Structural VAR: Proxy-VAR

Gu, Xin*

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1 Introduction

A central challenge in empirical macroeconomics is the **identification of structural shocks** from reduced-form innovations. In standard Vector Autoregression (VAR), the observed residuals (u_t) are typically correlated across equations, reflecting a mixture of various structural disturbances (ϵ_t). To recover the causal impact of a specific policy - such as a monetary policy - one must isolate the "pure" structural shock from the endogenous movements in the data.

To address this, the **proxy-SVAR framework**, popularize by Stock and Watson (2012) and Mertens and Ravn (2013), utilizes **external instruments** (Z_t) to achieve identification. Unlike traditional identification schemes that rely on short-run zero restrictions (Cholesky) or sign restrictions, the proxy-SVAR approach leverages information outside the VAR system. An external variable is considered a valid proxy if it satisfies two fundamental conditions:

- **Relevance**: The instrument must be correlated with the structural shock of interest, i.e., $\mathbb{E}[Z_t \epsilon'_{i,t}] \neq 0$.
- **Exogeneity**: The instrument must be uncorrelated with all other structural shocks in the system, i.e., $\mathbb{E}[Z_t \epsilon'_{-i,t}] = 0$.

2 Proxy-SVAR Estimation

The Proxy-SVAR relies on the relationship $u_t = B\epsilon_t$. We focus on identifying the first column of $B(b_1)$, which represents the impact of structural shock ($\epsilon_{1,t}$) of interest, as Gertler and Karadi (2015).

We partition the n variable in the VAR into the **policy variable** (p_t, the first variable) and all **other variables** (q_t, the remaining $n-1$ variables). We partition the residuals and the B matrix accordingly:

$$\begin{bmatrix} u_{pt} \\ u_{qt} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \epsilon_{pt} \\ \epsilon_{qt} \end{bmatrix},$$

*PhD in Economics, School of Finance (School of Zheshang Asset Management), Zhejiang Gongshang University, Hangzhou, Zhejiang, China. Email: richardgu26@zgjsu.edu.cn.

2.1 2SLS Stage

We start the **Two-Stage-Least-Square (2SLS)**. First, we regress u_{pt} on the instrument Z_t:

$$u_{pt} = \beta_{11} + \beta_{12} Z_t + e_{pt}\,, \tag{1}$$

To include intercept, we ensure that β_{12} captures only the **variation** in the policy residual that moves with the instrument, regardless of any temporary level shifts in the sub-sample. The predicted value $\hat{u}_{pt}$ represents the part of policy residual driven purely by the instrument (and thus by the structural shock ϵ_{pt}).

Next, we **identify the relative ratio**. We regress the other residuals u_{qt} on the predicted $\hat{u}_{pt}$:

$$u_{qt} = \beta_{21} + \beta_{22}\hat{u}_{pt} + e_{qt}\,, \tag{2}$$

The coefficient β_{22} is a vector of **relative ratios**:

$$\beta_{22} = \frac{b_{21}}{b_{11}}\,, \tag{3}$$

It uncovers the "direction" of the shock.

The reduced-form covariance matrix $\Sigma = \mathbb{E}[u_t u_t']$ is partitioned to match our variables:

$$\Sigma = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

where $S_{11} = \mathbb{E}[u_{pt}^2]$, the variance of the policy residual, $S_{21} = \mathbb{E}[u_{qt}u_{pt}']$, the covariance between policy and others, and $S_{22} = \mathbb{E}[u_{qt}u_{qt}']$, the covariance matrix of other variables.

Now, we map covariance matrix Σ to the structural B matrix. Since $\Sigma = BB'$, we expand the partitioned multiplication:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} b_{11}' & b_{12}' \\ b_{21}' & b_{22}' \end{bmatrix} \tag{4}$$

This gives us three identity equations:

$$\begin{aligned} S_{11} &= b_{11}^2 + b_{12}b_{12}'\,, & (5)\\ S_{21} &= b_{21}b_{11}' + b_{22}b_{12}'\,, & (6)\\ S_{22} &= b_{21}b_{21}' + b_{22}b_{22}'\,, & (7) \end{aligned}$$

From 2SLS, we have the identified vector $\beta_{22} = b_{21}b_{11}^{-1}$, which mean we can write $b_{21} = \beta b_{11}$. Plug it into equations (5) and (6), and obtain

$$\begin{aligned} b_{12}b_{12}' &= S_{11} - b_{11}^2\,, \\ S_{21} &= \beta_{22}b_{11}^2 + b_{22}b_{12}'\,, \end{aligned}$$

Q is defined as the covariance matrix of the term $(u_{qt} - \beta_{22}u_{pt})$:

$$Q = \mathbb{E}[(u_{qt} - \beta_{22}u_{pt})(u_{qt} - \beta_{22}u_{pt})']\,,$$

We have $S_{11} = \mathbb{E}[u_{pt}u_{pt}']$, $S_{21} = \mathbb{E}[u_{qt}u_{pt}']$, and $S_{22} = \mathbb{E}[u_{qt}u_{qt}']$. Then,

$$\begin{aligned} Q &= \mathbb{E}[u_{qt}u_{qt}' - u_{qt}u_{pt}'\beta_{22}' - \beta_{22}u_{pt}u_{qt}' + \beta_{22}u_{pt}u_{pt}'\beta_{22}']\,, \\ &= S_{22} - \left(S_{21}\beta_{22}' + \beta_{22}S_{21}'\right) + \beta_{22}S_{11}\beta_{22}'\,, \end{aligned}$$

$$b_{11}^2 = S_{11} - (S_{21} - \beta_{22}S_{11})'Q^{-1}(S_{21} - \beta_{22}S_{11})\,, \tag{8}$$

References

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