Abstract algebra I

Homework 5

Due: 15th October 2025

1) Let G be a finite group that acts on a finite set S. We (again) define the *orbit* of $s \in S$ to be
$$O(s) = \{g \cdot s : g \in G\},$$
and the *stabilizer* of s in G to be
$$G_s = \{g \in G : g \cdot s = s\}.$$

   (a) Check that for any two distinct elements $s, t \in S$, we either have $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.

   (b) Verify that G_s is a subgroup of G, and that the map given by
$$\{\text{cosets of } G_s \text{ in } G\} \to O(s), \quad gG_s \mapsto g \cdot s$$
   is a well-defined bijection.

   (c) Conclude that $|G_s| \cdot |O(s)| = |G|$. *(This is called the orbit-stabilizer theorem.)*

2) Consider the action of G on itself given by $(g, h) \mapsto g^{-1}hg$, this action is called *conjugation.* In this case the orbit of $h \in G$ is called the *conjugacy class* of h and we shall denote it by
$$\text{class(h)} = \{g^{-1}hg : g \in G\},$$
and the stabilizer is called the *centralizer* of h and is denoted by
$$C_G(h) = \{g \in G : g^{-1}hg = h\}.$$
Elements in the same conjugacy class are called *conjugates (of one another).*

   (a) Let $\text{class}(h_1), ..., \text{class}(h_n)$ be the distinct conjugacy classes of G, i.e., $\bigcup_{i=1}^{n} \text{class}(h_i) = G$. Derive the *class equation*
$$|G| = \sum_{i=1}^{n} \frac{|G|}{|C_G(h_i)|}.$$

   (b) Furthermore, suppose for each $i = m+1, ..., n$, we have $|\text{class}(h_i)| = 1$, while for $i = 1, ..., m$, we have $|\text{class}(h_i)| > 1$. Show that
$$|G| = |Z(G)| + \sum_{i=1}^{m} \frac{|G|}{|C_G(h_i)|},$$
   where $Z(G)$ denotes the center of G. *(Usually the class equation is written in this second form.)*

   (c) Let p be a prime, and $n \geq 1$. If $|G| = p^n$, deduce that $Z(G) \neq \{e\}$.

The rest of the homework has nothing to do with group actions.

3) Let G be a finite group and H a subgroup. The *index* of H in G, denoted $[G : H]$, is the quantity $|G|/|H|$ (indeed I could have introduced this at the start of the assignment). We usually say *H is a subgroup of index $[G : H]$*. For another subgroup $K \leq G$, we are interested in questions about $HK = \{hk : h \in H, k \in K\}$, or KH.

   (a) Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$.

   (b) Prove that $[G : H \cap K] \leq [G : H][G : K]$, with equality if and only if $G = HK$.

   (c) Prove that HK is a subgroup of G if and only if $HK = KH$.

   (d) Prove that if $[G : H]$ and $[G : K]$ are relatively prime, then $G = HK$.

   (In general, $[G : H]$ is defined to be the number of distinct left cosets of H in G, since our current definition does not make sense if G is an infinite group. And then the problem of interest would be when would H be a subgroup of finite index, but we do not go into that here...)

4) Let G be an abelian group of order $2n$. If n is odd, prove that there is only one element of order 2.