Standard Incomplete Market (SIM) Model

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1 Introduction

The **Bewley-Huggett-Aiyagari** framework is considered as the backbone of modern macroeconomics. The model draws from three seminal papers that moved macroeconomics beyond the **representative agent paradigm**. Huggett (1993) introduces the pure exchange economy, and shows that uninsurable idiosyncratic risk causes people to over-save for self-insurance, pushing the equilibrium risk-free risk rate r below the rate of time preference $1/(\beta - 1)$. This endowment economy model is extended by Aiyagari (1994) to include production with capital. It proves that in a steady state, the aggregate capital stock is higher than in a world with complete markets because of the **precautionary savings motive**. Bewley (1986) provides a foundational mathematics for models where agents use a single asset to smooth consumption against income shocks when insurance markets are **incomplete**.

2 Bewley-Huggett-Aiyagari Model

2.1 Individual Sequential Problem

The economy is populated with a continuum of infinitely lived households. Each household i seeks to maximize her life-time expected utility:

$$\max_{a_{i,t}, c_{i,t}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_{i,t}) \tag{1}$$

The utility function $u(\cdot)$ is typically assumed to be strictly increasing, strictly concave, and satisfies the Inada conditions. This maximization is subject to period-by-period budget constraint:

$$a_{i,t} + c_{i,t} \leq (1+r)a_{i,t-1} + y(e_{i,t}), \tag{2}$$

where r is the risk-free interest rate and $y(e_{i,t})$ is the stochastic labor income (reasonably assumed to be bounded away from zero). The exogenous state e follows a first-order Markov chain. This defines the evolution of individual wealth. Assets tomorrow $a_{i,t}$ are determined by current wealth, interest income r, and labor income, minus current consumption.

Unlike the Arrow-Debreu framework, markets in this model are **incomplete**: households are unable to trade assets that pay out contingent on the realization of e. They can only use a single non-contingent asset a to self-insure. Furthermore, households face an ad-hoc borrowing constraint:

$$a_{i,t} \geq \underline{a}, \tag{3}$$

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where $\underline{a}$ is often set to 0 (no borrowing) or the "**natural borrowing limit**", which is the maximum debt an agent can surely repay in the worst-case income scenario. Economically, it creates a group of **constrained agents** who cannot smooth consumption, leading to high marginal propensities to consume (MPC).

Next, we convert the problem (1), and two constraints (2) and (3) into a **Bellman equation**

$$V(e,a) = \max_{c,a'} u(c) + \beta \mathbb{E} V\left[V(e',a') \mid e\right] , \tag{4}$$

$$\text{s.t. } a' + c = (1+r)a + y(e) \tag{5}$$

$$a' \geq \underline{a} \tag{6}$$

In this setup, (4) contains two state variables: exogenous state e, and endogenous asset a. The solution yields two critical **policy functions**: the savings rule $a'(e,a)$ and the consumption rule $c(e,a)$.

Policy functions satisfy standard **first-order condition**:

$$u'(c) \geq \beta \mathbb{E}\left[V_a(e',a') \mid e\right] , \tag{7}$$

where equality holds unless borrowing constraint binds. We also could obtain derivative on right from **envelope condition**:

$$V_a(e,a) = (1+r)u'(c) , \tag{8}$$

Based on (7), if $\beta(1+r) \geq 1$, then $u'(c_{i,t})$ is **supermartingale**, since then

$$\mathbb{E}_t[u'(c_{i,t+1})] \leq u'(c_{i,t}) ,$$

that is, in expectation, it is decreasing. Supermartingale convergence theorem: if bounded, then $u'(c_{i,t})$ will converge almost surely to some random variable $u^{'*}$.

If $\beta(1+r) \geq 1$, then households tend toward infinite assets and consumption. Intuitively, if $\beta(1+r) = 1$, no uncertainty would make consumption c over time constant. Uncertainty and borrowing constraint create upward drift, and then need $\beta(1+r) < 1$ to cancel this out. Otherwise, consumption and assets would drift up unboundedly. We want steady states with finite assets, so assume $\beta(1+r) < 1$[1].

2.2 Aggregate Problem

Generally, we would contemplate economies with a **continuum** of such households, and consider aggregate outcomes. At this stage, let us focus on total asset demand implied by this model. It is a **heterogeneous agent economy**, with a **distribution** of households across the two states, exogenous e and endogenous assets a.

To describe the distribution of households, we adopt a **measure** μ. If finitely many e, we could define $\mu(e,\mathbb{A})$ separately for each e, as a measure on subset $\mathbb{A}$ of the asset space.

The law of motion (also known as the **Kolmogorov forward equation**)is given by

$$\mu_{t+1}(e',\mathbb{A}) = \sum_e \mu_t\left(e,(a')^{-1}(e,\mathbb{A})\right) \cdot \mathbb{P}(e,e') , \tag{9}$$

where $\mathbb{P}(e,e')$ is **transition probability** and $(a')^{-1}(e,\cdot)$ is the inverse of policy $a'(e,\cdot)$.

2.3 The Steady State

The steady state of the model consists of **policy functions** that solve Bellman, and **measure** that satisfies steady-state law of motion. In such model economy, aggregate assets and consumption

$$A = \int a d\mu = \int a'(e,a) d\mu , \tag{10}$$

$$C = \int c d\mu , \tag{11}$$

[1] For more formal proof refers to Chamberlain and Wilson (2002).