# Calculus: Homework #5

Due on October 7, 2024 at 5:30 p.m.

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# Problem 1

(Sec 3.6 - 58) Find $y\prime$ if $x^y = y^x$.

**Solution**
First, we apply natural logarithm to both side of the equation.

$$\begin{aligned} x^y &= y^x \\ y \ln x &= x \ln y \end{aligned}$$

Then, we use implict differentiation to solve this problem:

$$\begin{aligned} & y \ln x = x \ln y \\ & y\prime \ln x + \frac{y}{x} = \ln y + \frac{xy\prime}{y} \\ & (\ln x - \frac{x}{y})y\prime = \ln y - \frac{y}{x} \\ & (xy \ln x - x^2)y\prime = xy \ln y - y^2 \\ & y\prime = \frac{xy \ln y - y^2}{xy \ln x - x^2} \end{aligned}$$

# Problem 2

(Sec 3.6 - 78) Find the derivative of the following function, and simplify where possible.

$$y = \arctan\sqrt{\frac{1-x}{1+x}}$$

**Solution**

By the differential rules of inverse trigonometric functions and chain rules, we can easily find $y\prime$.

$$\begin{aligned}
y\prime &= (\arctan\sqrt{\frac{1-x}{1+x}})\prime \\
&= \frac{1}{1+\frac{1-x}{1+x}} \cdot (\sqrt{\frac{1-x}{1+x}})\prime \\
&= \frac{x+1}{2} \cdot \frac{1}{2\cdot\sqrt{\frac{1-x}{1+x}}} \cdot (\frac{1-x}{1+x})\prime \\
&= \frac{x+1}{2} \cdot \frac{1}{2\cdot\sqrt{\frac{1-x}{1+x}}} \cdot \frac{-2}{(1+x)^2} \\
&= \frac{-1}{2\sqrt{1-x^2}}
\end{aligned}$$

# Problem 3

(Sec 3.6 - 85) In exercise 83, we got formula for finding derivatives of inverse functions.

If $f(x) = x + e^x$, find $(f^{-1})\prime(1)$.

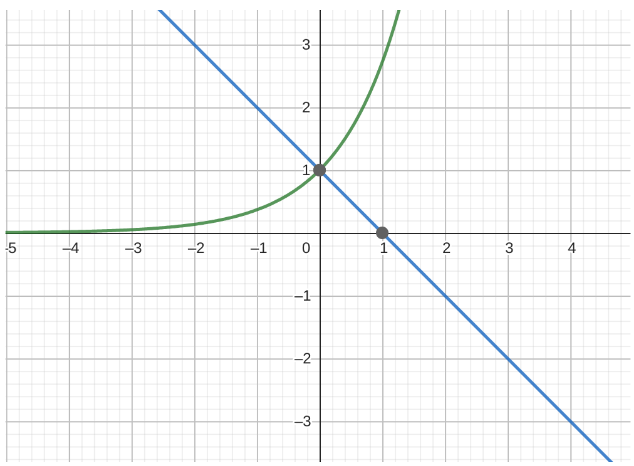
**Solution**
By the definition of inverse function and the formula for finding derivatives of inverse functions:

$$f(f\prime(x)) = x$$
$$(f^{-1})\prime(x) = \frac{1}{f\prime(f^{-1}(x))}$$

Thus, we find:

$$f(f^{-1}(1)) = 1$$

Let $f^{-1}(1)$ be t, and $e^t + t = 1$. We draw the graph of $g(x) = e^x$ (Green line) and $k(x) = 1 - x$ (Blue line)




Obviously, the only solution for $t$ is 0. $f^{-1}(1) = 0$. Then we use the formula to solve this problem:

$$f\prime(x) = e^x + 1$$

$$\begin{aligned}(f^{-1})\prime(1) &= \frac{1}{f\prime(f^{-1}(1))} \\ &= \frac{1}{f\prime(0)} \\ &= \frac{1}{2}\end{aligned}$$

# Problem 4

(Sec 3.10 - 52)

$$g(2) = -4$$
$$g\prime(x) = \sqrt{x^2 + 5}$$

(a) Use a linear approximation to estimate g(1.95) and g(2.05).
(b) Are your estimates in (a) too small or too large?

### Solution

**(a)** The linear approximation near x=2 can be written as below:

$$f(x) = g(2) + g\prime(2)(x - 2)$$
$$f(x) = 3x - 10$$

Thus:

$$f(1.95) \approx f(1.95) = 3 * 1.95 - 10 = -4.15$$
$$f(2.05) \approx f(2.05) = 3 * 2.05 - 10 = -3.85$$

**(b)**

The formula $g\prime(x) = \sqrt{x^2 + 5}$ indicates that $g\prime(x)$ is always positive when $x \in \mathbb{R}$.

Also, $g\prime(x)$ is increasing while $x \geq 0$. The fact means that the slope of $g(x)$ is positive and getting steeper.

Thus, the tangent line $f(x)$ lies below g(x), and the estimates in part(a) are too small.

# Problem 5

(Sec 4.1 - 63) Find the absolute maximum and absolute minimum values of f(x) in given interval:

$$f(x) = x^{-2} \ln x,\ x \in [\frac{1}{2}, 4]$$

**Solution**

First, we find the derivative of f(x):

$$f\prime(x) = -2x^{-3} \ln x + x^{-3} = \frac{1 - 2\ln x}{x^3}$$

$\forall x \in [\frac{1}{2}, 4],\ x^3 > 0.$ We should discuss the sign of $1 - 2\ln x$:

$$\begin{aligned} 1 - 2\ln x &= 0 \\ \ln x &= \frac{1}{2} \\ x &= \sqrt{e} \end{aligned}$$

Thus:

If $x \in [\frac{1}{2}, \sqrt{e})$, $f(x)$ is increasing; else if $x \in (\sqrt{e}, 4]$, $f(x)$ is decreasing.

So $f(x)$ has a absolute maximum value in $x = \sqrt{e}$ for $x \in [1/2, 4]$. The value is $f(\sqrt{e}) = \frac{1}{2e}$

Also, the absolute minimum value must falls on either $x = 1/2$ or $x = 4$. But, we find $f(1/2) = -4\ln 2 < 0$ and $f(4) > 0$. Hence, the absolute minimum value of $f(x)$ for $x \in [1/2, 4]$ is $-4\ln 2$ when $x = 1/2$