Stat 394 Probability I

Lecture 2

Richard Li

June 25, 2017

Note: work out the same result using
1 − P(all 3 are the same)
− P(all 3 are different)

Problem 4

In the game of odd man out, each player tosses a fair coin. If all the coins come out the same, except for one, the minority coin is declared "odd man out" and is out of the game. Suppose that three people play odd man out. What is the probability that on the first toss someone will be eliminated? What if they play the same game by rolling a die?

Recap:

H H T — Switch H to the odd man

Coin: $\binom{3}{1} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times 2$

die: $\binom{3}{1} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times 6$

Step 3. count how many different step 2 there is

Step 1: choose the odd man

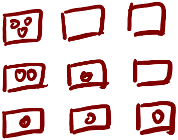
Step 2A prob for two outcomes of the same

Step 2B prob for a third and different outcome

Problem 5

Three balls are dropped at random into three boxes. What is the probability that exactly one box is empty?

Recap (what's the problem with these counts?)



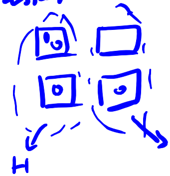
$\binom{3}{1} = 3 \times 1 = 3$

$3 \times 2 = 6 \times \binom{3}{2} = 18$

$1 \times 3 \times 2 = 6$

Solution 1.

Easier



HH TT

HT

Solution 2.

(ordered drop)

First: 3

Second: 2 | 1

Third: 2 | 2

$3 \times 2 \times 2 + 3 \times 2 = 18$

Solution 3.

choose 2 balls in the same $\binom{3}{2} = 3$

put these 2 in one box: 3

put final ball in another: 2

$3 \times 3 \times 2$

Problem 6

Suppose you draw one card at the time from a deck of 52 cards. What is the probability that the second card drawn is an ace?

$$\frac{4 \times 3 + (52-4) \times 4}{52 \times 51} = \frac{204}{52 \times 51} = \ldots$$

$$\frac{4 \times 51}{52 \times 51}$$

Problem 7

The probability that there are n insured losses throughout a year obey the rule $p_{n+1} = p_n/5$. What is the probability that there are two or more insured losses?

$$\sum_{i=2}^{\infty} P_i = 1 - P_0 - P_1 = 1 - P_0 - \frac{P_0}{5}$$

$$\sum_{i=0}^{\infty} P_i = 1 \Rightarrow \sum_{i=0}^{\infty} \frac{1}{5^i} P_0 = 1$$

$$\Rightarrow \frac{5}{4} P_0 = 1$$

$$\Rightarrow P_0 = 0.8$$

Problem 8

Without calculation, use logic and definition to explain

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

LHS: choose k people out of n

RHS: 1. choose $k-1$ from first $n-1$, add last

2. choose k from first $n-1$, not add last.

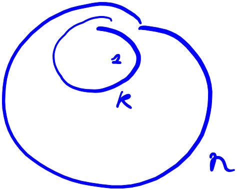
Problem 9

Without calculation, use logic and definition to explain

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

LHS: choose k as candidates, then pick 1 to be president

RHS: [fill-in-yourself]



Problem 10

A woman has n keys, of which one will open her door. If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try? What if she does not discard any of the tried keys (i.e., in this case, she might try the same key multiple times in a row)?

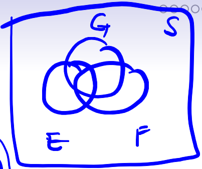
Discard.

$$P(\underbrace{0, 0, 0, \cdots,}_{(k-1)\ 0\text{'s}}\ 1) = \frac{n-1}{n} \times \frac{n-2}{n-1} \times \cdots \times \frac{n-(k-1)}{n-(k-2)} \times \frac{1}{n-(k-1)} = \frac{1}{n}$$

Not discard.

$$\frac{n-1}{n} \times \frac{n-1}{n} \times \cdots \cdots \frac{n-1}{n} \times \frac{1}{n} = \frac{(n-1)^{k-1}}{n^k}$$

Propositions



Recap

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

$\hookrightarrow P(E) + P(F) + P(G) - P(EF) - P(FG) - P(EG) + P(EFG)$

More sets?

Thm $\cdot$ $$P(E_1 \cup \cdots \cup E_n) = \sum_{r=1}^{n} (-1)^{r+1} \underbrace{\sum_{i_1 < \cdots < i_r}}_{} P(E_{i_1} \cap \cdots \cap E_{i_r})$$

Sum prob of all possible intersections of size r.

Example

Pick an integer in $[1, 1000]$ at random. Compute the probability that it is divisible neither by 12 nor by 15.

$S = \{1, 2, \ldots, 1000\}$

E_r = subset of S consisting of all integers divisible by r

least common multiplier

$E_r \cap E_s = E_{\mathrm{lcm}(r,s)} \Rightarrow E_6 \cap E_4 = E_{12}$

$$1 - P(E_{12} \cup E_{15}) = 1 - \left(P(E_{12}) + P(E_{15}) - P(E_{12} \cap E_{15})\right)$$

$$= 1 - \left(P(E_{12}) + P(E_{15}) - P(E_{60})\right)$$

$$= 1 - \left(\frac{83}{1000} + \frac{66}{1000} - \frac{16}{1000}\right) = \cdots$$

Example

Sit 3 men and 4 women at random in a row. What is the probability that either all the men or all the women end up sitting together?

E_1 : all men together

E_2 : all women together

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

$$= \frac{3!\,5!}{7!} + \frac{4!\,4!}{7!} - \frac{3!\,4!\,2!}{7!}$$

$$= \ldots$$

Example

A group of 3 Democrats, 4 Republicans, and 5 Independents is seated at random ~~around a table~~ in a row. Compute the probability that at least one of the three groups ends up sitting together.

$$P(E_D \cup E_R \cup E_I) = P(E_D) + P(E_R) + P(E_I) - P(E_{DR}) - P(E_{RI}) - P(E_{ID}) + P(E_{DRI}).$$

$$P(E_{DRI}) = \frac{3!\,4!\,5!\,3!}{(3+4+5)!}$$

.....

Example

A large company with n employees has a scheme according to which each employee buys a Christmas gift and the gifts are then distributed at random to the employees. What is the probability that someone gets his or her own gift?

E_i : i-th employer get his own gift.

$$P\left(\bigcup_{i=1}^{n} E_i\right) = ?$$

$$P(E_i) = \frac{1}{n} \quad \forall i = 1, \dots, n$$

$$P(E_i \cap E_j) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)} \quad \forall i \neq j$$

$$P(E_i \cap E_j \cap E_k) = \frac{1}{n(n-1)(n-2)} \quad \forall i \neq j \neq k$$

$$\vdots$$

$$P(E_1 \cap \dots \cap E_n) = \frac{1}{n!}$$

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \overbrace{n \times \frac{1}{n}}^{\sum_{i=1}^{n} P(E_i)} - \underbrace{\binom{n}{2} \frac{1}{n(n-1)}}_{\sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2})} + \binom{n}{3} \frac{1}{n(n-1)(n-2)} - \dots$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \dots + (-1)^{n-1} \frac{1}{n!}$$

$$\to 1 - \frac{1}{e} \approx 0.63$$

Example

Just a review

A good time to recall some common mathematical series:

$$\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
log(1-x) &= -\sum_{n=1}^{\infty} \frac{x^n}{n} \\
log(1+x) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \\
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n
\end{aligned}$$

Binomial series, or binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

pf see textbook Chap 1

Binomial series, or binomial theorem

Example

How many subsets are there of a set consisting of n elements?

for sets of size k : $\binom{n}{k}$

total $\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k}$

$= (1+1)^n$

$= 2^n$

Example: Birthday Problem

Assume that there are k people in the room. What is the probability that there are two who share a birthday? We will ignore leap years, assume all birthdays are equally likely.

$$P(\text{a shared birthday}) = 1 - P(\text{no shared birthday})$$
$$= 1 - \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{(365-k+1)}{365}$$

k	P
10	0.1169
23	0.5073
41	0.9032

How many pairs can be formed by 23 people?

$$\binom{23}{2} = \frac{23 \times 22}{2} = 253$$

More on the birthday problem

- Each day, the Massachusetts lottery chooses a four digit number at random, with leading 0's allowed.
- On February 6, 1978, the Boston Evening Globe reported that

"During [the lotterys] 22 months existence [...], no winning number has ever been repeated. [...] doesnt expect to see duplicate winners until about half of the 10, 000 possibilities have been exhausted."

- What if k people enter the room one by one, which person has the highest probability of being the first to share the same birthday with the people in the room? Come back to this in Chap 4.

Coupon collector problem

Within the context of the birthday problem, assume that $k \geq n$ and compute P(all n birthdays are represented).

Will come back to this in a few lectures.

Coupon collector problem