Stat 394 Probability I

Lecture 6

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Review: binomial distribution

Theorem

If $X \sim Bin(n, p)$, then as $n \to \infty$,

$$P(X = i) \to \frac{e^{-\lambda}\lambda^i}{i!}, \quad \text{for } i = 0, 1, 2, \ldots$$

where $\lambda = np$.

Pf. $P(X=i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}$

$= \frac{n(n-1)(n-2)\cdots(n-i+1)}{i!} \cdot \frac{\lambda^i}{n^i} \left(1-\frac{\lambda}{n}\right)^n \cdot \left(1-\frac{\lambda}{n}\right)^{-i}$

$= \frac{\lambda^i}{i!} \cdot \underbrace{\frac{n(n-1)\cdots(n-i+1)}{n^i}}_{A} \cdot \underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{B} \cdot \underbrace{\frac{1}{\left(1-\frac{\lambda}{n}\right)^i}}_{C}$

$\lim_{n\to\infty} A = \lim_{n\to\infty}\left(\frac{n}{n}\cdot\frac{n-1}{n}\cdot\ldots\cdot\frac{n-i+1}{n}\right) = \lim_{n\to\infty}\left(1\cdot\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{i-1}{n}\right)\right) = 1$

Theorem

$$\lim_{n\to\infty} B = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

$$\lim_{n\to\infty} C = \lim_{n\to\infty} \frac{1}{\left(1-\frac{\lambda}{n}\right)^{i}} = 1$$

Poisson distribution

Definition: We say $X \sim Poisson(\lambda)$ with $\lambda > 0$ if it has the following p.m.f

$$P(X = i) = \frac{e^{-\lambda}\lambda^i}{i!}, \text{ for } i = 0, 1, 2, \ldots$$

Example: Suppose the number of typos on a single page of my lecture slides has a Poisson distribution with parameter $\lambda = 0.2$. Calculate the probability that there is at leest one typo on this page.

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - \frac{e^{-0.2} \cdot (0.2)^0}{0!} = 1 - e^{-0.2} \approx 0.181$$

Example

Suppose that the probability that a person is killed by lightning in a year is, independently, 1/(500 million). Assume that the US population is 300 million.

- Compute P(3 or more people will be killed by lightning next year) exactly.

X: # of people killed by lightning.

$X \sim Bin\left(300m, \frac{1}{500m}\right)$

$P(X \geq 3) = 1 - P(X=0) - P(X=1) - P(X=2) = 1 - (1-p)^n - np(1-p)^{n-1} - \binom{n}{2} p^2 (1-p)^{n-2}$

$= 0.02311530$

- Approximate the above probability.

$X \sim Poi\left(\frac{3}{5}\right)$ (Approx), $\lambda = \frac{3}{5}$

$P(X \geq 3) = 1 - e^{-\lambda} - \lambda e^{-\lambda} - \frac{\lambda^2}{2} e^{-\lambda}$

≈ 0.02311529

Expectation

$Y \sim Bin(n,p)$

$E(Y) = np$

Approx by $X \sim Poi(\lambda)$

$\lambda = np$

$X \sim Poi(\lambda)$

$$E(X) = \sum_{i=0}^{\infty} i \cdot p(i) = \sum_{i=0}^{\infty} \frac{i \cdot e^{-\lambda} \lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{i \cdot \lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=1}^{\infty} \frac{i \cdot \lambda^i}{i!}$$

$$= e^{-\lambda} \underbrace{\sum_{i=1}^{\infty} \frac{\lambda^i}{(i-1)!}}$$

$$= e^{-\lambda} \cdot \lambda \cdot \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda}$$

$$= \lambda$$

$$e^x = \begin{cases} \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \\ \sum_{i=0}^{\infty} \frac{x^i}{i!} \end{cases}$$

Variance

$$= \sum_{i=0}^{\infty} i^2 p(i)$$

$$E(X^2) = \sum_{i=0}^{\infty} \frac{i^2 \cdot e^{-\lambda} \lambda^i}{i!}$$

$$= 0 + \frac{1^2 \cdot e^{-\lambda} \cdot \lambda^1}{1!} + \sum_{i=2}^{\infty} \frac{i^2 e^{-\lambda} \lambda^i}{i!}$$

$$= \lambda e^{-\lambda} + e^{-\lambda} \sum_{i=2}^{\infty} \frac{i \lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} + e^{-\lambda} \sum_{i=2}^{\infty} \frac{(i-1+1)\lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} + e^{-\lambda} \left(\underbrace{\sum_{i=2}^{\infty} \frac{(i-1)\lambda^i}{(i-1)!}}_{\sum_{i=2}^{\infty} \frac{\lambda^i}{(i-2)!} = \lambda^2 \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!}} + \overbrace{\sum_{i=2}^{\infty} \frac{\lambda^i}{(i-1)!}}^{\lambda \sum_{i=2}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}} \right)$$

$Y \sim Bin(n, p)$

$Var(Y) = np(1-p)$

$\lambda = np$

$Var(Y) = \lambda\left(1 - \frac{\lambda}{n}\right) = \lambda - \frac{\lambda^2}{n}$

$\lim_{n \to \infty} Var(Y) = \lambda$

Variance

$$= \lambda e^{-\lambda} + e^{-\lambda}\left(\lambda^2 \sum_{i=2}^{\infty} \frac{\lambda^{i-2}}{(i-2)!} + \lambda \sum_{i=2}^{\infty} \frac{\lambda^{i-1}}{(i-1)!}\right)$$

$$= \lambda e^{-\lambda} + e^{-\lambda}\left(\lambda^2 \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^{j}}{j!}} + \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k}}{k!}\right)$$

$$= \lambda e^{-\lambda} + e^{-\lambda}\left(\lambda^2 \cdot \underbrace{e^{\lambda}} + \lambda\left(\underbrace{\sum_{k=0}^{\infty} \frac{\lambda^{k}}{k!}} - \frac{\lambda^0}{0!}\right)\right)$$

$$= \lambda e^{-\lambda} + e^{-\lambda}\left(\lambda^2 e^{\lambda} + \lambda(e^{\lambda} - 1)\right)$$

$$= \lambda e^{-\lambda} + e^{-\lambda}\left(\lambda^2 e^{\lambda} + \lambda e^{\lambda} - \lambda\right)$$

$$= \cancel{\lambda e^{-\lambda}} + \lambda^2 + \lambda - \cancel{\lambda e^{-\lambda}}$$

$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Additional OH

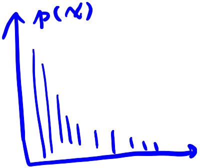
Fri	1-2	7
	2-3	8
	3-4	8

Poisson paradigm

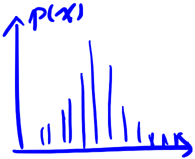
This Friday

Consider n events, with p_i equal to the probability that event i occurs, $i = 1, \ldots, n$. If all the p_i are "small" and the trials are either independent or at most "weakly dependent", then the number of these events that occur approximately has a Poisson distribution with mean $\lambda = \sum_{i=1}^{n} p_i$.

- $p \approx 0.5$, moderate n
- $p \to 0$, large n



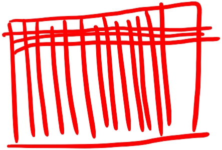
$p = 0.01$



$p = 0.5$

Examples

- the number of winners in a lottery
- the number of people entering a store between 9:00 and 10:00 on weekdays
- the number of wrong numbers dialed each day in Seattle
- the number of raindrops on one brick on the red square at a given second



brick

$P(X=0)$
$P(X=1) \to P(X \le 1) = \ldots$
$P(X=2) \to P(X \le 2) = \ldots$
$P(X=3) \to$
$\vdots$

Example

A company which sells flood insurance has three groups of clients.

- The first group, $N_1 = 10000$, is low-risk: each client has probability $p_1 = 0.01\%$ of a flood, independently of others.
- The second group, $N_2 = 1000$ clients, is medium-risk: $p_2 = 0.05\%$.
- The third group, $N_3 = 100$ clients, is high-risk, $p_3 = 0.5\%$.

$k = 5$

For every flood, a company pays $100,000$. How much should it charge its clients so that is does not go bankrupt with probability at least 95%? ~~The third group, $N_3 = 100$ clients, is high-risk, $p_3 = 0.5\%$~~

X: # of floods

$X \sim Poi(\lambda)$, $\lambda = \sum \tilde{p}_i = N_1 p_1 + N_2 p_2 + N_3 p_3 = 2$

Find k $P(X \le k) \ge 95\% \Rightarrow P(100000 X \le 100000 k) \ge 95\%$

Example: People v. Collins, 1968

Suppose there are $n = 5$ million couples in the LA area, and the probability that a randomly chosen couple fits the descriptions of the witness is $p = 1/12$ million. One such couple is found. What is the probability that there exist other couple(s) who also fit the descriptions.

X : # of such couples

$\cancel{X \sim Bin(n, p)}$

$X \sim Poi(\underbrace{np}_{\frac{5}{12} \to \lambda})$

$$P(X \geq 2 \mid X \geq 1) = \frac{P(X \geq 2 \cancel{\ \& \ X \geq 1})}{P(X \geq 1)}$$

$$= \frac{1 - P(X=0) - P(X=1)}{1 - P(X=0)} = \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{1 - e^{-\lambda}} \approx 0.19$$

Continuous time interpretation of Poisson r.v.

For events happening in continuous time at a rate of λ, the number of event happening at any interval of length t follows a Poisson distribution with parameter λt if the following (informal) conditions are satisfied:

$o(h) = f(h)$ s.t. $\lim_{h \to 0} \frac{f(h)}{h} = 0$

1. The probability of exactly one occurrence of the event in a given time interval h is $\lambda h + o(h)$.
2. The probability of two or more occurrences of the event in a very small time interval is negligible.
3. The numbers of occurrences of the event in disjoint time intervals are mutually independent.

Example

- Number of flying-bomb hits in the south of London during World War II
- Number of wars per year
- Number of earthquakes occurring during a fixed time span

Example

Suppose that the probability that a person is killed by lightning in a year is, independently, 1/(500 million). Assume that the US population is 300 million.

- Approximate P(two or more people are killed by lightning within the first 6 months of next year).

rate : 3/5 per year

X : # killed within 6 months

$\Rightarrow X \sim Poi(\lambda t) \rightarrow Poi(3/5 \times \frac{1}{2}) \rightarrow Poi(0.3)$

$P(X \geq 2) = 1 - P(X=0) - P(X=1) = \ldots$

Combining multiple distributions together

Note: assume year-to-year independent.

- Approximate P(in exactly 3 of next 10 years exactly 3 people are killed by lightning).

X: # killed within a year $\to X \sim Poi(3/5)$

$\Rightarrow P(X=3) = \frac{\lambda^3 e^{-\lambda}}{3!} = p$ where $\lambda = 3/5$

Y: # years where 3 kills happen.

$\hookrightarrow Y \sim Bin(10, P(X=3)) \Rightarrow P(Y=3) = \binom{10}{3} p^3 (1-p)^{10-3}$

- Compute the expected number of years, among the next 10, in which 2 or more people are killed by lightning.

Y: # such years in the ten

$Y \sim Bin(10, \underbrace{P(X \geq 2)}_{= 1 - e^{-\lambda} - \lambda e^{-\lambda}}) \Rightarrow E(Y) = 10 P(X \geq 2)$

$= \ldots$

More example

The number of students coming to the Suzzallo and Allen library in a given hour follow a Poisson distribution with parameter λ. Each student choose to stay at Suzzallo library with probability $1/4$, or stay at Allen library with probability $3/4$. What is the distribution of the number of students in Allen library in a given hour?

More example

Final thoughts about Poisson distribution

- This use of Poisson random variable in continuous time is often times referred to as "Poisson process".
- It goes much deeper beyond what we have looked at in this course.
    - e.g., how do we characterize the interarrival times between two events? *(You will learn in 395, I think)*
- It has a lot of cool applications in finance, physics, environmental science, etc.
- The example we just talked about is called "thinning/splitting of Poisson process".