Abstract algebra I Homework 2

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1)

(a)

Take the sum of 14 and 13, and it's 27 modulo 30, $27 \notin G_1$ thus, G_1 is not a subgroup of G.

(b)

We can check some necessary properties a subgroup must follow:

- All elements of G_2 are also in G, $G_2 \in G$
- $\exists e$ such that $\forall g \in G_2, g + e = e + g = g$. There $e = 0$.
- $0 + 0 \equiv 0 \pmod{30}$, thus, $g^{-1} = 0$ when $g = 0$. All the other non-zero elements $\in G_2$ can be written as the form $2k, k \in [1, 14], k \in \mathbb{N}$. Assume $g = 2k$, then there must exists an element $h = 2(15 - k) \in G_2$ such that $g + k \equiv 0 \pmod{30}$. Thus, every element in G_2 has an inverse element.

(c)

Take the sum of 1 and 29, and it's 0 modulo 30, $0 \notin G_3$ thus, G_3 is not a subgroup of G.

2)

(i)

Proof. Since H is not empty, we can choose $x, y \in G$.

By the closedness of inverse, the inverse of x exists and belongs to the H, let $y = x^{-1}$.

By the closedness of $*$, $x * x^{-1} = e \in H$, where e is the identity element of H. Thus, the identity of H exists.

Since H is closed under products, and inverse for each element exists, and the identity for H exists. It's a group and $H \subset G$, so H is a subgroup of G. □

(ii)

For simplicity, I denote the determinant of a n by n matrix A as $|A|$.

Since the determinant of an identity matrix I_n is 1, $SL_n(\mathbb{R}) \neq \emptyset$.

For any matrices $a, b \in SL_n(\mathbb{R})$, suppose $c = ab$, then c must be a real matrix (all entries are real), also, $|c| = |a||b| = 1 * 1 = 1$, the determinant of c is also 1. Thus, $c \in SL_n(\mathbb{R})$. Here proves the closedness of matrix multiplication.

Claim 1: Real $n \times n$ matrix A is invertible if and only if $|A| \neq 0$

Proof. Suppose A is invertible, then there exists a matrix B such that $AB = I$. $|I| = |A||B| = 1$, $|A|$ can't be zero.

Assume $|A| \neq 0$, then $B = \frac{1}{|A|}\text{adj}(A)$ (B is also a real $n \times n$ matrix) satisfies $AB = BA = I$ where $\text{adj}(A)$ is the classical adjoint matrix of A and I is the identity matrix.

Thus, $|A| \neq 0$ is necessary and sufficient. □

By claim 1, every element in H has its inverse due to their non-zero determinant. Suppose A is any matrix in H, and its inverse is A^{-1}, then $AA^{-1} = A^{-1}A = I, |A||A^{-1}| = |I| = 1$, thus, $|A^{-1}| = 1$.

Hence, the inverse of A, i.e., A^{-1} is also in H. Here the closedness of inverse is proved. By the subgroup criterion proved in 2(i), $\mathrm{SL}_n\mathbb{R}$ is a subgroup of $\mathrm{GL}_n\mathbb{R}$.

3)

(a)

Proof. Let's call the two sets A and B. $A = \{1, 2, \ldots, n\}, B = \{1, 2, \ldots, n\}$. And A is mapped to B.

Since the map is bijective, for 1 in A, there are n choices to be mapped, after 1 is mapped, 2 in A has $n-1$ choices to be mapped, and so on.

Thus, there are $n(n-1)(n-1)\ldots 1 = n!$ types of bijection, i.e., the order of S_n is $n!$. □

(b)

(c)

The identity element in the group for matrices multiplication is the identity matrix $I_{2\times 2}$.

For a, $a = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$,$a^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$,$a^3 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$,$a^1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = I_{2\times 2}$. Thus, $o(a) = 4$. For b, $b = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, $b^2 = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$,$b^3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = I_{2\times 2}$, thus, $o(b) = 3$.

4)

(a)

(b)

(c)

5)

(a)

Proof. □

(b)

(c)

Since elements in the abelian group $(G, *)$ are commutative, i.e., for any $a, b \in G$, we have $a * b = b * a$.

Let's choose one arbitrariry elements g, consider the subgroup as $B = \{a_1, a_2, \ldots, a_m\}$. Then $gB = \{g * a_1, g * a_2, \ldots, g * a_m\}$, and $Bg = \{a_1 * g, a_2 * g, \ldots, a_m * g\}$. since $g * a_i = a_i * g$ for all i, we have $gB = Bg$.

Hence, by the definition, every subgroup of an abelian group is normal.

(d)