On the Constructive Constructible Universe

Richard Matthews

University of Leeds

Logic Colloquium 2022

Joint work with Michael Rathjen

Constructing the Constructible Universe Constructively

Arxiv: 2206.08283. June 2022

Some of the results appear in my thesis:

Large Cardinals in Weakened Axiomatic Theories

University of Leeds, 2021.

Aims

Gödel's Constructible Universe in Constructive Set Theories:

- How can we construct it?
- What theory does it satisfy?

More generally, what is an *inner model* intuitionistically?

Non-constructive Principles

- (Law of Excluded Middle) $\varphi \vee \neg\varphi$
- (Double Negation Elimination) $\neg\neg\varphi \rightarrow \varphi$
- (Some Classical Logical Equivalences) $(\varphi \rightarrow \psi) \rightarrow (\neg\varphi \vee \psi)$

- Foundation: $\forall a(\exists x(x \in a) \rightarrow \exists x \in a\ \forall y \in a(y \notin x))$
- "Minimal elements" of sets
- Axiom of Choice / Well-Ordering Principle
- Definition by cases which differentiate between successor and limit ordinals

Remarks (BHK interpretation)

- $\varphi \rightarrow \psi$ should be read as *"from a proof of φ we can construct a proof of ψ"*.
- $\neg\varphi$ is interpreted as $\varphi \rightarrow (0 = 1)$.

The Joys of Intuitionism

Many intuitionistic theories satisfy "pleasing mathematical properties" for example they:

- Give insight on when Excluded Middle is needed in a given proof,
- Give "*more direct*" proofs of mathematical statements,
- Satisfy the Disjunction Property (if $T \vdash \varphi \vee \psi$ then either $T \vdash \varphi$ or $T \vdash \psi$),
- Satisfy the Numerical Existence Property (if $T \vdash \exists x \in \omega \varphi(x)$ then $T \vdash \varphi(n)$ for some $n \in \omega$),
- Are the internal logic of toposes with a natural number object,
- Have a type theoretic interpretation.

The Joys of Intuitionism

The following principles can consistently hold intuitionistically:

- Every function $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous,
- Church's Thesis (All total functions are computable),
- The Uniformity Principle holds ($\forall x \exists n \in \omega \varphi \to \exists n \in \omega \forall x \varphi$),
- Every set is subcoutable (the surjective image of a subset of ω).

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$
$\downarrow$ + *Strong Infinity*
IKP
$\downarrow$ + *Strong Collection*
CZF^{-}
$\downarrow$ + *Subset Collection*
CZF
$\downarrow$ + *Power Set*
$\mathrm{CZF}_{\mathcal{P}}$
$\downarrow$ + *Full Separation*
IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Strong Infinity

$$\exists a\ (Ind(a)\ \wedge\ \forall b\ (Ind(b) \to \forall x \in a(x \in b))),$$

$$Ind(a) \equiv \emptyset \in a\ \wedge\ \forall x \in a\ (x \cup \{x\} \in a).$$

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$

$\downarrow$ + *Strong Infinity*

IKP

$\downarrow$ + *Strong Collection*

CZF^-

$\downarrow$ + *Subset Collection*

CZF

$\downarrow$ + *Power Set*

$\mathrm{CZF}_{\mathcal{P}}$

$\downarrow$ + *Full Separation*

IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Strong Collection

For any formula $\varphi(u, v)$ and set a,
$$\forall x \in a\ \exists y\ \varphi(x, y) \to \exists b \big(\forall x \in a\ \exists y \in b\ \varphi(x, y) \wedge \forall y \in b\ \exists x \in a\ \varphi(x, y)\big).$$

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$
$\downarrow$ + *Strong Infinity*
IKP
$\downarrow$ + *Strong Collection*
CZF^{-}
$\downarrow$ + *Subset Collection*
CZF
$\downarrow$ + *Power Set*
$\mathrm{CZF}_{\mathcal{P}}$
$\downarrow$ + *Full Separation*
IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Subset Collection

Equivalent to the axiom of *Fullness*.

Implies Exponentiation ($\forall a, b \;\; {}^{a}b$ is a set).

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$
$\downarrow$ + *Strong Infinity*
IKP
$\downarrow$ + *Strong Collection*
CZF^{-}
$\downarrow$ + *Subset Collection*
CZF
$\downarrow$ + *Power Set*
$\mathrm{CZF}_{\mathcal{P}}$
$\downarrow$ + *Full Separation*
IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Power Set

$\forall a\ \mathcal{P}(a)$ is a set.

Intuitionistic Theories

Idea

IKP (IZF) is the theory KP (ZF) with intuitionistic logic instead of classical logic.

$\mathrm{IKP}^{-\mathit{Inf}}$

$\downarrow$ + *Strong Infinity*

IKP

$\downarrow$ + *Strong Collection*

CZF^{-}

$\downarrow$ + *Subset Collection*

CZF

$\downarrow$ + *Power Set*

$\mathrm{CZF}_{\mathcal{P}}$

$\downarrow$ + *Full Separation*

IZF

$\mathrm{IKP}^{-\mathit{Inf}}$

- Extensionality
- Empty Set
- Pairing
- Unions
- Set Induction
- Bounded Separation
- Bounded Collection

Full Separation

For any formula $\varphi(u)$ and set a, $\{x \in a \mid \varphi(x)\}$ is a set.

What is an Ordinal?

Definition

$\langle A, \prec \rangle$ is a well-ordering if it is a strict total order such that any non-empty subset X of A has an $\prec$-least element.

Definition (Classical)

An ordinal α is a transitive set which is well-ordered by $\in$.

Proposition (Folklore)

α is an ordinal iff it is a transitive set of transitive sets.

Definition (More Constructive)

An ordinal is a transitive set of transitive sets.
Let $\textsc{Ord}$ denote the class of ordinals.

Intuitionistic Ordinals

Nice Properties

- If α is an ordinal then so is $\alpha + 1 := \alpha \cup \{\alpha\}$,
- If X is a set of ordinals then $\bigcup X$ is an ordinal,
- We can perform definitions by transfinite recursion and make stratified hierarchies (e.g. rank function, V_α, L_α ...).

Ordinary Ordinal Oddities

- $\beta \in \alpha \not\Rightarrow \beta + 1 \in \alpha + 1$,
- $\forall \alpha\ (0 \in \alpha + 1)$ implies excluded middle!

Truth Values

Given a formula φ, an important ordinal is

$$\alpha_\varphi := \{0 \in 1 \mid \varphi\}.$$

Naively, if we don't assume $\varphi \vee \neg\varphi$ then α_φ is neither 0 nor 1.

In general, we let

$$\Omega := \mathcal{P}(1) = \{x \mid x \subseteq 1\}$$

be the class of *truth values*.

If $\Omega = 2$ then the Law of Excluded Middle holds.

Note that

$$(0 \in \alpha_\varphi + 1) \implies (0 \in \alpha_\varphi \vee 0 = \alpha_\varphi) \implies (\varphi \vee \neg\varphi).$$

Inner Models

Definition

Let T be a theory (in the language of set theory) and suppose that $\mathrm{V} \models \mathrm{T}$. A transitive class M is said to be an *inner model* if

- $\mathrm{M} \subseteq \mathrm{V}$,
- $\mathrm{M} \models \mathrm{T}$,
- $\textsc{Ord} \cap \mathrm{M} = \textsc{Ord} \cap \mathrm{V}$.

Example

If T is KP or ZF then L is an inner model.

Theorem (M., Rathjen)

1. $\mathrm{IZF} \not\vdash \textsc{Ord} \cap \mathrm{L} = \textsc{Ord} \cap \mathrm{V}$,
2. $\mathrm{CZF} \not\vdash (\mathrm{CZF})^{\mathrm{L}}$.

The Constructible Universe

- The constructible universe was developed by Gödel in papers published in 1939 and 1940 to show the consistency of the Axiom of Choice and the Generalised Continuum Hypothesis with ZF.
- As with the standard V_α cumulative hierarchy, one builds the L_α hierarchy in stages where one replaces the power set operation with a "*definable*" version.
- There are 2/3 equivalent approaches to building L, all of which are formalisable in KP:[1]
  - Syntactically as the set of definable subsets of M (See Devlin - *Constructibility*),
  - Using Gödel functions (See Barwise - *Admissible Sets*) or
  - Using Rudimentary Functions (See Gandy, Jensen, Mathias).

[1]In fact significantly weaker systems - see Mathias: *Weak Systems of Gandy, Jensen and Devlin*, 2006

Constructing the Constructible Universe Constructively

- The syntactic approach was then modified for IZF by Lubarsky (*Intuitionistic L* - 1993).
- And then for IKP by Crosilla (*Realizability models for constructive set theories with restricted induction* - 2000).
- Rudimentary functions were extended to constructive theories by Aczel (*Rudimentary and arithmetical constructive set theory* - 2012).
- (M.) Using an extended set of Gödel functions, one can construct L in IKP^{-Inf}.

Gödel Functions

Theorem

If T *is one of* IKP^{-Inf}, IKP *or* IZF *then*

$$\mathrm{T} \vdash (\mathrm{T})^{\mathrm{L}}.$$

Constructibility

Definition ($\mathrm{IKP}^{-\mathit{Inf}}$)

- Let $\mathcal{F}_1, \ldots, \mathcal{F}_N$ denote the (extended) Gödel functions.
- Let $\mathfrak{E}(b) := b \cup \{\mathcal{F}_i(x, y) \mid x, y \in b \ \wedge \ i \leq N\}$ be the 1-step closure of b under these functions.
- Let $\mathbb{L}_\alpha := \bigcup_{\beta \in \alpha} \mathfrak{E}(\mathbb{L}_\beta \cup \{\mathbb{L}_\beta\})$.
- Let $\mathbb{L} := \bigcup_\alpha \mathbb{L}_\alpha$.

Lemma (Lubarsky / M.)

For every ordinal α in V there is an ordinal α^* in $\mathbb{L}$ such that $\mathbb{L}_\alpha = \mathbb{L}_{\alpha^*}$.

Theorem (Axiom of Constructibility)

If T is one of $\mathrm{IKP}^{-\mathit{Inf}}$, IKP *or* IZF *then*

$$\mathrm{T} \vdash (\mathrm{V} = \mathbb{L})^{\mathbb{L}}.$$

Theorem (M.)

It is consistent to have a model of IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

Full Kripke Model (Hendtlass, Lubarsky)

Details

Given a Kripke structure $\mathscr{K}$ and a model of ZF assigned to each node, one can build a model structure $\mathrm{V}(\mathscr{K})$ and interpretation $\Vdash_{\mathscr{K}}$ such that

$$\mathrm{V}(\mathscr{K}) \models \varphi \iff \forall p \in \mathscr{K} \ p \Vdash_{\mathscr{K}} \varphi.$$

Moreover, this model will satisfy IZF.

Sketch.

Theorem (M.)

It is consistent to have a model of IZF *such that*

$$\textsc{Ord} \cap \mathrm{V} \neq \textsc{Ord} \cap \mathrm{L}.$$

Sketch.

The desired model will be $\mathrm{V}(\mathscr{K})$ where

- $\mathcal{K}$ is the two node Kripke structure $\{\mathbb{1}, \alpha\}$,
- $\mathcal{D}(\mathbb{1}) = \mathcal{D}(\alpha) = \mathrm{L}[c]$,
- c is a Cohen real over L.

$$\mathcal{K} \quad = \quad \begin{array}{lcl} \alpha \bullet & \quad & \mathrm{L}[c] \\ \quad | & & \\ \mathbb{1} \bullet & & \mathrm{L}[c] \end{array}$$

Theorem (M.)

It is consistent to have a model of IZF *such that*

$$\mathrm{ORD} \cap \mathrm{V} \neq \mathrm{ORD} \cap \mathrm{L}.$$

Sketch.

- Let c^p be the interpretation of c at node p. Details
- Then $p \Vdash c^p \notin \mathrm{L}$.
- So, $\mathrm{V}(\mathscr{K}) \models c \notin \mathrm{L}$.
- Let 1_α be the ordinal in $\mathrm{V}(\mathscr{K})$ which looks like 0 at $\mathbb{1}$ and 1 at α.

$$1_\alpha : \mathcal{K} \to 2 \qquad 1_\alpha(p) = \begin{cases} 0, & \text{if } p = \mathbb{1} \\ 1, & \text{if } p = \alpha. \end{cases}$$

- Then, in $\mathrm{V}(\mathscr{K})$, $1_\alpha \subseteq 1$ and $\mathrm{L}_{1_\alpha} = 1_\alpha$.

Theorem (M.)

It is consistent to have a model of IZF *such that*

$$\mathrm{Ord} \cap \mathrm{V} \neq \mathrm{Ord} \cap \mathrm{L}.$$

Sketch.

- Define δ_c to be an ordinal encoding c, for example,

$$\begin{aligned}
\delta_c &= \bigcup_{n \in \omega} (1_\alpha \cup n) + c(n) \\
&= \{1_\alpha \cup n \mid c(n) = 0\} \cup \{1_\alpha \cup n \cup \{1_\alpha \cup n\} \mid c(n) = 1\} \\
&= \{1_\alpha \cup n \mid n \in \omega\} \cup \{\{1_\alpha \cup n\} \mid c(n) = 1\}.
\end{aligned}$$

- Then $c(n) = 1$ if and only if $(1_\alpha \cup n) \in \delta_c$,
- Note that $c \in \mathrm{L} \iff \delta_c \in \mathrm{L}$,
- Therefore, $\delta_c \notin \mathrm{L}$. $\square$

$E_\wp$-recursion

- E-recursion or set recursion is a generalisation of recursion theory to be able to apply arbitrary sets to one another (see Sack's *Higher Recursion Theory* or Normann's *Set Recursion*).
- $[s](t)$ can be viewed as "s *is some kind of Turing machine that takes as input the set* t *and outputs* $[s](t)$".
- It will contain designated integers which provide indices for special functions computing basic operations (e.g. pairing, succesors, unions, intersections, ω, $\dots$). Details
- $E_\wp$ recursion was developed by Rathjen (From the weak to the strong existence property, 2012) to also have the power set operation be "*computable*".
- An $E_\wp$*-recursive partial function* is essentially a $\Sigma^{\mathcal{P}}$ function which computes a term built using $E_\wp$-recursion.

Realizability with truth

Details

Next combine $E_\wp$ computability with a notion of realizability that preserves truth, $\Vdash^{\wp}_{\mathfrak{wt}}$. Realizers for existential statements will provide a set of witnesses for the existential quantifier.

Proposition (Rathjen 2012)

Let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables among $x_1, \ldots, x_n$. Then

$$\mathrm{CZF}_{\mathcal{P}} \vdash (\exists e\, e \Vdash^{\wp}_{\mathfrak{wt}} \varphi(x_1, \ldots, x_n)) \rightarrow \varphi(x_1, \ldots, x_n).$$

Theorem (Rathjen 2012)

Let $\varphi(x_1, \ldots, x_n)$ be a formula with free variables among $x_1, \ldots, x_n$. If $\mathrm{CZF}_{\mathcal{P}} \vdash \varphi(x_1, \ldots, x_n)$ then one can effectively construct an index of an $E_\wp$-recursive function f such that

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall a_1 \ldots \forall a_n\, f(a_1, \ldots a_n) \Vdash^{\wp}_{\mathfrak{wt}} \varphi(a_1, \ldots, a_n).$$

Conservativity

Using a very similar notion of realizability, one can show

Theorem (Rathjen, 2012)

1. CZF^- *is conservative over* IKP *for* Π_2 *sentences,*
2. $\mathrm{CZF}_{\mathcal{P}}$ *is conservative over* $\mathrm{IKP}(\mathcal{P})$[2] *for* $\Pi_2^{\mathcal{P}}$ *sentences.*

Next, via a proof-theoretic interpretation of $\mathrm{KP}(\mathcal{P})$ to an extension of Zermelo set theory, we have

Theorem (Rathjen, 2014)

Let φ *be a* $\Pi_2^{\mathcal{P}}$ *sentence. If* $\mathrm{IKP}(\mathcal{P}) \vdash \varphi$ *then* $\mathrm{V}_{\mathrm{BH}} \models \varphi$ *where* BH *is the Bachmann-Howard ordinal.*

Corollary (Rathjen)

1. IKP and $\mathrm{IKP}(\mathcal{P})$ have the existence property for Σ and $\Sigma^{\mathcal{P}}$ formulas respectively.
2. CZF^- and $\mathrm{CZF}_{\mathcal{P}}$ have the existence property.

[2] IKP + *Power Set* and both Separation and Collection for $\Sigma_0^{\mathcal{P}}$ formulas.

Exponentiation and $\mathrm{CZF}_{\mathcal{P}}$

Theorem (M., Rathjen)

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{CZF})^{\mathrm{L}}$.

- Suppose that $\mathrm{CZF}_{\mathcal{P}}$ proved that ${}^{\omega}\omega \cap \mathrm{L} \in \mathrm{L}$.
- Then

$$\mathrm{CZF}_{\mathcal{P}} \vdash \exists a \big(a \in \mathrm{ORD} \ \wedge \ \omega \in a \ \wedge \\ \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \rightarrow f \in \mathrm{L}_a)\big)$$

- Construct an index for an $E_{\wp}$-recursive function g such that $\mathrm{CZF}_{\mathcal{P}} \vdash \forall x \; g(x)$ *is defined* and

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall x \; g(x) \Vdash^{\wp}_{\mathfrak{wt}} \exists a \big(a \in \mathrm{ORD} \ \wedge \ \omega \in a \ \wedge \\ \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \rightarrow f \in \mathrm{L}_a)\big).$$

Exponentiation and $\mathrm{CZF}(\mathcal{P})$

Theorem

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{V} = \mathrm{L})^{\mathrm{L}}$.

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall x\, g(x) \Vdash^{\wp}_{\mathfrak{wt}} \exists a(a \in \textsc{Ord} \ \wedge \ \omega \in a \ \wedge \\ \forall f \in {}^{\omega}\omega(f \in \mathrm{L} \to f \in \mathrm{L}_a)).$$

Note: $a \Vdash^{\wp}_{\mathfrak{wt}} \exists x \varphi(x)$ iff $\exists u(u \in a) \wedge \forall d \in a(\mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d])$,

- Unpacking ... $\mathrm{CZF}_{\mathcal{P}} \vdash \exists w(w \in g(0))$ and

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\ \mathbf{p}_1 y \Vdash^{\wp}_{\mathfrak{wt}} (\mathbf{p}_0 y \in \textsc{Ord} \ \wedge \ \omega \in \mathbf{p}_0 \ \wedge \\ \forall f \in {}^{\omega}\omega(f \in \mathrm{L} \to f \in \mathrm{L}_{\mathbf{p}_0 y})).$$

- Since realizability preserves truth,

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\ (\mathbf{p}_0 y \in \textsc{Ord} \ \wedge \ \omega \in \mathbf{p}_0 \ \wedge \\ \forall f \in {}^{\omega}\omega(f \in \mathrm{L} \to f \in \mathrm{L}_{\mathbf{p}_0 y})).$$

Exponentiation and $\mathrm{CZF}(\mathcal{P})$

Theorem

$\mathrm{CZF}_{\mathcal{P}} \nvdash \mathrm{Exp}^{\mathrm{L}}$. *Thus* $\mathrm{CZF}_{\mathcal{P}} \nvdash (\mathrm{V} = \mathrm{L})^{\mathrm{L}}$.

- Since g was $E_{\wp}$ recursive, "$\forall x\, g(x)$ *is defined*" is $\Pi_2^{\mathcal{P}}$ definable.
- Now $\mathrm{CZF}_{\mathcal{P}}$ is conservative over $\mathrm{IKP}(\mathcal{P})$ for $\Pi_2^{\mathcal{P}}$ sentences, and these reflect to V_{BH} where BH is the *Bachmann-Howard* ordinal.
- Therefore, $g(0) \in \mathrm{V}_{\mathrm{BH}}$. So

$$\mathrm{CZF}_{\mathcal{P}} \vdash \forall y \in g(0)\, \exists a \in \mathrm{V}_{\mathrm{BH}}$$
$$\Big(a \in \textsc{Ord} \ \wedge\ \omega \in a \ \wedge\ \forall f \in {}^{\omega}\omega (f \in \mathrm{L} \rightarrow f \in \mathrm{L}_a)\Big).$$

- Which means that ${}^{\omega}\omega \cap \mathrm{L} \in \mathrm{L}_{\mathrm{BH}}$ which is a contradiction.

Inner Models

Intuitionistically:

- L is a definable class,
- If V is a model of $\mathrm{IKP}^{-\mathit{Inf}}$, IKP or IZF then so is L,
- But if V only satisfies CZF / $\mathrm{CZF}_{\mathcal{P}}$, L may not satisfy Exponentiation,
- Moreover, we may not have $\mathrm{V} \cap \textsc{Ord} = \mathrm{L} \cap \textsc{Ord}$.

- So what can inner models do ...?

External Cumulative Hierarchies

Definition

Let $\mathrm{M} \subseteq \mathrm{N}$. We say that M is almost universal in N if for any $x \in \mathrm{N}$, if $x \subseteq \mathrm{M}$ then there exists some $y \in \mathrm{M}$ such that $x \subseteq y$.

Theorem (M.)

Suppose that N is a model of IZF and $\mathrm{M} \subseteq \mathrm{N}$ is a transitive (proper) class with an external cumulative hierarchy in N.[3] Then M is a model of IZF if and only if M is closed under the extended Gödel functions and is almost universal in N.

Remark

So, under some "*nice*" condition, we can express being a model of IZF by a single sentence.

[3] Essentially, $\mathrm{M} = \bigcup_{\alpha \in \mathrm{ORD} \cap \mathrm{N}} \mathrm{M}_\alpha$.

Gödel Functions

Back

Definition

- $\mathcal{F}_p(x,y) := \{x,y\}$,
- $\mathcal{F}_\cap(x,y) := x \cap \bigcap y$, $(\cap y = \{u \mid \forall v \in y\ (u \in v)\})$
- $\mathcal{F}_\cup(x,y) := \bigcup x$,
- $\mathcal{F}_\setminus(x,y) := x \setminus y$,
- $\mathcal{F}_\times(x,y) := x \times y$,
- $\mathcal{F}_\rightarrow(x,y) := x \cap \{z \mid \textit{y is an ordered pair} \wedge (z \in 1^{st}(y) \rightarrow z \in 2^{nd}(y))\}$,
- $\mathcal{F}_\forall(x,y) := \{x``\{z\} \mid z \in y\}$, $(x``u = \{v \mid v \in 2^{nd}(x) \wedge \langle u,v\rangle \in x\})$

Gödel Functions

Back

Definition

- $\mathcal{F}_{\text{dom}}(x, y) := \text{dom}(x) = \{1^{st}(z) \mid z \in x \wedge$ *z is an ordered pair*$\}$,
- $\mathcal{F}_{\text{ran}}(x, y) := \text{ran}(x) = \{2^{nd}(z) \mid z \in x \wedge$ *z is an ordered pair*$\}$,
- $\mathcal{F}_{123}(x, y) := \{\langle u, v, w\rangle \mid \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{132}(x, y) := \{\langle u, w, v\rangle \mid \langle u, v\rangle \in x \ \wedge \ w \in y\}$,
- $\mathcal{F}_{=}(x, y) := \{\langle v, u\rangle \in y \times x \mid u = v\}$,
- $\mathcal{F}_{\in}(x, y) := \{\langle v, u\rangle \in y \times x \mid u \in v\}$.

Notation

Let $\mathcal{I}$ be the finite set indexing the above operations.

Kripke Models

Definition

A *Kripke model* is an ordered quadruple $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ where

- $\mathcal{K}$ is a non-empty set of "*nodes*",
- $\mathcal{D}$ is a function on $\mathcal{K}$,
- $\mathcal{R}$ is a binary, reflexive relation between elements of $\mathcal{K}$,
- ι is a set of functions $\iota_{p,q}$ for each pair $p, q \in \mathcal{K}$ with $p\mathcal{R}q$

such that the following hold.

- For each $p \in \mathcal{K}$, $\mathcal{D}(p)$ is an inhabited class structure,
- If $p\mathcal{R}q$ then $\iota_{p,q} \colon \mathcal{D}(p) \to \mathcal{D}(q)$ is a homomorphism,
- If $p\mathcal{R}q$ and $q\mathcal{R}r$ then $\iota_{p,r} = \iota_{q,r} \circ \iota_{p,q}$.

Forcing(ish)

Now, for atomic formulae φ, let $p \Vdash \varphi$ denote that $\mathcal{D}(p) \models \varphi$. Then $\Vdash$ can be extended to arbitrary formulae by the following prescription:

- For no p do we have $p \Vdash \perp$,
- $p \Vdash \varphi \wedge \psi$ iff $p \Vdash \varphi$ and $p \Vdash \psi$,
- $p \Vdash \varphi \vee \psi$ iff $p \Vdash \varphi$ or $p \Vdash \psi$,
- $p \Vdash \varphi \to \psi$ iff for any $r \in \mathcal{K}$ with $p\mathcal{R}r$, if $r \Vdash \varphi$ then $r \Vdash \psi$,
- $p \Vdash \forall x \ \varphi(x)$ iff whenever $p\mathcal{R}q$ and $d \in \mathcal{D}(q)$, $q \Vdash \varphi(d)$,
- $p \Vdash \exists x \ \varphi(x)$ iff there is some $d \in \mathcal{D}(p)$ such that $p \Vdash \varphi(d)$.

Validity

Definition

Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model and $p \in \mathcal{K}$.

- A formula φ is said to be *valid at* p iff $p \Vdash \varphi$.
- A formula φ is *valid in the full Kripke model*, written $\mathscr{K} \Vdash \varphi$, if for every $p \in \mathcal{K}$, $p \Vdash \varphi$.

Fact (Hendtlass, Lubarsky)

It is possible to add a model structure to $\mathscr{K}$, $\mathrm{V}(\mathscr{K})$ such that

$$\mathrm{V}(\mathscr{K}) \models \varphi \Longleftrightarrow \forall p \in \mathcal{K}\ p \Vdash \varphi.$$

Theorem (Hendtlass, Lubarsky)

If for each $p, q \in \mathcal{K}$, $\mathcal{D}(p) \models \mathrm{ZF}$ *and* $\textsc{Ord} \cap \mathcal{D}(p) = \textsc{Ord} \cap \mathcal{D}(q)$, *then* $\mathrm{V}(\mathscr{K}) \models \mathrm{IZF}$.

The Model

Back

Suppose that $\mathscr{K}$ is a Kripke model and that for each node p, $\mathcal{D}(p)$ is a model of ZF. We shall simultaneously define the set of objects at p, $\mathrm{M}^p := \bigcup_\alpha \mathrm{M}^p_\alpha$, inductively through the ordinals.
So suppose that $\{\mathrm{M}^p_\beta \mid p \in \mathcal{K}\}$ has been defined for each $\beta \in \alpha$ along with transition functions $k_{p,q} \colon \mathrm{M}^p_\beta \to \mathrm{M}^q_\beta$ for each pair $p\mathcal{R}q$.
The objects of M^p_α are then the collection of functions g such that

- $\mathrm{dom}(g) = \mathcal{K}^p$,
- $g \restriction \mathcal{K}^q \in \mathcal{D}(q)$,
- $g(q) \subseteq \bigcup_{\beta \in \alpha} \mathrm{M}^q_\beta$,
- If $h \in g(q)$ and $q\mathcal{R}r$ then $k_{q,r}(h) \in g(r)$.

Finally, extend $k_{p,q}$ to M^p_α by setting $k_{p,q}(g) := g \restriction \mathcal{K}^q$. Then the objects at node p are $\bigcup_\alpha \mathrm{M}^p_\alpha$.
We now define truth at node p for formulae by the following:

- $p \Vdash g \in h \iff g \restriction \mathcal{K}^p \in h(p)$,
- $p \Vdash g = h \iff g \restriction \mathcal{K}^p = h \restriction \mathcal{K}^p$,
- For logical connectives and quantifiers we use the rules for $\Vdash$.

Interpreting the initial node

Back

Let $\mathscr{K} = \langle \mathcal{K}, \mathcal{R}, \mathcal{D}, \iota \rangle$ be a Kripke model.

Definition

Define $\mathscr{K}^p$ to be the *truncation* of the Kripke model to $\mathcal{K}^p := \{q \in \mathcal{K} \mid p\mathcal{R}q\}$. So $\mathcal{K}^p$ is the cone of nodes which are related to p.

Fact

Given $p \in \mathcal{K}$ and $x \in \mathcal{D}(p)$ we can define an interpretation x^p such that if $p\mathcal{R}q$ then $q \Vdash x^p = x^q$.

This gives us a way to talk about the past worlds in the current one.

$E_\wp$-recursive functions

Back

We inductively define a class $\mathbb{E}_\wp$ of triples $\langle e, x, y\rangle$ Instead of saying $\langle e, x, y\rangle \in \mathbb{E}_\wp$ we shall say $[e](x) \simeq y$. The relation is then defined by the following clauses:

Indices for applicative structures (APP):

$$\begin{aligned}
[\mathbf{k}](x, y) &\simeq x \\
[\mathbf{p}](x, y) &\simeq \langle x, y\rangle \\
[\mathbf{p_N}](0) &\simeq 0 \\
[\mathbf{p_0}](x) &\simeq 1^{st}(x) \\
[\mathbf{d_N}](n, m, x, y) &\simeq x \text{ if } n, m \in \mathbb{N} \text{ and } n = m \\
[\mathbf{d_N}](n, m, x, y) &\simeq y \text{ if } n, m \in \mathbb{N} \text{ and } n \neq m
\end{aligned}$$

$$\begin{aligned}
[\mathbf{s}](x, y, z) &\simeq [[x](z)]([y](z)) \\
[\mathbf{s_N}](n) &\simeq n + 1 \text{ if } n \in \mathbb{N} \\
[\mathbf{p_N}](n + 1) &\simeq n \text{ if } n \in \mathbb{N} \\
[\mathbf{p_1}](x) &\simeq 2^{nd}(x) \\
[\bar{\mathbf{0}}](x) &\simeq 0 \\
[\bar{\boldsymbol{\omega}}](x) &\simeq \omega
\end{aligned}$$

$E_\wp$-recursive functions

Back

We inductively define a class $\mathbb{E}_\wp$ of triples $\langle e, x, y\rangle$ Instead of saying $\langle e, x, y\rangle \in \mathbb{E}_\wp$ we shall say $[e](x) \simeq y$. The relation is then defined by the following clauses:

Indices for set-theoretic axioms:

$$
\begin{aligned}
[\boldsymbol{\pi}](x, y) &\simeq \{x, y\} \\
[\boldsymbol{\gamma}](x, y) &\simeq x \cap \bigcap y \\
[\boldsymbol{\wp}](x, y) &\simeq \mathcal{P}(x)
\end{aligned}
\qquad
\begin{aligned}
[\boldsymbol{\nu}](x) &\simeq \bigcup x \\
[\boldsymbol{\rho}](x, y) &\simeq \{[x](u) \mid u \in y\} \\
&\quad \text{if } [x](u) \text{ is defined} \\
&\quad \text{for all } u \in y
\end{aligned}
$$

Indices for equality axioms:

$$
\begin{aligned}
[\mathbf{i_1}](x, y, z) &\simeq \{u \in x \mid y \in z\} \\
[\mathbf{i_2}](x, y, z) &\simeq \{u \in x \mid u \in y \to u \in z\} \\
[\mathbf{i_3}](x, y, z) &\simeq \{u \in x \mid u \in y \to z \in u\}
\end{aligned}
$$

$E_\wp$-recursive functions

Back

Definition ($E_\wp$-*Application terms*)

1. The above constants are application terms,
2. Variables are application terms,
3. If s and t are application terms then so is $[s](t)$.

An application term is *closed* if it does not contain any variables.

Definition

A partial n-place class function Υ is called an $E_\wp$-*recursive partial function* if there exists a closed $E_\wp$-application term t_Υ such that

$$\mathrm{dom}(\Upsilon) = \{(a_1, \ldots, a_n) \mid [t_\Upsilon](a_1, \ldots, a_n) \downarrow\}$$

and for all sets $(a_1, \ldots, a_n) \in \mathrm{dom}(\Upsilon)$,

$$[t_\Upsilon](a_1, \ldots, a_n) \; \simeq \; \Upsilon(a_1, \ldots, a_n).$$

Realizers

Back

$[a](b) \simeq x$ will denote $\langle a, b, x \rangle \in \mathbb{E}_\wp$. Then $\Vdash^{\wp}_{\mathfrak{wt}}$ is defined recursively by:

$$
\begin{array}{lll}
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi & \text{iff} & \varphi \text{ is true, whenever } \varphi \text{ is an atomic formula,} \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \wedge \psi & \text{iff} & \mathbf{p}_0 a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \wedge \mathbf{p}_1 a \Vdash^{\wp}_{\mathfrak{wt}} \psi, \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \vee \psi & \text{iff} & \exists u(u \in a) \wedge \forall d \in a \big((\mathbf{p}_0 d = 0 \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi) \\
& & \qquad \vee (\mathbf{p}_0 d = 1 \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \psi)\big), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \varphi \to \psi & \text{iff} & (\varphi \to \psi) \wedge \forall c(c \Vdash^{\wp}_{\mathfrak{wt}} \varphi \to [a](c) \Vdash^{\wp}_{\mathfrak{wt}} \psi), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \forall x \in b\, \varphi(x) & \text{iff} & \forall c \in b\, ([a](c) \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/c]), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \exists x \in b\, \varphi(x) & \text{iff} & \exists u(u \in a) \wedge \forall d \in a \\
& & \qquad (\mathbf{p}_0 d \in b \wedge \mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d]), \\
a \Vdash^{\wp}_{\mathfrak{wt}} \forall x\, \varphi(x) & \text{iff} & \forall c\, [a](c) \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/c], \\
a \Vdash^{\wp}_{\mathfrak{wt}} \exists x\, \varphi(x) & \text{iff} & \exists u(u \in a) \wedge \forall d \in a(\mathbf{p}_1 d \Vdash^{\wp}_{\mathfrak{wt}} \varphi[x/\mathbf{p}_0 d]), \\
\Vdash^{\wp}_{\mathfrak{wt}} \varphi & \text{iff} & \exists a\, a \Vdash^{\wp}_{\mathfrak{wt}} \varphi.
\end{array}
$$