Stat 394 Probability I

Lecture 5

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Example

- Toss a coin 10 times and let X be the number of Heads.
- Choose a random point in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ and let X be its distance from the origin.
- Choose a random person in a class and let X be the height of the person, in inches.

Random variables

A **random variable** is a number whose value depends upon the outcome of a random experiment.

$S \xrightarrow{X} \{1, 2, 3, 5\}$

A random variable X is a real-valued function on sample space S:

$$X : S \to \mathcal{R}$$

Example

Toss 2 coins. $X = \#$ of H's.

$\{HH\}$, $\{HT\}$, $\{TH\}$, $\{TT\}$.

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$

2 1 1 0

$P(X=0) = \frac{1}{4}$

$s \in S$

$P(X=x) \longrightarrow P(s)$ where $X(s) = x$

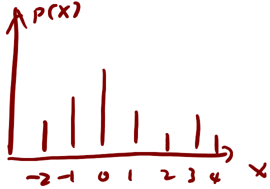
$\underbrace{X=x}$ correspond to some events.

Discrete random variables

A **discrete random variable** X has finitely or countably many values $x_i, i = 1, 2, ...$

$p(x_i) = P(X = x_i)$ with $i = 1, 2, ...$ is called the **probability mass function** (or p.m.f) of X.

$$\sum_{i=1}^{\infty} p(x_i) = 1$$



Using r.v. to solve probability problems

4 balls are to be randomly selected without replacement from an urn containing 10 balls numbered 1 through 10. If we bet that at least one of the balls that are drawn has a number as large as or larger than 7, what is the probability that we win the bet?

X : largest # selected

$$P(X=i) = \frac{\binom{i-1}{3}}{\binom{10}{4}} \qquad i = 4, 5, 6, \ldots, 10$$

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= \ldots$$

Bernoulli random variables

(and the probability of success is p)

Suppose an experiment where the outcome can be considered as either a *success* or a *failure*, let $X = 1$ when outcome is success and $X = 0$ when outcome is failure, then X is said to be a **Bernoulli random variable** with parameter p.

p.m.f.

$$\begin{cases} p(0) = 1-p \\ p(1) = p \end{cases}$$

Def 2.

a r.v. is a Bernoulli r.v if its p.m.f is as on the left.

And we can say $X \sim \text{Bern}(p)$

Binomial random variables

- $X \sim Bern(p) \iff X \sim Bin(1, p)$
- Example: Toss a fair coin 3 times, let X denote # of heads $\Rightarrow X \sim Bin(3, 1/2)$

Suppose for n independent trials, each of which results in a *success* with probability p and *failure* with probability $1-p$, and let X represent the number of success that occur in the n trials, then X is said to be a **binomial random variable** with parameter (n, p).

$(X \sim Bin(n, p))$

p.m.f: $p(i) = P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}, \ i = 0, 1, \cdots, n$

We can check

$$\sum_{i=0}^{n} p(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} = (p + (1-p))^n = 1$$

binomial formula.

Bernoulli and Binomial distribution

$X \sim \text{Bin}(n, p)$

- X is # of success out of n trials with success prob p
- $X = X_1 + X_2 + \cdots + X_n$, $X_i = \begin{cases} 1 & \text{if trial } i \text{ is success.} \\ 0 & \text{otherwise.} \end{cases}$ $i = 1, 2, 3, \ldots, n$.

i.e. $X_1, X_2, \ldots, X_n \overset{\text{i.i.d}}{\sim} \text{Bern}(p)$

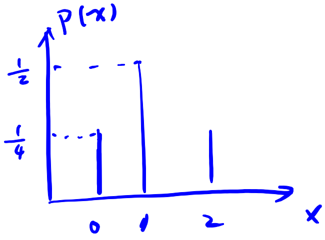
↳ independently and identically distributed

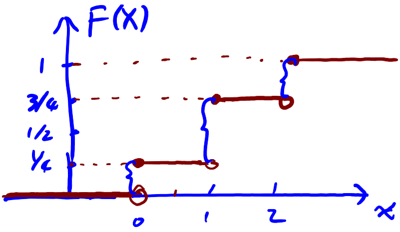
Cumulative distribution function (CDF)

Def.

$$F(x) = P(\underbrace{X \le x}_{\text{event.}}) \quad , \quad -\infty < x < \infty$$

- $\lim_{x \to -\infty} F(x) = 0$
- $\lim_{x \to \infty} F(x) = 1$





$F(0) = P(X \le 0) = P(X=0) = \frac{1}{4}$

$F(0.5) = P(X \le 0.5) = P(X=0)$

Example

$X \sim Bin(2k-1, p)$

Example

Let X: # of functional comp. in the first $(2k-1)$ comp

A communication system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. In general, when is a $(2k+1)$-component system better than a $(2k-1)$-component system?

Ans: when $p > 1/2$

$p_1 = P(\text{2k+1 system is effective}) = P(X \geq k+1) + P(X=k)\cdot(1-(1-p)^2) + P(X=k-1)\cdot p^2$

$\Rightarrow$ at least $k+1$ comp. function

$p_2 = P(\text{2k-1 system is effective}) = P(X \geq k) = P(X=k) + P(X \geq k+1)$

$$p_1 - p_2 = -P(X=k)(1-p)^2 + P(X=k-1)\cdot p^2$$

$$= -\binom{2k-1}{k} p^k (1-p)^{k-1} \cdot (1-p)^2 + \binom{2k-1}{k-1} p^{k-1}(1-p)^k \cdot p^2$$

$\binom{2k-1}{k}$

$$= \binom{2k-1}{k}\left[-p^k(1-p)^{k+1} + p^{k+1}(1-p)^k\right] = \binom{2k-1}{k} p^k (1-p)^k \left[-(1-p)+p\right]$$

$2p-1$

First, why do we want to study random variables

A quote:

“Once you get what a random variable is, it can be hard to explain. Now that I understand what a random variable is, it is difficult to remember what was difficult to understand about it”

First, why do we want to study random variables

But seriously, the way I see the advantage of using random variables

- Describe more complicated events without cumbersome notations
- Focus on the quantities we care about
- Consider problems beyond calculating probabilities
- Handle continuity and infinity more easily

more on 395

Expectation

Variance

...

Expectation

Assume X is a discrete random variable with possible values $x_i, i = 1, 2, \ldots$. Then the **expected value**, or **expectation**, of X is

$$E(X) \text{ or } EX = \sum_i x_i p(x_i)$$

X : # H for 2 tosses

$p(0) = 1/4$
$p(1) = 1/2$
$p(2) = 1/4$.

$E(X) = 0 \times 1/4 + 1 \times 1/2 + 2 \times 1/4$
$= 1/2 + 1/2 = 1$

Example

A box contains 5 red balls and 3 blue balls. Two balls are withdrawn randomly, and if they are of the same color, you win \$2, otherwise you lose \$1. What is the amount you expect to win?

X : amount of \$ you walk away with.

$$P(X=2) = \frac{\binom{5}{2}+\binom{3}{2}}{\binom{8}{2}} = \frac{13}{28}$$

$$P(X=-1) = 1-\frac{13}{28} = \frac{15}{28}$$

$$E(X) = 2 \times \frac{13}{28} + (-1) \times \frac{15}{28} = \frac{11}{28}$$

Chocolate store

A candy store has M boxes of chocolates. The number of chocolates in each box vary. Suppose for $k = 1, 2, \ldots, N$, there are n_k boxes containing k chocolates, and $\sum_{k=1}^{N} n_k = M$. The store owner keeps a magic book where all the chocolates labeled $1, 2, \ldots, \sum_{k=1}^{N} kn_k$ are listed.

c_3 c_4 c_{100} c_{40}

$\boxed{3}$ $\boxed{4}$ $\boxed{10}$ $\cdots$ $\boxed{2}$ $\rightarrow$ 100 chocolates

The store owner lets you to take one box of chocolate for free, and gives you two options:

1 2 3 4
5 . . .
100

1. Take one box without knowing how many chocolates are in the box. X_1 : # of c you will get from opt 1.
2. Pick one chocolate from the book, and you can take the box containing that chocolate. X_2 : # you get from opt 2.

Chocolate store

$$P(X_1 = i) = \frac{n_i}{M}$$

$$P(X_2 = i) = \frac{\#\text{ chocolates in boxes containing } i \text{ chocolates}}{\#\text{ chocolates}}$$

$$= \frac{i \cdot n_i}{\sum_{k=1}^{N} k n_k}$$

$$E(X_1) = \sum_{i=1}^{N} i \cdot \frac{n_i}{M}$$

$$E(X_2) = \sum_{i=1}^{N} i \cdot \frac{i n_i}{\sum_{k=1}^{N} k n_k}$$

$E(X_1)/E(X_2) < 1$

pf omitted.

Expectation of function of a r.v.

$\{y_1, y_2, \dots\}$ denotes the possible values of $g(X)$

Assume X is a discrete random variable with possible values $x_i, i = 1, 2, \dots$. Then the **expected value**, or **expectation**, of $g(X)$, where g is some real-valued function, is

$$E(g(X)) \text{ or } Eg(X) = \sum_i g(x_i)p(x_i)$$

pf. $\sum_i g(x_i)p(x_i) = \sum_j \left(\sum_{i:g(x_i)=y_j} y_j \, p(x_i) \right)$

$= \sum_j y_j \underbrace{\left(\sum_{i:g(x_i)=y_j} p(x_i) \right)}_{}$

$= \sum_j y_j \; P(g(X) = y_j) = E(g(X))$ $\blacksquare$

Example

Roll a die. When you roll k, you will win (k^2-9) dollars.

X : value you roll

$$E(X^2-9) = \sum_{i=1}^{6} (i^2-9) \cdot p(i)$$

$$= \frac{1}{6}\left((1^2-9) + (2^2-9) + \cdots + (6^2-9) \right)$$

$$= \frac{37}{6}$$

Variance

If X is a random variable, we say the **variance** of X is defined by

$$Var(X) = E((X - E(X))^2)$$

and the square root of the variance is usually called the **standard deviation** of X

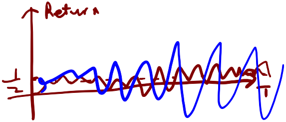
Example

A box contains 5 red balls and 3 blue balls. There are two games offered to you. Both games ask you to take one ball out,

- Game 1: if it is a red ball, you win \$2, otherwise you lose \$2.
- Game 2: if it is a red ball, you win \$5, otherwise you lose \$7.

$$E(X_1) = \frac{5}{8} \times 2 + \frac{3}{8}(-2) = \frac{4}{8} = \frac{1}{2}$$

$$E(X_2) = \frac{5}{8} \times 5 + \frac{3}{8}(-7) = \frac{4}{8} = \frac{1}{2}$$



$$Var(X_1) = E\left((X_1 - E(X_1))^2\right) = \frac{5}{8} \times \left(2 - \frac{1}{2}\right)^2 + \frac{3}{8} \times \left(-2 - \frac{1}{2}\right)^2 = 3.75$$

$$Var(X_2) = E\left((X_2 - E(X_2))^2\right) = \frac{5}{8} \times \left(5 - \frac{1}{2}\right)^2 + \frac{3}{8} \times \left(-7 - \frac{1}{2}\right)^2 = 33.75$$

Alternative formula for variance

$$Var(X) = E((X - E(X))^2) = E(X^2) - [E(X)]^2$$

Pf.

$$= \sum_x (x - E(X))^2 p(x)$$

$E^2(X)$

$$= \sum_x (x^2 - 2xE(X) + E(X)^2)\, p(x)$$

$$= \sum_x x^2 p(x) - \sum_x 2x E(X) p(x) + \sum_x E(X)^2 p(x)$$

$$= \sum_x x^2 p(x) - 2E(X) \underbrace{\sum_x x p(x)} + E(X)^2 \sum_x p(x)$$

$$= E(X^2) - 2E(X) \cdot \underbrace{E(X)} + E(X)^2 \cdot 1$$

$$= E(X^2) - E(X)^2$$

Expectation and variance under linear transformation

$X, \quad aX+b$

$$E(aX+b) = aE(X)+b$$

Pf. $LHS = \sum_{x_i} (ax_i+b)\,p(x_i) = \sum_{x_i} a x_i p(x_i) + b\,p(x_i)$

$= a\sum_{x_i} x_i p(x_i) + b\sum_{x_i} p(x_i)$

$= aE(X)+b$

$$Var(aX+b) = a^2 Var(X)$$

Pf. $LHS = E\left(\left(aX+b - E(aX+b)\right)^2\right)$

$= E\left(\left(aX+b-(aE(X)+b)\right)^2\right)$

$= E\left(\left(a(X-E(X))\right)^2\right)$

$= E\left(a^2(X-E(X))^2\right)$

$= a^2 E\left((X-E(X))^2\right)$

$= a^2 Var(X)$

Binomial random variable revisit

$X \sim Bin(n, p)$

$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \cdots, n$

$E(X) = np$

x from 1 to n
x' from 0 to n-1
x' from 0 to n'

Pf. $E(X) = \sum_{x=0}^{n} \underbrace{x\binom{n}{x}}_{n\binom{n-1}{x-1}} p^x (1-p)^{n-x}$

$= \sum_{x=1}^{n} x\binom{n}{x} p^x (1-p)^{n-x} + \underbrace{0 \cdot \binom{n}{0} p^0 (1-p)^{n-0}}_{=0}$

Let $x' = x-1$, $n' = n-1$

$= \sum_{x=1}^{n} n\binom{n-1}{x-1} p^x (1-p)^{n-x}$

$= \sum_{x'=0}^{n'} (n'+1)\binom{n'}{x'} p^{x'+1} (1-p)^{n'-x'}$

$= (n'+1) \sum_{x'=0}^{n'} \binom{n'}{x'} p^{x'+1} (1-p)^{n'-x'}$

$= (n'+1)p \sum_{x'=0}^{n'} \binom{n'}{x'} p^{x'} (1-p)^{n'-x'}$

$= (n'+1)p$

$= np$

Binomial random variable revisit

$$\mathrm{Var}(X) = np(1-p)$$

$Y \sim \mathrm{Bin}(n', p)$

$E(Y) = n'p$

pf. $\mathrm{Var}(X) = E(X^2) - E(X)^2$

$$E(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} nx \binom{n-1}{x-1} p^x (1-p)^{n-x}$$

$$= np \sum_{x=1}^{n} x \binom{n-1}{x-1} p^{x-1} (1-p)^{n-x} \leftarrow \begin{cases} x' = x-1 \\ n' = n-1 \end{cases}$$

$$= np \sum_{x'=0}^{n'} (x'+1) \binom{n'}{x'} p^{x'} (1-p)^{n'-x'}$$

$$= np \sum_{x'=0}^{n'} x' \binom{n'}{x'} p^{x'} (1-p)^{n'-x'} + np \sum_{x'=0}^{n'} \binom{n'}{x'} p^{x'} (1-p)^{n'-x'}$$

$$= np\big((n-1)p\big) + np\big(1\big) = n^2p^2 + np(1-p)$$

Then

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 = E(X^2) - n^2p^2 = np(1-p)$$

Example

Let X be the number of Heads in 50 tosses of a fair coin. Determine $E(X)$, $Var(X)$ and $P(X \leq 10)$.

$$X \sim Bin(50, 1/2)$$

$$E(X) = 50 \times \frac{1}{2} = 25$$

$$Var(X) = 50 \times \frac{1}{2} \times \left(1 - \frac{1}{2}\right) = 12.5$$

$$P(X \leq 10) = \sum_{i=0}^{10} p(i) = \sum_{i=0}^{10} \binom{50}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{50-i}$$

$F(10)$

$$= \sum_{i=0}^{10} \binom{50}{i} \left(\frac{1}{2}\right)^{50}$$

Newsboy

Textbook 4b

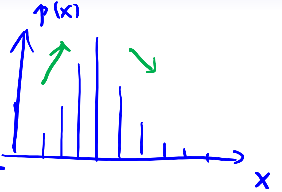
A newsboy buy papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand is a binomial random variable with $n = 10$ and $p = \frac{1}{3}$, approximately how many papers should he purchase so as to maximize his expected profit?

Newsboy

If $X \sim Bin(n,p)$, which x maximizes $p(x)$

Binomial p.m.f

http://shiny.albany.edu/stat/binomial/

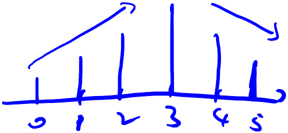


$$\frac{P(X=k)}{P(X=k-1)} = \frac{\frac{\cancel{n!}}{\cancel{k!}(n-k)!} p^k (1-p)^{n-k}}{\frac{\cancel{n!}}{\cancel{(k-1)!}(n-k+1)!} p^{k-1} (1-p)^{n-k+1}}$$

$$= \frac{\frac{1}{k} \cdot p}{\frac{1}{n-k+1} \cdot (1-p)} \quad (*)$$

e.g $n=10$, $p=0.3$

$k < 11 \times 0.3 = 3.3$



$(*) > 1$

$$\frac{p}{k} > \frac{1-p}{n-k+1}$$

$$\Rightarrow \quad k < (n+1)p$$

$n = 365$

Birthday problem revisit

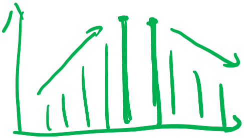
X : the place in the line that's the first match.

$$P(X = x) = 1 \times \frac{n-1}{n} \times \frac{n-2}{n} \times \cdots \times \frac{n-(x-2)}{n} \times \frac{x-1}{n}$$

$$= \frac{x-1}{n^{x-1}} \cdot \frac{(n-1)!}{(n-(x-1))!}$$

$$\frac{P(X = x)}{P(X = x-1)} = \frac{\frac{x-1}{n^{x-1}} \cdot \frac{(n-1)!}{(n-(x-1))!}}{\frac{x-2}{n^{x-2}} \cdot \frac{(n-1)!}{(n-(x-2))!}}$$

$$= \cdots$$



$$\frac{P(X = x)}{P(X = x-1)} > 1 \Rightarrow x < \cancel{19.61}$$

20.61

Birthday problem revisit