Stat 394 Probability I

Lecture 2

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How do we define probability?

How do we define probability?

Example: Roll a die 4 times, what is the probability that you get different numbers?

outcomes: $\{(a, b, c, d)\}$, $a, b, c, d \in \{1, 2, \cdots, 6\}$.

total # $= 6^4$

good outcome $= 6 \times 5 \times 4 \times 3$

$$P = \frac{6 \times 5 \times 4 \times 3}{6^4}$$

Why is the example too simple?

- The number of total outcomes needs to be finite
- We assume that all outcomes are equally likely.

Probability space

A probability space is a triple $(S, \mathcal{F}, P)$ (FYI, in many other books, notation S is replaced by Ω)

- S: "sample space"
  arbitrary set representing all outcomes

(- $\mathcal{F}$: ~~a set of subsets of S~~)
  ~~"σ-algebra"~~

- P: probability attached to each event E.
  "probability measure function"
  - P is a function/mapping

Probability axioms

A1 For each event E, $P(E) \geq 0$

A2 $P(S) = 1$

$E_i \cap E_j = \phi \quad \forall i \neq j$

A3 If $E_1, E_2, \cdots,$ is a sequence of pairwise disjoint events.

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

Example

$$\sum_{j=1}^{\infty}\left(\sum_{i\in E_j}\frac{1}{2^i}\right)$$

$$=\sum_{j=1}^{\infty}P(E_j)$$

- Let $S = \{1, 2, ...\}$
- For any $E \subset S$,

$$P(E) = \sum_{i \in E} \frac{1}{2^i}$$

- Is it a probability space?

✓ A1 $\forall E \quad P(E) = \sum_{i\in E}\frac{1}{2^i} \geq 0$

✓ A2 $P(S) = \sum_{i\in S}\frac{1}{2^i} = \sum_{i=1}^{\infty}\frac{1}{2^i} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2^i} = \lim_{n\to\infty} 1-\frac{1}{2^n}$

$= 1$

✓ A3 For mutually exclusive $E_1, E_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i\in\bigcup E_i}\frac{1}{2^i} = \sum_{i\in E_1}\frac{1}{2^i} + \sum_{i\in E_2}\frac{1}{2^i} + \cdots$$

Propositions

1. $P(\phi) = 0$

Pf. $E_1 = S$, $E_2 = E_3 = \cdots\cdots = \phi$

$$P(S) = \sum_{i=1}^{\infty} P(E_i) = P(S) + \sum_{i=2}^{\infty} P(E_i)$$

$$\Rightarrow \sum_{i=2}^{\infty} P(E_i) = 0 \xRightarrow{A1} P(E_i) = 0 \quad \forall i = 2, \cdots$$

$$\Rightarrow P(\phi) = 0$$

2. Finite many $E_1, E_2, \cdots, E_n$ mutually exclusive.

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Pf. $F_1 = E_1$, $F_2 = E_2$, $\cdots$, $F_n = E_n$, $F_{n+1} = F_{n+2} = \cdots = \phi$

Propositions

3. $P(E) \leq 1$

Pf. If $P(E) > 1$

$S = E \cup E^c \Rightarrow \overset{1}{\overset{\parallel}{P(S)}} = P(E \cup E^c)$

(prop 2) $= \underbrace{P(E)}_{>1} + P(E^c)$

$\Rightarrow P(E^c) < 0$ violates (A1)

4. $P(E^c) = 1 - P(E)$

Propositions

5. If $E \subseteq F$, $P(E) \leq P(F)$

<u>pf</u>. $F = E \cup (E^c \cap F)$

$\Rightarrow P(F) = P(E) + \underbrace{P(E^c \cap F)}_{\geq 0} \Rightarrow P(F) \geq P(E)$

6. $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

<u>pf</u>. $P(E \cup F) = P(E \cup (E^c \cap F))$

$= P(E) + P(E^c \cap F)$

$P(F) = P((E \cap F) \cup (E^c \cap F)) = P(E \cap F) + P(E^c \cap F)$

Example

Pick an integer in $[1, 1000]$ at random. Compute the probability that it is divisible neither by 12 nor by 15.

Example

Sit 3 men and 4 women at random in a row. What is the probability that either all the men or all the women end up sitting together?

Example

A group of 3 Democrats, 4 Republicans, and 5 Independents is seated at random around a table. Compute the probability that at least one of the three groups ends up sitting together.

Example

A large company with n employees has a scheme according to which each employee buys a Christmas gift and the gifts are then distributed at random to the employees. What is the probability that someone gets his or her own gift?

Example

Just a review

A good time to recall some common mathematical series:

$$
\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
log(1-x) &= -\sum_{n=1}^{\infty} \frac{x^n}{n} \\
log(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n
\end{aligned}
$$

Binomial series, or binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Binomial series, or binomial theorem

Example

How many subsets are there of a set consisting of n elements?