Stat 394 Probability I

Lecture 3

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What we have learned and what we are going to

- Counting
  - There is an event,
  - We can count how many elements fall in that event
- Probability
  - a function that maps any event to a number between 0 and 1
  - In all examples we have learned so far, we need **equally likely events** to calculate probability
- Set operations, e.g., exclusion-Inclusion properties
  - For a complicated event, we can break it down into smaller pieces
  - Mostly addition and subtraction
- Conditional probability
  - More complicated events
  - More multiplication and division

More review

- Combinatorial is still important for most of the problems
- Counting and probability calculation are still going to be a key part
- Example: Toss two coins
    - Counting with identical coins: 3 outcomes
    - Counting with distinguishable coins: 4 outcomes
    - Probability of seeing different outcomes: $1/2$
- Why it does not matter if the coins are identical?

$$P = P(\{HT\}) = P(\{1 = H, 2 = T\} \cup \{1 = T, 2 = H\})$$

More review

- Why not $1/3$?
- In counting, we removed repeated outcomes depending on our sample space, i.e., $\{HT\}, \{TH\}$ are considered one outcome.
- In probability calculation, this leads to problems, as the outcomes we considered are not equally likely.
- So we can't use the ratio of the sizes of two sets to get probability.
- which means, when calculating probability, many times we need to put labels back in, so that we can define a sample space where each outcome is equally likely.

Motivation

Consider the simple scenario of rolling 2 dice - we know there are 36 outcomes - What is the probability that the sum of the two numbers we get is 3? Will this change after we roll the first die?

Conditional on you get a 5.

Conditional probability

Definition

Let two events E, F, call conditional probability that E occurs given F occurs

is

$$P(E|F) = \frac{P(E\cap F)}{P(F)} = \frac{P(EF)}{P(F)}$$

if $P(F) > 0$

Example

	red	blue
cube	3	4
ball	2	2

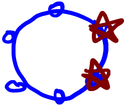
$$P(C) = \frac{7}{11}$$

$$P(CR) = \frac{3}{11}$$

$$P(R \mid C) = \frac{3}{7} = \frac{P(CR)}{P(C)}$$

Example: Russian roulette

Lets play a game of Russian roulette. You are tied to your chair. Heres a gun, a revolver. Heres the barrel of the gun, six chambers, all empty. Now watch me as I put two bullets into the barrel, into two adjacent chambers. I close the barrel and spin it. I put a gun to your head and pull the trigger. Click. Lucky you! Now Im going to pull the trigger one more time. Which would you prefer: that I spin the barrel first or that I just pull the trigger?



Multiplication rule

$$P(E_1E_2\cdots E_n) = P(E_1)\,P(E_2|E_1)\,P(E_3|E_1E_2)\cdots P(E_n|E_1E_2\cdots E_{n-1})$$

证.

observe

$$P(E_2|E_1) = \frac{P(E_1E_2)}{P(E_1)}$$

$$P(E_3|E_1E_2) = \frac{P(E_1E_2E_3)}{P(E_1E_2)}$$

$\cdots$

Previous examples revisited

"oddman-out", 3 players, flip coins

prob of C is out.

$$P(\text{finish}) = P(A) + P(B) + P(C)$$

$$P(A) = P(\{H\} \mid \{TT\})\,P(\{TT\}) + P(\{T\} \mid \{HH\})\,P(\{HH\})$$

$$= \frac{1}{2} \times \left(\frac{1}{2} \times \frac{1}{2}\right) + \frac{1}{2} \times \left(\frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{4}$$

$$P(B) = P(C) = \frac{1}{4}$$

Previous examples revisited

52 Cards $\Pr(\text{second draw is an } A)$

(draw without replacement).

$$P(X_2=A) = \overbrace{P(X_2=A \mid X_1=A)\,P(X_1=A)}^{P(X_2=A,\ X_1=A)} + \underbrace{P(X_2=A \mid X_1\neq A)\,P(X_1\neq A)}_{P(X_2=A,\ X_1\neq A)}$$

$$= \frac{3}{51}\times\frac{4}{52} + \frac{4}{51}\times\frac{48}{52}$$

$$= \ldots$$

Matching problem

$P(EG) = \frac{(n-k)!}{n!} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$

$P(k \text{ match}) = \binom{n}{k} P(EG)$

$= \cdots$

N people. P(exact k people get their own gift)

Recall P(exist one person getting his/her own)

$$= 1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

Fix k people, E: these k people get their own

G: other (n-k) people do not get their own

$P(\text{exact these } k) = P(EG) = P(E)P(G|E)$

F_i : i-th person get his/her own, $i=1,\cdots,k$

$P(E) = P(F_1)P(F_2|F_1)P(F_3|F_1F_2)\cdots = \frac{1}{n} \times \frac{1}{n-1} \times \cdots \times \frac{1}{n-k+1} = \frac{(n-k)!}{n!}$

$P(G|E) = 1 - P(\text{exist one person getting his own for } (n-k) \text{ people})$

$$= \sum_{i=0}^{n-k} \frac{(-1)^i}{i!}$$

Bayes's formula

First Bayes's formula

Let $F_1, F_2, \cdots, F_n$ pairwise disjoint.

and $F_1 \cup F_2 \cup \cdots \cup F_n = S$,

$\forall$ event $A \subseteq S$, $P(A) = P(F_1)P(A|F_1) + P(F_2)P(A|F_2) + \cdots + P(F_n)P(A|F_n)$

Pf.

$$\begin{aligned} RHS &= P(AF_1) + P(AF_2) + \cdots + P(AF_n) \\ &= P(AF_1 \cup AF_2 \cup \cdots \cup AF_n) \quad \rightarrow \text{axiom 3} \\ &= P(A \cap (F_1 \cup F_2 \cup \cdots \cup F_n)) \quad \rightarrow \text{distributive law} \\ &= P(A \cap S) = P(A) \end{aligned}$$

Bayes's formula

A special case

$$P(E) = P(F)P(E|F) + P(F^c)P(E|F^c)$$

Second Bayes's formula

$$P(F_j | A) = \frac{P(A|F_j)P(F_j)}{\underbrace{P(F_1)P(A|F_1) + P(F_2)P(A|F_2) + \cdots + P(F_n)P(A|F_n)}_{P(A)}}$$

Example

Flip a fair coin. If you toss Heads, roll 1 die. If you toss Tails, roll 2 dice. Compute the probability that you roll exactly one 6.

W: winning the game.

$$P(W) = P(W|H)P(H) + P(W|T)P(T)$$

$$= \frac{1}{6} \times \frac{1}{2} + \frac{5\times 2}{36} \times \frac{1}{2}$$

$$= \frac{2}{9}$$

Example

Roll a die, then select at random, without replacement, as many cards from the deck as the number shown on the die. What is the probability that you get at least one Ace?

E_i : getting an i from roll, $i = 1, \cdots, 6$

$P(A \mid E_1) = \frac{4}{52}$

$P(A \mid E_2) = 1 - \frac{\binom{48}{2}}{\binom{52}{2}}$

$P(A \mid E_3) = 1 - \frac{\binom{48}{3}}{\binom{52}{3}}$

$\vdots$

$P(A) = P(A \mid E_1) P(E_1) + P(A \mid E_2) P(E_2) + \cdots$

$= \frac{1}{6}\left(P(A \mid E_1) + \cdots + P(A \mid E_6)\right)$

$= \cdots$

Optimism in probability

A test of a disease presents a rate of 5% false positives, and no false negatives. The disease strikes 1/1,000 of the population. People are tested at random, regardless of whether they are suspected of having the disease. A patient's test is positive. What is the probability of the patient being stricken with the disease?

S : having disease

T : test positive.

$P(T|S^c) = 0.05$, $P(T|S) = 1$

$$P(S|T) = \frac{P(T|S)P(S)}{P(T)} = \frac{P(T|S)P(S)}{P(T|S)P(S) + P(T|S^c)P(S^c)}$$

$$= \frac{1 \times \frac{1}{1000}}{1 \cdot \frac{1}{1000} + 0.05 \times (1 - \frac{1}{1000})} \approx 0.02$$

OJ Simpson

The following statement is from one of the lawyers of O.J.,

> *We knew we could prove, if we had to, that an infinitesimal percentage - certainly fewer than 1 of 2,500 - of men who slap or beat their domestic partners go on to murder them.*

S: all women abused by her husband

H: a woman in S killed by her husband

M: a woman in S killed by someone.

$P(H) = \frac{1}{2500}$, $P(M|H) = 1$, $P(M|H^c) \approx$ murder rate in 1995 ≈ 0.0001

OJ Simpson

$$P(H|M) = \frac{P(M|H)P(H)}{P(M|H)P(H) + P(M|H^c)P(H^c)}$$

$$= \frac{1 \times \frac{1}{2500}}{1 \times \frac{1}{2500} + 0.0001 \times \left(1 - \frac{1}{2500}\right)}$$

$$\approx 0.8$$

Intuitively

100,000 women $\rightarrow$ 40 murdered by husband

$\downarrow$

10 murdered