Stat 394 Probability I

Lecture 4

Richard Li

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Last Friday

A deck of 52 playing cards is shuffled, and the cards are turned up one at a time until the first ace appears. Is the next card, i.e., the card following the first ace, more likely to be the ace of hearts or the two of spades?

total number of ordering of 52 cards : $52!$

Take out $A\heartsuit$, total # of 51 card ordering : $51!$

$$P(A\heartsuit \cdots) = \frac{51!}{52!} = \frac{1}{52}$$

Take out $2\spadesuit$

$$P(2\spadesuit \cdots) = \frac{51!}{52!} = \frac{1}{52}$$

Exercise.

$$P(A\heartsuit \cdots) = P(A\heartsuit \text{ follow } A\spadesuit \mid A\spadesuit \text{ is the first A}) \times P(A\spadesuit \text{ is the first A}) + \_\_\_ + \_\_\_$$

Also last Friday

$P(\text{both A \& have at least one A})$

Randomly draw 2 cards from the deck, if you know you have an Ace in the two cards, what is the probability that both cards are Aces? If you know you have an Ace of heart, what is the probability that both cards are Aces?

$$P(\text{both A's} \mid \text{have at least one A}) = \begin{cases} \dfrac{\binom{4}{2} / \binom{52}{2}}{1 - \binom{48}{2} / \binom{52}{2}} = \dfrac{1}{33} \\[2ex] \dfrac{\binom{4}{2}}{\binom{48}{1}\binom{4}{1} + \binom{4}{2}} \end{cases}$$

$P(\text{have one A})$

$$P(\text{both A's} \mid \text{have A}\heartsuit) = \begin{cases} \dfrac{\binom{3}{1} / \binom{52}{2}}{\binom{51}{1} / \binom{52}{2}} = \dfrac{1}{17} \\[2ex] \dfrac{\binom{3}{1}}{\binom{51}{1}} = \dfrac{3}{51} = \dfrac{1}{17} \end{cases}$$

$A\heartsuit + 48$

$A\spadesuit + 48$

$A\diamondsuit + 48$

$A\clubsuit + 48$

$A\heartsuit\spadesuit$ ___

$\heartsuit\diamondsuit$ ___

$\heartsuit\clubsuit$ ___

Review: Mutual Independence

$\{E_i\}_{i=1,\dots,n}$ are mutually indep. if for every

subset $\{E_{i_1}, \dots, E_{i_r}\}$, $P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r})$

e.g. Take $r=2$ $\{E_{i_1}, E_{i_2}\} \rightarrow P(E_{i_1} E_{i_2}) = P(E_{i_1}) P(E_{i_2})$

Proposition

If E, F, and G are mutually independent, then E is also independent of $F \cup G$.

Pf.

$$P(E(F\cup G)) = P(EF \cup EG)$$

$$= P(EF) + P(EG) - \overbrace{P(EF \cap EG)}^{P(EFG)}$$

$$= P(E)P(F) + P(E)P(G) - P(E)P(FG)$$

$$= P(E)\big(P(F) + P(G) - P(FG)\big)$$

$$= P(E)P(F\cup G)$$

Example

Proposition

Once we start to think about conditional relationships involving more than two events, there is a helpful fact to know about: Conditional probabilities are probabilities.

$$P_1(\cdot \mid E) \longrightarrow P_2(\cdot)$$

A1 $\quad 0 \le P(E|F) \le 1$

A2 $\quad P(S|F) = 1$

A3 $\quad$ For mutually exclusive E_i, $i=1,\ldots$:

$$P\left(\bigcup_{i=1}^{\infty} E_i \,\Big|\, F\right) = \sum_{i=1}^{\infty} P(E_i|F)$$

e.g. $\quad P(E_1 \cup E_2 | F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$

$$P(E_1|F) = P(E_1|E_2F)\,P(E_2|F) + P(E_1|E_2^c F)\,P(E_2^c|F)$$

Proposition

Monty Hall problem

choose door A

host opens door B → there's a goat.

$E_A\ E_B\ E_C$: participant select door A/B/C.

$F_A\ F_B\ F_C$: host opens door A/B/C.

$G_A\ G_B\ G_C$: car behind door A/B/C.

$$P(G_A) = P(G_B) = P(G_C) = 1/3$$

$$P(F_A | E_A) = 0$$

$$P(F_C | E_A G_C) = 0, \quad P(F_B | E_A G_C) = 1$$

$$P(F_B | E_A G_A) = P(F_C | E_A G_A) = 1/2$$

(*)

$$P(G_C | E_A F_B) = \frac{P(G_C E_A F_B)}{P(E_A F_B)} = \frac{P(F_B | E_A G_C) \cancel{P(E_A)} P(G_C)}{P(F_B | E_A) \cancel{P(E_A)}}$$

$$(*) = \frac{1 \times \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Now since. $P(F_B | E_A) = P(F_B | E_A G_C) P(G_C) + P(F_B | E_A G_B) P(G_B) + P(F_B | E_A G_A) P(G_A)$

$$= 1 \times 1/3 + 0 \times \tfrac{1}{3} + 1/2 \times \frac{1}{3} = \frac{1}{2}$$

Prosecutor's fallacy

$P(A|B)$, $P(B|A)$

$$\frac{1}{8500} \times \frac{1}{8500} = \frac{1}{73 \times 10^6}$$

$\uparrow$

p (unexpected death)

Other common mistakes

(1) confusing prior probability with posterior prob.

$P(A)$

$P(A|A)=1$

(2) confusing independence with conditional independence.

Def. E and F are conditionally independent conditional on G, if
given

$$P(EF|G)=P(E|G)P(F|G)$$

More on conditional independence

Fact.

1. conditional indep → indep ? No
2. indep → conditional indep ? No

Example.

2 coins : A: regular coin, B: both H's.

A : a toss an unknown coin gives H

B : a second toss also gives H (of the same coin)

C : it is coin A

A and B are indep given C.

But A and B are dependent.

Example.

A: alarm goes off if {there's fire / alien invasion.

E: alien invasion.

F: Fire

E and F are independent.

F and E^c are dependent given A

$P(F \mid E^c A) = 1 \neq P(F \mid A) \Rightarrow F$ and E....

Simpson's paradox

2 doctors, 2 types of surgery. Let X as success.

A	success	fail
I	8	2
II	12	18

B	success	fail
I	21	9
II	3	7

I: $P(X|A,I)=\frac{8}{10}=0.8$

II: $P(X|A,II)=\frac{12}{30}=0.4$

overall: $P(X|A)=\frac{20}{40}=0.5$

I: $P(X|B,I)=\frac{21}{30}=0.7$

II: $P(X|B,II)=\frac{3}{10}=0.3$

overall: $P(X|B)=\frac{24}{40}=0.6$

$$\begin{cases} P(X|AF) > P(X|BF) \\ P(X|AF^c) > P(X|BF^c) \end{cases} \not\Rightarrow P(X|A) > P(X|B)$$

Simpson's paradox