

Abstract algebra I Homework 5

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1)

(a)

Suppose that for any two distinct elements $s, t \in S$, we have $O(s) \neq O(t)$ and $O(s) \cap O(t) \neq \emptyset$. There exists $g_1, g_2 \in G$ such that $g_1s = g_2t \Rightarrow t = g_2^{-1}g_1s$, hence, for every $g \in G$, we have $gt = gg_2^{-1}g_1s \in O(s)$, i.e., there's always one corresponding element in $O(s)$ for every element in $O(t)$, similarly we have $gs = gg_1^{-1}g_2t \in O(t)$, so $O(s) = O(t)$, which contradicts our assumption. Thus $O(s) \neq O(t)$ **and** $O(s) \cap O(t) \neq \emptyset$ is impossible, we either have $O(s) = O(t)$ or $O(s) \cap O(t) = \emptyset$.

(b)

$e \in G_s$ trivially, also for any $u, v \in G_s$, $(uv)s = us = s \Rightarrow (uv) \in G_s$, also $us = s, s = u^{-1}s \Rightarrow u^{-1} \in G_s$, thus G_s is closed under taking products and inverses, $G_s \leq G$.

Let the map be ϕ , if $g_1G_s = g_2G_s$ and $g_1, g_2 \in G$, then $g_2^{-1}g_1G_s = G_s \Rightarrow g_2^{-1}g_1 \in G_s$. Hence, $g_2^{-1}g_1s = s, g_1s = g_2s, \phi(g_1G_s) = \phi(g_2G_s)$, we know ϕ is well-defined.

To prove the injectivity of ϕ , if $g_1s = g_2s$ ($g_1, g_2 \in G$), then $g_1^{-1}g_2s = s \Rightarrow g^{-1}g_2 \in G_s$, so $g^{-1}g_2G_s = G_s$, we have $g_1G_s = g_2G_s$.

For every $u \in O(s)$, it can be written as the form: $u = gs$ for some $g \in G$, so $u = \phi(gG_s)$. ϕ is surjective. Since ϕ is both injective and surjective, ϕ is a well-defined bijection.

(c)

By the result in (b), we know $|G : G_s| = |G|/|G_s| = |O(s)|$, hence $|G_s||O(s)| = |G|$.

2)

(a)

(b)

(c)

3)

(a)

(b)

(c)

If $HK = KH$, every element hk in HK can be written as $k'h'$ for some $k, k' \in K, h, h' \in H$. Clearly, HK contains e , the identity element in G . Suppose $u, v \in HK = KH$, $u = h_1k_1, v = k_2h_2$, $uv = h_1k_1k_2h_2$ ($h_1, h_2 \in H$ and $k_1, k_2 \in K$), let $k_1k_2 = k_3 \in K$, then $uv = h_1k_3h_2 = h_1hk = h'k \in HK$ for some $h, h' \in H, k \in K, hk = k_3h_2, h' = h_1h$, thus HK is closed under taking products. Also, $u^{-1} = k_1^{-1}h_1^{-1}$ and every element in KH must can be written as ab for some $a \in H, b \in K$, since $k^{-1} \in K, h^{-1} \in H$, so $u^{-1} \in HK$, HK is closed under taking inverse. In conclusion, $HK \leq G$ if $HK = KH$.

(d)

4)