Abstract algebra I Homework 5

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1)

(a)

Suppose that for any two distinct elements $s, t \in S$, we have $O(s) \neq O(t)$ and $O(s) \cap O(t) \neq \emptyset$. There exists $g_1, g_2 \in G$ such that $g_1 s = g_2 t \Rightarrow t = g_2^{-1} g_1 s$, hence, for every $g \in G$, we have $gt = g g_2^{-1} g_1 s \in O(s)$, i.e., there's always one corresponding element in $O(s)$ for every element in $O(t)$, similarity we have $gs = g g_1^{-1} g_2 t \in O(t)$, so $O(s) = O(t)$, which contradicts our assumption. Thus $O(s) \neq O(t)$ **and** $O(s) \cap O(t) \neq \emptyset$ is impossible, we either have $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.

(b)

$e \in G_s$ trivially, also for any $u, v \in G_s$, $(uv)s = us = s \Rightarrow (uv) \in G_s$, also $us = s, s = u^{-1}s \Rightarrow u^{-1} \in G_s$, thus G_s is closed under taking products and inverses, $G_s \leq G$.

Let the map be ϕ, if $g_1 G_s = g_2 G_s$ and $g_1, g_2 \in G$, then $g_2^{-1} g_1 G_s = G_s \Rightarrow g_2^{-1} g_1 \in G_s$. Hence, $g_2^{-1} g_1 s = s, g_1 s = g_2 s, \phi(g_1 G_s) = \phi(g_2 G_s)$, we know ϕ is well-defined.

To prove the injectivity of ϕ, if $g_1 s = g_2 s$ $(g_1, g_2 \in G)$, then $g_1^{-1} g_2 s = s \Rightarrow g^{-1} g_2 \in G_s$, so $g^{-1} g_2 G_s = G_s$, we have $g_1 G_s = g_2 G_s$.

For every $u \in O(s)$, it can be written as the form: $u = gs$ for some $g \in G$, so $u = \phi(gG_s)$. ϕ is surjective. Since ϕ is both injective and surjective, ϕ is a well-defined bijection.

(c)

By the result in (b), we know $|G : G_s| = |G|/|G_s| = |O(s)|$, hence $|G_s||O(s)| = |G|$.

2)

(a)

(b)

(c)

3)

(a)

(b)

(c)

(d)

4)