Abstract algebra I

Homework 4

Due: 8th October 2025

1) For two groups G, H with identities e_G, e_H respectively, define the *direct product* of G and H to be the group whose underlying set is $G \times H$ and whose binary operation is given by
$$(g, h)(g', h') = (gg', hh'), \quad g, g' \in G, h, h' \in H.$$
   (a) Let $p \neq q$ be two prime numbers. Prove that $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \simeq \mathbb{Z}/pq\mathbb{Z}$ by constructing an isomorphism explicitly.
   (b) Let G, H be cyclic groups. Prove that $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.
   (c) Deduce that S_3 is not a direct product of any of its proper subgroups.

2) Find the order of the following group:
$$G = \langle a, b : a^6 = 1, b^2 = a^3, ba = a^{-1}b \rangle.$$
And then show that it is **not** isomorphic to the group
$$H = \langle r, s : r^6 = s^2 = 1, srs^{-1} = r^{-1} \rangle.$$

3) The symmetric group S_4 has a natural action on the set $T = \{1, 2, 3, 4\}$. For a subgroup $H \leq S_4$ and $n \in T$, we define the *orbit* $O(n)$ of n to be the set $\{\sigma(n) : \sigma \in H\}$. In this exercise the orbit of H refers to the set $\{O(n) : n \in T\}$.
   (a) Find the orbits of the following subgroups of S_4 defined by their action on T; $\langle (12) \rangle, \langle (123) \rangle, V$. *(Recall that V is the unique Klein 4-group in S_4, and we saw in class that $V \triangleleft S_4$.)*
   (b) Find another proper subgroup of S_4 with the same set of orbits as that of V.
   (c) Prove that the following subgroup of S_4 is trivial:
$$Z(S_4) = \{\sigma \in S_4 : \tau^{-1}\sigma\tau = \sigma, \textit{ for all } \tau \in S_4\}.$$
   (Recall that we encountered such a subgroup in class. It is called the center (of S_4) and $Z(S_4) \triangleleft S_4$. The notation $Z(G)$ for the center of a group G is standard.)

4) Let G be a group and $H \leq G$ a subgroup. For a set S, denote by $\mathrm{Perm}(S)$ the group of all permutations of S.
   (a) If G acts on S, show that one has an induced homomorphism $G \to \mathrm{Perm}(S)$.
   (b) Now let $S = \{gH : g \in G\}$. Show that the kernel of the induced homomorphism $G \to \mathrm{Perm}(S)$ is contained in H.
   (c) Suppose $|G|/|H| = n$ and that no nontrivial normal subgroup of G is contained in H, prove that G is isomorphic to a subgroup of S_n.