Stat 394 Probability I

Lecture 2

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Review

Counting

permutation → Count different ordering of n object where some of them are alike.

combination → Choose k out of n objects.

permutation

AABBB

Combination

— — — — —

$$\frac{5!}{2!\,3!} \longleftrightarrow \binom{5}{2} = \binom{5}{3}$$

Example

You are at a Poke place, from 5 kinds of fish and 10 choices of toppings, how many different combinations consisting of 3 fish and 5 toppings can be formed (assume you can't choose the same fish more than once). What if 2 of the toppings are very spicy, and you don't want to have them both together?

$$\# \text{ fish} = \binom{5}{3} = 10 \qquad 8 \quad 2$$

$$\# \text{ toppings} = \binom{10}{5} - \underbrace{\binom{8}{3}\binom{2}{2}}_{\#\text{ combinations that are too spicy}} = 252 - 56 = 196$$

$$\# \text{ poke} = 196 \times 10 = 1960$$

Multinomial coefficients

A set of n distinct items is to be divided into r distinct groups of size $n_1, n_2, \ldots, n_r$, where $\sum_{i=1}^{r} n_i = n$. How many different divisions are possible?

$$\binom{n}{n_1\, n_2 \cdots n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

recap. (permutation)
the # of permutations of n objects where n_1 are alike, n_2 —"—, ..., n_r —"—.

Example

\# of outcomes

$$\binom{64}{2\cdots 2} \times \frac{1}{32!} \times 2^{32} = \frac{64!}{\cancel{2^{32}}} \times \frac{1}{32!} \times \cancel{2^{32}}$$

In a knockout tournament involving 64 players. In each round, the players are divided into pairs, and the winners go on the next round. How many possible outcomes are there for the first round?

Solution 1.

(*) How many different ordered pairings can there be

$$\binom{64}{\underbrace{2\ 2 \cdots 2}_{32\ 2's}} \times \frac{1}{32!}$$

→ # of pairings

(**) For each fixed pairing. # of outcome $= 2^{32}$

Example

Solution 2

(*) Pick 32 winners $\binom{64}{32} = \frac{64!}{32!\,32!}$

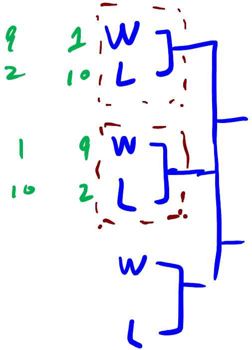
(**) For each fixed set of 32 winners.

$$32 \times 31 \times \cdots \times 1 = 32!$$

$$\Rightarrow \#\text{ outcomes} = \frac{64!}{32!\,\cancel{32!}} \times \cancel{32!} = \frac{64!}{32!}$$

Example

Solution 3.



64 slots: put 64 players into 64 slots $= 64!$

32 slots: put 32 pairs into 32 slots $= 32!$

$\Rightarrow$ # outcomes $= \dfrac{64!}{32!}$

Summary

- Permutation and combination are strongly related, as we have seen in examples.
- We will see more of these in the chapters to come.
- Read Chapter 1 on your own (especially multinomial theorem in Sec 1.5).
- Homework 1 due next Monday.

Introduction

- Before we start talking about probability, we need to define it.
- To do that, we need to review some set theory and definitions.

Sample space

- Experiment

  can be repeated
  well-defined outcome
  uncertainty as to which of the possible outcomes occur

  x : outcome of an experiment

- Sample space

  is the set of all possible outcomes of an experiment
  "S"

- Event

  is a subset of the sample space. "E"

  $E \subseteq S$

Examples

outcome of an Experiment = { sex of a new born baby }

$S = \{ \text{girl}, \text{boy} \}$

$E_1 = \{ \text{girl} \}$ $\quad E_2 = \{ \text{boy} \}$

Experiment: tossing two coins

$S = \{ (HH), (HT), (TH), (TT) \}.$

$E_1 = \{ \text{first toss is head} \}$

$= \{ (HH), (HT) \}$

Set theory

Suppose that E and F are subsets of a set S

$$E \subseteq S$$
$$F \subseteq S$$

- The union of events E and F is

$$E \cup F = \{ x \in S : x \in E \text{ or } x \in F \}$$

- The intersection of events E and F is

$$(EF) = E \cap F = \{ x \in S : x \in E \text{ and } x \in F \}$$

- E and F are disjoint (or mutually exclusive) if

$$E \cap F = \emptyset$$

Set theory

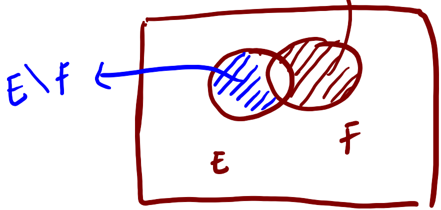
- The complement of E is

$$E^c = \{x \in S : x \notin E\}$$

- The set difference of E and F is

$$F \setminus E = \{x \in S : x \in F,\ x \notin E\}$$

- Venn diagram

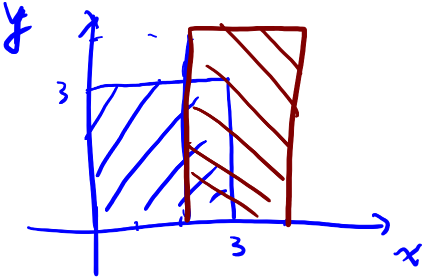


Example

Let $S = \mathbb{R}^2$, and two events

$$E = \{(x, y) \in S : 0 < x < 3, 0 < y < 3\}$$

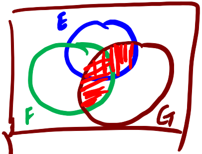
$$F = \{(x, y) \in S : 2 < x < 4, 0 < y < 4\}$$



Basic rules

- Commutative laws

$$E \cup F = F \cup E, \quad E \cap F = F \cap E$$



- Associative laws

$$(E \cup F) \cup G = E \cup (F \cup G)$$
$$(E \cap F) \cap G = E \cap (F \cap G)$$

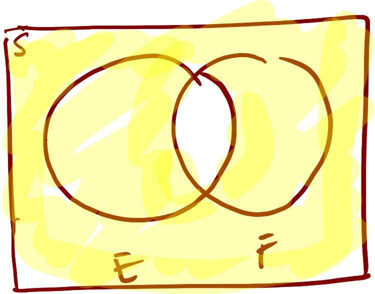
- Distributive laws

$$(E \cup F) \cap G = (E \cap G) \cup (F \cap G)$$
$$(E \cap F) \cup G = (E \cup G) \cap (F \cup G)$$

DeMorgan's laws

$$(E \cup F)^c = E^c \cap F^c$$

$$(E \cap F)^c = E^c \cup F^c$$



DeMorgan's laws

$$\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \quad (*), \qquad \left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c \quad (**)$$

<u>proof of (*)</u>

"$\Rightarrow$" $\forall x \in \left(\bigcup_{i=1}^{n} E_i\right)^c \rightarrow x \notin \bigcup_{i=1}^{n} E_i$

$\rightarrow x \notin E_i, \; i=1,2,\cdots,n$

$\rightarrow x \in E_i^c, \; i=1,2,\cdots,n$

$\rightarrow x \in \bigcap_{i=1}^{n} E_i^c$

"$\Leftarrow$" reverse all the steps above $\blacksquare$

~~How do we define probability?~~

De Morgan's law.

(**)

$$\left(\bigcap_{i=1}^{n} E_i\right)^c = \bigcup_{i=1}^{n} E_i^c$$

Pf. $\left(\bigcup_{i=1}^{n} E_i\right)^c = \bigcap_{i=1}^{n} E_i^c \Rightarrow \left(\bigcup_{i=1}^{n} E_i^c\right)^c = \bigcap_{i=1}^{n} (E_i^c)^c$

using (*)

$$\Rightarrow \left(\bigcup_{i=1}^{n} E_i^c\right)^c = \bigcap_{i=1}^{n} E_i$$

$$\Rightarrow \bigcup_{i=1}^{n} E_i^c = \left(\bigcap_{i=1}^{n} E_i\right)^c$$

~~How do we define probability?~~

DeMorgan's Law

Second proof of (**) similar to pf of (*)

"$\Rightarrow$"

$\forall x \in \left(\bigcap_{i=1}^{n} E_i\right)^c \Rightarrow x \notin \bigcap_{i=1}^{n} E_i$

$\Rightarrow \exists k \in \{1, 2, \cdots, n\}$ s.t. $x \notin E_k$

$\Rightarrow$ then $x \in E_k^c$ for some k

$\Rightarrow x \in \bigcup_{i=1} E_i^c$

"$\Leftarrow$" is just reversing the steps above.