

# **Bayesian Nonparametrics**

## **A tutorial**

### **with applications in Brain Imaging Analysis**

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O'Bayes 2022

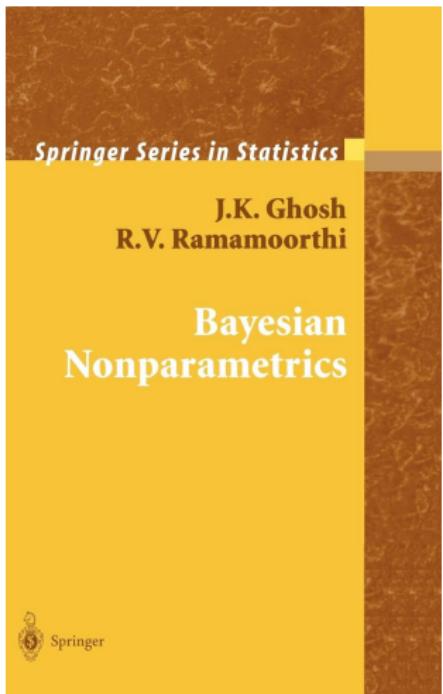
Slides available here: [http://bit.ly/BNP\\_OBayes2022](http://bit.ly/BNP_OBayes2022)



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*Bayesians believe that all inference and more is Bayesian territory.*

*So, it is natural that a Bayesian should explore non-parametrics and other infinite-dimensional problems.*

*However, putting a prior, which is always a delicate and difficult exercise in Bayesian analysis, poses special conceptual, mathematical, and practical difficulties in infinite-dimensional problems.*

*Can one really have a subjective prior based on knowledge and belief, in an infinite-dimensional space?*

*Even if one settles for a largely non-subjective prior, it is mathematically difficult to construct prior distributions on such sets as the space of all distribution functions or the space of all probability density functions and ensure that they have large support, which is a minimum requirement because a largely non-subjective prior should not put too much mass on a small set.*

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- 👉 Bayesian Additive Regression Trees (BART): flexible modeling of the relationships between covariates and outcomes
- 👉 Density Estimation: provide the flexibility necessary to analyze complex data beyond simple parametric assumptions:
  - Dirichlet Processes (DP)
  - Polya Tree priors  
and their generalizations (Dependent DP, Normalized Random Measures...)

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- 👉 A parametric model  $X|\theta \sim p_\theta$  for  $\theta \in \Theta \subset \mathbb{R}^d$  with a prior specification  $\theta \sim \pi$  can be also described as follows:

$$X|p \sim p, \quad p \sim \Pi$$

where  $\Pi$  is a prior distribution on the set of all possible densities with the property that  $\Pi(\{p_\theta : \theta \in \Theta\}) = 1$ .

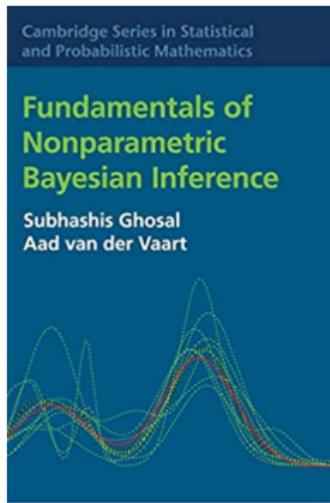
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- !! Thus parametric modeling insists on a prior that assigns probability one to a very small subset of all densities.

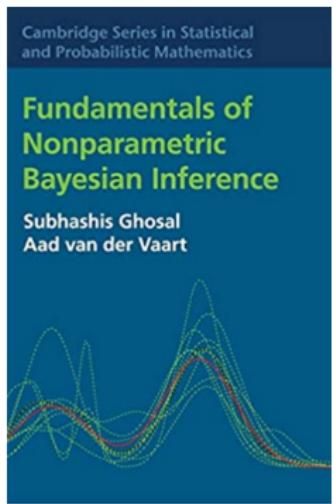
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Foundational and  
Theoretical Aspects

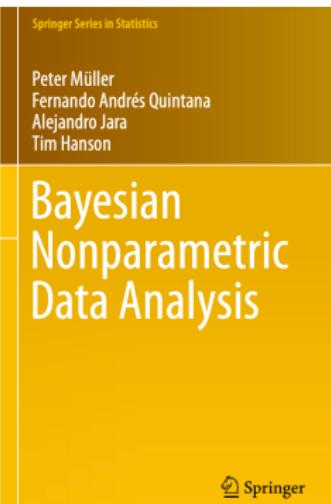
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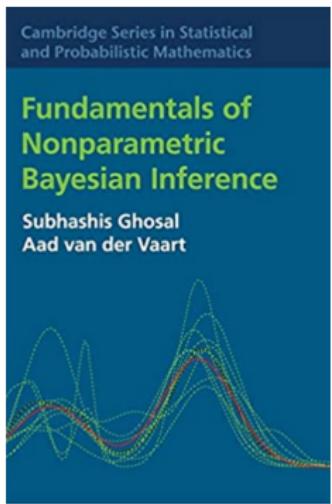
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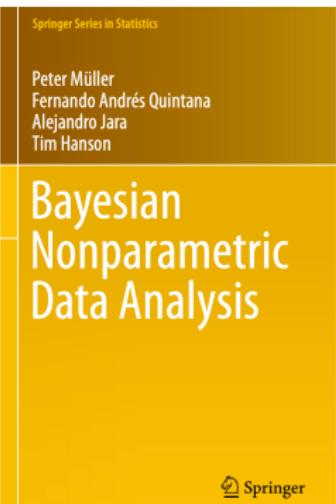
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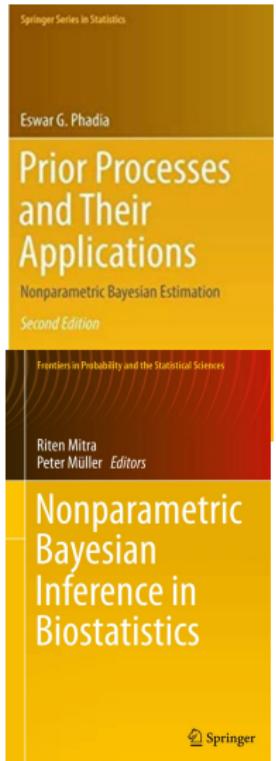
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Bayesian Analysis (2013) 8, Number 2, pp. 269–302

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**Abstract.** We review inference under models with nonparametric Bayesian (BNP) priors. The discussion follows a set of examples for some common inference problems. The examples are chosen to highlight problems that are challenging for standard parametric inference. We discuss inference for density estimation, clustering, regression and for mixed effects models with random effects distributions. While we focus on arguing for the need for the flexibility of BNP models, we also review some of the more commonly used BNP models, thus hopefully answering a bit of both questions, why and how to use BNP.

**Keywords:** Nonparametric models, Dirichlet process, Polya tree, dependent Dirichlet process



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Stat Methods Appl (2018) 27:175–206  
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CrossMark

ORIGINAL PAPER

## Nonparametric Bayesian inference in applications

Peter Müller<sup>1</sup> · Fernando A. Quintana<sup>2</sup> ·  
Garritt Page<sup>3</sup>

Springer Series in Statistics

Eswar G. Phadia

## Prior Processes and Their Applications

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Second Edition

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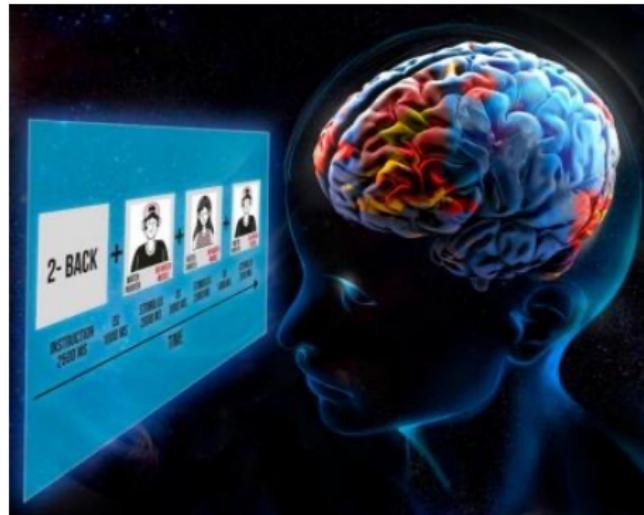
## Nonparametric Bayesian inference in applications

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The collage consists of three video frames. The top frame shows a YouTube video titled "Chinese restaurant process" with a woman standing next to a whiteboard. The middle frame shows a man in a white shirt standing at a whiteboard with mathematical equations and diagrams related to the Chinese Restaurant Process. The bottom frame shows a portrait of a woman with the text "TOPIC: Bayesian nonparametrics" and "By Sinead Williamson".

There is an increasing recognition that brain functioning is heterogeneous and varies greatly both **within and between individuals**:

- differences in **activation** to different stimuli
- differences in **connectivity** to different stimuli
- the different brain activity patterns may be **associated to a clinical outcome or different behaviors** (e.g., large brain responses to food-related cues predict cue-induced eating, Versace et al, 2019)



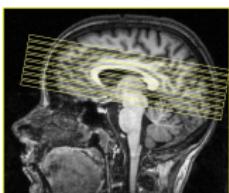
A Bayesian Nonparametric Approach can be used to account for such heterogeneity

Capturing within-subject heterogeneity

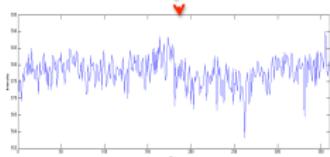
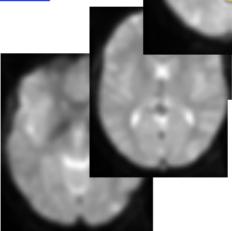
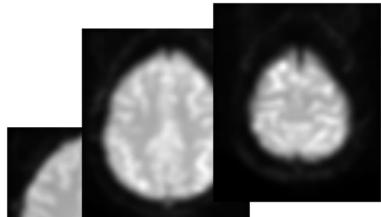
# Capturing Activation patterns in fMRI data



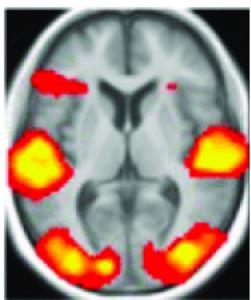
Source: [photos.uc.wisc.edu](http://photos.uc.wisc.edu)



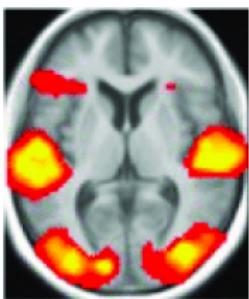
Source: [www.anc.ed.ac.uk](http://www.anc.ed.ac.uk)



- ☛ Indirect measure of brain activity as changes in blood flow, typically collected during a sensorimotor task.
- ☐ Observed data ☛ time series of the blood oxygenation level dependent (BOLD) response, at each voxel in the brain.



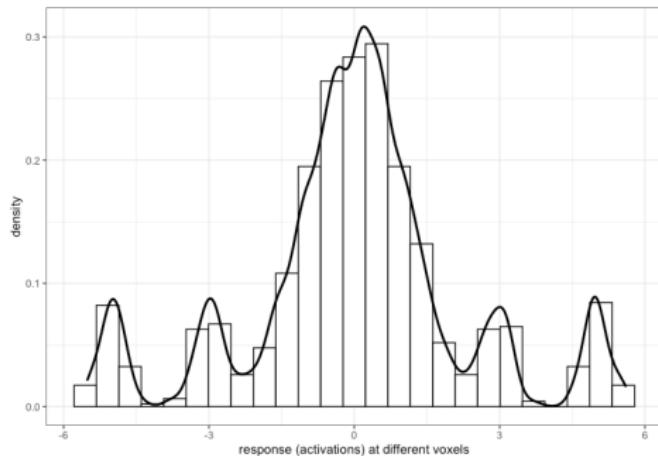
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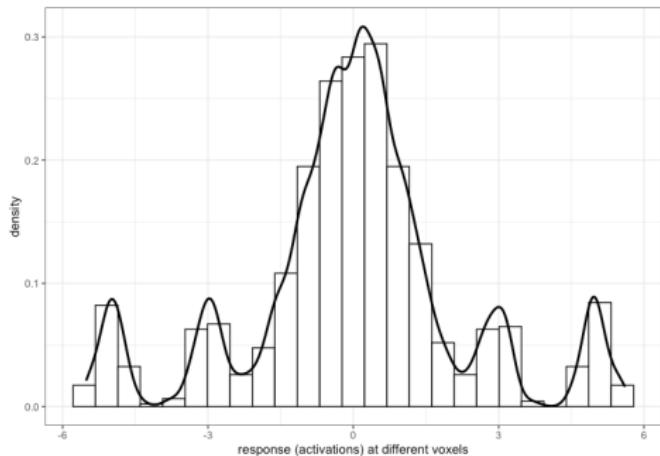
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- $Y_v$ , BOLD response summarized at the  $v$ th voxel in a subject
- $\mu_v$ , random effect to capture activations at different voxels
- $\varepsilon_v$ , is an error term.

- To fix ideas, we can think that the response is characterized by different levels of activations at different voxels:



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👉 We can think at a mixture model to describe the responses:

$$f(y_\nu | \sigma) = \int \mathcal{N}(y_\nu; \mu_\nu, \sigma) d\tilde{p}(\mu_\nu)$$

We can rewrite the previous model in hierarchical form as:

$$\begin{aligned} \mathbf{Y}_\nu \mid \mu_\nu &\stackrel{\text{ind}}{\sim} \mathcal{N}(\mathbf{Y}_\nu; \mu_\nu), \quad \nu = 1, \dots, V \\ \mu_\nu \mid \tilde{p} &\stackrel{\text{iid}}{\sim} \tilde{p} \end{aligned}$$

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- ③ The hyperparameters in the model should be easily **interpretable**.

(Ferguson, 1973)

- One natural choice is to assume:

$$\tilde{p} = \sum_{k=1}^K \pi_k \delta_{\mu_k^*}$$

where  $1 \leq K \leq \infty$ ,  $\pi_k$  are weights ( $\sum_{k=1}^K \pi_k = 1$ ) and the  $\mu_k^*$ 's can be thought of as "*centroids*" of the set of responses

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😊 "The Pony Process"

The stick-breaking formulation for the weights generalizes the Sethuraman's (1994) construction of the weights of the Dirichlet Process. Indeed, setting  $K = \infty$ ,

- 👉  $a_k = 1$  and  $b_k = \alpha$ 
  - ⇒  $DP(\alpha, G_0)$  (Ferguson, 1973; Sethuraman, 1994)
- 👉  $a_k = 1 - a, b_k = b + ka$ , with discount parameter  $0 \leq a < 1$  and strength parameter  $b > -a$ 
  - ⇒  $\mathcal{PY}(a, b, G_0)$  (Pitman and Yor, 1997)
- The discount parameter plays a role on the induced distribution of the number of clusters in the data, the larger being  $a$  the flatter and less informative the prior

👉 Based on the priors above, the model for the data becomes

$$\mathbf{y}_\nu \stackrel{iid}{\sim} \sum_{k=1}^K \pi_k f(y_\nu | \mu_k^*, \sigma)$$

that is, a univariate location PY mixture model (Ferguson, 1983)

- We can assume  $G_0 \equiv N(m_0, \sigma_0)$ , and  $\sigma \sim \pi(\sigma)$ , e.g.  $\sigma^2 \sim \text{IGa}(a_0, b_0)$ .
- We can further assume  $m_0 | \sigma_0^2 \sim N(m_1, \sigma_0^2/k_1)$  and  $\sigma_0^2 \sim \text{IGa}(a_1, b_1)$ .

Two major types of MCMC algorithms have been proposed:

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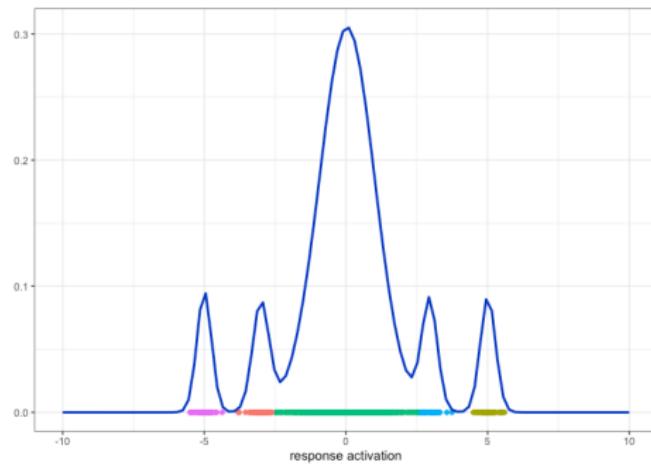
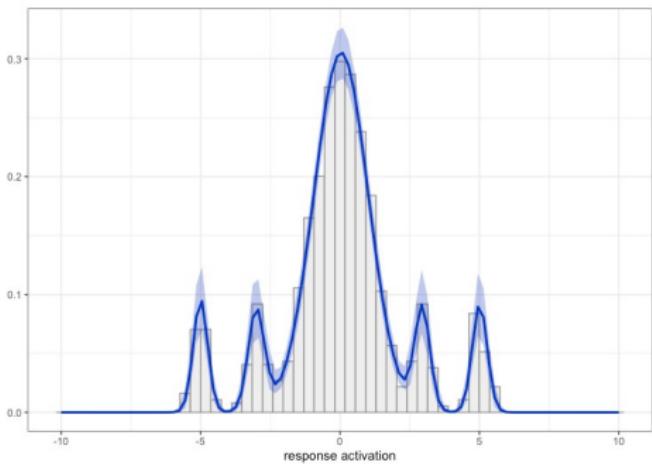
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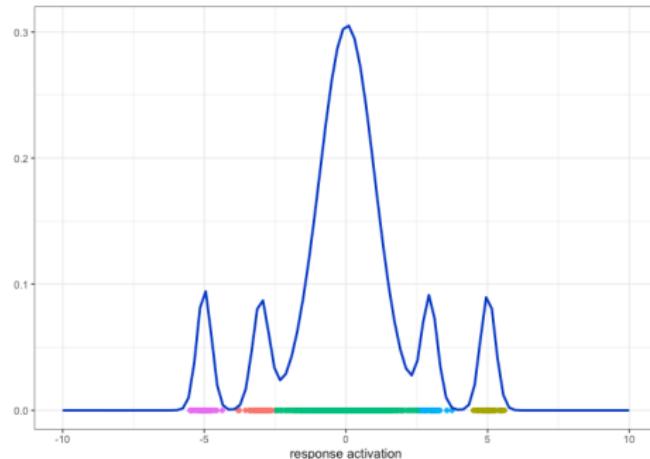
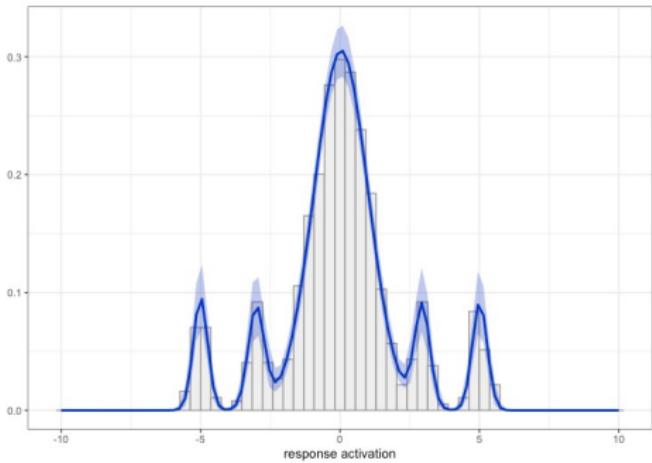
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💻 BNPmix (Corradin, R., Canale, A., and Nipoti, B. 2021) (C++, Rcpp)

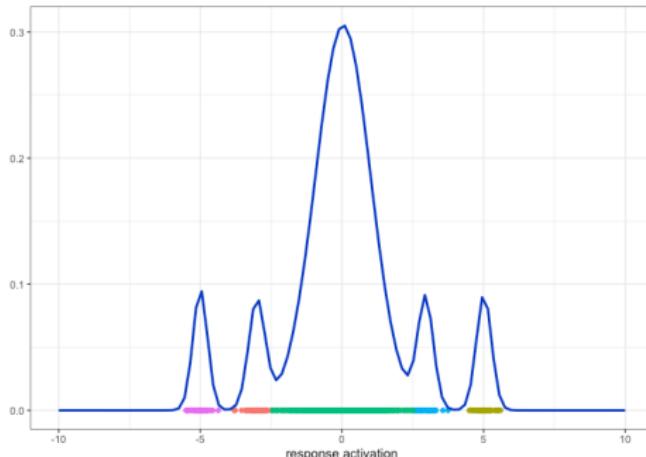
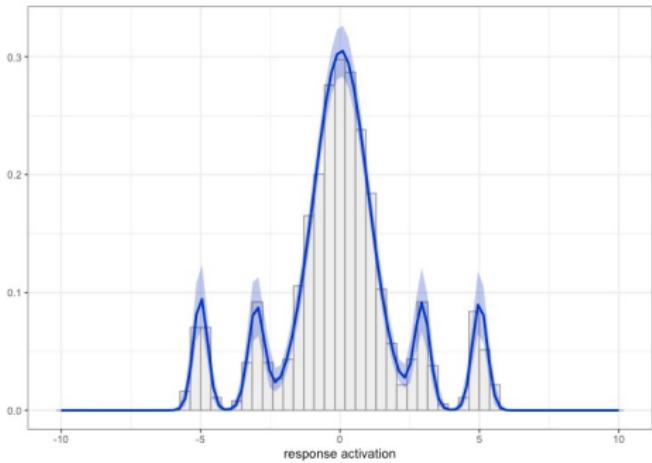




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Induced Posterior on the number of components is inconsistent (Miller and Harrison, 2014)



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III-posed problem ? → Finding the number of clusters is essentially a decision problem

- ?
- What is an appropriate point estimate of the clustering structure based on the posterior distribution?
- ?
- What is an appropriate loss function on the space of clusterings?
- Let  $L(c, \hat{c})$  be a loss function which measures the loss of estimating the true clustering  $c$  with  $\hat{c}$ .
- Since the true clustering is unknown, and the posterior weights the possible clustering configurations, the optimal cluster configurations can be obtained as

$$c^* = \operatorname{argmin}_{\hat{c}} \mathbb{E} [L(c, \hat{c}) | y] = \operatorname{argmin}_{\hat{c}} \sum_c L(c, \hat{c}) p(c | y)$$

- Binder's loss (1978) is invariant to permutations of the data points indices and cluster labels
- It penalizes the two errors of allocating two observations to different clusters when they should be in the same cluster or allocating them to the same cluster when they should be in different clusters:

$$B(c, \hat{c}) = \sum_{n < n'} l_1 \mathbf{1}(c_n = c_{n'}) \mathbf{1}(\hat{c}_n \neq \hat{c}_{n'}) + l_2 \mathbf{1}(c_n \neq c_{n'}) \mathbf{1}(\hat{c}_n = \hat{c}_{n'})$$

If the errors have the same penalty, then it results in a quadratic function of the counts in the two clusters penalized by disagreements between the true and estimated clusterings.

- The variation of information loss (VI) has been proposed by Wade and Ghahramani (2018) and Meilă (2007)
- It measures the amount of information lost and gained in changing from one clustering partition to another

$$\text{VI}(c, \hat{c}) = H(c) + H(\hat{c}) - 2I(c, \hat{c})$$

where  $H(\cdot)$  measures the entropy of a partition (zero if there is only one cluster) and  $I(\cdot)$  is a measure of mutual information between the two clustering (sort of distance between the two clustering structures)

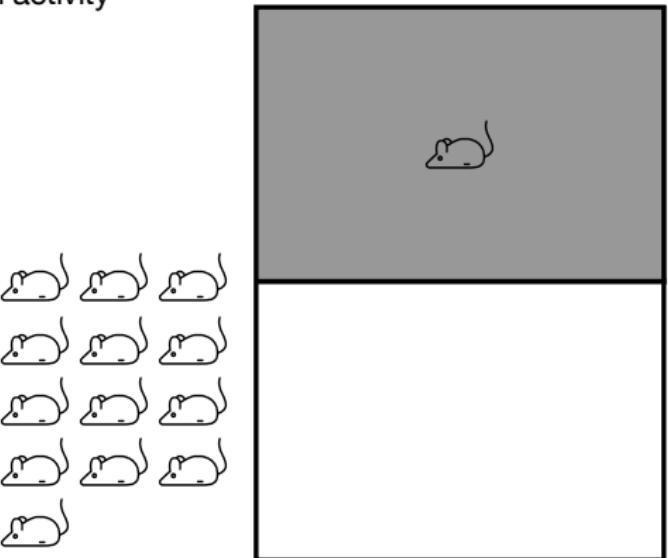
- WG (2018) show how the VI is able to better represent the idea of closest set of partitions to a true partition
- ⇒ They obtain point estimates and credible balls to reflect uncertainty on the partitions.

Implemented in the packages `mcclust` and `BNPmix`.

Bayesian NP mixtures for screening in large-scale testing

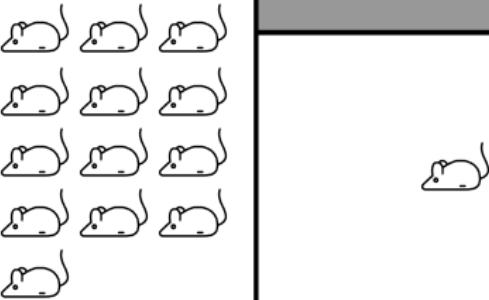
## The light-sheet fluorescence microscopy dataset

- Fourteen mice were **individually housed in the dark** for 24 hours to establish baseline visual activity



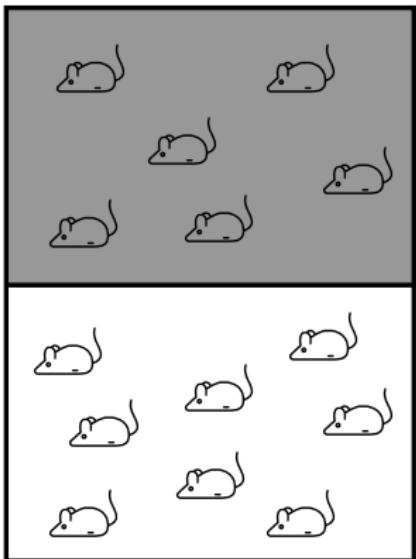
## The light-sheet fluorescence microscopy dataset

- Fourteen mice were **individually housed in the dark** for 24 hours to establish baseline visual activity
- Mice were then transferred into a new cage **exposed to ambient light**

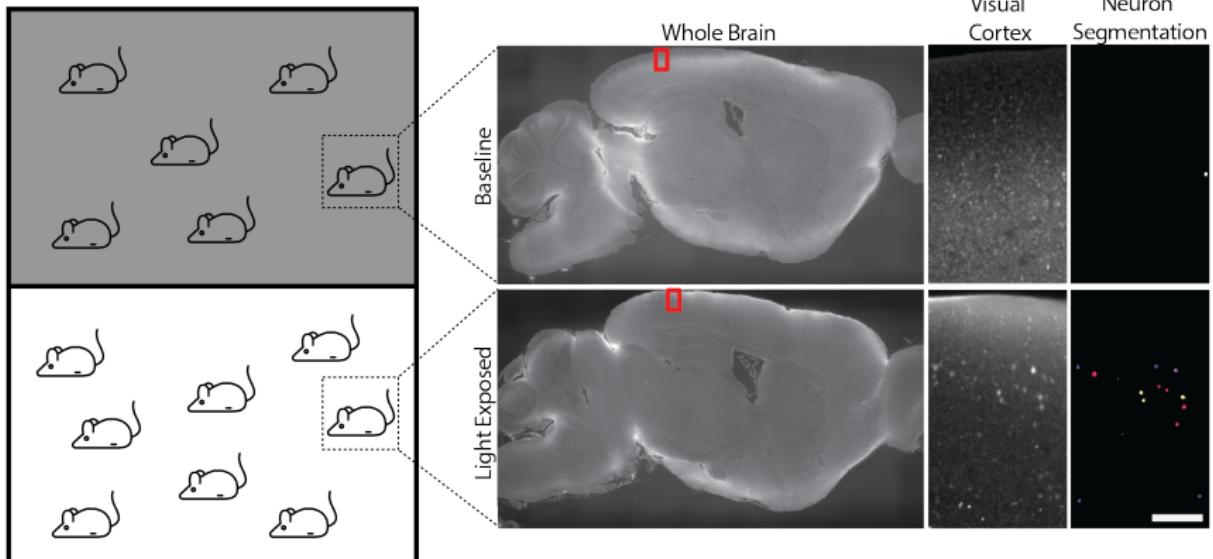


## The light-sheet fluorescence microscopy dataset

- Fourteen mice were **individually housed in the dark** for 24 hours to establish baseline visual activity
- Mice were then transferred into a new cage **exposed to ambient light**
- The brains of six mice were examined **0-15 minutes** (i.e., no light) after light exposure to serve as the **baseline** group
- The brains of another eight mice were examined 30-120 minutes after light exposure, within the window of **Npas4** protein up-regulation (Ramamoorthi et al., 2011)



## The light-sheet fluorescence microscopy dataset

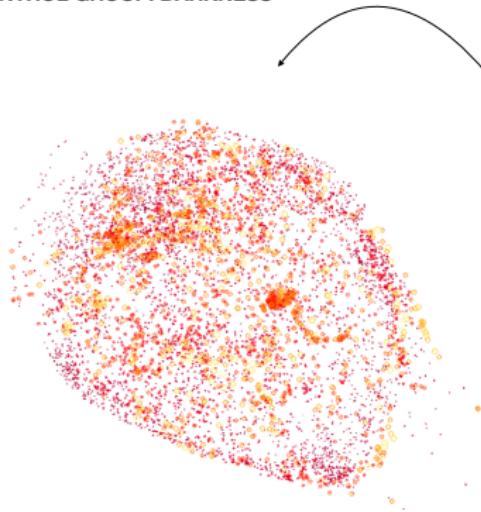


- The light-sheet fluorescence microscopy imaging techniques allows the detection of activated cells at high resolution *in vivo* in the whole-brain of the mouse.

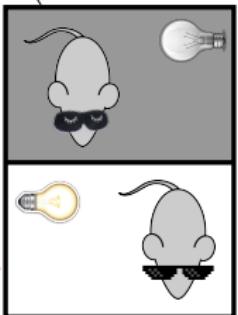
# Light-sheet fluorescence microscopy (LSFM)

UCLA

CONTROL GROUP: DARKNESS



CASE GROUP: LIGHT-EXPOSED



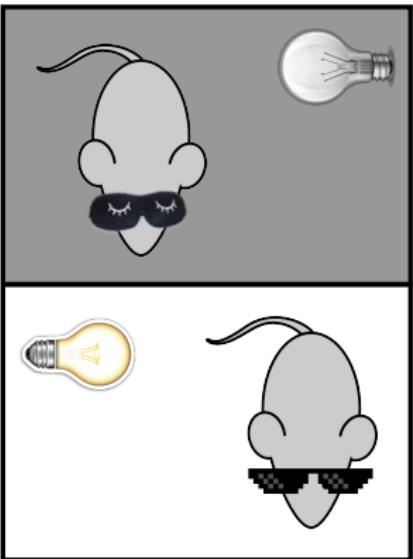
- “Activation” seems to be linked to both fluorescence intensity and frequency of neurons

## The light-sheet fluorescence microscopy dataset

- **GOAL of the study:**

- Assess **differentially activated regions** by comparing the baseline and light-exposed groups
- The activation is measured in terms of Npas4 expression (we will refer to this as **fluorescence**)
- We expect that light exposure induces widespread, visually evoked activity in terms of **fluorescence intensity**
- Data are pre-processed eventually organized into 281 brain regions of interest and z-scores

$$Z_\nu = \beta_\nu + \varepsilon_\nu, \quad \varepsilon_\nu \sim N_T(0, \sigma)$$



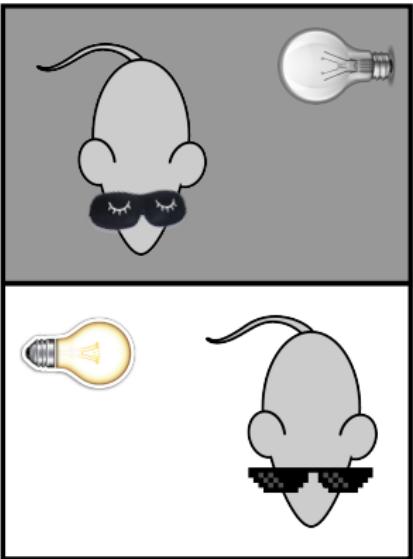
## The light-sheet fluorescence microscopy dataset

- **GOAL of the study:**

- Assess **differentially activated regions** by comparing the baseline and light-exposed groups
- The activation is measured in terms of Npas4 expression (we will refer to this as **fluorescence**)

BH discovers 142 regions (50%) too liberal!

The local FDR method (Efron, 2004) flags only 38 brain regions as relevant, however missing many regions known to be associated with the visual task.

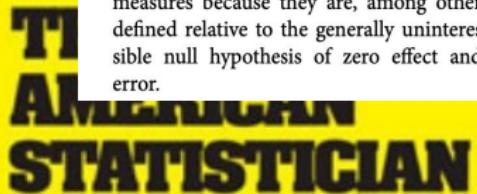


## 7.4. Adopting More Holistic Approaches

**McShane, B., Gal, D., Gelman, A., Robert, C., and Tackett, J.,  
Abandon Statistical Significance**

1. Treat  $p$ -values (and other purely statistical measures like confidence intervals and Bayes factors) continuously rather than in a dichotomous or thresholded manner. In doing so, bear in mind that it seldom makes sense to calibrate evidence as a function of  $p$ -values or other purely statistical measures because they are, among other things, typically defined relative to the generally uninteresting and implausible null hypothesis of zero effect and zero systematic error.

5. Accept uncertainty and embrace variation in effects: we can learn much (indeed, more) about the world by forsaking the false promise of certainty offered by dichotomous declarations of truth or falsity—binary statements about there being “an effect” or “no effect”—based on some  $p$ -value or other statistical threshold being attained.



RONALD L. WASSERSTEIN, ALLEN L. SCHIRM & NICOLE A. LAZAR  
THE AMERICAN STATISTICIAN  
2019, VOL. 73, NO. S1, 1–19: Editorial  
<https://doi.org/10.1080/00031305.2019.1583913>

Moving to a World Beyond " $p < 0.05$ "

👉 Continuous scale mixtures of Gaussians (Carvalho et al, 2010, Polson et al 2012) do not lead to an immediate “selection” of relevant parameters

$$\beta_\nu \mid \tau, \lambda_\nu \sim \mathcal{N}_1(0, \tau^2 \cdot \lambda_\nu^2)$$

with

$\tau \sim g$  a global shrinkage parameter

and

$\lambda_\nu \sim h_\nu$  a local shrinkage parameter

- However, the decisions on the “significance” of the  $\beta$ 's coefficients are typically dichotomized (e.g., based on 90% credible intervals or shrinkage factor)  
or other decision theoretic-based procedures (Chandra, Mueller, Sarkar, 2022+; Lee et al, 2022+)

- We can consider a mixture:

$$\beta_\nu \mid \tau, \lambda_K, \pi, \sigma^2 \sim \sum_{k=1}^K \pi_k \phi(\beta_\nu; 0, \sigma^2 \cdot \tau^2 \cdot \lambda_k^2)$$

where  $\lambda_k^2$  is a mixture shrinkage component.

The smallest variance component is typically such that  $\tau \lambda_{(1)} \approx 0$  and represents the null distribution

The other components can be sorted according to the magnitudes of  $\lambda_k$ 's.

The alternative distribution gets segmented into different levels

- 👉 One can rank the  $\beta_\nu$ 's into **tiers of relevance**

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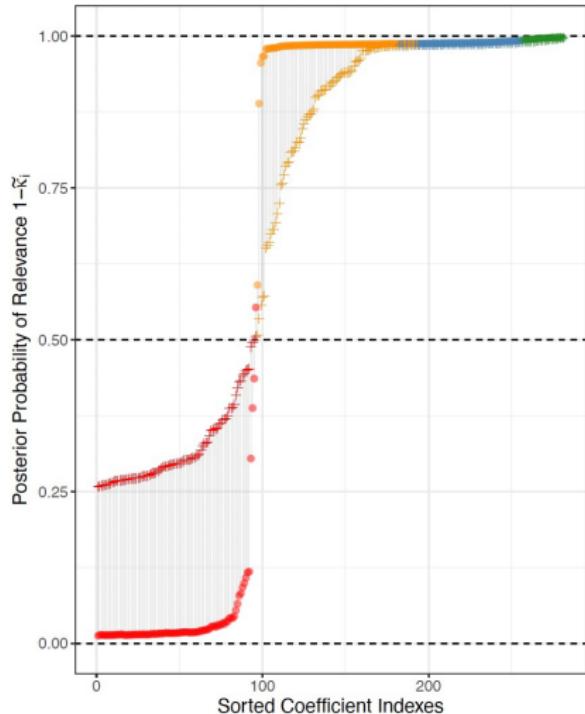
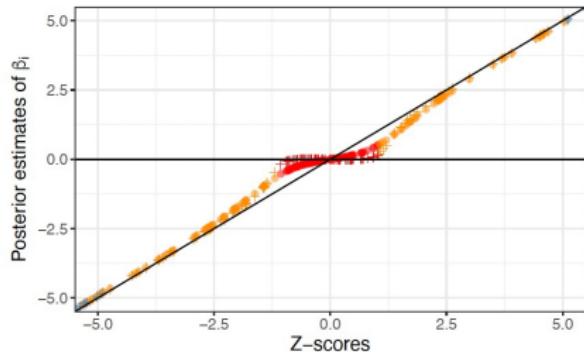
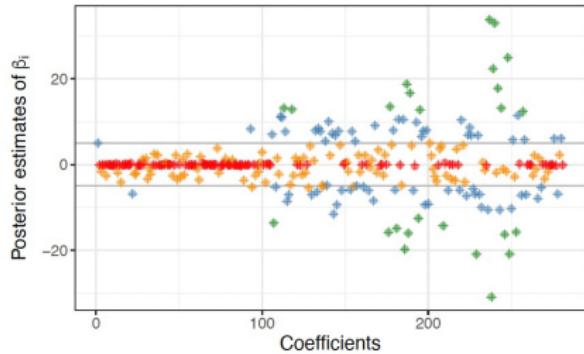
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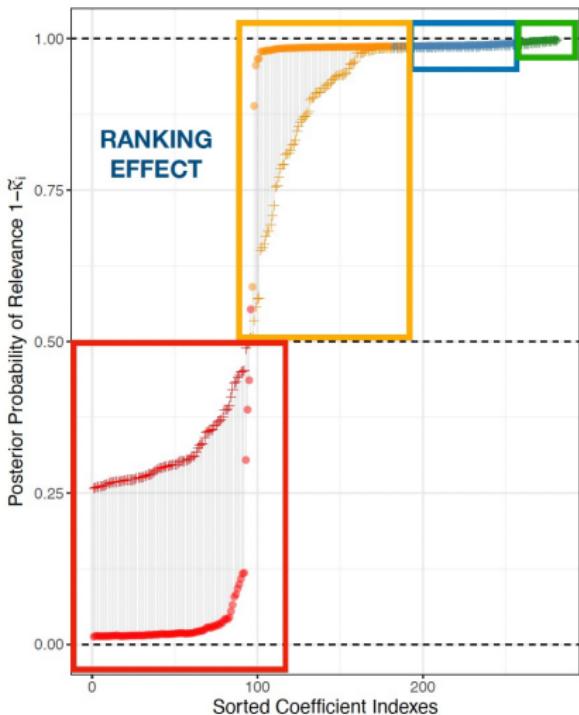
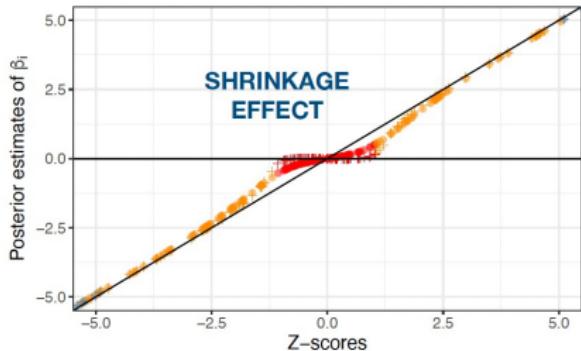
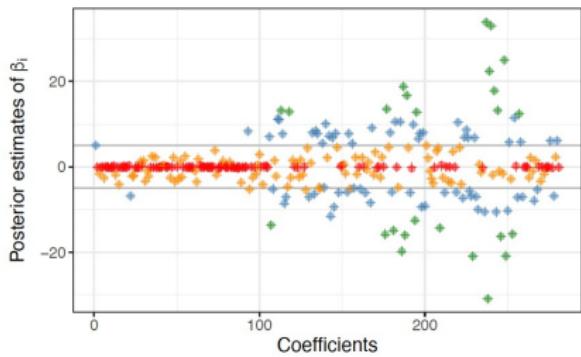
- 👉 One can rank the  $\beta_\nu$ 's into **tiers of relevance**
- 👉 One can choose a **Half-Cauchy prior** for the mixture shrinkage component,

$$\lambda_l \sim \text{Cauchy}^+(0, 1), \forall l \quad (\text{Horseshoe pit})$$

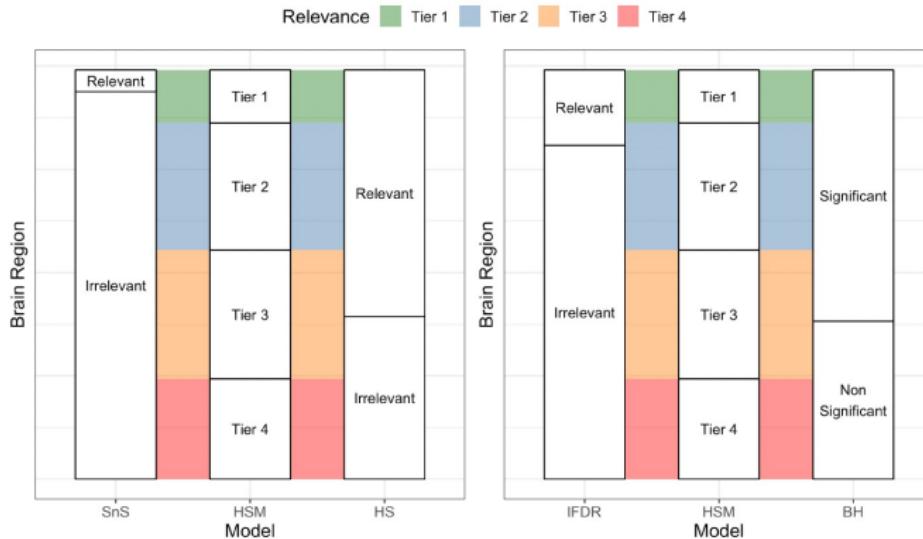
We can segment the results into 4 tiers of activation, from high-activity (**Tier 1**) to no activity (**Tier 4**)



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We compare the findings with other well-known methods: Local-FDR (IFDR), Horseshoe prior (HS), Spike-and-Slab (SnS), and Benjamini-Hochberg (BH)

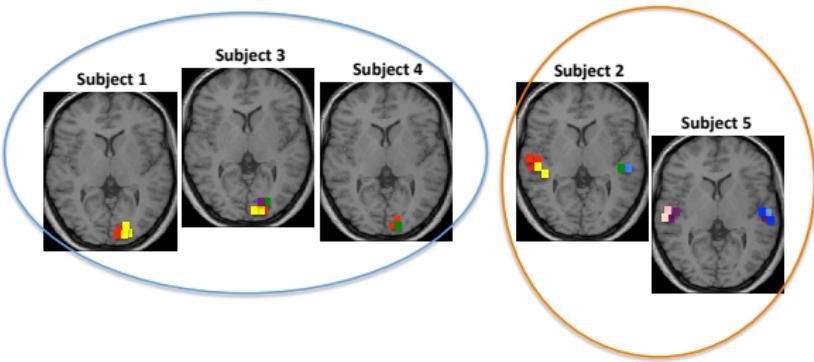


- The HSM model mediates between the more conservative IFDR and SnS methods and the numerous discoveries of the BH and HS models.
- Denti et al (2022+), *A Horseshoe mixture model for Bayesian screening with an application to light sheet fluorescence microscopy in brain imaging*, Submitted. <https://arxiv.org/abs/2106.08281> ↗

Capturing Between-subjects heterogeneity

- Hierarchical mixtures are widely used in Bayesian Nonparametrics to cluster together observations from different groups (Camerlenghi et al, 2019; Bassetti et al, 2020; Argiento et al, 2019)

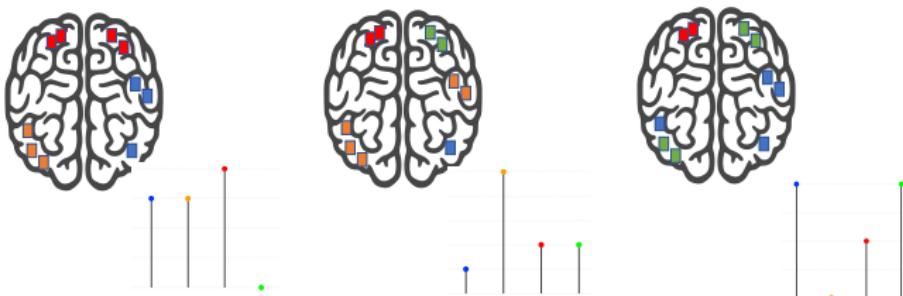
- Hierarchical mixtures are widely used in Bayesian Nonparametrics to cluster together observations from different groups (Camerlenghi et al, 2019; Bassetti et al, 2020; Argiento et al, 2019)
- **Basic Idea: Two-level mixtures:** a mixture is used to cluster subjects showing similar brain patterns; a lower-level mixture captures individual specific features



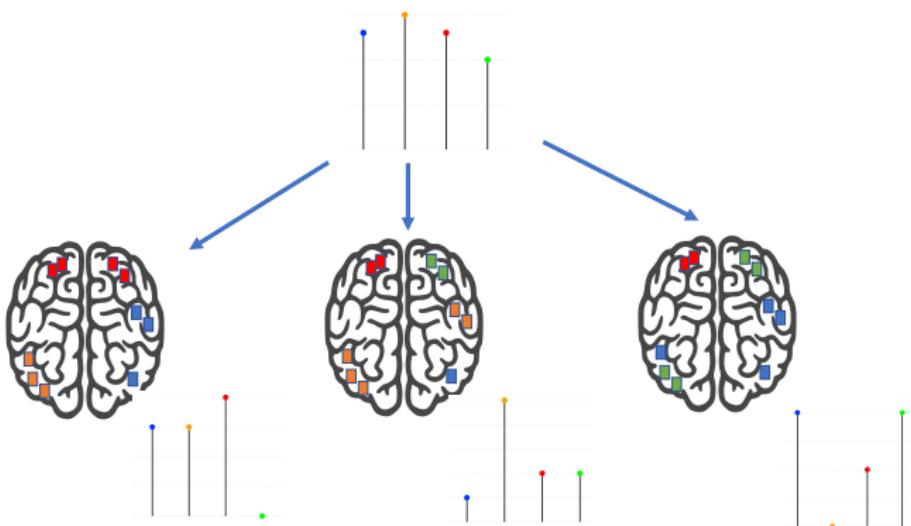
Zhang, G., Versace, Engelmann, Vannucci, *Annals of Applied Statistics*, 2016

$$Y_{iv} = X_{iv}\beta_{iv} + \varepsilon_{iv}, \quad \varepsilon_{iv} \sim N_T(0, \Sigma_{iv})$$

- **Objective:** capture activation patterns in response to a stimulus within and across subjects.



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- More specifically, we use a Hierarchical Dirichlet Process (HDP, Teh et al, 2006) to define a multi-subject spike-and-slab nonparametric prior,

$$\beta_{iv} | \gamma_{iv}, G_i \sim \gamma_{iv} G_i + (1 - \gamma_{iv})\delta_0$$

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The  $G_i$  are built by "picking" hierarchically the atoms in their support from a **common underlying (discrete) distribution**

$$G_i | \eta_1, G_0 \sim DP(\eta_1, G_0)$$

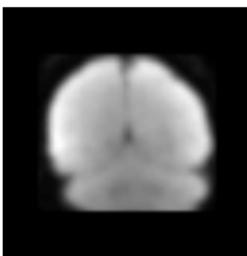
$$G_0 | \eta_2, P_0 \sim DP(\eta_2, P_0)$$

$$P_0 = N(0, \tau)$$

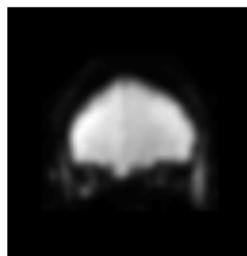
- $\star \eta_1, \eta_2$ : concentration parameters, controlling the variability  
 $P_0$ : base measure, generating the global components which are shared within and across subjects

- ❑ Real fMRI data collected by Versace's lab (MDACC):
  - Data Dimension: 27 subjects, 286 time points, 2 slices of interest,  $64 \times 64$  voxels per slice

Occipital Slice  
(y = -60 mm)

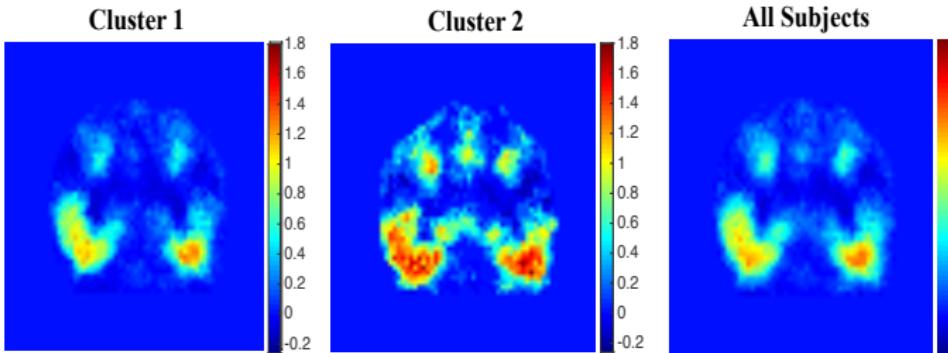


Frontal Slice  
(y = +38 mm)



- Event-related design
- **Goal:** detecting (differential) brain activity in response to visual scenes: emotional pictures ( vs neutral pictures)

# Results of subject-level clustering



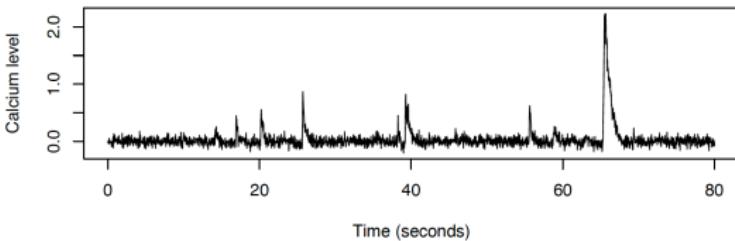
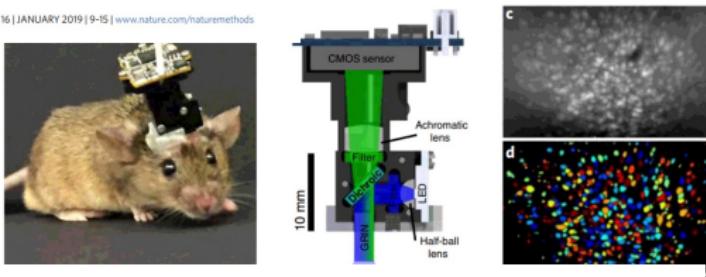
- Two groups of subjects characterized by different levels of activations
- Subjects who show decreased responses to certain emotional pictures may show lower reward sensitivity in surveys' responses (e.g., higher dissatisfaction, affecting mechanisms connected to reward processing)

Capturing Distributional heterogeneity

- ⇒ The Hierarchical Dirichlet Process assumes subject-specific distributions but does not allow **clustering distributions**

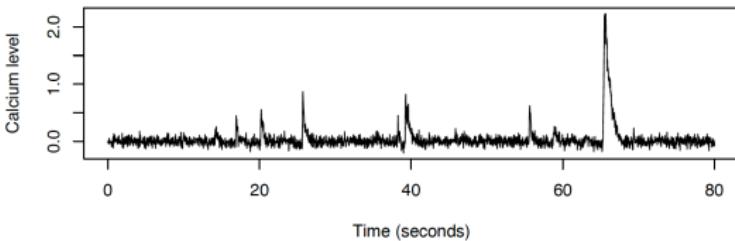
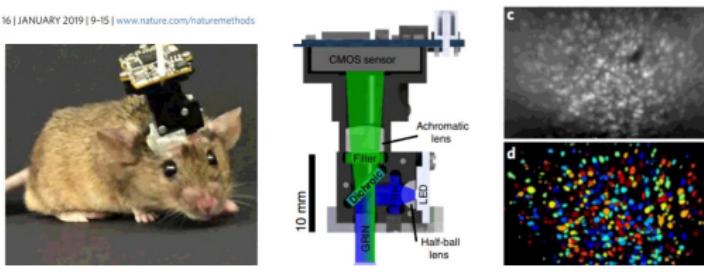
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NATURE METHODS | VOL 16 | JANUARY 2019 | 9–15 | www.nature.com/naturemethods



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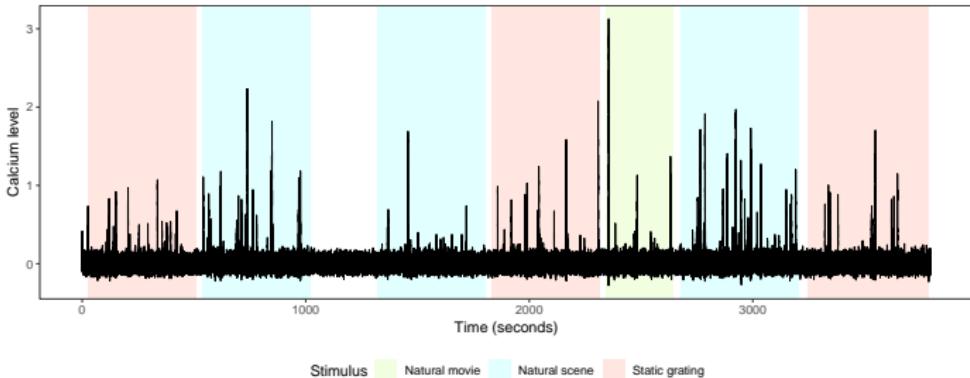
NATURE METHODS | VOL 16 | JANUARY 2019 | 9–15 | www.nature.com/naturemethods



- ➡ Study **how individual neurons react to stimulation** and how they encode information by **deconvolving the calcium traces** and identify the **precise spike times** of the observable neurons

Usually, the experiment involves multiple stimuli (e.g. visual stimuli, or odors):

- the interest is to understand how the different types of stimuli affect the neuronal activity  $\Rightarrow$  investigate similarities and differences in the **distribution of spikes** over time and conditions.



The distribution of the spikes can be very similar across the conditions of an experiment  $\Rightarrow$  Clustering

- Approaches for clustering distributional features directly are sparse.

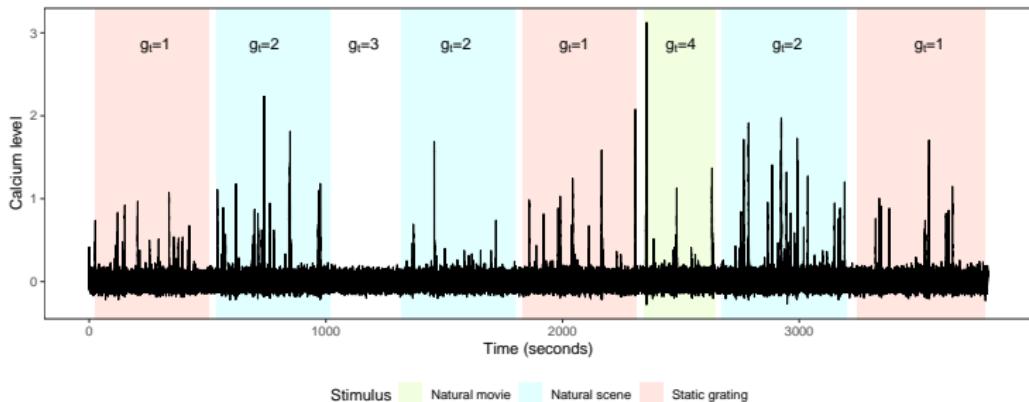
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- Approaches for clustering distributional features directly are sparse.
- Clustering methods in symbolic statistics (Irpino and Verde, 2015; Batagelj et al., 2015) do not allow for a probabilistic assessment of cluster uncertainty.
- The Nested Dirichlet process (nDP, Rodriguez et al, 2008) and its extensions have been widely employed to identify distributional groups in Bayesian Nonparametric model-based approaches.
- The nDP leads to a **two-layered clustering**: first, it allows grouping together similar distributions (**distributional clustering**), and then it clusters similar observations within each distributional cluster (**observational clustering**).

We are interested in characterizing the neural activity under different experimental conditions.

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We introduce a categorical variable  $g_t$  taking values in  $\{1, \dots, J\}$ , with  $J$  the number of different experimental settings.



For each  $t = 1, \dots, T$ ,  $g_t = j$  indicates that the neural activity at time  $t$  is observed under condition  $j$ .

## Model for the calcium dynamics

A popular model<sup>1</sup> to relate the observed trace  $y_t$  to the underlying true calcium concentration  $c_t$ , and to the neuronal activity  $A_t$ :

$$y_t = b + c_t + \epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

$$c_t = \gamma c_{t-1} + A_t + \omega_t \quad \omega_t \sim N(0, \tau^2)$$

for  $t = 1, \dots, T$ ; with  $b$  baseline level,  $\epsilon_t$  measurement error.

- In absence of neuronal activity:  $A_t = 0$  and the calcium level follows a AR(1) process controlled by the parameter  $\gamma$ ;
- when a spike occurs:  $A_t > 0$  and the concentration increases instantaneously with the spike amplitude  $A_t$ .

---

<sup>1</sup> Vogelstein et al. (2010). Fast nonnegative deconvolution for spike train inference from population calcium imaging. *Journal of Neurophysiology* **104**, 3691–3704

To allow the response to vary according to the condition, we assume that the spikes  $A_t$  come from **stimulus-specific distributions**: for  $j = 1, \dots, J$

$$A_t \mid g_t = j, G_j \sim G_j.$$

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To model the  $G_j$ 's we adopt a Bayesian **nested** finite mixture model:

**nested structure** → reconstruct the distribution within each experimental condition + borrow information between groups (distributional clustering)

**mixture formulation** → cluster the  $A_t$  across and within distributions  
⇒ discover similarities in the activation response to different stimuli.

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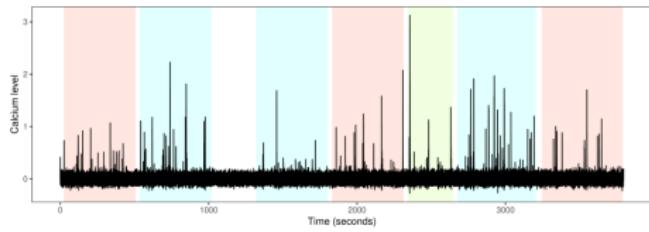
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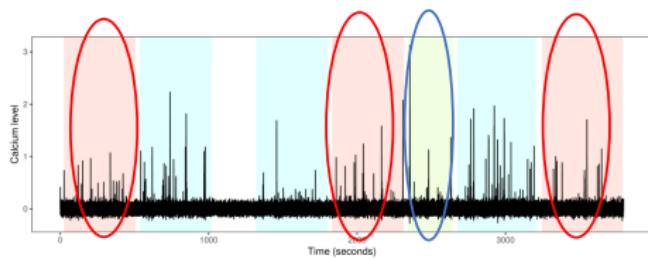
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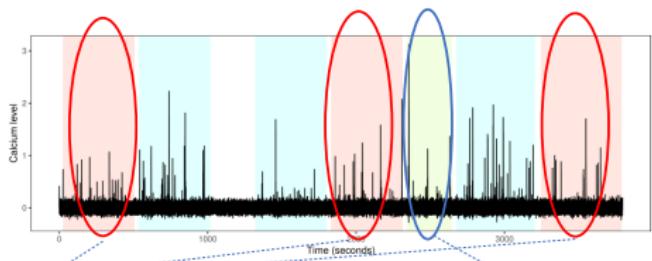
The model allows to represent the data through two-layers: at the first level clusters of distributions across conditions, and at the second level a convenient representation of the distributions via models

# Visual idea of the NDP



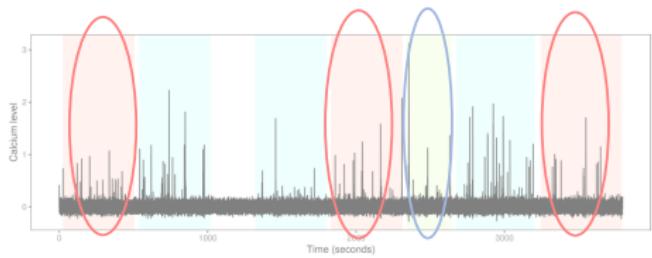
# Visual idea of the NDP





$$A_t \mid g_t = 1 \sim G_1.$$

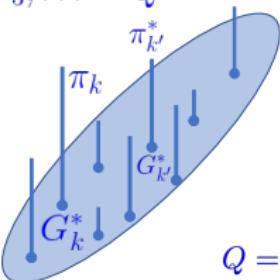
$$A_t \mid g_t = 3 \sim G_3$$



$$A_t \mid g_t = 1 \sim G_1.$$

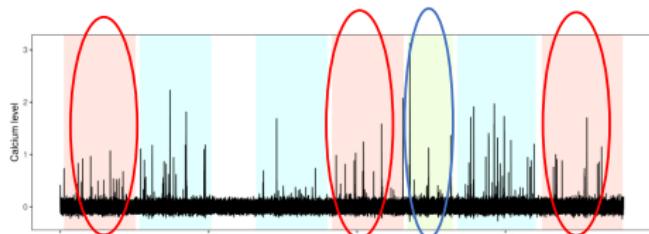
$$A_t \mid g_t = 3 \sim G_3$$

$$G_1, G_2, G_3, \dots \sim Q$$



$$Q = \sum_{k=1}^K \pi_k \delta_{G_k^*}$$

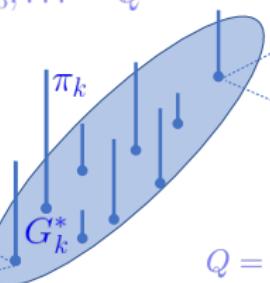
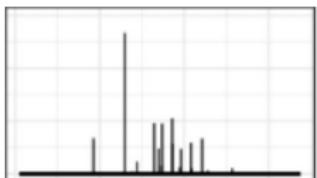
# Visual idea of the NDP



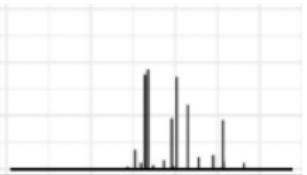
$$A_t \mid g_t = 1 \sim G_1.$$

$$A_t \mid g_t = 3 \sim G_3$$

$$G_1, G_2, G_3, \dots \sim Q$$



$$Q = \sum_{k=1}^K \pi_k \delta_{G_k^*} \quad G_k^*(\cdot) = \sum_{l=1}^{\infty} \omega_{lk} \delta_{\theta_{lk}}(\cdot)$$



- Camerlenghi et al (2019) have recently proved that the inference obtained using the nDP may be affected by a *degeneracy* property:
  - ⇒ If two distributions share **even only one atom in their support**, the two distributions are automatically assigned to the same cluster.

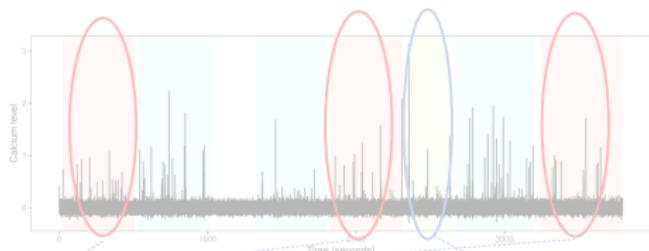
More precisely, the partially exchangeable partition probability function (pEPPF), i.e. the function which describes the probability of each clustering allocation for partially exchangeable data modeled with a nDP, collapses to a fully exchangeable case when ties are present among the observational atoms.

The problem persists with nDP mixture model formulations

- Camerlenghi et al (2019) propose a class of latent nested processes, which relies on estimating a latent **mixture of shared and idiosyncratic processes**  $\Rightarrow$  very computationally complex, only small datasets with few groups.

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- Beraha et al (2021) propose a variation of the hierarchical DP, where the **baseline distribution** is itself a **mixture of a DP and a non-atomic measure** (semi-HDP). They further combine the semi-HDP prior with a random partition model that allows different populations to be grouped in clusters that are internally homogeneous, i.e. arising from the same distribution.
- Denti, Camerlenghi, Guindani & Mira (2022+) show that the degeneracy is avoided if the prior explicitly models **commonality of atoms** between groups.
- Lijoi, Pruenster, Rebaudo (2022+) move this idea further along by essentially combining the NDP and the HDP into a *hidden hierarchical Dirichlet Process*.

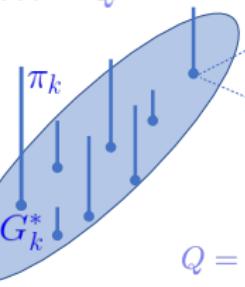
# Visual idea of the Common Atom Model



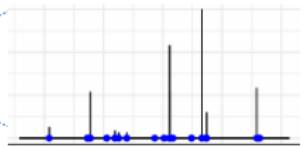
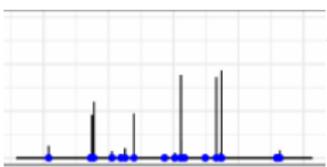
$$A_t \mid g_t = 1 \sim G_1.$$

$$A_t \mid g_t = 3 \sim G_3$$

$$G_1, G_2, G_3, \dots \sim Q$$



$$Q = \sum_{k=1}^K \pi_k \delta_{G_k^*} \quad G_k^*(\cdot) = \sum_{l=1}^{\infty} \omega_{lk} \delta_{\theta_l}(\cdot)$$



- For computational efficiency (long time series), one can employ the **generalized mixtures of finite mixtures (gMFM)** of Frühwirth-Schnatter et al. (*BA*, 2021) where the nested structure is based on the **common atom model**:

$$A_t \mid g_t = j, G_j \sim G_j.$$

$$G_1, \dots, G_J \mid Q \sim Q, \quad Q = \sum_{k=1}^K \pi_k \delta_{G_k^*}$$

where  $G_k^*$  are distributions (identifying clusters of distributions across conditions/experimental settings)

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- More specifically, we assume:

$$\pi_1, \dots, \pi_K \mid K, \alpha \sim Dir_K\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right)$$

$$K - 1 \sim \text{beta-negative-binomial}$$

$$\alpha \sim F$$

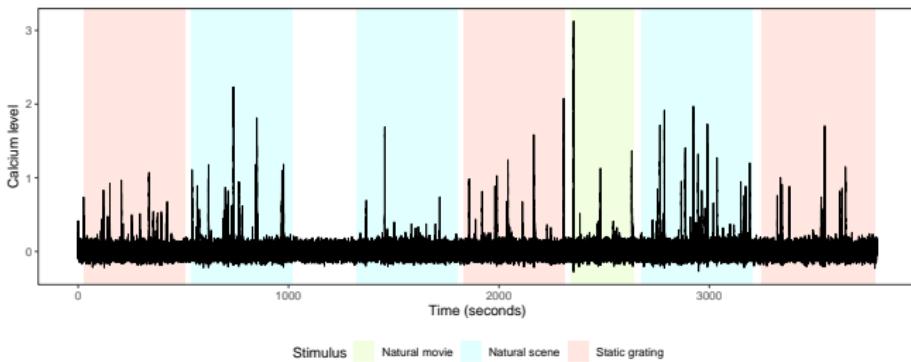
(Frühwirth-Schnatter et al., 2021)

- Also for the **distributional atoms**  $G_k^*$ , for  $k = 1, \dots, K$  we assume a mixture

$$G_k^* = \sum_{l=1}^L \omega_{l,k} \delta_{A_l^*}$$

where the set of atoms  $A^*$  is **common** across the distributions  $G_1^*, \dots, G_K^*$  and they are obtained as i.i.d. draws from a base measure  $G_0$ .

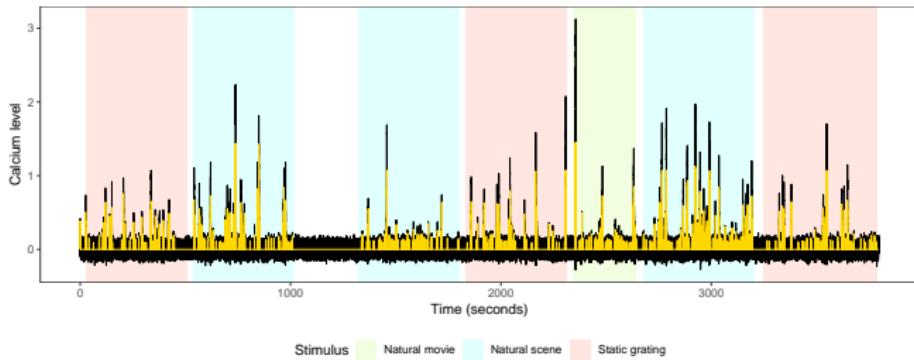
- The distributions  $G_k^*$  differ by the weight given to each atom (some weights  $\omega_{l,k}$  can be zero for some k)

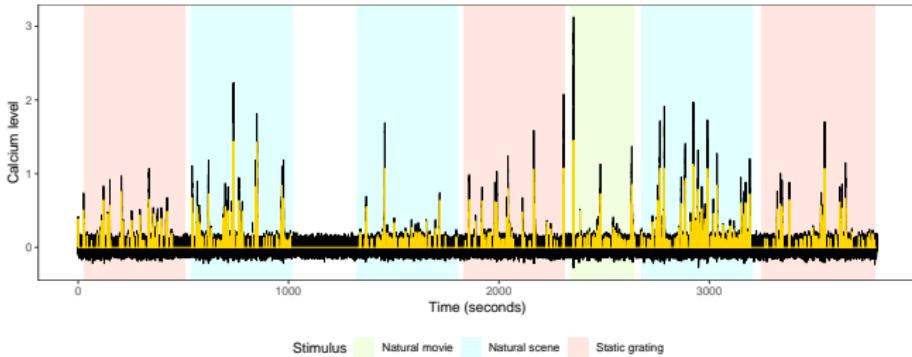


#### 4 experimental conditions:

- 3 stimuli of increasing complexity (static grating, natural scene, natural movie)
- period of spontaneous activity (absence of stimuli)

<sup>6</sup>Allen Institute for Brain Science (2016). Allen brain observatory.  
<http://observatory.brain-map.org/visualcoding>.

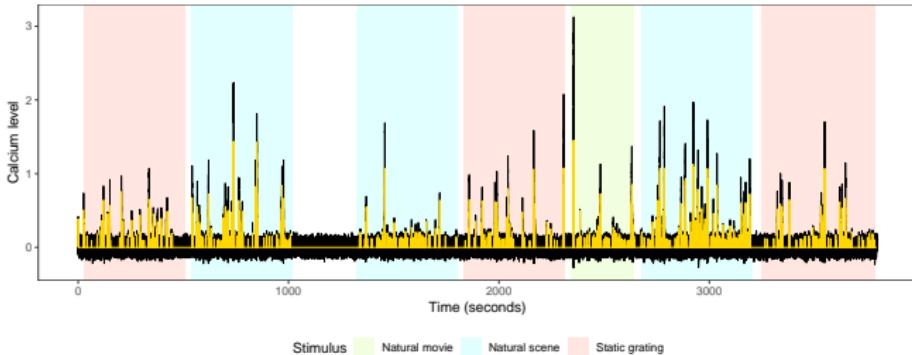




Cluster 1 = { Natural scene, Natural movie }

Cluster 2 = { Static grating }

Cluster 3 = { Absence of stimuli }



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D'Angelo et al (2022+) Bayesian nonparametric analysis for the detection of spikes in noisy calcium imaging data, BiometRICS, to appear

<https://arxiv.org/abs/2102.09403>

- We have discussed old and recent modeling frameworks in BNP with an emphasis on applications to neuroimaging.
- **What did I leave out?**

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  - Smith and Allenby (2020), Paganin et al (2021), Dhal, Warr et al (2022+)
- **Dependent random measures** (MacEachern, 2000; Quintana et al, 2022)
  - e.g., adding covariates/clustering dependent on external stimuli/information about the environment.
- **Computational Challenges**
  - Dimension reduction
  - Approximate computational methods

# Collaborators - Thanks



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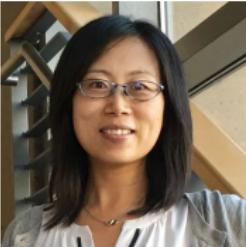
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**B**ayesian  
**BOREDOM**

**N**on  
**oT**

**P**arametric  
**PERMITTED**