Big Classes and Class Forcings

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(Prikry) Forcing Online

Motivation

From *"Taking Reinhardt's Power Away", arXiv:2009.01127*

Suppose that

- V is a model of ZFC *without Power Set*,[1]
- $M \subseteq V$ is a transitive class containing $\textsc{Ord}$,
- $j : V \to M$ is a non-trivial elementary embedding with critical point κ.

Questions

- Is V_κ a set in V?
- What about $\mathcal{P}(\omega)$?

[1] To be defined on the next slide

ZFC without Power Set[2]

Under ZF the following three principles are equivalent:

- The Reflection Principle
- The Collection Scheme
- The Replacement Scheme.

However, without Power Set the reverse implications break down.

Definition (ZF−)

Let ZF− denote the theory consisting of the following axioms:

- Empty set, Extensionality, Pairing, Unions, Infinity,
- the Foundation Scheme, the Separation Scheme,
- the Replacement Scheme.

[2] See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

ZFC without Power Set[2]

Definition

- ZF^- denotes the theory ZF$-$ plus the Collection Scheme.
- ZFC^- denotes the theory ZF^- plus the Well-Ordering Principle.
- ZFC^-_{Ref} denotes the theory ZFC^- plus the Reflection Principle.

Remarks

- For μ regular, $\mathrm{H}_\mu \models \mathrm{ZFC}^-_{Ref}$.
- Models of ZFC$-$ can behave very counter-intuitively.

[2]See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

Big Classes

Definition

A proper class A is called *Big* if for every non-zero ordinal α there is a surjection of A onto α.

Proposition

If $j : \mathrm{V} \to \mathrm{M}$ is elementary and if $j \restriction (\mathcal{C} \cup \{\mathcal{C}\})$ is the identity then $\mathcal{C}$ does not surject onto κ.

Corollary

If $j : \mathrm{V} \to \mathrm{M}$ is elementary, $\mathcal{P}(\omega)$ is not big.

Proposition

Under ZF, every proper class is big.

Easy Examples 1

Proposition

Under ZF*, every proper class is big.*

Proof.

- Given a proper class $\mathcal{C}$, define

$$S := \{\gamma \in \textsc{Ord} : \exists x \in \mathcal{C} \ \ \operatorname{rank}(x) = \gamma\}.$$

- S must be unbounded in the ordinals.
- So, given an ordinal α, we can take the first α many elements of S, $\{\gamma_\beta : \beta \in \alpha\}$.
- Then

$$f(x) = \begin{cases} \beta, & \textit{if } \operatorname{rank}(x) = \gamma_\beta \\ 0, & \textit{otherwise} \end{cases}$$

defines a surjection of $\mathcal{C}$ onto α. □

Easy Examples 2

Theorem (Gitman, Hamkins, Johnstone)

Suppose that $\mathrm{V} \models \mathrm{ZFC}$, κ *is a regular cardinal with* $2^\omega < \aleph_\kappa$ *and that* $G \subseteq \mathrm{Add}(\omega, \aleph_\kappa)$ *is* V*-generic. If* $\mathrm{W} = \bigcup_{\gamma<\kappa} \mathrm{V}[G_\gamma]$ *where* $G_\gamma = G \cap \mathrm{Add}(\omega, \aleph_\gamma)$, *(that is* G_γ *is the first* $\aleph_\gamma$ *many of the Cohen reals added by* G*) then* $\mathrm{W} \models \mathrm{ZFC}-$ *has the same cardinals as* V *and the* DC_α*-Scheme holds in* W *for all* $\alpha < \kappa$, *but the* DC_κ*-Scheme and the Collection Scheme fail.*

- W will have the same cardinals as V and $\mathrm{V}[G]$.
- In $\mathrm{V}[G]$, $2^\omega = \aleph_\kappa$.
- Therefore there is no surjection of $\mathcal{P}(\omega)$ onto $\aleph_{\kappa+1}$.
- Hence there is no such surjection in W.
- So W is a model of ZFC− with a proper class that is not big.

Dependent Choice

Under ZF, Set dependent choice of length μ is defined as follows:

Definition (DC_μ)

- Let S be a non-empty set and R a binary relation.
- Suppose that for every $\alpha \in \mu$ and every α-sequence $s = \langle x_\beta \colon \beta \in \alpha \rangle$ of elements of S there exists some $y \in S$ such that sRy.
- Then there is a function $f\colon \mu \to S$ such that for every $\alpha \in \mu$, $(f \restriction \alpha) R f(\alpha)$.

S is the domain and R is the relation

Dependent Choice

Under ZF^-, Class dependent choice of length μ is defined as follows:

Definition (DC_μ-Scheme)

- Let ψ and φ be formulae and u and w be sets such that for some z, $\psi(z, u)$.
- Suppose that for every $\alpha \in \mu$ and every α-sequence $s = \langle x_\beta \colon \beta \in \alpha \rangle$ satisfying $\psi(x_\beta, u)$ for each β there exists some y satisfying $\psi(y, u)$ and $\varphi(s, y, w)$.
- Then there is a function f with domain μ such that for every $\alpha \in \mu$, $\psi(f(\alpha), u)$ and $\varphi((f \restriction \alpha), f(\alpha), w)$.

ψ is the domain and φ is the relation

Reflection

Definition

$\mathrm{ZF}^-_{\mathrm{DC}_{<\mathrm{Ord}}}$ is the theory ZF^- plus the DC_μ-Scheme for every cardinal μ.

Theorem (Gitman, Hamkins and Johnstone)

Over ZFC^-, the $\mathrm{DC}_{\aleph_0}$-Scheme is equivalent to the Reflection Principle.

Theorem (Friedman, Gitman and Kanovei)

The Reflection Principle is not provable in ZFC^-.

Proper Classes are Big with Dependent Choice

Theorem (M.)

Suppose that $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$ *for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, which is definable over* V*, there is a subset* b *of* $\mathcal{C}$ *of cardinality* μ*.*

Proof

Corollary

If $\mathrm{V} \models \mathrm{ZF}^-_{\mathrm{DC}_{<\mathrm{ORD}}}$ *then every proper class is big.*

Corollary

If $j : \mathrm{V} \to \mathrm{M}$ *is an elementary embedding then both* $\mathcal{P}(\omega)$ *and* V_κ *are sets.*

An attempt

- Start with a model of ZFC.
- Consider the forcing $\mathbb{P} = \mathrm{Add}(\omega, \textsc{Ord} \times \omega)$ to add $\textsc{Ord}$ many ω blocks of Cohen reals.
- Let $G \subseteq \mathbb{P}$ be generic. Then $\mathrm{M}[G] \models \mathrm{ZFC}^-$.
- Take the symmetric model N such that the blocks form an amorphous proper class.[3]

Assertion

N *is a model of* ZF^- *with an infinite class which doesn't surject onto* ω.

[3] That is an infinite class A such that for any subclass B either B or $A \setminus B$ is finite.

A Contradiction

Theorem

Suppose that $\langle \mathrm{N}, A\rangle$ *satisfies;*

1. N *models* ZF− *in the language with a predicate for* A,
2. $A \subseteq \mathrm{N}$ *and* $\langle \mathrm{N}, A\rangle \models$ *"A is a proper class"*,
3. $\langle \mathrm{N}, A\rangle \models$ *"if $B \subseteq A$ is infinite then B is a proper class"*.

Then the Collection Scheme fails in $\langle \mathrm{N}, A\rangle$. *In fact,* $\langle \mathrm{N}, A\rangle$ *does not have a cumulative hierarchy and therefore the Power Set also fails.*

To prove that the Collection Scheme fails consider the sentence

$$\forall n \in \omega\ \exists y\ (|y| = n\ \wedge\ y \subseteq A).$$

What Does this mean?

- Suppose that $\mathrm{M} \models \mathrm{ZF}^-$.
- A class forcing $\mathbb{P} \in \mathrm{M}$ is *pretame* if for any $\mathbb{P}$-generic G, $\mathrm{M}[G] \models \mathrm{ZF}^-$.
- Adding $\textsc{Ord}$ many Cohen reals is pretame.
- But the Collection Scheme failed in N.
- Therefore the symmetric submodel of a pretame class forcing need not preserve the Collection Scheme.

Remarks

- In fact, it is unclear what the symmetric submodel actually satisfies!
- In Gitik's model where every cardinal in singular, the forcing is pretame (and Power Set fails) but the symmetric submodel satisfies ZF!

Symmetric Class Forcing

- Suppose that $\langle \mathrm{M}, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \rangle$[4] is a model of fourth order ZFC and $\mathbb{P}$ is a pretame class forcing. $(\mathrm{Add}(\omega, \textsc{Ord}))$
- Let $\mathcal{G} \subseteq \mathcal{C}_2$ be a group of order preserving automorphisms of $\mathbb{P}$. (*The automorphisms generated by bijections of* $\textsc{Ord}$)
- Let $\mathcal{K} \in \mathcal{C}_3$ denote the collection of subclasses of $\mathcal{G}$.
- $\mathcal{F} \in \mathcal{C}_3$ is a *normal filter of subgroups of* $\mathcal{G}$ if
  - $\mathcal{F} \subseteq \mathcal{K}$,
  - If $H \in \mathcal{F}$ and $K \in \mathcal{F}$ then $H \cap K \in \mathcal{F}$,
  - If $H \in \mathcal{F}$ and $H \subseteq K$ where $K \in \mathcal{K}$ then $K \in \mathcal{F}$,
  - (Normality) If $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$ then $\pi H \pi^{-1} \in \mathcal{F}$.

  (*The filter generated by fixing finite subsets of* $\textsc{Ord}$)
- We shall then call the triple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a *symmetric system*.

[4] $\mathcal{C}_1$ are the classes, $\mathcal{C}_2$ the hyper-classes and $\mathcal{C}_3$ the hyper-hyper-classes.

The Symmetric Model

Definition

Say that a name $\dot{x}$ is *symmetric* if

$$sym(\dot{x}) := \{\pi \in \mathcal{G} : \pi\dot{x} = \dot{x}\} \in \mathcal{F}.$$

Let $\mathrm{HS}_{\mathcal{F}}$ denote the class of *hereditarily symmetric names.*

Definition

The *symmetric model* given by $\mathcal{F}$ is $\langle \mathrm{N}, \mathcal{C} \rangle$ where

$$\mathrm{N} := \{\dot{x}^G : \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}$$

$$\mathcal{C} := \{\dot{x}^G : \dot{x} \in \mathcal{C}_1^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}.$$

The Symmetry Lemma

Let φ *be a formula,* $p \in \mathbb{P}$, $\pi \in \mathcal{G}$, $\dot{x} \in \mathrm{M}^{\mathbb{P}}$ *and* $\dot{\Gamma} \in \mathcal{C}_1$. *Then*

$$p \Vdash \varphi(\dot{x}, \dot{\Gamma}) \iff \pi p \Vdash \varphi(\pi\dot{x}, \pi\dot{\Gamma}).$$

Replacement

- Let N be the symmetric submodel.
- Suppose that $p \Vdash \dot{f}$ is a total function on $\dot{a}$ where $\dot{f}$ and $\dot{a}$ are hereditarily symmetric names.
- We want a name for the range of f.
- Using collection, we can find some set of hereditarily symmetric names c containing witnesses to elements being in the range of $\dot{f}$.
- Let

$$\dot{b} = \{\langle \dot{y}, s\rangle : \dot{y} \in c \wedge \exists \langle \dot{x}, r\rangle \in \dot{a}\ (s \in d_{\dot{x},r}{}^{5} \wedge s \Vdash \dot{f}(\dot{x}) = \dot{y})\}.$$

- Want to conclude that for any $\pi \in \mathrm{sym}(\dot{a}) \cap \mathrm{sym}(\dot{f})$, $\pi\dot{b} = \dot{b}$.
- However, in general, $\{\pi(\langle \dot{y}, s\rangle) : \pi \in \mathrm{sym}(\dot{a}) \cap \mathrm{sym}(\dot{f})\}$ will not be a set!

[5] These sets are determined using pretameness

Hereditary Respect

Definition

Say that a name $\dot{x}$ is *respected* if $\{\pi \in \mathcal{G} : \mathbb{1} \Vdash \pi\dot{x} = \dot{x}\} \in \mathcal{F}$. Let $\mathrm{HR}_{\mathcal{F}}$ denote the class of *hereditarily respected names*.

Definition

The *respected model* given by $\mathcal{F}$ is $\langle \mathrm{N}, \mathcal{C} \rangle$ where

$$\mathrm{N} := \{\dot{x}^G : \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HR}_{\mathcal{F}}\}$$

$$\mathcal{C} := \{\dot{x}^G : \dot{x} \in \mathcal{C}_1^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HR}_{\mathcal{F}}\}.$$

Remark

If $\dot{a}$ and $\dot{f}$ are hereditarily respected and $\{\pi : \pi p = p\} \in \mathcal{F}$ then so is

$$\dot{b} = \{\langle \dot{y}, s \rangle : \dot{y} \in c \ \wedge \ \exists \langle \dot{x}, r \rangle \in \dot{a} \ (s \in d_{\dot{x},r} \ \wedge \ s \Vdash \dot{f}(\dot{x}) = \dot{y})\}.$$

The Respected Model

Theorem (M.)

Suppose that $\mathbb{M}$ is a model of (fourth-order) GB^-. Let $\mathbb{P}$ be a pretame class forcing and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a tenacious[6] *symmetric system. Then for any $\mathbb{P}$-generic G, the hereditarily respected model $\langle \mathrm{N}, \mathcal{C} \rangle$ is a model of $\mathrm{GB}-$.*

Proposition

Suppose that $\mathbb{M}$ is a model of (fourth-order) GB. Let $\mathbb{P}$ be a pretame class forcing and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ a tenacious symmetric system. Suppose further that for any $\dot{x} \in \mathrm{M}^{\mathrm{HR}}$ and any $H \in \mathcal{F}$, $\{\pi\dot{x} : \pi \in H\} \in \mathrm{M}$.

Then $\dot{x} \in \mathrm{HR}$ iff there is some $\dot{y} \in \mathrm{HS}$ such that $\mathbb{1} \Vdash \dot{x} = \dot{y}$. Therefore $\mathrm{N} = \{\dot{x}^G : \dot{x} \in \mathrm{M}^{\mathbb{P}} \ \wedge \ \dot{x} \in \mathrm{HS}_{\mathcal{F}}\}$.

[6] That is, for every $p \in \mathbb{P}$, $\{\pi \in \mathcal{G} : \pi p = p\} \in \mathcal{F}$.

Union of ZF^- models

Construction (Zarach)

Suppose that $\mathcal{M} = \langle \mathrm{M}, \in \rangle \models \mathrm{ZFC}$, $\mathbb{P} \in \mathrm{M}$, $\omega(\mathbb{P})$ *is the finite support product of* ω *many copies of* N *and* $h : \mathbb{P} \cong \omega(\mathbb{P})$ *be an order isomorphism. Let* G *be* $\mathbb{P}$*-generic over* $\mathcal{M}$ *and* $H = h``G$ *be* $\omega(\mathbb{P})$*-generic. Let* $G_n = H \restriction \{n\}$ *be the* n^{th} *generic and let* $\mathrm{M}_n = \mathrm{M}[G_0 \times \cdots \times G_{n-1}]$*. Consider*

$$\mathrm{N} = \bigcup_n \mathrm{M}_n.$$

Theorem

$\langle \mathrm{N}, \in \mathrm{M} \rangle$ *is a model of* $\mathrm{ZFC}^-_{Ref} + \neg \mathrm{DC}_{|\mathcal{P}^{\mathrm{V}[G]}(\mathbb{P})|^+}$*. In particular,* $\mathcal{P}(\mathbb{P})$ *is a proper class that does not surject onto every ordinal!*

A solution

Theorem

$\langle \mathrm{N}, \in \mathrm{M}\rangle$ is a model of $\mathrm{ZFC}^-_{Ref} + \neg\mathrm{DC}_{|\mathcal{P}^{\mathrm{V}[G]}(\mathbb{P})|^+}$. In particular, $\mathcal{P}(\mathbb{P})$ is a proper class that does not surject onto every ordinal!

Corollary

One can have models V of ZFC^-_{Ref} with an elementary embedding $j : \mathrm{V} \to \mathrm{M}$ for which $\mathcal{P}(\omega)$ is a proper class.

Bibliography

[FGK] Sy-David Friedman, Victoria Gitman and Vladimir Kanovei,
A model of second-order arithmetic satisfying AC but not DC,
Journal of Mathematical Logic, Volume 19, 2019.

[GHJ] Victoria Gitman, Joel David Hamkins and Thomas A. Johnstone,
What is theory ZFC without power set?,
Mathematical Logic Quarterly, Volume 62, 2016

[HK] Peter Holy, Regula Krapf and Philipp Schlicht,
Characterizations of pretameness and the Ord-cc,
Annals of Pure and Applied Logic, Volume 169, 2018.

[Mat] Richard Matthews,
Taking Reinhardt's Power Away,
arxiv preprint arXiv:2009.01127, 2020.

[Zar] Andrezj Zarach,
Unions of ZF^- models which are themselves ZF^- models,
Studies in Logic and the Foundations of Mathematics, Volume 108, 1982.

A note on Injections

Theorem (Monro)

Let $\mathrm{ZF(K)}$ *be the theory with the language of* ZF *plus a one-place predicate* K *and let* M *be a countable transitive model of* ZF*. Then there is a model* N *such that* N *is a transitive model of* $\mathrm{ZF(K)}$ *and*

$$\begin{aligned} \mathrm{N} \quad &\models \quad \mathrm{K} \textit{ is a proper class which is Dedekind-finite} \\ &\qquad \textit{and can be mapped onto the universe.} \end{aligned}$$

Proper Classes Are Big with Reflection

Theorem (M.)

Suppose that $\mathrm{V} \models \mathrm{ZF}^- + \mathrm{DC}_\mu$ *for* μ *an infinite cardinal. Then for any proper class* $\mathcal{C}$*, which is definable over* V*, there is a subset* b *of* $\mathcal{C}$ *of cardinality* μ*.*

Proof

- We shall prove that for any $\nu \leq \mu$ there is a subset b of $\mathcal{C}$ and a bijection between b and ν.
- Suppose not and let δ be the least cardinal for which this fails.
- Let $\varphi(s, y) \equiv (s \cup \{y\} \subseteq \mathcal{C} \ \wedge \ y \notin s)$.
- This satisfies the hypothesis of DC_δ.
- So there is a function f with domain δ and whose range gives a subset of $\mathcal{C}$ of cardinality δ. Contradiction. $\square$

Back