Stat 394 Probability I

Lecture 5

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Example

- Toss a coin 10 times and let X be the number of Heads.
- Choose a random point in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ and let X be its distance from the origin.
- Choose a random person in a class and let X be the height of the person, in inches.

Random variables

A **random variable** is a number whose value depends upon the outcome of a random experiment.

$S \xrightarrow{X} \{1, 2, 3, 5\}$

A random variable X is a real-valued function on sample space S:

$$X : S \to \mathcal{R}$$

Example

Toss 2 coins. $X = \#$ of H's.

$\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$.

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

2 1 1 0

$P(X=0) = \frac{1}{4}$

$s \in S$

$\underbrace{P(X=x)}$ $\longrightarrow$ $P(s)$ where $X(s) = x$

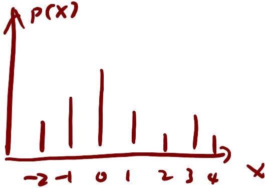
correspond to some events.

Discrete random variables

A **discrete random variable** X has finitely or countably many values $x_i, i = 1, 2, \dots$

$p(x_i) = P(X = x_i)$ with $i = 1, 2, \dots$ is called the **probability mass function** (or p.m.f) of X.

$$\sum_{i=1}^{\infty} p(x_i) = 1$$



Using r.v. to solve probability problems

4 balls are to be randomly selected without replacement from an urn containing 10 balls numbered 1 through 10. If we bet that at least one of the balls that are drawn has a number as large as or larger than 7, what is the probability that we win the bet?

X : largest # selected

$$P(X=i) = \frac{\binom{i-1}{3}}{\binom{10}{4}} \qquad i = 4, 5, 6, \dots, 10$$

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \dots$$

Bernoulli random variables

(and the probability of success is p)

Suppose an experiment where the outcome can be considered as either a *success* or a *failure*, let $X = 1$ when outcome is success and $X = 0$ when outcome is failure, then X is said to be a **Bernoulli random variable** with parameter p.

p.m.f.

$$\begin{cases} p(0) = 1 - p \\ p(1) = p \end{cases}$$

Def 2.

a r.v. is a Bernoulli r.v if its p.m.f is as on the left.

And we can say $X \sim \text{Bern}(p)$

Binomial random variables

- $X \sim Bern(p) \Leftrightarrow X \sim Bin(1, p)$
- Example: Toss a fair coin 3 times, let X denote # of heads $\Rightarrow X \sim Bin(3, 1/2)$

Suppose for n independent trials, each of which results in a *success* with probability p and *failure* with probability $1-p$, and let X represent the number of success that occur in the n trials, then X is said to be a **binomial random variable** with parameter (n, p).

$(X \sim Bin(n, p))$

p.m.f: $p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$, $i = 0, 1, \cdots, n$

We can check

$$\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \overset{\text{binomial formula.}}{=} (p + (1-p))^n = 1$$

Bernoulli and Binomial distribution

$X \sim Bin(n, p)$

- X is # of success out of n trials with success prob p
- $X = X_1 + X_2 + \cdots + X_n$, $X_i = \begin{cases} 1 & \text{if trial } i \text{ is success.} \\ 0 & \text{otherwise.} \end{cases}$ $i = 1, 2, 3, \ldots, n$.

i.e. $X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} Bern(p)$

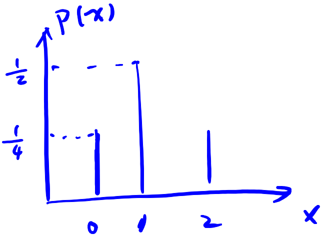
↳ independently and identically distributed

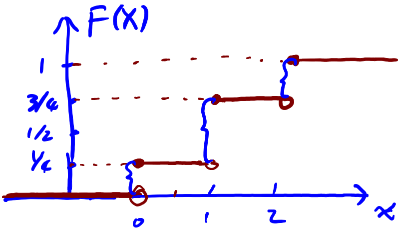
Cumulative distribution function (CDF)

Def.

$$F(x) = P(\underbrace{X \le x}_{\text{event}}) \,, \quad -\infty < x < \infty$$

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$





$$F(0) = P(X \le 0) = P(X = 0) = \frac{1}{4}$$

$$F(0.5) = P(X \le 0.5) = P(X = 0)$$

Example

$X \sim Bin(2k-1, p)$

Example

Let X : # of functional comp. in the first $(2k-1)$ comp

A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. In general, when is a $(2k+1)$-component system better than a $(2k-1)$-component system?

Ans: when $p > 1/2$

$p_1 = P(2k+1 \text{ system is effective}) = P(X \geq k+1) + P(X=k)\cdot(1-(1-p)^2) + P(X=k-1)\cdot p^2$

$\Rightarrow$ at least $k+1$ comp. function

$p_2 = P(2k-1 \text{ system is effective}) = P(X \geq k) = P(X=k) + P(X \geq k+1)$

$p_1 - p_2 = -P(X=k)(1-p)^2 + P(X=k-1)\cdot p^2$

$= -\binom{2k-1}{k} p^k (1-p)^{k-1} \cdot (1-p)^2 + \binom{2k-1}{k-1} p^{k-1}(1-p)^k \cdot p^2$

$\binom{2k-1}{k}$

$= \binom{2k-1}{k}\left[-p^k(1-p)^{k+1} + p^{k+1}(1-p)^k\right] = \binom{2k-1}{k} p^k (1-p)^k \left[-(1-p)+p\right]$

$2p-1$

First, why do we want to study random variables

A quote:

"Once you get what a random variable is, it can be hard to explain. Now that I understand what a random variable is, it is difficult to remember what was difficult to understand about it"

First, why do we want to study random variables

But seriously, the way I see the advantage of using random variables

- Describe more complicated events without cumbersome notations
- Focus on the quantities we care about
- Consider problems beyond calculating probabilities
- Handle continuity and infinity more easily

↳ more on 395

↳ Expectation

↳ Variance

↳ ...

Expectation

Assume X is a discrete random variable with possible values $x_i, i = 1, 2, \ldots$. Then the **expected value**, or **expectation**, of X is

$$E(X) \text{ or } EX = \sum_i x_i p(x_i)$$

X : # H for 2 tosses

$p(0) = 1/4$
$p(1) = 1/2$
$p(2) = 1/4$.

$E(X) = 0 \times 1/4 + 1 \times 1/2 + 2 \times 1/4$
$= 1/2 + 1/2 = 1$

Example

A box contains 5 red balls and 3 blue balls. Two balls are withdrawn randomly, and if they are of the same color, you win \$2, otherwise you lose \$1. What is the amount you expect to win?

X : amount of \$ you walk away with.

$$P(X=2) = \frac{\binom{5}{2}+\binom{3}{2}}{\binom{8}{2}} = \frac{13}{28}$$

$$P(X=-1) = 1-\frac{13}{28} = \frac{15}{28}$$

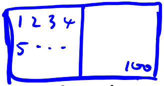
$$E(X) = 2 \times \frac{13}{28} + (-1) \times \frac{15}{28} = \frac{11}{28}$$

Chocolate store

A candy store has M boxes of chocolates. The number of chocolates in each box vary. Suppose for $k = 1, 2, \ldots, N$, there are n_k boxes containing k chocolates, and $\sum_{k=1}^{N} n_k = M$. The store owner keeps a magic book where all the chocolates labeled $1, 2, \ldots, \sum_{k=1}^{N} kn_k$ are listed.

$\boxed{3}$ $\boxed{4}$ $\boxed{10}$ $\cdots$ $\boxed{1}$ $\rightarrow$ 100 chocolates

The store owner lets you to take one box of chocolate for free, and gives you two options:



1. Take one box without knowing how many chocolates are in the box. X_1 : # of c you will get from opt 1.
2. Pick one chocolate from the book, and you can take the box containing that chocolate. X_2 : # you get from opt 2.

Chocolate store

$$P(X_1 = i) = \frac{n_i}{M}$$

$$P(X_2 = i) = \frac{\#\text{ chocolates in boxes containing } i \text{ chocolates}}{\#\text{ chocolates}}$$

$$= \frac{i \cdot n_i}{\sum_{k=1}^{N} k n_k}$$

$$E(X_1) = \sum_{i=1}^{N} i \cdot \frac{n_i}{M}$$

$$E(X_2) = \sum_{i=1}^{N} i \cdot \frac{i n_i}{\sum_{k=1}^{N} k n_k}$$

$$E(X_1)/E(X_2) < 1$$

pf omitted

Expectation of function of a r.v.

$\{y_1, y_2, \dots\}$ denotes the possible values of $g(X)$

Assume X is a discrete random variable with possible values $x_i, i = 1, 2, \dots$. Then the **expected value**, or **expectation**, of $g(X)$, where g is some real-valued function, is

$$E(g(X)) \text{ or } Eg(X) = \sum_i g(x_i)p(x_i)$$

pf. $\sum_i g(x_i)p(x_i) = \sum_j \left(\sum_{i: g(x_i) = y_j} y_j \, p(x_i) \right)$

$= \sum_j y_j \underbrace{\left(\sum_{i: g(x_i) = y_j} p(x_i) \right)}$

$= \sum_j y_j \, P(g(X) = y_j) = E(g(X))$ $\blacksquare$

Example

Roll a die. When you roll k, you will win (k^2-9) dollars.

X : value you roll

$$E(X^2-9) = \sum_{i=1}^{6} (i^2-9)\cdot p(i)$$

$$= \frac{1}{6}\left((1^2-9) + (2^2-9) + \cdots + (6^2-9) \right)$$

$$= \frac{37}{6}$$

Variance

If X is a random variable, we say the **variance** of X is defined by

$$Var(X) = E((X - E(X))^2)$$

and the square root of the variance is usually called the **standard deviation** of X

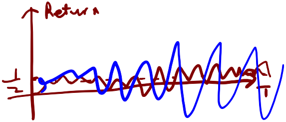
Example

A box contains 5 red balls and 3 blue balls. There are two games offered to you. Both games ask you to take one ball out,

- Game 1: if it is a red ball, you win \$2, otherwise you lose \$2.
- Game 2: if it is a red ball, you win \$5, otherwise you lose \$7.

$$E(X_1) = \frac{5}{8} \times 2 + \frac{3}{8}(-2) = \frac{4}{8} = \frac{1}{2}$$

$$E(X_2) = \frac{5}{8} \times 5 + \frac{3}{8}(-7) = \frac{4}{8} = \frac{1}{2}$$



$$Var(X_1) = E\left((X_1 - E(X_1))^2\right) = \frac{5}{8} \times \left(2 - \frac{1}{2}\right)^2 + \frac{3}{8} \times \left(-2 - \frac{1}{2}\right)^2 = 3.75$$

$$Var(X_2) = E\left((X_2 - E(X_2))^2\right) = \frac{5}{8} \times \left(5 - \frac{1}{2}\right)^2 + \frac{3}{8} \times \left(-7 - \frac{1}{2}\right)^2 = 33.75$$

Alternative formula for variance

$$\mathrm{Var}(X) = E((X - E(X))^2) = E(X^2) - [E(X)]^2$$

Pf.

$$= \sum_x (x - E(X))^2 p(x)$$

$E^2(X)$

$$= \sum_x \left(x^2 - 2x E(X) + E(X)^2\right) p(x)$$

$$= \sum_x x^2 p(x) - \sum_x 2x E(X) p(x) + \sum_x E(X)^2 p(x)$$

$$= \sum_x x^2 p(x) - 2E(X) \underbrace{\sum_x x p(x)} + E(X)^2 \sum_x p(x)$$

$$= E(X^2) - 2E(X) \cdot \underbrace{E(X)} + E(X)^2 \cdot 1$$

$$= E(X^2) - E(X)^2$$

Expectation and variance under linear transformation

Binomial random variable revisit

Binomial random variable revisit

Example

Let X be the number of Heads in 50 tosses of a fair coin. Determine $E(X)$, $Var(X)$ and $P(X \leq 10)$.

Newsboy

A newsboy buy papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10$ and $p = \frac{1}{3}$, approximately how many papers should he purchase so as to maximize his expected profit?

Newsboy

Binomial p.m.f

http://shiny.albany.edu/stat/binomial/

Birthday problem revisit

Birthday problem revisit