Abstract algebra I Homework 5

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1)

(a)

Suppose that for any two distinct elements $s, t \in S$, we have $O(s) \neq O(t)$ and $O(s) \cap O(t) \neq \emptyset$. There exists $g_1, g_2 \in G$ such that $g_1 s = g_2 t \Rightarrow t = g_2^{-1} g_1 s$, hence, for every $g \in G$, we have $gt = g g_2^{-1} g_1 s \in O(s)$, i.e., there's always one corresponding element in $O(s)$ for every element in $O(t)$, similarity we have $gs = g g_1^{-1} g_2 t \in O(t)$, so $O(s) = O(t)$, which contradicts our assumption. Thus $O(s) \neq O(t)$ **and** $O(s) \cap O(t) \neq \emptyset$ is impossible, we either have $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.

(b)

$e \in G_s$ trivially, also for any $u, v \in G_s$, $(uv)s = us = s \Rightarrow (uv) \in G_s$, also $us = s, s = u^{-1}s \Rightarrow u^{-1} \in G_s$, thus G_s is closed under taking products and inverses, $G_s \leq G$.

Let the map be ϕ, if $g_1 G_s = g_2 G_s$ and $g_1, g_2 \in G$, then $g_2^{-1} g_1 G_s = G_s \Rightarrow g_2^{-1} g_1 \in G_s$. Hence, $g_2^{-1} g_1 s = s, g_1 s = g_2 s, \phi(g_1 G_s) = \phi(g_2 G_s)$, we know ϕ is well-defined.

To prove the injectivity of ϕ, if $g_1 s = g_2 s$ $(g_1, g_2 \in G)$, then $g_1^{-1} g_2 s = s \Rightarrow g^{-1} g_2 \in G_s$, so $g^{-1} g_2 G_s = G_s$, we have $g_1 G_s = g_2 G_s$.

For every $u \in O(s)$, it can be written as the form: $u = gs$ for some $g \in G$, so $u = \phi(gG_s)$. ϕ is surjective. Since ϕ is both injective and surjective, ϕ is a well-defined bijection.

(c)

By the result in (b), we know $|G : G_s| = |G|/|G_s| = |O(s)|$, hence $|G_s||O(s)| = |G|$.

2)

(a)

(b)

(c)

3)

(a)

Note that HK is the union of left cosets of K, namely, $HK = \bigcup_{h \in H} hK$. Suppose $h_1K = h_2K$, then $h_2^{-1}h_1K = K \Rightarrow h_2^{-1}h_1 \in K \Rightarrow h_2^{-1}h_1 \in H \cap K$. $H \cap K$ is trivially a subgroup of H, since for any $u, v \in H \cap K$, then $uv \in H$ and $uv \in K$, $u^{-1} \in H$ and $u^{-1} \in K$, we have $uv, u^{-1} \in H \cap K$, also $H \cap K$ contains the identity, thus $H \cap K \leq H$.

Since $h_2^{-1}h_1 \in H \cap K$ implies $h_1(H \cap K) = h_2(H \cap K)$, the number of left cosets of K equals to the left cosets of $H \cap K$ in H. By Lagrange's Theorem, the number is $\frac{|H|}{|H \cap K|}$, also each left cosets of K have the size of $|K|$. Hence, $|HK| = \frac{|H||K|}{|H \cap K|}$.

(b)

If $G = HK$, then $|G| = |HK| = \frac{|H||K|}{|H \cap K|}$ by (a). We obtain that $[G : H \cup K] = \frac{|H||K|}{|H \cap K|^2}$ and $[G : H][G : K] = \frac{|G|^2}{|K||H|} = \frac{|H||K|}{|H \cap K|^2}$. Thus, $[G : H \cup K] = [G : H][G : K]$.

not finish yet!!!

(c)

If $HK = KH$, every element hk in HK can be written as $k^{'}h^{'}$ for some $k, k^{'} \in K, h, h^{'} \in H$. Clearly, HK contains e, the identity element in G. Suppose $u, v \in HK = KH$, $u = h_1k_1, v = k_2h_2$, $uv = h_1k_1k_2h_2$ ($h_1, h_2 \in H$ and $k_1, k_2 \in K$), let $k_1k_2 = k_3 \in K$, then $uv = h_1k_3h_2 = h_1hk = h'k \in HK$ for some $h, h^{'} \in H, k \in K, hk = k_3h_2, h^{'} = h_1h$, thus HK is closed under taking products. Also, $u^{-1} = k_1^{-1}h_1^{-1}$ and every element in KH must can be written as ab for some , $a \in H, b \in K$, since $k^{-1} \in K, h^{-1} \in H$, so $u^{-1} \in HK$, HK is closed under taking inverse. In conclusion, $HK \leq G$ if $HK = KH$.

Conversely, if $HK \leq G$. Obviously, $K \in HK$ and $H \in HK$ by the closure property of subgroups, $KH \subseteq HK$. Suppose $hk \in HK$, since HK is a subgroup of G, let $a = h_1k_1$ be its inverse, then $hk = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} \in KH$. Since HK and KH contains each other, we have $HK = KH$. So HK is a subgroup of G if and only if $HK = KH$.

(d)

4)