Abstract algebra I Homework 3 uwu

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1)

(a)

Claim: If G is cyclic and H is isomorphic to G, then H is cylic.

Proof. Suppose the isomorphism is $\varphi : G \to H$ and $G = \langle x \rangle \infty$, then every element of H can be written as the form $\varphi(x^d)$ for some integer d, also: $\varphi(x^d) = \varphi(x)^d$. Hence, H is generated by $\varphi(x)$, i.e., by claim above, $H = \langle \varphi(x) \rangle$ is cyclic. $\square$

$(\mathbb{Z}/15\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \bar{8}, \bar{1}1, \bar{1}3, \bar{1}4\}$ and $\mathbb{Z}/8\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$. Since $\mathbb{Z}/8\mathbb{Z}$ is a cyclic group of order 8 under addition, and $(\mathbb{Z}/15\mathbb{Z})^\times$ is not cyclic, they are not isomorphic.

(b)

Both group are cyclic groups with order 4, so they are isomorphic. Define $u_4 = \{z \in \mathbb{C}\backslash\{0\} : z^4 = 1\} = \{-1, 1, -i, i\}$. Define:

$$\varphi : \mathbb{Z}/4\mathbb{Z} \to u_4, \varphi(\bar{k}) = i^k$$

.

Suppose $a, b \in \bar{k}$ then $(a - b)$ is a multiple of 4, so $i^a = i^b$. Thus φ is well-defined. For simplicity, I denote the member of $\bar{k}$ class as $[k]$. For any classes a, b, we have $\varphi([a] + [b]) = \varphi([a + b]) = i^{a+b} = i^a i^b = \varphi([a])\varphi([b])$, thus, this is an isomorphism.

Since $\varphi([k]) = 1$ if and only if $i^k = 1, k \equiv 0 \pmod 4$, so $\ker \varphi = [0]$, only contains the identity element in $\mathbb{Z}/4\mathbb{Z}$, so φ is injective, also the cardinality of the two groups are the same, so φ must be a bijection, φ is an isomorphism.

(c)

Define:

$$\varphi : \mathbb{Z} \to 3\mathbb{Z}, \varphi(x) = 3x, x \in \mathbb{Z}$$

Since $\varphi(x+y) = 3x + 3y = \varphi(x) + \varphi(y)$, it's trivially a homomorphism. Also $\ker\varphi = \{0\}$ where 0 is the identity of $\mathbb{Z}$, also the only element mapped to 0 (the identity in $3\mathbb{Z}$), thus it's injective. Also for every element y in $3\mathbb{Z}$, we can always find an corresponding element $\frac{y}{3}$ in $\mathbb{Z}$ (since y is divisible by 3), so φ is surjective. Hence, φ is one isomorphism.

(d)

The second group contains infinite elements, but all the elements have finite order, hence, it's not cyclic. Since $\mathbb{Z}$ with additive operation is infinitly cyclic, the two groups can't be isomorphic.

(e)

$D_3 = \{e, s, r, r^2, sr, sr^2\}$ and $S_3 = \{e, (12), (23), (13), (123), (132)\}$, we can define $\varphi : D_3 \to S_3$ by:

$$\begin{cases} s \mapsto (12) \\ r \mapsto (123) \end{cases}$$

This is one isomorphism. Instead of checking all 36 pairs, we use generators and relations of D_3 to check the homomorphism. Since $\varphi(r^3) = \varphi(r)^3 = (123)(123)(123) = e, \varphi(s^2) = \varphi(s)^2 = (12)(12) = e, \varphi(srs^{-1}) = \varphi(s)\varphi(r)\varphi(s^{-1}) = (12)(123)(12) = (132) = (123)^{-1}$, φ preserves the structure of D_3.

Since $\ker\varphi = \{e\}$ and all permutations in S_3 can be generated using operations (flip and rotations) in D_3, i.e., $\varphi(D_3) = S_3$. Thus, φ is bijective. φ is an isomorphism.

(f)

Since $|S_4| = 4! = 24$ and $|D_4| = |\{e, s, r, r^2, r^3, rs, r^2s, r^3s\}| = 8$, their order are different, there can't be an bijection. Hence, they are not isomorphic.

(g)

$Q = \{1, -1, i, j, k, -i, -j, -k\}$ and $T = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$

An isomorphism is defined as:

$$f : Q \to T, f(i) = a, f(j) = b$$

Let's check if f a homomorphism, preserving the relations given in T:

$$
\begin{aligned}
f(-1) = f(i^2) = (f(i))^2 = a^2 = f(j^2) = (f(j))^2 = b^2 \\
f(k) = f(ij) = ab \\
f(-k) = f(ji) = f(j)f(i) = ba \\
f(-k) = f(-1)f(k) = a^3b \\
ba = a^3b \\
f(i^4) = f(1) = (f(i))^4 = a^4 = 1
\end{aligned}
$$

Thus, since f preserves the relations in T, it's a homomorphism. Also:

$$
\begin{aligned}
f(1) = 1 \\
f(-1) = a^2 \\
f(i) = a \\
f(j) = b \\
f(k) = f(ij) = ab \\
f(-i) = f(-1)f(i) = a^3 \\
f(-j) = f(-1)f(j) = a^2b \\
f(-k) = f(-1)f(k) = a^3b
\end{aligned}
$$

The homomorphism f is an 1 to 1 function and all elements in T can be mapped from one unique element in Q. Hence f is bijective, and f is an isomorphism!

(h)

They are isomorphic. Since any subgroups of a cyclic group is cyclic. Let $G = \langle x \rangle$, and nontrivial subgroup has the form $H = \langle x^m \rangle$ for some integer m. Define:

$$\varphi : G \to H, \varphi(g) = g^m, g \in G$$

Since $\varphi(ab) = (ab)^m = a^m b^m = \varphi(a)\varphi(b), \forall a, b \in G$, it's a homomorphism. If $\varphi(x^k) = e, x^{km} = e$, the only possible k is zero, so $\ker \varphi = e$, where e is the identity element in G and H, φ is injective. Also every element in H has the form $(x^m)^k = \varphi(x^k)$, it's also surjective. Thus, φ is an isomorphism between G and H.

2)

Suppose $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$ are two elements in G. Then we have:

$$\begin{aligned}
\varphi(\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix}) &= \begin{pmatrix} a+c & -b-d \\ b+d & a+c \end{pmatrix} \\
&= (a+c) + (b+d)i \\
&= (a+bi) + (c+di) \\
&= \varphi(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}) + \varphi(\begin{pmatrix} c & -d \\ d & c \end{pmatrix})
\end{aligned}$$

Thus, $\varphi : G \to \mathbb{C}$ is a homomorphism. To prove φ is isomorphic, we need to prove the injectivity and surjectivity. Suppose $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$ For the injectivity, suppose $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)$ are distinct elements in G, i.e., $a \neq c \vee b \neq d$. Since $\varphi(\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right)) = c + di$, $\varphi(\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)) = a + bi \neq \varphi(\left(\begin{smallmatrix} c & -d \\ d & c \end{smallmatrix}\right))$, the φ is injective.

For every element $x = r + si$ in $\mathbb{C}$, where $r, s \in \mathbb{R}$, it can correspond to a unique real matrix $\left(\begin{smallmatrix} r & -s \\ s & r \end{smallmatrix}\right)$ in G. Thus, $\varphi(G) = \mathbb{C}$, the surjectivity is proved. Hence, G and $\mathbb{C}$ are isomorphic!

3)

(a)

Suppose $H = \langle h \rangle, h \in G$ is a normal subgroup of G. Since any subgroup of a cyclic group is cyclic, the subgroup N of H can be written as the form $\langle h^d \rangle, d \in \mathbb{Z}$.

Suppose $g \in G$, since H is normal in G, $g^{-1}hg = h^i \in H$ for some integer i in $[1, n]$ Then for any integer r, we have:

$$g^{-1}(h^d)^r g = (g^{-1}hg)^{rd} = (h^d)^{ir} \in N$$

Thus, for all $g \in G$ and elements in $N = \langle x^d \rangle$ (Let $n \in N$), we have $g^{-1}ng \in N$, N is normal in G.

(b)

Let $G = S_4$, the symmetry group on 4 letters. Let $H = V = \{e, (12)(34), (14)(23), (13)(24)\}$, the Klein four group. Obviously H is a non-empty subset of G. Since every non-identity element in H is a product of two disjoint transposition, their order is 2, i.e, their inverses are themselves, so also in H. We need to show that for any two elements x, y in H, $xy^{-1} = xy \in H$. By some calculating, we can find that the product of any element and itself is the identity, and the product of any two distinct elements is also a double transposition (two disjoint transposition), thus in H. By subgroup criterion, $H \leq G$.

Since G can be generated using $u = (12)$ and $v = (1234)$ by definition. To show H is normal in G, checking $uHu^{-1} = H$ and $vHv^{-1} = H$ is enough, since any elements can be expressed as the product of u and v. And by some easy calculations we know $uHu^{-1} = H$ and $vHv^{-1} = H$ are both true. Hence, H is normal in G.

Let $N = \{e, (12)(34)\}$, since $hNh^{-1} = N \forall h \in H$, N is a normal subgroup of H (N is non-empty, it's closed under taking inverse and product/compositions, so $N \leq H$). Taking $g = (123) \in G$, since $gNg^{-1} = \{e, (13)(24)\} \neq N$, N is not normal in H. This is a counterexample.

4)

Claim: Left cosets are in bijection via left multiplication. In other words, given a group G, a subgroup H, and two left cosets xH, yH of H, where $x, y \in G$, left multiplication by yx^{-1} creates a bijection between xH and yH.

Proof. We can prove the correctness, injectivity, surjectivity of the mapping. Suppose x, y are in G.

First note that if $g = xh,\ h \in H,\ x \in G$ then $(yx^{-1})g = yh$, thus the left multiplication of yx^{-1} can map any elements in xH to yH. Here proves the correctness.

Given two distinct elements $xh_1, xh_2 \in xH,\ h_1, h_2 \in H$, $(yx^{-1})xh_1 = yh_1$ and $(yx^{-1})xh_2 = yh_2$ are also distinct since if $yh_1 = yh_2$ then we will get $xh_1 = xh_2$ (by cancelling yx^{-1}), contradiction appeared. Thus, the map is injective.

Every element in yH takes the form as $yh = (yx^{-1})xh, h \in H$, it arises as the image of left multiplication by yx^{-1}. Thus, the map is surjective. From these properties, we know left cosets are in bijection via left multiplication. □

Take any $g \in G$ and $n \in N$. Since φ is a homomorphism and H is abelian, we have:

$$\begin{aligned}\varphi(gng^{-1}n^{-1}) &= \varphi(g)\varphi(n)\varphi(g^{-1})\varphi(n^{-1}) \\ &= \varphi(g)\varphi(g^{-1})\varphi(n)\varphi(n^{-1}) \\ &= \varphi(gg^{-1})\varphi(nn^{-1}) = e_H\end{aligned}$$

Where e_H is the identity element of H, thus, $gng^{-1}n^{-1} \in \ker\varphi$. By hypothesis, $\ker\varphi \in N$, so $gng^{-1}n^{-1} \in N$. Since $gng^{-1} = (gng^{-1}n^{-1})n \in N$, we can conclude that $gNg^{-1} \subset N\ \forall g \in G$. Since the left coset, right coset and the conjugate have the same size with subgroup, i.e. $|gNg^{-1}| = N$, $gNg^{-1} = N\ \forall g \in G$ is true, and N is normal subgroup of G. By the claim above, we can easily know the size of left coset of subgroup N is the same as N, so is the right coset (The bijectivity is also proved in the same way as claim.). Thus, $|gNg^{-1}| = N$, which implies $gNg^{-1} = N$, N is the normal subgroup of G.