x_i + \underline{\delta}/2) && \text{(iv)}
\end{aligned}$$

$\forall i \in \{2, ..., d-1\}$, where $-$ and $+$ above denote left and right derivatives respectively. Note that

$$\begin{aligned}
\text{(i)} \quad g_i(x_i - \underline{\delta}/2) &= \frac{1}{2}\left(\hat{f}(x_i) + \hat{f}(x_i - \underline{\delta})\right) = \frac{1}{2}\left(f_i + f_{i-1} + \Delta_{i-1}\boldsymbol{f}(x_i - x_{i-1} - \underline{\delta})\right) \\
&= f_{i-1} + \Delta_{i-1}\boldsymbol{f}(x_i - \underline{\delta}/2 - x_{i-1}) = \hat{f}(x_i - \underline{\delta}/2) \\
\text{(ii)} \quad g_i(x_i + \underline{\delta}/2) &= \frac{1}{2}\left(\hat{f}(x_i) + \hat{f}(x_i + \underline{\delta})\right) = \frac{1}{2}\left(2f_i + \Delta_i\boldsymbol{f}(x_i + \underline{\delta} - x_i)\right) \\
&= f_i + \Delta_i\boldsymbol{f}(x_i + \underline{\delta}/2 - x_i) = \hat{f}(x_i + \underline{\delta}/2) \\
\text{(iii)} \quad g'_i(x_i - \underline{\delta}/2) &= \frac{\hat{f}(x_i) - \hat{f}(x_i - \underline{\delta})}{\underline{\delta}} = \Delta_{i-1}\boldsymbol{f} = \hat{f}'_-(x_i - \underline{\delta}/2)
\end{aligned}$$

$$\text{(iv)}\quad g_i'(x_i+\underline{\delta}/2) = \Delta_{i-1}\boldsymbol{f} + \frac{\hat{f}(x_i+\underline{\delta}) - \hat{f}(x_i)}{\underline{\delta}} - \frac{\hat{f}(x_i) - \hat{f}(x_i-\underline{\delta})}{\underline{\delta}}$$
$$= \Delta_i\boldsymbol{f} = \hat{f}_+'(x_i+\underline{\delta}/2)$$

To see that s is concave, we note that since $\hat{f}$ is concave by corollary 3.1.1, $\hat{f}(x_i+\underline{\delta}) - 2\hat{f}(x_i) + \hat{f}(x_i-\underline{\delta}) \le 0$ so that s' is decreasing on $[0,1)$ and we may apply lemma 3.3. □

Note that lemma 3.6 implies that $s(x) > 0\ \forall x \in [0,1]$ since by concavity

$$s(x) \ge (1-x)s(0) + xs(1) \ge s(0) \wedge s(1) = \exp(\ell_1 \wedge \ell_d) > 0$$

In particular, $\log(s(x)) \in C^1([0,1])$.

Lemma 3.7. *If $\boldsymbol{\ell}$ satisfies* (3.1)-(3.4), then

$$\sup_{x\in[0,1]} \left|\log(s(x))' - \hat{\ell}'(x)\right| \le \left(\frac{7}{2}\exp\left(2\sqrt{\beta}\delta\right) + 1\right)\beta\delta$$

where $'$ denotes the right derivative except at $x = 1$, where we take it to be the left derivative.

Proof:
Let $\overline{\ell_i} \equiv \ell_i \vee \ell_{i+1}$, $\underline{\ell_i} \equiv \ell_i \wedge \ell_{i+1}$, $\delta_i \equiv |x_{i+1} - x_i|$ for $i \in \{1, ..., d-1\}$ and j be the index such that $x_j = 1/2$. We will proceed in three cases:

1. If $x \in [0, x_2 - \underline{\delta}/2), [x_{d-1} + \underline{\delta}/2, 1]$ or $[x_j + \underline{\delta}/2, x_{j+1} - \underline{\delta}/2)$ for some $j \in \{2, ..., d-2\}$, then clearly for some $i \in \{1, ..., d-1\}$ we have

$$\left|\log(s(x))' - \hat{\ell}'(x)\right| = \frac{1}{\delta_i}\left|\frac{e^{\overline{\ell_i}} - e^{\underline{\ell_i}}}{s(x)} - (\overline{\ell_i} - \underline{\ell_i})\right|$$

   and note that $\exp\left(\underline{\ell_i}\right) \le s(x) \le \exp\left(\overline{\ell_i}\right)$ (since s is a linear interpolation of these upper and lower bounds on the interval in question), so that

$$\frac{1}{\delta_i}\left(1 - e^{\underline{\ell_i} - \overline{\ell_i}} - (\overline{\ell_i} - \underline{\ell_i})\right) \le \frac{1}{\delta_i}\left(\frac{e^{\overline{\ell_i}} - e^{\underline{\ell_i}}}{s(x)} - (\overline{\ell_i} - \underline{\ell_i})\right) \le \frac{1}{\delta_i}\left(e^{\overline{\ell_i} - \underline{\ell_i}} - 1 - (\overline{\ell_i} - \underline{\ell_i})\right)$$
$$\implies \frac{e^{c_1}\left(\overline{\ell_i} - \underline{\ell_i}\right)^2}{\delta_i} \le \frac{1}{\delta_i}\left(\frac{e^{\overline{\ell_i}} - e^{\underline{\ell_i}}}{s(x)} - (\overline{\ell_i} - \underline{\ell_i})\right) \le \frac{e^{c_2}\left(\overline{\ell_i} - \underline{\ell_i}\right)^2}{\delta_i}$$

   for some $c_1, c_2 \in \left[0, \overline{\ell_i} - \underline{\ell_i}\right]$, by Taylor's theorem. Letting $c \equiv c_1 \vee c_2$, we have

$$\left|\log(s(x))' - \hat{\ell}'(x)\right| \le \frac{e^{c}\left(\overline{\ell_i} - \underline{\ell_i}\right)^2}{\delta_i} \le \beta e^{\sqrt{\beta}\delta_i}\delta_i \le \beta e^{\sqrt{\beta}\delta}\delta$$

2. If $x \in [x_i - \underline{\delta}/2, x_i)$ for some $i \in \{2, ..., d-1\}$ then

$$\log(s(x))' - \hat{\ell}'(x) = \frac{1}{s(x)}\left(\Delta_{i-1}\boldsymbol{f} + \frac{\Delta_i\boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}}{\underline{\delta}}(x - x_i + \underline{\delta}/2)\right) - \frac{\ell_i - \ell_{i-1}}{\delta_{i-1}}$$

Since $x \in [x_i - \underline{\delta}/2, x_i + \underline{\delta}/2]$, we have $s(x) \geq \exp\left(\underline{\ell_{i-1}} \wedge \underline{\ell_i}\right)$ by concavity and

$$\begin{aligned} s(x) &= e^{\ell_{i-1}} + \Delta_{i-1}\boldsymbol{f}(x - x_{i-1}) + \underbrace{\frac{\Delta_i \boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}}{2\underline{\delta}}(x - x_i + \underline{\delta}/2)^2}_{\leq 0} \\ &\leq \exp\left(\overline{\ell_{i-1}}\right) \end{aligned}$$

By applying Taylor's theorem multiple times, we note $\exists c_1, c_2 \in \left[\underline{\ell_{i-1}}, \overline{\ell_{i-1}}\right]$, $c_3 \in \left[\underline{\ell_i}, \overline{\ell_i}\right]$, $c_4 \in \left[c_2 \wedge c_3, c_2 \vee c_3\right]$ and c_5 betweeen 0 and $c_1 - \log(s(x))$ such that

$$\begin{aligned} \implies \left|\log(s(x))' - \hat{\ell}'(x)\right| &= \left|\frac{1}{s(x)}\left(e^{c_1}\Delta_{i-1}\boldsymbol{\ell} + \frac{\Delta_i \boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}}{\underline{\delta}}(x - x_i + \underline{\delta}/2)\right) - \Delta_{i-1}\boldsymbol{\ell}\right| \\ &\leq \frac{1}{2s(x)}\left|\Delta_i \boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}\right| + \left|\Delta_{i-1}\boldsymbol{\ell}\right|\left|e^{c_1 - \log(s(x))} - 1\right| \\ &= \frac{1}{2s(x)}\left|e^{c_3}\Delta_i \boldsymbol{\ell} - e^{c_2}\Delta_{i-1}\boldsymbol{\ell}\right| + \left|\Delta_{i-1}\boldsymbol{\ell}\right|\left|e^{c_1 - \log(s(x))} - 1\right| \\ &\leq \frac{e^{c_3}}{2s(x)}\left|\Delta_i \boldsymbol{\ell} - \Delta_{i-1}\boldsymbol{\ell}\right| + \frac{\left|\Delta_{i-1}\boldsymbol{\ell}\right|}{2s(x)}\left|e^{c_3} - e^{c_2}\right| + \left|\Delta_{i-1}\boldsymbol{\ell}\right|\left|e^{c_1 - \log(s(x))} - 1\right| \\ &= \frac{e^{c_3}}{2s(x)}\left|\Delta_i \boldsymbol{\ell} - \Delta_{i-1}\boldsymbol{\ell}\right| + \frac{\left|\Delta_{i-1}\boldsymbol{\ell}\right| e^{c_4}}{2s(x)}\left|c_3 - c_2\right| + \left|\Delta_{i-1}\boldsymbol{\ell}\right| e^{c_5}\left|c_1 - \log(s(x))\right| \\ &\leq \frac{\beta\left(\frac{\delta_{i-1} + \delta_i}{2}\right) e^{2\sqrt{\beta}\delta}}{2} + e^{2\sqrt{\beta}\delta}\beta\delta + 2e^{2\sqrt{\beta}\delta}\beta\delta \\ &= \frac{7}{2}e^{2\sqrt{\beta}\delta}\beta\delta \end{aligned}$$

where the last inequality follows from (3.2) and the fact that c_1, c_2, c_3, c_4, c_5 and $\log(s(x))$ all lie in the interval $\left[\ell_{i-1} \wedge \ell_i \wedge \ell_{i+1}, \ell_{i-1} \vee \ell_i \vee \ell_{i+1}\right]$ which has length at most $2\sqrt{\beta}\delta$.

3. If $x \in [x_i, x_i + \underline{\delta}/2)$ for some $i \in \{2, ..., d-1\}$ then

$$\log(s(x))' - \hat{\ell}'(x) = \frac{1}{s(x)}\left(\Delta_{i-1}\boldsymbol{f} + \frac{\Delta_i \boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}}{\underline{\delta}}(x - x_i + \underline{\delta}/2)\right) - \frac{\ell_{i+1} - \ell_i}{\delta_i}$$

so that applying (3.2) again along with the result from case 2., we see easily that

$$\left|\log(s(x))' - \hat{\ell}'(x)\right| \leq \left(\frac{7}{2}e^{2\sqrt{\beta}\delta} + 1\right)\beta\delta$$

As $x \in [0, 1]$ must lie in one of these intervals, this yields the stated result. □

Corollary 3.7.1. $\forall \alpha \in (0, 1)$,

$$\left|\log\left(s(x) + \delta^\alpha\right)' - \hat{\ell}'(x)\right| \leq \left(1 + \exp\left(\sqrt{\beta}\right)\delta^\alpha\right)\left(\frac{7}{2}e^{2\sqrt{\beta}\delta} + 1\right)\beta\delta + \exp\left(\sqrt{\beta}\right)\sqrt{\beta}\delta^\alpha$$

Proof:

Note that

$$\left|\log\left(s(x)+\delta^\alpha\right)' - \log(s(x))'\right| = \left|\frac{s'(x)}{s(x)}\right|\left|\frac{\delta^\alpha}{(s(x)+\delta^\alpha)}\right| \le e^{\sqrt{\beta}}\left(\left(\frac{7}{2}e^{2\sqrt{\beta}\delta}+1\right)\beta\delta + \sqrt{\beta}\right)\delta^\alpha$$

where the inequality follows from applying lemma 3.7 and the fact that $|\Delta_i \boldsymbol{\ell}| \le \sqrt{\beta}$, $s(x) \ge e^{-\sqrt{\beta}/2}$. The result follows from applying the triangle inequality and lemma 3.7 a second time. □

The following is a another very basic lemma which we will cite in subsequent results. The proof may be skipped but is included here for the sake of completeness.

Lemma 3.8. *Suppose $\ell \in C^1([0,1])$ is exponentially concave with $\ell(1/2) = 0$. If its right second derivative, denoted by $\ell''_+(x)$, exists on $[0,1)$ and satisfies*

$$\ell''_+(x) \ge -\beta \qquad \forall x \in [0,1)$$

then $\ell \in \mathcal{E}^1_\beta$.

Proof:
By definition, we only need to check that ℓ' is β-Lipschitz continuous: Fix $0 < x_0 < x_1 < 1$ and consider the function

$$h(x) \equiv \frac{\ell'(x_1)-\ell'(x_0)}{x_1-x_0}(x-x_0) - \big(\ell'(x)-\ell'(x_0)\big)$$

We claim there must exist some $y \in [x_0, x_1]$ such that $h'_+(y) \ge 0$: Suppose not. Then $h'_+(x) < 0$ for every $x \in [x_0, x_1]$, which implies that h is monotone decreasing on $[x_0, x_1]$ as seen in the proof of lemma 3.3.

So, we have

$$\begin{aligned}
&\implies 0 = h(x_0) \ge h(x) \ge h(x_1) = 0 = h(x_0) \quad \forall x \in (x_0, x_1)\\
&\implies h'_+(x) = 0 \quad \forall x \in (x_0, x_1)\\
&\implies \text{contradiction}
\end{aligned}$$

Therefore,

$$\begin{aligned}
h'_+(y) &= \frac{\ell'(x_1)-\ell'(x_0)}{x_1-x_0} - \ell''_+(y) \ge 0\\
\implies &\frac{\ell'(x_1)-\ell'(x_0)}{x_1-x_0} \ge \ell''_+(y) \ge -\beta
\end{aligned}$$

By taking limits, the above inequality holds for $0 \le x_0 \le x_1 \le 1$. By the concavity of ℓ, its first derivative is nonincreasing so this yields the result. □

Lemma 3.9. *Suppose that $\delta - \underline{\delta} \le M\underline{\delta}^2$ for some $M \ge 0$. Define $c_\alpha \equiv -\log\left(s(1/2)+\delta^\alpha\right)$. $\exists \delta^* > 0$ depending only on α, β and M such that $\forall \delta < \delta^*$*

$$\ell(x) \equiv \log\left(s(x)+\delta^\alpha\right) + c_\alpha \in \mathcal{E}^1_\beta$$

Proof:

First, we prove a better upper bound on the ℓ_i. Let $g(x) \equiv \log\left(\hat{f}(x)\right)$ and note that by lemma 2.4

$$-\frac{1}{1-x} \leq g'(x) \leq \frac{1}{x} \quad \forall x \in [0,1]$$

(since g is differentiable at $x = 1$). Noting that $g(1/2) = 0$ and integrating both sides of the above inequality between $x \in (0,1)$ and $1/2$ yields

$$g(x) \leq \log(2x \vee (2-2x)) \leq \log(2)$$

By continuity, the bound holds $\forall x \in [0,1]$. In particular, $\ell_i = g(x_i) \leq \log(2)\ \forall i$.

Clearly ℓ is exponentially concave by lemma 3.6 and satisfies $\ell(1/2) = 0$ by construction, so it suffices to show that its first derivative is β-Lipschitz continuous. Fix $x \in [0,1)$. We proceed in two cases, using lemma 3.8:

1. If $x \in [0, x_2 - \underline{\delta}/2), [x_{d-1} + \underline{\delta}/2, 1]$ or $[x_j + \underline{\delta}/2, x_{j+1} - \underline{\delta}/2)$ for some $j \in \{2, ..., d-2\}$, then we have, for some $i \in \{1, ..., d-1\}$

$$\sqrt{-\ell''_+(x)} = \left|\frac{s'(x)}{s(x)+\delta^\alpha}\right| = \left|\frac{\Delta_i \boldsymbol{f}}{e^{\ell_i}(1+R_1)+\delta^\alpha}\right| = \left|\frac{e^{\ell_i}(\Delta_i \boldsymbol{\ell} + R_2)}{e^{\ell_i}(1+R_1)+\delta^\alpha}\right|$$

where $R_1 \equiv e^{c_1}\Delta_i\ell(x - x_i)$ and $R_2 \equiv \frac{(\ell_{i+1}-\ell_i)^2 e^{c_2}}{2\delta_i}$ for some c_1, c_2 between 0 and $\ell_{i+1} - \ell_i$, by Taylor's theorem. Note that the bounds on the ℓ_j established in lemma 3.5 imply that

$$|R_1| \leq e^{\sqrt{\beta}\delta}\sqrt{\beta}\delta \qquad \text{and} \qquad 0 \leq R_2 \leq e^{\sqrt{\beta}\delta}\beta\delta/2$$

Thus, applying Taylor's theorem again to the function $h(x) \equiv 1/(1+x)$ we see that for some c_3 between 0 and $R_1 + \delta^\alpha e^{-\ell_i}$

$$\begin{aligned}
\sqrt{-\ell''_+(x)} &= \left|\frac{\Delta_i \boldsymbol{\ell} + R_2}{1 + R_1 + \delta^\alpha e^{-\ell_i}}\right| \\
&= \left|\Delta_i\ell\left(1 - \delta^\alpha e^{-\ell_i}\right) + R_2\left(1 - \delta^\alpha e^{-\ell_i}\right) + (\Delta_i\ell + R_2)\left(\frac{\left(R_1 + \delta^\alpha e^{-\ell_i}\right)^2}{(1+c_3)^3} - R_1\right)\right| \\
&\leq \left|\Delta_i\ell\left(1 - \delta^\alpha e^{-\ell_i}\right)\right| + |o(\delta^\alpha)|
\end{aligned}$$

where $o(\delta^\alpha)$ is generic notation for a function of δ which satisfies $\lim_{\delta\to 0} o(\delta^\alpha)/\delta^\alpha = 0$ uniformly across the index i.

Let δ_1 such that $\delta_1^\alpha e^{\sqrt{\beta}/2} < 1$ and $|o(\delta^\alpha)| \leq \sqrt{\beta}\delta^\alpha/2\ \forall \delta < \delta_1$ and noting that $e^{\sqrt{\beta}/2} \geq e^{-\ell_i} \geq 1/2$, we have

$$\sqrt{-\ell''_+(x)} \leq |\Delta_i\ell|\left(1 - \frac{\delta^\alpha}{2}\right) + \sqrt{\beta}\frac{\delta^\alpha}{2} \leq \sqrt{\beta} \implies \ell''_+(x) \geq -\beta$$

whenever $\delta < \delta_1$.

2. If $x \in [x_i - \underline{\delta}/2, x_i + \underline{\delta}/2)$ for some $i \in \{2, ..., d-1\}$, then

$$-\ell''_+(x) = \left(\frac{s'(x)}{s(x)+\delta^\alpha}\right)^2 - \frac{s''(x)}{s(x)+\delta^\alpha}$$

We first establish some relationships between s, s' and its right second derivative s''. We have

$$s(x) = e^{\ell_i} + \Delta_{i-1}\boldsymbol{f}(x - x_i) + \frac{\Delta_i \boldsymbol{f} - \Delta_{i-1}\boldsymbol{f}}{2\underline{\delta}}(x - x_i + \underline{\delta}/2)^2$$

and by Taylor's theorem there exist constants c_1, c_2 between 0 and $\ell_{i-1} - \ell_i$ and 0 and $\ell_{i+1} - \ell_i$ respectively, such that

$$\begin{aligned}\Delta_{i-1}\boldsymbol{f} &= e^{\ell_i}\left(\Delta_{i-1}\boldsymbol{\ell} - \frac{(\ell_i - \ell_{i-1})^2}{2\delta_{i-1}} - \frac{(\ell_i - \ell_{i-1})^3 e^{c_1}}{6\delta_{i-1}}\right) \qquad \text{and}\\ \Delta_i \boldsymbol{f} &= e^{\ell_i}\left(\Delta_i \boldsymbol{\ell} + \frac{(\ell_{i+1} - \ell_i)^2}{2\delta_i} + \frac{(\ell_{i+1} - \ell_i)^3 e^{c_2}}{6\delta_i}\right)\end{aligned}$$

Again, the bounds established on the ℓ_j in lemma 3.5 imply that

$$\left|\frac{(\ell_i - \ell_{i-1})^3 e^{c_1}}{6\delta_{i-1}}\right| \vee \left|\frac{(\ell_{i+1} - \ell_i)^3 e^{c_2}}{6\delta_i}\right| \leq \frac{e^{\sqrt{\beta}\delta}\beta^{3/2}\delta^2}{6}$$

We also note the existence of $c \in \left[\underline{\delta}, \frac{\delta_{i-1}+\delta_i}{2}\right]$ such that

$$\begin{aligned}0 \leq \frac{\Delta_{i-1}\boldsymbol{\ell} - \Delta_i \boldsymbol{\ell}}{\underline{\delta}} &= 2\frac{\Delta_{i-1}\boldsymbol{\ell} - \Delta_i \boldsymbol{\ell}}{x_{i+1} - x_{i-1}}\left(1 + \frac{\left(\frac{\delta_{i-1}+\delta_i}{2}\right)\left(\frac{\delta_{i-1}+\delta_i}{2} - \underline{\delta}\right)}{c^2}\right) \leq \beta\left(1 + M\delta\right)\\ \implies 0 \leq \frac{\Delta_{i-1}\boldsymbol{f} - \Delta_i \boldsymbol{f}}{\underline{\delta}} &\leq e^{\ell_i}\left(\beta(1 + M\delta) - \frac{(\ell_i - \ell_{i-1})^2}{2\delta_{i-1}\underline{\delta}} - \frac{(\ell_{i+1} - \ell_i)^2}{2\delta_i \underline{\delta}} + o(\delta^\alpha)\right)\\ &\leq e^{\ell_i}\left(\beta - (\Delta_{i-1}\ell)^2 + o(\delta^\alpha)\right)\end{aligned}$$

where the last inequality on the first line above follows from the assumption that $\delta - \underline{\delta} \leq M\underline{\delta}^2$ and the last inequality on the third line follows as a direct consequence of (3.2) and (3.3).

$$\implies -\ell_+''(x) = \left(\frac{\Delta_{i-1}\boldsymbol{\ell} + R_2}{1 + R_1 + e^{-\ell_i}\delta^\alpha}\right)^2 + \frac{\Delta_{i-1}\boldsymbol{f} - \Delta_i \boldsymbol{f}}{\underline{\delta}\left(1 + R_1 + e^{-\ell_i}\delta^\alpha\right)}$$

where $|R_1|, |R_2| \leq o(\delta^\alpha)$. By expanding $1/\left(1 + R_1 + e^{-\ell_i}\delta^\alpha\right)$ exactly as in part 1., we obtain

$$\begin{aligned}\implies -\ell_+''(x) &\leq (\Delta_{i-1}\ell)^2\left(1 - 2e^{-2\ell_i}\delta^\alpha + o(\delta^\alpha)\right) + \left(\beta - (\Delta_{i-1}\boldsymbol{\ell})^2 + o(\delta^\alpha)\right)\left(1 - e^{-\ell_i}\delta^\alpha + o(\delta^\alpha)\right)\\ &\leq \beta\left(1 - e^{-\ell_i}\delta^\alpha\right) + o(\delta^\alpha)\end{aligned}$$

Choosing δ_2 such that $\delta < \delta_2$ implies $|o(\delta^\alpha)| \leq \delta^\alpha\beta/2$ and once again noting that $e^{-\ell_i} \geq 1/2$, we have

$$-\ell_+''(x) \leq \beta$$

whenever $\delta < \delta_2$.

Letting $\delta < \delta^* \equiv \delta_1 \wedge \delta_2$ yields $\ell_+''(x) \geq -\beta \; \forall x \in [0, 1)$, so applying lemma 3.8 concludes the proof. □

Lemma 3.10. *If $\ell \in \mathcal{E}_\beta^1$, then*

$$\boldsymbol{\ell} \equiv \left(\ell(x_1), ..., \ell(x_d)\right)^T \in \mathbb{R}^d \textit{ satisfies } (3.1) \text{ - } (3.4).$$

Proof:
Noting that $x_{i+1}w_i + x_{i-1}(1-w_i) = x_i$, the fact that $\boldsymbol{\ell}$ satisfies (3.1) follows immediately from exponential concavity.

Note that for any $x, y \in [0,1]$, we have

$$
\begin{aligned}
|\ell(y) - \ell(x) - \ell'(x)(y-x)| \leq \int_{x\wedge y}^{x\vee y} |\ell'(x) - \ell'(u)|\, du \leq \int_{x\wedge y}^{x\vee y} \beta|x-u|du = \frac{\beta(y-x)^2}{2} \\
\implies \Delta_i\boldsymbol{\ell} - \Delta_{i+1}\boldsymbol{\ell} = \Delta_i\boldsymbol{\ell} - \ell'(x_i) + \ell'(x_i) - \Delta_{i+1}\boldsymbol{\ell} \\
\leq \frac{\beta\delta_i}{2} + \frac{\beta\delta_{i+1}}{2} = \frac{\beta}{2}(x_{i+2} - x_i)
\end{aligned}
$$

and this is precisely (3.2) upon rearranging.

That $\boldsymbol{\ell}$ satisfies (3.3) is an immediate consequence of the mean value theorem and lemma 2.7, and (3.4) holds by definition of $\mathcal{E}^1_\beta$. □

Theorem 3.11. *Assume that $\delta - \underline{\delta} \leq M\underline{\delta}^2$ for some $M \geq 0$. For every $\alpha \in (0,1), \beta \geq 0$, $\exists\delta^* > 0$ such that $\forall\delta < \delta^*$,*

$$
-4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\beta\delta \leq \hat{S}_1(\delta) - S_1 \leq 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta)
$$

where

$$
\epsilon_1(\delta) = \left(1 + \exp\left(\sqrt{\beta}\right)\delta^\alpha\right)\left(\frac{7}{2}e^{2\sqrt{\beta}\delta} + 1\right)\beta\delta + \exp\left(\sqrt{\beta}\right)\sqrt{\beta}\delta^\alpha
$$

Proof:
We provide the proof for the ranked based case only, since the proof for the non rank based case is identical apart from notation.

Let $\{\boldsymbol{\ell}^*_j\}_{j=1}^J \in (\hat{\mathcal{E}}^1_\beta)^{\otimes J}$ be a maximizer for problem 3.4 and $\{\ell^*_j\}_{j=1}^J \in (\mathcal{E}^1_\beta)^{\otimes J}$ be such that

$$
g^*(y) \equiv \sum_{j=1}^J \lambda_j \sum_{m\in C_j} \ell^*_j(y_m)
$$

is a maximizer for problem 2.1 with $\mathcal{G} = \mathcal{G}_1$.

Define $\boldsymbol{\ell}_j \equiv \left(\ell^*_j(x_1), ..., \ell^*_j(x_d)\right)^T \in \mathbb{R}^d$ and note by lemma 3.10 that each $\boldsymbol{\ell}_j$ satisfies (3.1)-(3.4).

Define $s_j(x)$ using equation (3.6) for each vector $\boldsymbol{\ell}^*_j$, $j \in \{1, ..., J\}$ and note by lemma 3.9 that $\exists\delta_1 > 0$ be such that

$$
\ell_j(x) \equiv \log\left(s_j(x) + \delta^\alpha\right) - \log\left(s_j(1/2) + \delta^\alpha\right) \in \mathcal{E}^1_\beta
$$

for every $\delta < \delta_1$ and each $j \in \{1, ..., J\}$.

Lastly, define

$$\hat{g}(y) \equiv \sum_{j=1}^{J} \lambda_j \sum_{m \in C_j} \hat{\ell}_j(y_m)$$
$$\hat{g}^*(y) \equiv \sum_{j=1}^{J} \lambda_j \sum_{m \in C_j} \hat{\ell}_j^*(y_m)$$
$$g(y) \equiv \sum_{j=1}^{J} \lambda_j \sum_{m \in C_j} \ell_j(y_m)$$

for $y = (y_1, ..., y_m)^T \in \Delta^n$ where the "hat" ($\hat{}$) notation refers to linear interpolation of the $\boldsymbol{\ell}_j$, $\boldsymbol{\ell}_j^*$ across the x_i.

Now, for arbitrary $h_j \in \mathcal{E}_\beta^1$, $j \in \{1, ..., J\}$, $H(x) \equiv \sum_{j=1}^{J} \lambda_j \sum_{m \in C_j} h_j(y_m)$, we have

$$\begin{aligned}
\log R(H, p(t), q(t)) &= \log \left(1 + \sum_{j=1}^{J} \lambda_j \sum_{m \in C_j} h_j'(p_m(t)) \Big(q_m(t) - p_m(t) \Big) \right) \\
&\geq H(q(t)) - H(p(t)) \\
&\geq -\lambda_{(1)} \sqrt{\beta} \sum_{i=1}^{n} |q_i(t) - p_i(t)| \\
&\geq -2\lambda_{(1)} \sqrt{\beta} \qquad (3.7)
\end{aligned}$$

where the first inequality follows from the exponential concavity of H and the second follows by the $\sqrt{\beta}$ Lipschitz property of the h_j from lemma 2.7.

We will again abbreviate $R(H, p(t), q(t))$ by $R(H, t)$ throughout. Note that by the mean value theorem, for each $i \in \{1, ..., d-1\}$ and each $j \in \{1, ..., J\}$, $\exists c_{ij} \in [x_i, x_{i+1}]$ such that

$$\begin{aligned}
&\Delta_i \boldsymbol{\ell}_j = \ell_j^{*\prime}(c_{ij}) \\
&\implies \sup_{x \in [0,1]} \left| \ell_j^{*\prime}(x) - \hat{\ell}_j'(x) \right| \leq \beta\delta \\
&\implies |R(g^*, t) - R(\hat{g}, t)| \leq 2\lambda_{(1)}\beta\delta \quad \forall t \in \{1, ..., T-1\}
\end{aligned}$$

By corollary 3.7.1,

$$|R(\hat{g}^*, t) - R(g, t)| \leq 2\lambda_{(1)}\epsilon_1(\delta)$$

Let $\delta_2 > 0$ be such that $2\lambda_{(1)}\epsilon_1(\delta_2) \leq \frac{1}{2}\exp\left(-2\lambda_{(1)}\sqrt{\beta}\right)$. For $\delta < \delta^* \equiv \delta_1 \wedge \delta_2$, we have

$$R(\hat{g}^*, t) \geq \frac{1}{2}\exp\left(-2\lambda_{(1)}\sqrt{\beta}\right)$$
$$\text{and}$$
$$R(\hat{g}, t) \geq \frac{1}{2}\exp\left(-2\lambda_{(1)}\sqrt{\beta}\right) \qquad \forall t \in \{1, ..., T-1\}$$

where we have applied the bound on $\log R(g, t)$, $\log R(g^*, t)$ established above and the fact that $\beta\delta \leq \epsilon_1(\delta)$. Noting that $\log(x)$ is Lipschitz continuous with constant $2\exp\left(2\lambda_{(1)}\sqrt{\beta}\right)$ on the interval $\left[\frac{1}{2}\exp\left(-2\lambda_{(1)}\sqrt{\beta}\right), \infty\right)$, we have

$$|S_1 - V(\hat{g},T)| = |V(g^*,T) - V(\hat{g},T)| \leq \frac{1}{T}\sum_{t=1}^{T-1} |\log R(g^*,t) - \log R(\hat{g},t)| \leq 4\left(\frac{T-1}{T}\right) e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\beta\delta$$

and

$$\left|\hat{S}_1(\delta) - V(g,T)\right| = |V(\hat{g}^*,T) - V(g,T)| \leq \frac{1}{T}\sum_{t=1}^{T-1} |\log R(\hat{g}^*,t) - \log R(g,t)| \leq 4\left(\frac{T-1}{T}\right) e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta)$$

Therefore

$$\begin{aligned} S_1 - 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\beta\delta \leq J(\hat{g}) \leq V(\hat{g}^*,T) = \hat{S}_1(\delta) &\leq V(g,T) + 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta) \\ &\leq V(g^*,T) + 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta) \\ &= S_1 + 4e^{2\lambda_{(1)}\sqrt{\beta}}\lambda_{(1)}\epsilon_1(\delta) \end{aligned}$$

since the $\boldsymbol{\ell}_j \in \hat{\mathcal{E}}_\beta^1$ satisfy the constraints of problem 3.4 and the $\ell_j \in \mathcal{E}_\beta^1$ satisfy the constraints of problem 2.1 with $\mathcal{G} = \mathcal{G}_1$. Rearranging yields the result. □

Remark 3.12. This is problematic for practical purposes. If one chooses to use "large" β, δ needs to be astronomically small in order to guarantee that the difference between optimal values in problem 2.1 with $\mathcal{G} = \mathcal{G}_1$ and problem 3.4 is small, in which case problem 3.4 becomes incredibly computationally complex. For example, in [1], the authors use β as large as 10^8 for numerical examples, with the smallest value used being 10^4 in a market of $n = 100$ stocks with $K = 1, \boldsymbol{\lambda} = 1/n$. Based on the above analysis, we only know that we are able to ensure

$$\hat{S}_1 \geq S_1 - \epsilon$$

for some $\epsilon \in (0,1)$ and $\beta = 10^8$ when

$$\delta \leq \frac{\epsilon}{4e^{200}10^6} \leq \frac{1}{4e^{200}10^6}$$

which would require optimization in at least $\lceil 4e^{200}10^6 \rceil$ dimensions (!), computational complexity which is tremendously problematic. Hence, we would not necessarily expect the approximation to be good for any computationally reasonable selection of δ - i.e. we require β to be small if we wish to obtain a computationally feasible approximation to S_1 that is also close enough to it.

3.2 Optimizing Over $\mathcal{G}_1$ Under Additional Market Assumptions

There is considerably more that can be said with respect to this problem if we make the following two realistic assumptions about the market of stocks in which trades are conducted:

Market Assumption 1. $\exists \theta \in (0,1]$ such that

$$q_i(t) \geq \theta p_i(t) \quad \forall i \in \{1,...,n\}, t \in \{1,...,T-1\}$$

Market Assumption 2. $\exists \gamma > 0$ such that

$$p_1(t) \leq 1 - \gamma \quad \forall t \in \{1,...,T-1\}$$

Market assumption 1 can be explained economically as follows: at time $t+1$, we assume that the market weight for the stock which had rank i at time t is bounded below by some proportion of its market weight at time t. Loosely speaking, the stock with rank i at time t did not lose too much of its value relative to other stocks in the market over a one step time horizon. This is realistic in the case of large cap stocks, but this is of course not the case for low cap or penny stocks, whose prices are much more volatile over short time horizons. Market assumption 2 is commonly referred to as "market diversity" in the literature (maybe [2]) - specifically, the market cannot be dominated by a single stock.

We have the following result:

Theorem 3.13. *Suppose market assumption 1 and market assumption 2 hold and that* $n > \frac{(1-\theta)(1-\gamma)}{\theta\gamma}$. *If* $\boldsymbol{\lambda} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^J$, *then*

$$S_1 - C\delta \leq \hat{S}_1$$

where

$$C \equiv 4\left(1 - \frac{1}{T}\right)\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}\left(\frac{\beta(1-\theta)}{n}\right)$$

whenever

$$\frac{2(1-\theta)\beta\delta}{n} \leq \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)$$

Proof:
We again provide only the proof for the rank based case, as the proof for the non rank based case is identical apart from notation.

Let $\{\boldsymbol{\ell}_j^*\}_{j=1}^J \in (\hat{\mathcal{E}}_\beta^1)^{\otimes J}$ be a maximizer for problem 3.4 and $\{\ell_j^*\}_{j=1}^J \in (\mathcal{E}_\beta^1)^{\otimes J}$ be such that

$$g^*(y) \equiv \frac{1}{n}\sum_{j=1}^J \sum_{m \in C_j} \ell_j^*(y_m)$$

is a maximizer in problem 2.1 with $\mathcal{G} = \mathcal{G}_1$.

Define $\boldsymbol{\ell}_j \equiv \left(\ell_j^*(x_1), ..., \ell_j^*(x_d)\right)^T \in \hat{\mathcal{E}}_\beta^1$ and let

$$\hat{g}(y) \equiv \frac{1}{n}\sum_{j=1}^J \sum_{m \in C_j} \hat{\ell}_j(y_m)$$

$$\hat{g}^*(y) \equiv \frac{1}{n}\sum_{j=1}^J \sum_{m \in C_j} \hat{\ell}_j^*(y_m)$$

By lemma 2.4 and market assumption 2,

$$-\gamma^{-1} \leq -\frac{1}{1-p_i(t)} \leq \ell_j^{*\prime}(p_i(t)) \leq \frac{1}{p_i(t)} \qquad \forall i \in \{1, ..., n\}, t \in \{1, ...T-1\}$$

Thus, dropping the t arguments and abbreviating summations, we have

$$R(g^*, t) = 1 + \frac{1}{n}\sum_{\substack{j=1 \\ m \in C_j}}^J \ell_j^{*\prime}(p_m)(q_m - p_m) \geq 1 - \frac{1}{n}\sum_{q_m \geq p_m} \gamma^{-1}(q_m - p_m) + \frac{1}{n}\sum_{q_m < p_m}\left(\frac{q_m}{p_m} - 1\right)$$

$$
\begin{aligned}
&= 1 + \frac{1}{n} \sum_{q_m < p_m} \left(\frac{q_m}{p_m} - 1 + \gamma^{-1}(q_m - p_m) \right) \\
&\geq 1 + \frac{1}{n} \sum_{q_m < p_m} \left(\theta - 1 + \gamma^{-1}(\theta - 1) p_m \right) \\
&= 1 - \frac{1-\theta}{n} \sum_{q_m < p_m} \left(1 + \gamma^{-1} p_m \right) \\
&\geq 1 - \frac{(1-\theta)}{n} \left(n - 1 + \frac{1}{\gamma} \right) \\
&= \theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} > 0 \qquad (3.8)
\end{aligned}
$$

$\forall t \in \{1, ..., T-1\}$, where the second equality holds since $\sum_{i=1}^{n}(q_i - p_i) = 0$, the last inequality holds from the condition on n and we have applied market assumption 1.

For each $i \in \{1, ..., d-1\}$, $j \in \{1, ..., J\}$, $\exists c_{ij} \in [x_i, x_{i+1}]$ such that

$$
\begin{aligned}
\ell_j^{*\prime}(c_{ij}) &= \Delta_i \boldsymbol{\ell} \\
\implies \sup_{x \in [0,1]} \left| \ell_j^{*\prime}(x) - \hat{\ell}_j'(x) \right| &\leq \beta\delta
\end{aligned}
$$

Define

$$
\begin{aligned}
A &\equiv \{(j, m) : m \in C_j, \quad \ell_j^{*\prime}(p_m(t)) - \hat{\ell}_j'(p_m(t)) > 0, \quad q_m(t) - p_m(t) > 0\} \\
B &\equiv \{(j, m) : m \in C_j, \quad \ell_j^{*\prime}(p_m(t)) - \hat{\ell}_j'(p_m(t)) < 0, \quad q_m(t) - p_m(t) < 0\}
\end{aligned}
$$

We have

$$
\begin{aligned}
R(g^*, t) - R(\hat{g}, t) &\leq \frac{1}{n} \sum_{A \cup B} \left(\ell_j^{*\prime}(p_m(t)) - \hat{\ell}_j'(p_m(t)) \right) (q_m(t) - p_m(t)) \\
&\leq \frac{\beta\delta}{n} \left(\sum_{q_m > p_m} (q_m - p_m) + \sum_{p_m > q_m} (p_m - q_m) \right) \\
&= \frac{2\beta\delta}{n} \sum_{p_m > q_m} (p_m - q_m) \\
&\leq \frac{2\beta\delta(1-\theta)}{n} \leq \frac{1}{2} \left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} \right) \\
\implies R(\hat{g}, t) &\geq \frac{1}{2} \left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} \right)
\end{aligned}
$$

By the identical logic for $R(\hat{g}, t) - R(g^*, t)$, we have

$$
|R(g^*, t) - R(\hat{g}, t)| \leq \frac{2\beta(1-\theta)\delta}{n}
$$

Since log is Lipschitz continuous with constant $2\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}$ on the interval $\left[\frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right), \infty\right)$, we have

$$
\begin{aligned}
|S_1 - V(\hat{g}, T)| &\leq \frac{1}{T} \sum_{t=1}^{T-1} |\log(R(g^*, t)) - \log(R(\hat{g}, t))| \\
&\leq \frac{T-1}{T} \cdot 2 \left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} \right)^{-1} \cdot \frac{2\beta(1-\theta)\delta}{n} = C\delta
\end{aligned}
$$

$$\implies S_1 - C\delta \le V(\hat{g}, T) \le V(\hat{g}^*, T) = \hat{S}_1$$

since $\{\boldsymbol{\ell}_j\}_{j=1}^J \in (\hat{\mathcal{E}}^1_\beta)^{\otimes J}$ by lemma 3.10. This concludes the proof. $\square$

We can prove a (very slightly) stronger result (which I have not written here) if we impose the further assumption

Market Assumption 3. $\exists \gamma' > 0$ such that

$$\gamma' \le p_n(t) \qquad \forall t \in \{1, ...T-1\}$$

However, we are not sure that this is a reasonable assumption.

Remark 3.14. Theorem 3.13 reduces the problem to grow in computational complexity linearly in β, which allows us to form good approximations when $\frac{\beta}{n}$ is reasonably small. It is not unreasonable to take $\gamma = \theta = 1/2$ (and perhaps using stock data to estimate these constants could yield an even better Lipschitz constant). Additionally, there is no requirement on the mesh size δ other than the one associated with theorem 3.13.

3.3 Optimizing Over $\mathcal{G}_2$ Under Additional Market Assumptions

In this subsection, we consider discretizing the derivative of the log generating function instead of the function itself.

Definition 3.15. Let $d \ge 3$ and fix a partition $\mathcal{P} = (x_1, ..., x_d)^T$ of $[0,1]$ with $0 = x_1 < x_2 < ... < x_d = 1$. As before, let

$$\delta \equiv \max_{i \in \{1,...,d-1\}} |x_{i+1} - x_i|$$

Let $\hat{\mathcal{E}}^2_{\alpha,\beta}$ be the set of vectors $\boldsymbol{\phi} = (\phi_1, ..., \phi_d)^T \in \mathbb{R}^d$ such that

$$\Delta_i \boldsymbol{\phi} + \phi_{i+1}^2 \le 0 \quad , \quad \Delta_i \boldsymbol{\phi} \equiv \frac{\phi_{i+1} - \phi_i}{x_{i+1} - x_i} \quad , \quad \forall i \in \{1, ..., d-1\} \tag{3.9}$$

$$0 \le \phi_d \quad , \quad \phi_1 \le \beta \tag{3.10}$$

$$\left| \frac{\Delta_i \boldsymbol{\phi} - \Delta_{i+1} \boldsymbol{\phi}}{\frac{x_{i+2} - x_i}{2}} \right| \le \alpha \qquad \forall i \in \{1, ..., d-2\} \tag{3.11}$$

We will use functions $\hat{\phi}(x) \equiv \phi_i + \Delta_i \boldsymbol{\phi}(x - x_i)$ on $[x_i, x_{i+1}]$ to approximate the derivative of functions in $\mathcal{E}^2_{\alpha,\beta}$. Explicitly, we will be considering portfolios π of the following form in the non rank based case: for $i \in C_j$, we will have portfolio weight

$$\pi_i(t) = \mu_i(t) \left(\lambda_j \hat{\phi}_j(x_i) + 1 - \sum_{k=1}^J \sum_{m \in C_k} \lambda_k \hat{\phi}_k(x_m) \mu_m(t) \right)$$

where the $\{\hat{\phi}_j\}_{j=1}^J$ are interpolations of $\{\boldsymbol{\phi}_j\}_{j=1}^J \in (\hat{\mathcal{E}}^2_{\alpha,\beta})^{\otimes J}$. The portfolio weights are defined analogously in the rank based case. (3.9) approximates exponential concavity, since $\ell \in C^1([0,1])$ with second right derivative existing everywhere on $[0,1)$ is exponentially concave if and only if

$$\ell''_+(x) + \left(\ell'(x)\right)^2 \le 0 \quad \forall x \in [0,1)$$

(3.10) enforces the bounds on the first derivative in $\mathcal{E}^2_{\alpha,\beta}$ and (3.11) is an approximation of α-Lipschitz continuity for the second derivative.

We define

$$
\begin{aligned}
\ell_{\boldsymbol{\phi}}(x) &\equiv \int_0^x \hat{\phi}(u)du \\
&= \sum_{i=1}^{d-1} \left(\frac{(\phi_i + \phi_{i+1})(x_{i+1} - x_i)}{2} \mathbb{1}(x_{i+1} \leq x) + (x - x_i) \left(\phi_i + \frac{\Delta_i \phi (x - x_i)}{2} \right) \mathbb{1}(x_i \leq x < x_{i+1}) \right)
\end{aligned}
$$

We introduce the following optimization problem, which will approximate problem 2.1 in the case $\mathcal{G} = \mathcal{G}_2$:

Problem 3.16.

$$
\sup_{\{\boldsymbol{\phi}_j\}_{j=1}^J \in (\hat{\mathcal{E}}^2_{\alpha,\beta})^{\otimes J}} \frac{1}{T} \log V \left(\sum_{j=1}^J \sum_{i \in C_j} \lambda_j \ell_{\boldsymbol{\phi}_j}(x_i), T \right)
$$

Lemma 3.17. *A maximizer exists for problem 3.16. We will define $\hat{S}_2(\delta)$ to be the associated optimal value.*

Proof:
The proof is analogous to that of lemma 3.5. $\left((\hat{\mathcal{E}}^2_{\alpha,\beta})^{\otimes J}, \|\cdot\| \right)$ where $\|\{\boldsymbol{\phi}_j\}_{j=1}^J\| \equiv \sqrt{\sum_{j=1}^J \sum_{i=1}^d (\boldsymbol{\phi}_j)_i^2}$ is a compact normed space, and the function $\{\boldsymbol{\phi}_j\}_{j=1}^J \mapsto V\left(\sum_{j=1}^J \sum_{i \in C_j} \lambda_j \hat{\ell}_{\boldsymbol{\phi}_j}(x_i), T \right)$ is continuous, which shows a maximizer exists.

Finiteness of $\hat{S}_2(\delta)$ follows since the zero vector is in $\hat{\mathcal{E}}^2_{\alpha,\beta}$, so that $\hat{S}_2(\delta) \geq \frac{1}{T} \log V_\mu(T) = 0$ □

Lemma 3.18. *If $\ell \in \mathcal{E}^2_{\alpha,\beta}$, then $\boldsymbol{\phi} \equiv (\ell'(x_1), ..., \ell'(x_d))^T \in \hat{\mathcal{E}}^2_{\alpha,\beta}$. Additionally,*

$$
d(\ell, \ell_{\boldsymbol{\phi}}) \leq \frac{\alpha \delta^2}{2} \tag{3.12}
$$

Proof:
We first prove (3.12). Fix $x \in [0,1)$. Of course, $x \in [x_i, x_{i+1})$ for some $i \in \{1, ..., d-1\}$. By the α-Lipschitz continuity of ℓ'' and the mean value theorem, $\exists c_i \in [x_i, x_{i+1}]$ such that

$$
\begin{aligned}
|\ell'(x) - \ell'_{\boldsymbol{\phi}}(x)| &= |\ell'(x) - \ell'(x_i) - \Delta_i \boldsymbol{\phi}(x - x_i)| \\
&= \left| \int_{x_i}^x \left(\ell''(u) - \ell''(c_i) \right) du \right| \\
&\leq \int_{x_i}^x \alpha |u - c_i| du \\
&= \int \alpha (u - c_i) \mathbb{1}(c_i \leq u \leq x) du + \int \alpha (c_i - u) \mathbb{1}(x_i \leq u \leq c_i \wedge x) du \\
&= \frac{\alpha}{2} \left((x - c_i)^2 \mathbb{1}(c_i \leq x) + (c_i - x_i)^2 - (c_i - c_i \wedge x)^2 \right) \\
&= \frac{\alpha}{2} \left((x - c_i)^2 (2\mathbb{1}(c_i \leq x) - 1) + (c_i - x_i)^2 \right) \\
&\leq \frac{\alpha}{2} \left((x_{i+1} - c_i)^2 + (c_i - x_i)^2 \right)
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\alpha}{2}\Big((x_{i+1}-c_i)^2 + 2(x_{i+1}-c_i)(c_i-x_i) + (c_i-x_i)^2\Big) \\
&= \frac{\alpha}{2}(x_{i+1}-x_i)^2 \\
&\le \frac{\alpha\delta^2}{2}
\end{aligned}
$$

As $\ell'(1) = \ell'_{\boldsymbol{\phi}}(1)$, this yields (3.12). We proceed to prove the first claim:

$$
\begin{aligned}
\Delta_i \boldsymbol{\phi} + \phi_{i+1}^2 &= \ell''(c_i) + \phi_{i+1}^2 \\
&\le -(\ell'(c_i))^2 + \phi_{i+1}^2 \\
&= -(\ell'(c_i))^2 + (\ell'(x_i))^2 \\
&\le 0
\end{aligned}
$$

where the first inequality follows from exponential concavity. This is precisely (3.9). Trivially (3.10) holds. Again, denote $x_{i+1} - x_i$ by δ_i and fix $i \in \{1, ..., d-2\}$. To see (3.11), we note that

$$
\begin{aligned}
|\Delta_{i+1}\boldsymbol{\phi} - \Delta_i \boldsymbol{\phi}| &\le |\Delta_{i+1}\boldsymbol{\phi} - \ell''(x_{i+1})| + |\ell''(x_{i+1}) - \Delta_i \boldsymbol{\phi}| \\
&= \frac{1}{\delta_{i+1}} \left| \int_{x_{i+1}}^{x_{i+2}} \Big(\ell''(u) - \ell''(x_{i+1})\Big) du \right| + \frac{1}{\delta_i} \left| \int_{x_i}^{x_{i+1}} \Big(\ell''(u) - \ell''(x_i)\Big) du \right| \\
&\le \frac{\alpha\delta_{i+1}^2}{2\delta_{i+1}} + \frac{\alpha\delta_i^2}{2\delta_i} \\
&= \alpha \frac{x_{i+2} - x_i}{2}
\end{aligned}
$$

again by α-Lipschitz continuity. This yields (3.11) upon rearranging. □

We have the following theorem:

Theorem 3.19. *Suppose market assumption 1 and market assumption 2 hold and that* $n > \frac{(1-\theta)(1-\gamma)}{\theta\gamma}$*. If* $\boldsymbol{\lambda} = \frac{1}{n}\mathbf{1} \in \mathbb{R}^J$*, then*

$$
S_2 - K\delta^2 \le \hat{S}_2(\delta)
$$

where

$$
K \equiv 2\left(1 - \frac{1}{T}\right)\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}\left(\frac{\alpha(1-\theta)}{n}\right)
$$

whenever

$$
\frac{(1-\theta)\alpha\delta^2}{n} \le \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)
$$

Proof:
We again provide only the proof for the rank based case, as the proof for the non rank based case is identical apart from notation.

Let $\{\boldsymbol{\phi}_j^*\}_{j=1}^J \in (\hat{\mathcal{E}}_{\alpha,\beta}^2)^{\otimes J}$ be a maximizer for problem 3.16 and $\{\ell_j^*\}_{j=1}^J \in (\mathcal{E}_{\alpha,\beta}^2)^{\otimes J}$ be such that

$$
g^*(y) \equiv \frac{1}{n}\sum_{j=1}^J \sum_{m \in C_j} \ell_j^*(y_m)
$$

is a maximizer in problem 2.1 with $\mathcal{G} = \mathcal{G}_2$.

Define $\boldsymbol{\phi}_j \equiv \left(\ell_j^{*\prime}(x_1), ..., \ell_j^{*\prime}(x_d)\right)^T$ and let

$$\hat{g}(y) \equiv \frac{1}{n} \sum_{j=1}^{J} \sum_{m \in C_j} \ell_{\boldsymbol{\phi}_j}(y_m)$$
$$\hat{g}^*(y) \equiv \frac{1}{n} \sum_{j=1}^{J} \sum_{m \in C_j} \ell_{\boldsymbol{\phi}_j^*}(y_m)$$

Recall that the proof of (3.8) relied only on exponential concavity, so we have yet again

$$R(g^*, t) \geq \theta - \frac{(1-\theta)(1-\gamma)}{\gamma n} > 0$$

$\forall t \in \{1, ..., T-1\}$, where the strict inequality holds from the condition on n.

Define

$$A \equiv \{(j,m) : m \in C_j, \quad \ell_j^{*\prime}(p_m(t)) - \ell'_{\boldsymbol{\phi}_j}(p_m(t)) > 0, \quad q_m(t) - p_m(t) > 0\}$$
$$B \equiv \{(j,m) : m \in C_j, \quad \ell_j^{*\prime}(p_m(t)) - \ell'_{\boldsymbol{\phi}_j}(p_m(t)) < 0, \quad q_m(t) - p_m(t) < 0\}$$

We have

$$\begin{aligned}
R(g^*, t) - R(\hat{g}, t) &\leq \frac{1}{n} \sum_{A \cup B} \left(\ell_j^{*\prime}(p_m(t)) - \ell'_{\boldsymbol{\phi}_j}(p_m(t))\right)(q_m(t) - p_m(t)) \\
&\leq \frac{\alpha\delta^2}{2n} \left(\sum_{q_m > p_m} (q_m - p_m) + \sum_{p_m > q_m} (p_m - q_m) \right) \\
&= \frac{\alpha\delta^2}{n} \sum_{p_m > q_m} (p_m - q_m) \\
&\leq \frac{\alpha\delta^2(1-\theta)}{n} \leq \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right) \\
\implies R(\hat{g}, t) &\geq \frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)
\end{aligned}$$

where the first inequality follows from (3.12) and the rest follow from market assumptions 1 and 2. By the identical logic for $R(\hat{g}, t) - R(g^*, t)$, we have

$$|R(g^*, t) - R(\hat{g}, t)| \leq \frac{\alpha(1-\theta)\delta^2}{n}$$

Since log is Lipschitz continuous with constant $2\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1}$ on the interval $\left[\frac{1}{2}\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right), \infty\right)$, we have

$$\begin{aligned}
|S_2 - V(\hat{g}, T)| &\leq \frac{1}{T} \sum_{t=1}^{T-1} |\log(R(g^*, t)) - \log(R(\hat{g}, t))| \\
&\leq \frac{T-1}{T} \cdot 2\left(\theta - \frac{(1-\theta)(1-\gamma)}{\gamma n}\right)^{-1} \cdot \frac{\alpha(1-\theta)\delta^2}{n} = K\delta^2 \\
\implies S_2 - K\delta &\leq V(\hat{g}, T) \leq V(\hat{g}^*, T) = \hat{S}_2(\delta)
\end{aligned}$$

since $\{\boldsymbol{\phi}_j\}_{j=1}^J \in (\hat{\mathcal{E}}^2_{\alpha,\beta})^{\otimes J}$ by lemma 3.18. This concludes the proof. $\square$

Remark 3.20. Theorem 3.19 shows that we may let β be arbitrarily large without penalty to the lower bound of $\hat{S}_2(\delta)$.

4 Empirical Results and Open Market Considerations

The approach outlined in the previous sections requires us to fix a sub-market containing n stocks in which to trade over the finite time horizon. However, in practice for this section we will be considering an open market in which we trade only in the top n stocks under the rank-based case. We first need to augment our construction of relative wealth slightly, where our benchmark is no longer the market portfolio, but instead the portfolio which invests a proportion

$$p_i(t) \equiv \mu_{(i)}(t) \equiv \frac{X_{(i)}(t)}{\sum_{j=1}^n X_{(j)}(t)}$$

of its wealth in the ith ranked stock at time t, where $X_{(i)}(t)$ is the market capitalization of the ith ranked stock at time t. This is simply the weighted portfolio for the index consisting of the top n stocks in the market. We denote its wealth at time T by $Z_\mu(T)$ in an abuse of notation.

Let $\tilde{X}_i(t+1)$ be the market capitalization at time $t+1$ of the stock which had rank i at time t. It is easily seen that

$$Z_\mu(T) = \prod_{t=0}^{T-1} \frac{\sum_{i=1}^n \tilde{X}_i(t+1)}{\sum_{j=1}^n X_{(j)}(t)}$$

which implies that, for any rank based portfolio π, we have

$$V_\pi(T) \equiv \frac{Z_\pi(T)}{Z_\mu(T)} = \prod_{t=0}^{T-1} \left(\sum_{i=1}^n \pi_{\sigma_t(i)}(t) \frac{q_i(t)}{p_i(t)} \right)$$

where

$$q_i(t) \equiv \frac{\tilde{X}_i(t+1)}{\sum_{j=1}^n \tilde{X}_j(t+1)}$$

In particular,

$$V_\pi(T) = \prod_{t=0}^{T-1} \Big(1 + \nabla g(p(t)) \cdot (q(t) - p(t)) \Big)$$

for π which is rank based functionally generated by e^g. Hence the results of sections 2 and 3 apply identically to this new setup, provided the assumptions on p and q are the same as the assumptions on the closed market.

We are now ready to compare the performance of the method of [1] and our optimization approach. We have market capitalizations for the top $n = 100$ US securities for each trading day between January 1st, 1957 to June 29th, 2022, along with their capitalizations on the next trading day. In this section, we present the results of solving[1] the maximization problems 3.4 and 3.16 over the finite time horizon which spans January 1st, 1957 to December 31st, 2002, and testing the performance of the portfolios generated by the resulting functions on the remaining data. For this example (untuned, with resepct to J and $\boldsymbol{\lambda}$), we fix $J = 3$, $\boldsymbol{\lambda} = (1, 1, 1)^T/n$.

This gives the following output:

1. The implementation is performed in MATLAB using the CVX software package; see [4], [5] for details.

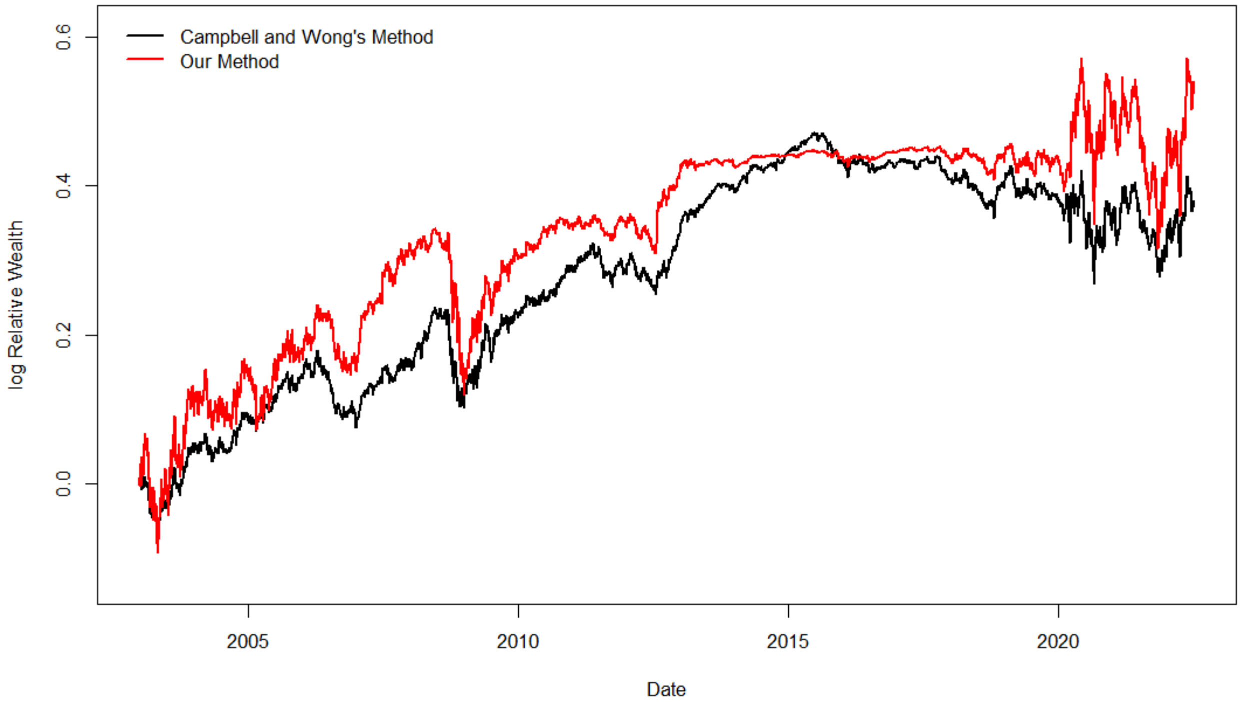

Figure 1: log of wealth relative to the weighted index portfolio of the top 100 US stocks, for the functionally generated portfolios created using our method and that of Campbell and Wong, [1]. The functionally generated portfolios are formed using the solution to problems 3.16 and 3.4, respectively, over the period between January 1st, 1957 and December 31st, 2002.

We remark that the function constructed by our optimization problem is considerably more volatile than that of problem 3.4, but both appear to follow nearly identical trends over time.

5 Future Research Goals

We desire to assess the long run properties of these types of functionally generated portfolios. In particular, we are interested in what type of statements can be made about long run average growth rate and the statistical properties of the wealth associated with these portfolios (relative, and non-relative to the market portfolio/weighted index portfolio). A very relevant question is how these portfolios perform under the imposition of transaction costs and costs associated with shorting stocks. This is partially addressed in [1]; however, it is not considered in the optimization algorithm and it is of great interest to incorporate such costs in the computation of the generating function We are also interested in the infinite dimensional optimization problems and whether or not they might be solved under additional assumptions on the structure/distributions of the underlying market capitalizations. The addition of more structural conditions on the market may also produce better results in terms of the finite dimensional approximations.

We also note the possibility of extending the definition of functional generation to functions which take inputs other than the market weights, such as the approach in continuous time of [6]. This idea is explored in [8], but with non-functionally generated portfolios in a machine learning context. Doing so would require optimizing with considerably more data.

Obviously, these goals are incredibly ambitious, and are simply listed to illustrate the areas which we would like to explore.

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