Stat 394 Probability I

Lecture 4

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Definition

Def. Event E, F. we say E and F are independent

if $P(EF) = P(E)P(F)$

$P(EF) = P(E|F)P(F)$

implication.

independence $\Leftrightarrow P(E|F) = P(E)$

Def. ______ " ______ dependent

if otherwise.

Fact: Mutually exclusive events (with non-zero probabilities) are dependent.

1, 6 4, 3
2, 5 5, 2
3, 4 6, 1

Example

1, 5 4, 2
2, 4 5, 1
3, 3

Roll a die twice, consider the event
$E_1 = \{\text{sum of the two values is } 6\}$, and $E_2 = \{\text{first value is } 4\}$.

$P(E_1) = \frac{5}{36}$

$P(E_2) = \frac{1}{6}$

$P(E_1 E_2) = \frac{1}{36}$

$E_3 = \{\text{sum of the two is } 7\}$.

$P(E_3) = \frac{1}{6}$

$P(E_2 E_3) = \frac{1}{36}$

$= P(E_2) P(E_3)$

Proposition

If event E and F are independent, then E and F^c are also independent (so is E^c and F, E^c and F^c).

Pf. $P(E)P(F) = P(EF)$

Notice $E = EF \cup EF^c$

$$P(E) = P(EF) + P(EF^c)$$

$$P(E)P(F) = P(E) - P(EF^c)$$

$$P(E)(1 - P(F)) = P(EF^c)$$

$$P(E)P(F^c) = P(EF^c)$$

$\blacksquare$

"Markov chain"

Example

You roll a die, your friend tosses a coin. (1) If you roll 6, you win outright. (2) If you do not roll 6 and your friend tosses Heads, you lose outright. (3) If neither, the game is repeated until decided.

$W = \{$you win$\}$. $D = \{$game is decided on the first round$\}$.

I claim W and D are independent.

$P(W \mid D^c) = P(W) \Rightarrow W$ and D^c are indep

$\Rightarrow W$ and D —"—

$$P(W \mid D) = \frac{P(WD)}{P(D)} = \frac{1/6}{1/6 + 5/6 \times \frac{1}{2}} = \frac{2}{7}$$

$$P(W) = P(W \mid D) = \frac{2}{7}$$

Pepys-Newton Problem (1693)

Which of the following is most likely?

- A • 6 fair dice are tossed independently and at least one 6 appears.
- B • 12 fair dice are tossed independently and at least two 6's appear.
- C • 18 fair dice are tossed independently and at least three 6's appear.

$P(A) = 1 - P(A^c) = 1 - \left(\frac{5}{6}\right)^6 \approx 0.665$

$P(B) = 1 - \left(\frac{5}{6}\right)^{12} - \underbrace{\binom{12}{1} \times \frac{1}{6} \times \left(\frac{5}{6}\right)^{11}} \approx 0.619$

binomial probability

$P(C) = 1 - \sum_{k=0}^{2} P(\text{exactly } k \text{ 6's}) = 1 - \underbrace{\sum_{k=0}^{2} \overbrace{\binom{18}{k}\left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{18-k}}}$

≈ 0.597

Multiple events

Fact

E & F indep, E & G indep

$\Rightarrow$ E and FG not necessarily indep.

E : { first is H }

F : { different outcomes }.

G : { second is H }.

HH
HT
TH
TT

FG : {TH}.

$$\underbrace{P(EFG)}_{=0} \neq \underbrace{P(E)P(FG)}_{\neq 0}$$

Mutual Independence

(3-event)

Def. E, F, G are independent (mutually independent)

(1) they are pairwise independent "regular" independence for two events.

(2) $P(EFG) = P(E)P(F)P(G)$

(n-event).

Def. $\{E_i\}$, $i = 1, 2, 3, \cdots, n$ are independent if for every subset of the sequence $\{E_{i_1}, E_{i_2}, \cdots, E_{i_r}\}$

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) P(E_{i_2}) \cdots P(E_{i_r})$$

Proposition

If E, F, and G are mutually independent, then E is also independent of $F \cup G$.

Example

Proposition

Once we start to think about conditional relationships involving more than two events, there is a helpful fact to know about: Conditional probabilities are probabilities.

Proposition

Monty Hall problem

Prosecutor's fallacy

Other common mistakes

More on conditional independence

Simpson's paradox

Simpson's paradox