On the Constructive Constructible Universe

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Joint work with Michael Rathjen

Constructing the Constructible Universe Constructively

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Some of the results appear in my thesis:

Large Cardinals in Weakened Axiomatic Theories

University of Leeds, 2021.

Aims

Gödel's Constructible Universe in Constructive Set Theories:

- How can we construct it?
- What theory does it satisfy?

More generally, what is an *inner model* intuitionistically?

Non-constructive Principles

- (Law of Excluded Middle) $\varphi \vee \neg\varphi$
- (Double Negation Elimination) $\neg\neg\varphi \rightarrow \varphi$
- (Some Classical Logical Equivalences) $(\varphi \rightarrow \psi) \rightarrow (\neg\varphi \vee \psi)$

- Foundation: $\forall a(\exists x(x \in a) \rightarrow \exists x \in a\ \forall y \in a(y \notin x))$
- "Minimal elements" of sets
- Axiom of Choice / Well-Ordering Principle
- Definition by cases which differentiate between successor and limit ordinals

- Built upon the *Brouwer-Heyting-Kolmogorov interpretation* which is an informal way to consider propositions as *"problems"* which one solves by breaking into simpler problems.

The Brouwer-Heyting-Kolmogorov Interpretation

- p proves $\perp$ is impossible, so there is no proof of $\perp$,
- p proves $\varphi \wedge \psi$ iff p is a pair $\langle q, r \rangle$ where q proves φ and r proves ψ,
- p proves $\varphi \vee \psi$ iff p is a pair $\langle n, q \rangle$ where $n = 0$ and q proves φ or $n = 1$ and q proves ψ,
- p proves $\varphi \to \psi$ iff p is a function which transforms any proof q of φ into a proof $p(q)$ of ψ,
- p proves $\exists x \in A\, \varphi(x)$ iff p is a pair $\langle a, q \rangle$ where a is a member of the set A and q is a proof of $\varphi(a)$,
- p proves $\forall x \in A\, \varphi(x)$ iff p is a function such that for each member a of A, $p(a)$ proves $\varphi(a)$.

Remark

$\neg\varphi$ is interpreted as $\varphi \to (0 = 1)$.

The Joys of Intuitionism

Many intuitionistic theories satisfy "pleasing mathematical properties" for example they:

- Give insight on when Excluded Middle is needed in a given proof,
- Give "*more direct*" proofs of mathematical statements,
- Satisfy the Disjunction Property (if $T \vdash \varphi \vee \psi$ then either $T \vdash \varphi$ or $T \vdash \psi$),
- Satisfy the Numerical Existence Property (if $T \vdash \exists x \in \omega \varphi(x)$ then $T \vdash \varphi(n)$ for some $n \in \omega$),
- Are the internal logic of toposes with a natural number object,
- Have a type theoretic interpretation.

The Joys of Intuitionism

The following principles can consistently hold intuitionistically:

- Every function $f \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous,
- Church's Thesis (All total functions are computable),
- The Uniformity Principle holds ($\forall x \exists n \in \omega \varphi \to \exists n \in \omega \forall x \varphi$),
- Every set is subcountable (the surjective image of a subset of ω).

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$
$\downarrow$ + *Strong Infinity*
IKP
$\downarrow$ + *Strong Collection*
CZF^{-}
$\downarrow$ + *Subset Collection*
CZF
$\downarrow$ + *Power Set*
$\mathrm{CZF}_{\mathcal{P}}$
$\downarrow$ + *Full Separation*
IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Strong Infinity

$\exists a\ (Ind(a)\ \wedge\ \forall b\ (Ind(b) \to \forall x \in a(x \in b))),$

$Ind(a) \equiv \emptyset \in a\ \wedge\ \forall x \in a\ (x \cup \{x\} \in a).$

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IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
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Strong Collection

For any formula $\varphi(u, v)$ and set a,
$\forall x \in a\ \exists y\ \varphi(x, y) \to \exists b$
$\big(\forall x \in a\ \exists y \in b\ \varphi(x, y) \wedge \forall y \in b\ \exists x \in a\ \varphi(x, y)\big)$.

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$\downarrow$ + *Power Set*

$\mathrm{CZF}_{\mathcal{P}}$

$\downarrow$ + *Full Separation*

IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
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Subset Collection

Equivalent to the axiom of *Fullness*.

Implies Exponentiation ($\forall a, b \ {}^{a}b$ is a set).

Intuitionistic Theories

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IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
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- Set Induction
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- Bounded Collection

Power Set

$\forall a\ \mathcal{P}(a)$ is a set.

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$$
\begin{array}{l}
\mathrm{IKP}^{-\mathit{Inf}} \\
\quad\downarrow \quad + \textit{ Strong Infinity} \\
\mathrm{IKP} \\
\quad\downarrow \quad + \textit{ Strong Collection} \\
\mathrm{CZF}^{-} \\
\quad\downarrow \quad + \textit{ Subset Collection} \\
\mathrm{CZF} \\
\quad\downarrow \quad + \textit{ Power Set} \\
\mathrm{CZF}_{\mathcal{P}} \\
\quad\downarrow \quad + \textit{ Full Separation} \\
\mathrm{IZF}
\end{array}
$$

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Full Separation

For any formula $\varphi(u)$ and set a, $\{x \in a \mid \varphi(x)\}$ is a set.

What is an Ordinal?

Definition

$\langle A, \prec \rangle$ is a well-ordering if it is a strict total order such that any non-empty subset X of A has an $\prec$-least element.

Definition (Classical)

An ordinal α is a transitive set which is well-ordered by $\in$.

Proposition (Folklore)

α is an ordinal iff it is a transitive set of transitive sets.

Definition (More Constructive)

An ordinal is a transitive set of transitive sets.
Let $\textsc{Ord}$ denote the class of ordinals.

Intuitionistic Ordinals

Nice Properties

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$,
- If X is a set of ordinals then $\bigcup X$ is an ordinal,
- We can perform definitions by transfinite recursion and make stratified hierarchies (e.g. rank function, V_α, L_α ...).

Ordinary Ordinal Oddities

- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$,
- $\forall \alpha\ (0 \in \alpha + 1)$ implies excluded middle!

Truth Values

Given a formula φ, an important ordinal is

$$\alpha_\varphi := \{0 \in 1 \mid \varphi\}.$$

Naively, if we don't assume $\varphi \vee \neg\varphi$ then α_φ is neither 0 nor 1.

In general, we let

$$\Omega := \mathcal{P}(1) = \{x \mid x \subseteq 1\}$$

be the class of *truth values*.

If $\Omega = 2$ then the Law of Excluded Middle holds.

Note that

$$(0 \in \alpha_\varphi + 1) \implies (0 \in \alpha_\varphi \vee 0 = \alpha_\varphi) \implies (\varphi \vee \neg\varphi).$$

Inner Models

Definition

Let T be a theory (in the language of set theory) and suppose that $\mathrm{V} \models \mathrm{T}$. A transitive class M is said to be an *inner model* if

- $\mathrm{M} \subseteq \mathrm{V}$,
- $\mathrm{M} \models \mathrm{T}$,
- $\mathrm{Ord} \cap \mathrm{M} = \mathrm{Ord} \cap \mathrm{V}$.

Example

If T is KP or ZF then L is an inner model.

Theorem (M., Rathjen)

1. $\mathrm{IZF} \not\vdash \mathrm{Ord} \cap \mathrm{L} = \mathrm{Ord} \cap \mathrm{V}$,
2. $\mathrm{CZF} \not\vdash (\mathrm{CZF})^{\mathrm{L}}$.

The Constructible Universe

- The constructible universe was developed by Gödel in papers published in 1939 and 1940 to show the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis with ZF.
- As with the standard V_α cumulative hierarchy, one builds the L_α hierarchy in stages where one replaces the power set operation with a "*definable*" version.
- There are $2/3$ equivalent approaches to building L, all of which are formalisable in KP:[1]
  - Syntactically as the set of definable subsets of M (See Devlin - *Constructibility*),
  - Using Gödel functions (See Barwise - *Admissible Sets*) or
  - Using Rudimentary Functions (See Gandy, Jensen, Mathias).

[1] In fact significantly weaker systems - see Mathias: *Weak Systems of Gandy, Jensen and Devlin*, 2006

Constructing the Constructible Universe Constructively

- The syntactic approach was then modified for IZF by Lubarsky (*Intuitionistic L* - 1993).
- And then for IKP by Crosilla (*Realizability models for constructive set theories with restricted induction* - 2000).
- Rudimentary functions were extended to constructive theories by Aczel (*Rudimentary and arithmetical constructive set theory* - 2012).
- (M.) Using an extended set of Gödel functions, one can construct L in IKP^{-Inf}.

Gödel Functions

Theorem

If T *is one of* IKP^{-Inf}, IKP *or* IZF *then*

$$\mathrm{T} \vdash (\mathrm{T})^{\mathrm{L}}.$$

Constructibility

Definition ($\mathrm{IKP}^{-\mathit{Inf}}$)

- Let $\mathcal{F}_1, \ldots, \mathcal{F}_N$ denote the (extended) Gödel functions.
- Let $\mathfrak{E}(b) := b \cup \{\mathcal{F}_i(x, y) \mid x, y \in b \ \wedge \ i \leq N\}$ be the 1-step closure of b under these functions.
- Let $\mathbb{L}_\alpha := \bigcup_{\beta \in \alpha} \mathfrak{E}(\mathbb{L}_\beta \cup \{\mathbb{L}_\beta\})$.
- Let $\mathbb{L} := \bigcup_\alpha \mathbb{L}_\alpha$.

Lemma (Lubarsky / M.)

For every ordinal α in V there is an ordinal α^* in $\mathbb{L}$ such that $\mathbb{L}_\alpha = \mathbb{L}_{\alpha^*}$.

Theorem (Axiom of Constructibility)

If T is one of $\mathrm{IKP}^{-\mathit{Inf}}$, IKP or IZF then

$$\mathrm{T} \vdash (\mathrm{V} = \mathbb{L})^{\mathbb{L}}.$$

$E_\wp$-recursion

- E-recursion or set recursion is a generalisation of recursion theory to be able to apply arbitrary sets to one another (see Sack's *Higher Recursion Theory* or Normann's *Set Recursion*).
- $[s](t)$ can be viewed as "s *is some kind of Turing machine that takes as input the set* t *and outputs* $[s](t)$".
- It will contain designated integers which provide indices for special functions computing basic operations (e.g. pairing, successors, unions, intersections, ω, ...). Details
- $E_\wp$ recursion was developed by Rathjen (From the weak to the strong existence property, 2012) to also have the power set operation be "*computable*".
- An $E_\wp$*-recursive partial function* is essentially a $\Sigma^{\mathcal{P}}$ function which computes a term built using $E_\wp$-recursion.

Realizability with truth

Details

Next combine $E_\wp$ computability with a notion of realizability that preserves truth, $\Vdash^{\wp}_{\mathfrak{wt}}$. Realizers for existential statements will provide a set of witnesses for the existential quantifier.

Proposition (Rathjen 2012)

Let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables among $x_1, \ldots, x_n$. Then

$$\mathrm{CZF}_{\mathcal{P}} \vdash (\exists e\, e \Vdash^{\wp}_{\mathfrak{wt}} \varphi(x_1, \ldots, x_n)) \rightarrow \varphi(x_1, \ldots, x_n).$$

Theorem (Rathjen 2012)

Let $\varphi(x_1, \ldots, x_n)$ *be a formula with free variables among* $x_1, \ldots, x_n$. *If* $\mathrm{CZF}_{\mathcal{P}} \vdash \varphi(x_1, \ldots, x_n)$ *then one can effectively construct an index of an* $E_\wp$*-recursive function* f *such that*

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall a_1 \ldots \forall a_n\, f(a_1, \ldots a_n) \Vdash^{\wp}_{\mathfrak{wt}} \varphi(a_1, \ldots, a_n).$$

Conservativity

Using a very similar notion of realizability, one can show

Theorem (Rathjen, 2012)

1. CZF^- *is conservative over* IKP *for* Π_2 *sentences,*
2. $\mathrm{CZF}_{\mathcal{P}}$ *is conservative over* $\mathrm{IKP}(\mathcal{P})$[2] *for* $\Pi_2^{\mathcal{P}}$ *sentences.*

Next, via a proof-theoretic interpretation of $\mathrm{KP}(\mathcal{P})$ to an extension of Zermelo set theory, we have

Theorem (Rathjen, 2014)

Let φ *be a* $\Pi_2^{\mathcal{P}}$ *sentence. If* $\mathrm{IKP}(\mathcal{P}) \vdash \varphi$ *then* $\mathrm{V}_{\mathrm{BH}} \models \varphi$ *where* BH *is the Bachmann-Howard ordinal.*

Corollary (Rathjen)

1. IKP and $\mathrm{IKP}(\mathcal{P})$ have the existence property for Σ and $\Sigma^{\mathcal{P}}$ formulas respectively.
2. CZF^- and $\mathrm{CZF}_{\mathcal{P}}$ have the existence property.

[2] IKP + *Power Set* and both Separation and Collection for $\Sigma_0^{\mathcal{P}}$ formulas.

Exponentiation and $\mathrm{CZF}_{\mathcal{P}}$

Theorem (M., Rathjen)

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{CZF})^{\mathrm{L}}$.

- Suppose that $\mathrm{CZF}_{\mathcal{P}}$ proved that ${}^{\omega}\omega \cap \mathrm{L} \in \mathrm{L}$.
- Then

$$\mathrm{CZF}_{\mathcal{P}} \vdash \exists a\big(a \in \textsc{Ord} \ \wedge\ \omega \in a \wedge \forall f \in {}^{\omega}\omega(f \in \mathrm{L} \to f \in \mathrm{L}_a)\big)$$

- Construct an index for an $E_{\wp}$-recursive function g such that $\mathrm{CZF}_{\mathcal{P}} \vdash \forall x\ g(x)$ *is defined* and

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall x\ g(x) \Vdash^{\wp}_{\mathfrak{wt}} \exists a\big(a \in \textsc{Ord} \ \wedge\ \omega \in a \wedge \forall f \in {}^{\omega}\omega(f \in \mathrm{L} \to f \in \mathrm{L}_a)\big).$$

Exponentiation and $\mathrm{CZF}(\mathcal{P})$

Theorem

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{V} = \mathrm{L})^{\mathrm{L}}$.

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall x\, g(x) \Vdash^{\wp}_{\mathfrak{wt}} \exists a (a \in \textsc{Ord} \;\wedge\; \omega \in a \;\wedge\; \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \to f \in \mathrm{L}_a)).$$

Note: $a \Vdash^{\wp}_{\mathfrak{wt}} \exists x \varphi(x)$ iff $\exists u(u \in a) \wedge \forall d \in a(\mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d])$,

- Unpacking ... $\mathrm{CZF}_{\mathcal{P}} \vdash \exists w(w \in g(0))$ and

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\, \mathbf{p}_1 y \Vdash^{\wp}_{\mathfrak{wt}} (\mathbf{p}_0 y \in \textsc{Ord} \;\wedge\; \omega \in \mathbf{p}_0 y \;\wedge\; \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \to f \in \mathrm{L}_{\mathbf{p}_0 y})).$$

- Since realizability preserves truth,

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\, (\mathbf{p}_0 y \in \textsc{Ord} \;\wedge\; \omega \in \mathbf{p}_0 \;\wedge\; \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \to f \in \mathrm{L}_{\mathbf{p}_0 y})).$$

Exponentiation and $\mathrm{CZF}(\mathcal{P})$

Theorem

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{V} = \mathrm{L})^{\mathrm{L}}$.

- Since g was $E_{\wp}$ recursive, "$\forall x\, g(x)$ *is defined*" is $\Pi_2^{\mathcal{P}}$ definable.
- Now $\mathrm{CZF}_{\mathcal{P}}$ is conservative over $\mathrm{IKP}(\mathcal{P})$ for $\Pi_2^{\mathcal{P}}$ sentences, and these reflect to V_{BH} where BH is the *Bachmann-Howard* ordinal.
- Therefore, $g(0) \in \mathrm{V}_{\mathrm{BH}}$. So

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\, \exists a \in \mathrm{V}_{\mathrm{BH}} \Big(a \in \mathrm{ORD} \ \wedge\ \omega \in a \ \wedge\ \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \to f \in \mathrm{L}_a) \Big).$$

- Which means that ${}^{\omega}\omega \cap \mathrm{L} \in \mathrm{L}_{\mathrm{BH}}$ which is a contradiction.

Intuitionistic Forcing

Essentially

Do forcing but only consider those sentences which are forced by every condition.

Let $\mathbb{P}$ be a partial order. Define the forcing relation by:

$$
\begin{array}{lll}
p \Vdash a = b & \text{iff} & \forall \langle c, q\rangle \in a \; \forall r \leq p, q \;\; r \Vdash c \in b \\
 & & \text{and } \forall \langle c, q\rangle \in b \; \forall r \leq p, q \;\; r \Vdash c \in a \\
p \Vdash a \in b & \text{iff} & \exists c \; \exists q \geq p \; (\langle c, q\rangle \in b \text{ and } p \Vdash a = c) \\
p \Vdash \varphi \wedge \psi & \text{iff} & p \Vdash \varphi \text{ and } p \Vdash \psi \\
p \Vdash \varphi \vee \psi & \text{iff} & p \Vdash \varphi \text{ or } p \Vdash \psi \\
p \Vdash \varphi \rightarrow \psi & \text{iff} & \forall q \leq p \;\; q \Vdash \varphi \Rightarrow q \Vdash \psi \\
p \Vdash \forall x \; \varphi(x) & \text{iff} & \forall a \; \forall q \leq p \;\; q \Vdash \varphi(a) \\
p \Vdash \exists x \; \varphi(x) & \text{iff} & \exists a \;\; p \Vdash \varphi(a)
\end{array}
$$

Intuitionistic Forcing

Essentially

Do forcing but only consider those sentences which are forced by every condition.

Lemma (Monotonicty)

If $p \Vdash \varphi$ and $q \leq p$ then $q \Vdash \varphi$.

Definition

We say that $\mathrm{V}(\mathbb{P}) \models \varphi$ iff $\forall p \in \mathbb{P}$, $p \Vdash \varphi$.

Theorem (Lipton)

For any formula φ, $\mathrm{IZF} \vdash \varphi \implies \mathrm{V}(\mathbb{P}) \models \varphi$.

Strange Ordinal

Suppose that $\mathrm{V} \models \mathrm{ZFC}$ and fix a partial order $\mathbb{P}$. Work in $\mathrm{V}(\mathbb{P})$. Suppose

- $\mathrm{V}(\mathbb{P}) \models f \in {}^{\omega}2$,
- $\mathrm{V}(\mathbb{P}) \models \alpha \in \textsc{Ord} \wedge \alpha \subseteq 1$.

Let $\delta_f := \bigcup_{n \in \omega} \Big((n+2) \cup \{\alpha\}\Big) + f(n)$.

Idea

If $\alpha \in \{0, 1\}$ then $\delta_f = \omega$.
Otherwise, δ_f "codes" f as an ordinal since $(n+2) \cup \{\alpha\} \in \delta_f$ if and only if $f(n) = 1$.

Formalising the Idea

Lemma

$$\mathrm{V}(\mathbb{P}) \models \forall n \in \omega\Big((n+2) \cup \{\alpha\} \in \delta_f \leftrightarrow f(n) = 1\Big) \vee \alpha \in \{0,1\}.$$

Observation

If z is in L then there is some ordinal γ, $u \in \mathrm{L}_\gamma$ and formula φ such that
$\mathrm{V}(\mathbb{P}) \models \forall t(t \in z \leftrightarrow \mathrm{L}_\gamma \models \varphi(t,u))$.
So, if $\delta_f \in \mathrm{L}$ then
$\mathrm{V}(\mathbb{P}) \models \forall n \in \omega\Big((n+2) \cup \{\alpha\} \in \delta_f \leftrightarrow \mathrm{L}_\gamma \models \varphi(n,u)\Big)$.

Theorem

$$\mathrm{V}(\mathbb{P}) \models \delta_f \in \mathrm{L} \longrightarrow f \in \mathrm{L} \vee \alpha \in \{0,1\}.$$

Non-constructive Ordinal

Theorem

$\mathrm{V}(\mathbb{P}) \models \delta_f \in \mathrm{L} \longrightarrow f \in \mathrm{L} \vee \alpha \in \{0,1\}$.

Theorem

It is possible to find a model V *of* ZFC $(+\mathrm{V} \neq \mathrm{L})$ *and a partial order* $\mathbb{P} \in \mathrm{V}$ *that such in* $\mathrm{V}(\mathbb{P})$*:*

- $\mathrm{V}(\mathbb{P}) \models \exists f \in {}^{\omega}2\; f \notin \mathrm{L}$,
- *There exists* α *such that* $\mathrm{V}(\mathbb{P}) \models \alpha \in \textsc{Ord}$ *but* $\mathrm{V}(\mathbb{P}) \not\models \alpha \in \{0,1\}$.

Corollary

$\mathrm{IZF} \nvdash \textsc{Ord} \cap \mathrm{V} = \textsc{Ord} \cap \mathrm{L}$.

Inner Models

Intuitionistically:

- L is a definable class,
- If V is a model of $\mathrm{IKP}^{-\mathit{Inf}}$, IKP or IZF then so is L,
- But if V only satisfies CZF / $\mathrm{CZF}_{\mathcal{P}}$, L may not satisfy Exponentiation,
- Moreover, we may not have $\mathrm{V} \cap \mathrm{ORD} = \mathrm{L} \cap \mathrm{ORD}$.

- So what can inner models do . . . ?

External Cumulative Hierarchies

Definition

Let $\mathrm{M} \subseteq \mathrm{N}$. We say that M is almost universal in N if for any $x \in \mathrm{N}$, if $x \subseteq \mathrm{M}$ then there exists some $y \in \mathrm{M}$ such that $x \subseteq y$.

Theorem (M.)

Suppose that N is a model of IZF and $\mathrm{M} \subseteq \mathrm{N}$ is a transitive (proper) class with an external cumulative hierarchy in N.[3] Then M is a model of IZF if and only if M is closed under the extended Gödel functions and is almost universal in N.

Remark

So, under some "*nice*" condition, we can express being a model of IZF by a single sentence.

[3] Essentially, $\mathrm{M} = \bigcup_{\alpha \in \mathrm{ORD} \cap \mathrm{N}} \mathrm{M}_\alpha$.

Gödel Functions

Back

Definition

- $\mathcal{F}_p(x, y) := \{x, y\}$,
- $\mathcal{F}_\cap(x, y) := x \cap \bigcap y$, $(\cap y = \{u \mid \forall v \in y\ (u \in v)\})$
- $\mathcal{F}_\cup(x, y) := \bigcup x$,
- $\mathcal{F}_\setminus(x, y) := x \setminus y$,
- $\mathcal{F}_\times(x, y) := x \times y$,
- $\mathcal{F}_\rightarrow(x, y) := x \cap \{z \mid \textit{y is an ordered pair} \wedge (z \in 1^{st}(y) \rightarrow z \in 2^{nd}(y))\}$,
- $\mathcal{F}_\forall(x, y) := \{x``\{z\} \mid z \in y\}$, $(x``u = \{v \mid v \in 2^{nd}(x) \wedge \langle u, v\rangle \in x\})$

Gödel Functions

Back

Definition

- $\mathcal{F}_{\mathrm{dom}}(x, y) := \mathrm{dom}(x) = \{1^{st}(z) \mid z \in x \wedge$ *z is an ordered pair*$\}$,
- $\mathcal{F}_{\mathrm{ran}}(x, y) := \mathrm{ran}(x) = \{2^{nd}(z) \mid z \in x \wedge$ *z is an ordered pair*$\}$,
- $\mathcal{F}_{123}(x, y) := \{\langle u, v, w\rangle \mid \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{132}(x, y) := \{\langle u, w, v\rangle \mid \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{=}(x, y) := \{\langle v, u\rangle \in y \times x \mid u = v\}$,
- $\mathcal{F}_{\in}(x, y) := \{\langle v, u\rangle \in y \times x \mid u \in v\}$.

Notation

Let $\mathcal{I}$ be the finite set indexing the above operations.

$E_\wp$-recursive functions

Back

We inductively define a class $\mathbb{E}_\wp$ of triples $\langle e, x, y\rangle$ Instead of saying $\langle e, x, y\rangle \in \mathbb{E}_\wp$ we shall say $[e](x) \simeq y$. The relation is then defined by the following clauses:

Indices for applicative structures (APP):

$$
\begin{aligned}
[\mathbf{k}](x,y) &\simeq x \\
[\mathbf{p}](x,y) &\simeq \langle x, y\rangle \\
[\mathbf{p_N}](0) &\simeq 0 \\
[\mathbf{p_0}](x) &\simeq 1^{st}(x) \\
[\mathbf{d_N}](n,m,x,y) &\simeq x \text{ if } n, m \in \mathbb{N} \text{ and } n = m \\
[\mathbf{d_N}](n,m,x,y) &\simeq y \text{ if } n, m \in \mathbb{N} \text{ and } n \neq m
\end{aligned}
\qquad
\begin{aligned}
[\mathbf{s}](x,y,z) &\simeq [[x](z)]([y](z)) \\
[\mathbf{s_N}](n) &\simeq n+1 \text{ if } n \in \mathbb{N} \\
[\mathbf{p_N}](n+1) &\simeq n \text{ if } n \in \mathbb{N} \\
[\mathbf{p_1}](x) &\simeq 2^{nd}(x) \\
[\bar{\mathbf{0}}](x) &\simeq 0 \\
[\bar{\boldsymbol{\omega}}](x) &\simeq \omega
\end{aligned}
$$

$E_\wp$-recursive functions

Back

We inductively define a class $\mathbb{E}_\wp$ of triples $\langle e, x, y\rangle$ Instead of saying $\langle e, x, y\rangle \in \mathbb{E}_\wp$ we shall say $[e](x) \simeq y$. The relation is then defined by the following clauses:

Indices for set-theoretic axioms:

$$
\begin{aligned}
[\boldsymbol{\pi}](x, y) &\simeq \{x, y\} \\
[\boldsymbol{\gamma}](x, y) &\simeq x \cap \bigcap y \\
[\boldsymbol{\wp}](x, y) &\simeq \mathcal{P}(x)
\end{aligned}
$$

$$
\begin{aligned}
[\boldsymbol{\nu}](x) &\simeq \bigcup x \\
[\boldsymbol{\rho}](x, y) &\simeq \{[x](u) \mid u \in y\} \\
&\quad \text{if } [x](u) \text{ is defined} \\
&\quad \text{for all } u \in y
\end{aligned}
$$

Indices for equality axioms:

$$
\begin{aligned}
[\mathbf{i_1}](x, y, z) &\simeq \{u \in x \mid y \in z\} \\
[\mathbf{i_2}](x, y, z) &\simeq \{u \in x \mid u \in y \to u \in z\} \\
[\mathbf{i_3}](x, y, z) &\simeq \{u \in x \mid u \in y \to z \in u\}
\end{aligned}
$$

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Definition ($E_\wp$-*Application terms*)

1. The above constants are application terms,
2. Variables are application terms,
3. If s and t are application terms then so is $[s](t)$.

An application term is *closed* if it does not contain any variables.

Definition

A partial n-place class function Υ is called an $E_\wp$-*recursive partial function* if there exists a closed $E_\wp$-application term t_Υ such that

$$\mathrm{dom}(\Upsilon) = \{(a_1, \ldots, a_n) \mid [t_\Upsilon](a_1, \ldots, a_n) \downarrow\}$$

and for all sets $(a_1, \ldots, a_n) \in \mathrm{dom}(\Upsilon)$,

$$[t_\Upsilon](a_1, \ldots, a_n) \ \simeq \ \Upsilon(a_1, \ldots, a_n).$$

Realizers

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$[a](b) \simeq x$ will denote $\langle a, b, x\rangle \in \mathbb{E}_{\wp}$. Then $\Vdash^{\wp}_{\mathfrak{wt}}$ is defined recursively by:

$$
\begin{array}{lll}
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi & \text{iff} & \varphi \text{ is true, whenever } \varphi \text{ is an atomic formula,} \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \wedge \psi & \text{iff} & \mathbf{p}_0 a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \wedge \mathbf{p}_1 a \Vdash^{\wp}_{\mathfrak{wt}} \psi, \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \vee \psi & \text{iff} & \exists u (u \in a) \wedge \forall d \in a \left((\mathbf{p}_0 d = 0 \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi) \right. \\
 & & \qquad\qquad \left. \vee\, (\mathbf{p}_0 d = 1 \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \psi) \right), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \to \psi & \text{iff} & (\varphi \to \psi) \wedge \forall c (c \Vdash^{\wp}_{\mathfrak{wt}} \varphi \to [a](c) \Vdash^{\wp}_{\mathfrak{wt}} \psi), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \forall x \in b\, \varphi(x) & \text{iff} & \forall c \in b\, ([a](c) \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/c]), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \exists x \in b\, \varphi(x) & \text{iff} & \exists u (u \in a) \wedge \forall d \in a \\
 & & \qquad (\mathbf{p}_0 d \in b \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d]), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \forall x\, \varphi(x) & \text{iff} & \forall c\, [a](c) \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/c], \\
a \Vdash^{\wp}_{\mathfrak{wt}} \exists x\, \varphi(x) & \text{iff} & \exists u (u \in a) \wedge \forall d \in a (\mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d]), \\
\Vdash^{\wp}_{\mathfrak{wt}} \varphi & \text{iff} & \exists a\, a \Vdash^{\wp}_{\mathfrak{wt}} \varphi.
\end{array}
$$