Abstract algebra I Homework 4

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1)

(a)

Define the homomorphism ϕ as below, where the congruent class modulo p is denoted as $[x]_p$:

$$\phi : \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/pq\mathbb{Z} \ \phi(([x]_p, [x]_q)) = [x]_{pq}$$

Suppose $[x]_p = [y]_p, [x]_q = [y]_q$, then $\phi(([x]_p, [x]_q)) = [x]_{pq}$. Since $p|(x-y)$ and $q|(x-y)$, we have $pq|(x-y), [x]_{pq} = [y]_{pq}$. Hence, $[x]_{pq} = [y]_{pq} = \phi(([y]_p, [y]_q)) = \phi(([x]_p, [x]_q))$. Thus ϕ is well-defined.

$\phi(([x]_p, [x]_q) + ([y]_p, [y]_q)) = \phi(([x+y]_p, [x+y]_q)) = [x+y]_{pq} = [x]_{pq} + [y]_{pq} = \phi(([x]_p, [x]_q)) + \phi(([y]_p, [y]_q))$, so ϕ is a homomorphism.

Suppose $\phi([x]_p, [x]_q) = 0$, x must be multiple of pq, hence, $([x]_p, [x]_q) = ([0]_p, [0]_q)$. Since $\ker \phi = \{([0]_p, [0]_q)\}$, ϕ is injective, and obviously $|\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}| = |\mathbb{Z}/pq\mathbb{Z}| = pq$, ϕ is also surjective. By proposition above, ϕ is an isomorphism.

(b)

Let $G = \langle g \rangle, H = \langle h \rangle, |G| = n, |H| = m$. If $G \times H$ is cyclic, there exists an integer d such that $(g,h)^d = (e_G, e_H)$. Since G, H are cyclic, we have $n|d, m|d \to \operatorname{lcm}(n,m)|d$. The minimum integer we can choose for d is $\operatorname{lcm}(n,m)$, it's also the order of $G \times H$. Since $|G \times H| = nm = \operatorname{lcm}(n,m)$, we conclude that $\gcd(n,m) = 1$.

Suppose $\gcd(n,m) = 1$, $\langle (g,h) \rangle$ can generate $G \times H$, since the least integer d such that $(g,h)^d = (e_G, e_H)$ is $\operatorname{lcm}(n,m) = nm$, which equals to the order of $G \times H$. Thus, $G \times H$ is cyclic if and only if $\gcd(|G|, |H|) = 1$.

(c)

$S_3 = (e, (12), (13), (23), (123), (132))$, the only proper subgroups are

$$\{e\}, \{e, (12)\}, \{e, (13)\}, \{e, (23)\}, \{e, (123), (132)\}$$

Since every two distinct subgroups follow the property: their orders are coprime and both are cyclic, their direct product should also be cyclic group by the result of last subproblem. But S_3 is not cyclic, thus it's not direct product of any of its proper subgroups.

2)

3)

(a)

(b)

(c)

4)

(a)

(b)

(c)