Playing Jenga with Infinite Cardinals

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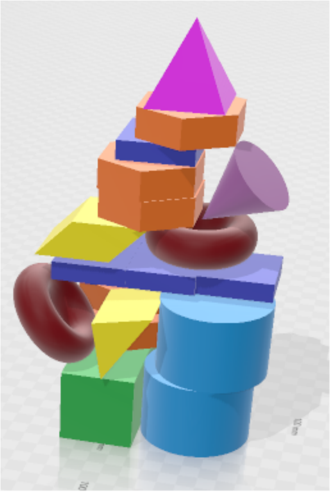
LMS Virtual Graduate Student Meeting

Jenga

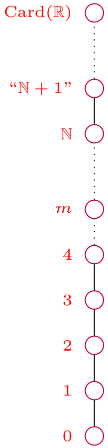
Consider a proof to be a jenga tower...

- **Maybe an axiom only appears once and can be easily removed.**
- **Maybe an axiom appears as an important bedrock to the proof.**
- **Maybe one axiom could be removed or another axiom, but not both.**
- **Maybe one axiom seems to be an important part of the proof but can be replaced by another.**



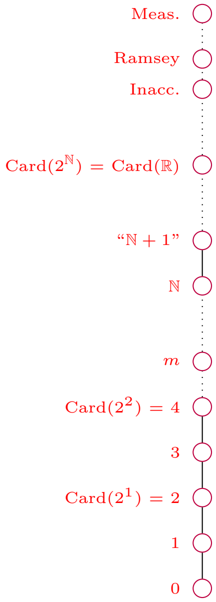
Cardinals

- We want to generalise properties of the natural numbers ($\mathbb{N}$). For example:
  1. If $m \in \mathbb{N}$ then so is every (natural) number less than m,
  2. $\mathbb{N}$ is linearly ordered with no infinite decreasing chain,
  3. No element of $\mathbb{N}$ is bijective with $\mathbb{N}$.
- A set X is *transitive* if for every $y \in X$, $y \subseteq X$.
- A set is an *ordinal* if it is transitive, linearly ordered (by $\in$) and has no infinite decreasing chain.
- A *cardinal* is an ordinal which is not bijective with any element of itself.



What makes them large?

- Properties of the natural numbers ($\mathbb{N}$) and real numbers ($\mathbb{R}$).
  1. No element of $\mathbb{N}$ can be mapped cofinally onto $\mathbb{N}$ and for any $m \in \mathbb{N}$, 2^m or the *power set* of m is still strictly smaller than $\mathbb{N}$.
  2. Lebesgue measurability on $\mathbb{R}$ ($\mathcal{P}(\omega)$). ($\mu(\bigcup_n X_n) = \sum_n \mu(X_n)$)
- A cardinal κ is *inaccessible* if no element of κ can be mapped cofinally onto κ and for any $\alpha \in \kappa$, the power set of α is still strictly smaller than κ.
- A cardinal κ is *measurable* if there is a κ-additive, non-trivial 0 - 1 measure on $\mathcal{P}(\kappa)$.
- The existence of inaccessible or measurable cardinals proves the consistency of ZFC (standard set theory) and therefore we can't prove they exist!

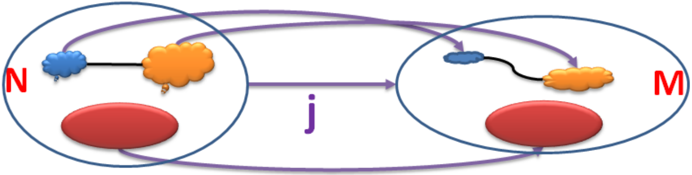


Elementarity

Definition

An *elementary embedding* between structures N and M is an injection $j : \mathrm{N} \to \mathrm{M}$ such that for any formula $\varphi(v)$ and $a \in \mathrm{N}$

$$\mathrm{N} \textit{ believes } \varphi(a) \textit{ is true} \iff \mathrm{M} \textit{ believes } \varphi(j(a)) \textit{ is true.}$$



Fact

A cardinal κ is measurable if and only if there exists some class $\mathrm{M} \subseteq \mathrm{V}$ [1] and an elementary embedding $j : \mathrm{V} \to \mathrm{M}$ such that

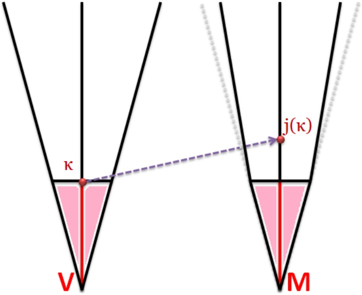
- Anything hereditarily of size smaller than κ is fixed by j.
- The size of $j(\kappa)$ is strictly bigger than that of κ.

[1] Where V denotes the mathematical universe

How close can M be to V?

- If κ is the least measurable cardinal then M will not be closed under arbitrary sequences indexed by subsets of κ. In particular $\{j(x) : x \subseteq \kappa\} \notin \mathrm{M}$.
- Asserting that κ is the smallest set moved by some embedding j with M close to V can add significant strength:

- Say that κ is *X-strong* if there is some $j : \mathrm{V} \to \mathrm{M}$ with $X \subseteq \mathrm{M}$
- Say that κ is *α-supercompact* if there is some $j : \mathrm{V} \to \mathrm{M}$ which is closed under sequences of length α.
- Say that κ is *n-huge* if M is closed under sequences of length $j^n(\kappa)$.



Question (Reinhardt)

Can we have an elementary embedding $j : \mathrm{V} \to \mathrm{V}$?

Jenga with Reinhardt

Theorem (Kunen 1971)

Under ZFC there is no elementary embedding $j : \mathrm{V} \to \mathrm{V}$!

Fact (Kunen 1971)

There exists some set W such that there is no non-trivial elementary embedding $j : \mathrm{W} \to \mathrm{W}$.

But if we weaken the theory an embedding could exist...

- If $\mathrm{V} = \{a, b\}$ then the map swapping a and b is an embedding
  - For example $x = a \ \& \ x \neq b$ becomes $x = b \ \& \ x \neq a$.
- If the theory is just the theory of some group G then the map $j : g \mapsto g^{-1}$ is an embedding.
  - For example, given x the statement $\exists g \ x \circ g = e$ becomes $\exists g \ x^{-1} \circ g = e$ which is clearly true.

The Naive Approach

Theorem (Suzuki, 1999)

There is no formula $\varphi(v)$ and "set of rules" p such that $\varphi(p)$ defines an elementary embedding $j : \mathrm{V} \to \mathrm{V}$.

Sketch

- Suppose there was such a formula $\varphi(v)$ and set p.
- Then there is a formula $\psi(v)$ such that $\psi(p)$ defines an elementary embedding j satisfying that there is some cardinal κ such that
  1. κ has size less than $j(\kappa)$,
  2. Everything smaller than κ is fixed by j,
  3. κ is least such that such an embedding exists.
- Since this is a formula, by elementarity $\psi(j(p))$ must also define an elementary embedding $k : \mathrm{V} \to \mathrm{V}$.
- But the smallest cardinal moved by k is $j(\kappa)$ which contradicts the third point. $\square$

Removing the Power Set

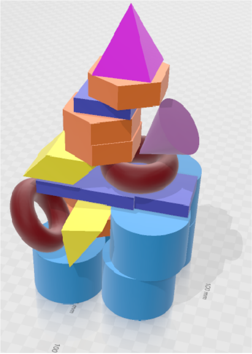
Theorem (M., 2020)

There is no non-trivial elementary embedding $j : \mathrm{V} \to \mathrm{V}$ where V satisfies ZFC without power set[2].

Basic Idea

There are two possible scenarios:

1. W exists in which case Kunen's result can be proved.
2. j is definable from some formula with parameters in V.



[2] Under the right assumptions / formalisations

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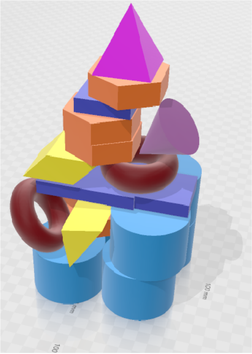

Thanks!

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