Abstract algebra I Homework 3 uwu

B13902022 賴昱錡

Due: 1st October 2025

1)

(a)

$(\mathbb{Z}/15\mathbb{Z})^\times = \{\bar{1}, \bar{2}, \bar{4}, \bar{7}, \bar{8}, \bar{11}, \bar{13}, \bar{14}\}$ and $\mathbb{Z}/8\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

(b)

(c)

(d)

(e)

(f)

(g)

(h)

2)

3)

(a)

(b)

4)

Claim: Left cosets are in bijection via left multiplication. In other words, given a group G, a subgroup H, and two left cosets xH, yH of H, where $x, y \in G$, left multiplication by yx^{-1} creates a bijection between xH and yH.

Proof. We can prove the correctness, injectivity, surjectivity of the mapping. Suppose x, y are in G.

First note that if $g = xh,\ h \in H,\ x \in G$ then $(yx^{-1})g = yh$, thus the left multiplication of yx^{-1} can map any elements in xH to yH. Here proves the correctness.

Given two distinct elements $xh_1, xh_2 \in xH,\ h_1, h_2 \in H$, $(yx^{-1})xh_1 = yh_1$ and $(yx^{-1})xh_2 = yh_2$ are also distinct since if $yh_1 = yh_2$ then we will get $xh_1 = xh_2$ (by cancelling yx^{-1}), contradiction appeared. Thus, the map is injective.

Every element in yH takes the form as $yh = (yx^{-1})xh, h \in H$, it arises as the image of left multiplication by yx^{-1}. Thus, the map is surjective. From these properties, we know left cosets are in bijection via left multiplication. □

Take any $g \in G$ and $n \in N$. Since ϕ is a homomorphism and H is abelian, we have:

$$\begin{aligned}\phi(gng^{-1}n^{-1}) &= \phi(g)\phi(n)\phi(g^{-1})\phi(n^{-1})\\ &= \phi(g)\phi(g^{-1})\phi(n)\phi(n^{-1})\\ &= \phi(gg^{-1})\phi(nn^{-1}) = e_H\end{aligned}$$

Where e_H is the identity element of H, thus, $gng^{-1}n^{-1} \in \ker\phi$. By hypothesis, $\ker\phi \in N$, so $gng^{-1}n^{-1} \in N$. Since $gng^{-1} = (gng^{-1}n^{-1})n \in N$, we can conclude that $gNg^{-1} \subset N\ \forall g \in G$. Since the left coset, right coset and the conjugate have the same size with subgroup, i.e. $|gNg^{-1}| = N$, $gNg^{-1} = N\ \forall g \in G$ is true, and N is normal subgroup of G. By the claim above, we can easily know the size of left coset of subgroup N is the same as N, so is the right coset (The bijectivity is also proved in the same way as claim.). Thus, $|gNg^{-1}| = N$, which implies $gNg^{-1} = N$, N is the normal subgroup of G.