Dependent Choice Schemes in Set Theory without Power Set

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ZFC without power set II: Reflection strikes back

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ZFC without power set

Under Zermelo set theory (ZF without replacement) the following three principles are equivalent:

- The reflection principle (*"any formula reflects to a transitive set."*)
- The collection scheme $(\forall x \in a\, \exists y\, \varphi(x, y) \to \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x, y))$
- The replacement scheme. $(\forall x \in a\, \exists! y\, \varphi(x, y) \to \exists b\, \forall x \in a\, \exists y \in b\, \varphi(x, y))$

However, without power set the reverse implications break down.

Definition

Let $\mathrm{ZF}-$ denote the theory consisting of the following axioms:

- empty set, extensionality, pairing, unions, infinity,
- the foundation scheme, the separation scheme,
- the replacement scheme.

ZFC without power set[1]

Definition

- ZF^- denotes the theory $\mathrm{ZF}-$ plus the collection scheme.
- ZFC^- denotes the theory ZF^- plus the well-ordering principle.

Remarks

- For μ regular, $\mathrm{H}_\mu \models \mathrm{ZFC}^-$.
- Any pretame class forcing over ZFC^- (GB^-) again satisfies ZFC^-.
- Models of $\mathrm{ZFC}-$ can behave very counter-intuitively.

[1] See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

Going wrong without collection[1]

Any of the following can occur in $\mathrm{ZFC}-$ models:

- ω_1 exists and is singular,
- ω_1 exists and every set of reals is countable,
- For every $n \in \omega$ there is a set of reals of size $\aleph_n$ but none of size $\aleph_\omega$,
- The Łoś ultrapower theorem fails,
- Gaifman's Theorem fails (there is a cofinal, Σ_1-elementary map $j\colon M \to N$ which is not fully elementary),
- The class of Σ_1 formulas is not closed under bounded quantification (i.e. φ is Σ_1 but $\forall x \in a\ \varphi$ is not).

Conclusion

All of these problems go away if we also assume collection. So ZFC^- is the "*correct*" way to state ZFC without power set.

[1] See *What is the Theory* ZFC *without Power Set?* by Gitman, Hamkins and Johnstone.

Dependent Choice Schemes

Definition

The DC_δ-scheme: For every formula $\varphi(x, y, a)$, if $\forall x \, \exists y \, \varphi(x, y, a)$ then there is a function f on δ such that $\forall \alpha < \delta \, \varphi(f \restriction \alpha, f(\alpha), a)$.

The $\mathrm{DC}_{<\mathrm{ORD}}$-scheme is the scheme asserting that the DC_δ-scheme holds for every cardinal δ.

Observations

ZFC proves every instance of the $\mathrm{DC}_{<\mathrm{ORD}}$-scheme.
For any regular μ, $\mathrm{H}_\mu \models \mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{ORD}}$-Scheme.

Theorem (Gitman, Hamkins, Johnstone 2016 & Friedman, Gitman, Kanovei 2019)

Over ZFC^-, the DC_ω-scheme is equivalent to the reflection principle.

Big Classes

Definition

A proper class is said to be δ-big if it surjects onto δ.

A proper class is said to be big if it surjects onto every non-zero ordinal.

Observation

Over ZF, every proper class is big.

A note on injections

Proposition

Over ZF^-, the DC_δ-scheme implies that every proper class is δ-big.

Moreover, the $\mathrm{DC}_{<\,\mathrm{ORD}}$-scheme implies that every proper class is big.

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Over ZF, every proper class is big.

A note on injections

Theorem (Friedman, Gitman, Kanovei)

There is a model of ZFC^- (plus every set is countable) in which every proper class is big but the DC_ω-scheme fails.

Taking Reinhardt's Power Away

Theorem (M.)

There is no non-trivial, cofinal, Σ_0-elementary embedding $j\colon \mathrm{W} \to \mathrm{W}$ such that $\mathrm{W} \models \mathrm{ZFC}^-_j$ and either:

- $(\sup\{j^n(\mathrm{crit}(j)) \mid n \in \omega\})^+$ *exists* or
- $\mathrm{V}_{\mathrm{crit}(j)}$ *is a set in* W.

Theorem (M.)

Suppose that $\mathrm{W} \models \mathrm{ZFC}^-$ and every proper class is big in W. If $j\colon \mathrm{W} \to \mathrm{M} \subseteq \mathrm{W}$ is an elementary embedding, then $\mathrm{V}_{\mathrm{crit}(j)}$ is a set in W.

Corollary

There is no non-trivial, cofinal Reinhardt embedding over $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{Ord}}$-scheme.

Motivating Questions

Corollary

There is no non-trivial, cofinal Reinhardt embedding over $\mathrm{ZFC}^- + \mathrm{DC}_{<\mathrm{ORD}}$*-scheme.*

Question 1

Is the following situation consistent: There is a non-trivial, cofinal, elementary embedding $j\colon \mathrm{W} \to \mathrm{W}$ such that $\mathrm{W} \models \mathrm{ZFC}_j^-$?

Question 2

Does ZFC^- prove that every proper class is big?

Question 3

Suppose that $W \models \mathrm{ZFC}_j^-$ for a non-trivial, cofinal $j\colon \mathrm{W} \to \mathrm{M}$ with $\mathrm{M} \subseteq \mathrm{W}$. Does $\mathrm{V}_{\mathrm{crit}(j)}$ exist in W? Does $\mathcal{P}(\omega)$ exist in W?

Zarach's Construction

$$
\begin{array}{ccccccccc}
 & & \mathrm{V}[G_1] & & \mathrm{V}[G_2] & & \mathrm{V}[G_{n+1}] & & \\
 & & \| & & \| & & \| & & \\
\mathrm{V} & \longrightarrow & \mathrm{V}_{[1]} & \longrightarrow & \mathrm{V}_{[2]} & - \cdots - & \mathrm{V}_{[n+1]} & - \cdots - & \mathrm{V}[G] \\
 & & & & & & & & \\
 & \bigcup_m (\mathrm{V}_{[1]})_{[m]} & & \prec & & \bigcup_m (\mathrm{V}_{[n]})_{[m]} & \prec & \bigcup_n \mathrm{V}_{[n]} &
\end{array}
$$

Set-up

- $\mathrm{V} \models \mathrm{ZFC} + \mathrm{CH}$,
- $\mathbb{P} = \mathrm{Add}(\omega, 1) \cong \mathrm{Add}(\omega, \omega)$,
- $G \subseteq \mathrm{Add}(\omega, \omega)$, V-generic,
- $G_n = G \restriction n$ is the restriction of G to the first n coordinates,
- $\mathrm{V}_{[n]} = \mathrm{V}[G_n]$,
- $\mathrm{W}_G^{\mathrm{V}} = \bigcup_{n \in \omega} V[G_n]$.

Zarach's Construction

$$\begin{array}{ccccccccc}
 & \mathrm{V}[G_1] & & \mathrm{V}[G_2] & & \mathrm{V}[G_{n+1}] & & \\
 & \| & & \| & & \| & & \\
\mathrm{V} \longrightarrow & \mathrm{V}_{[1]} & \longrightarrow & \mathrm{V}_{[2]} & - \cdots \longrightarrow & \mathrm{V}_{[n+1]} & - \cdots \longrightarrow & \mathrm{V}[G]
\end{array}$$

$$\bigcup_m (\mathrm{V}_{[1]})_{[m]} \quad \prec \quad \bigcup_m (\mathrm{V}_{[n]})_{[m]} \prec \bigcup_n \mathrm{V}_{[n]}$$

Theorem (Zarach)

1. $\mathrm{W}^{\mathrm{V}}_G \models \mathrm{ZFC}^- + \mathrm{DC}_\omega$-scheme,
2. V and $\mathrm{W}^{\mathrm{V}}_G$ have the same cardinals and cofinalities,
3. $\mathcal{P}(\omega)$ is a small proper class which does not surject onto ω_2,
4. $\mathrm{W}^{\mathrm{V}}_G \models \neg\mathrm{DC}_{\omega_2}$-scheme,
5. (Blass) $\mathrm{W}^{\mathrm{V}}_G \models \neg\mathrm{DC}_{\omega_1}$-scheme.

Embeddings with $\mathcal{P}(\omega)$ a proper class

- Suppose that $\mathrm{V} \models \mathrm{ZFC} + \mathrm{CH}$ and κ is a measurable cardinal with ultrapower embedding $j\colon \mathrm{V} \to \mathrm{M}$.
- Let $G \subseteq \mathrm{Add}(\omega, \omega)$ be generic and construct the union model $\mathrm{W}^{\mathrm{V}}_{G}$ as before.
- Then j lifts to some $j^\star\colon \mathrm{V}[G] \to \mathrm{M}[G]$.
- Let $j_n\colon \mathrm{V}[G_n] \to \mathrm{M}[G_n]$ be the lift given from restricting G to its first n coordinates.
- Let $j^{\mathrm{W}} = \bigcup_n j_n\colon \mathrm{W}^{\mathrm{V}}_{G} \to \mathrm{W}^{\mathrm{M}}_{G}$ be the restriction of $j^\star$.

Theorem (Gitman, M.)

1. $\mathrm{W}^{\mathrm{V}}_{G} \models \mathrm{ZFC}^- + \mathrm{DC}_\omega$*-scheme,*
2. *The* DC_{ω_1}*-scheme fails in* $\mathrm{W}^{\mathrm{V}}_{G}$*,*
3. $\mathcal{P}(\omega)$ *(and therefore* V_κ*) is not a set in* $\mathrm{W}^{\mathrm{V}}_{G}$*,*
4. $\mathrm{W}^{\mathrm{V}}_{G}$ *has a definable elementary embedding with critical point* κ*.*

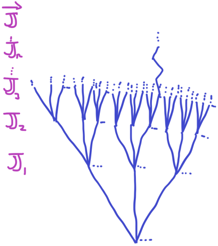
Jensen reals

- Start with a model of $\mathrm{ZFC} + \diamondsuit$ (e.g. L).
- (Jensen 1970) There is a subposet $\mathbb{J}$ of Sacks forcing (perfect trees ordered by $\subseteq$) which:
  - has the ccc,
  - adds a unique generic real.
- (Lyubetsky, Kanovei 2017) Also, we can construct iterations of Jensen forcing $\mathbb{J}_n$ for any $n \in \omega$ satisfying:
  - $\mathbb{J}_n$ has the ccc,
  - If $\langle r_1, \ldots, r_n \rangle$ is a V-generic sequence of reals for $\mathbb{J}_n$ then it is the unique generic sequence for $\mathbb{J}_n$ in $\mathrm{V}[\langle r_1, \ldots, r_n \rangle]$.

Jensen reals on trees

Set-up

- X is any set or class ($X = \textsc{Ord}$).
- $\mathcal{T}$ is a subtree of $X^{<\omega}$ (the tree of finite sequences from X ordered by extension).
- $\vec{\mathbb{J}} = \langle \mathbb{J}_n \mid n \in \omega \rangle$.
- $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ is the tree iteration of Jensen reals along X.



Theorem (Friedman, Gitman, Kanovei)

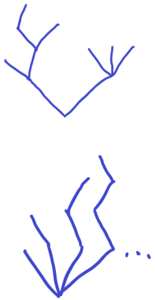
For any set or class X and subtree $\mathcal{T}$ of $X^{<\omega}$,

1. *The poset $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ has the ccc,*
2. *The poset $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ is pretame (so preserves ZFC^-),*
3. *Suppose that $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$ is V-generic. Then the V-generic sequences $\langle r_1, \ldots, r_n \rangle$ for $\mathbb{J}_n$ in $\mathrm{V}[G]$ are precisely the sequences added by nodes of $\mathcal{T}$ on level n.*

Tree union model

Set-up

- Work in L,
- Let $G \subseteq \mathbb{P}(\vec{\mathbb{J}}, \textsc{Ord}^{<\omega})$ be L-generic,
- Let $\mathbb{T}$ consist of all infinite (set) trees $\mathcal{T} \subseteq \textsc{Ord}^{<\omega}$ which do not have a cofinal branch,
- Given $\mathcal{T} \in \mathbb{T}$, let $G_\mathcal{T}$ be the restriction of G to $\mathbb{P}(\vec{\mathbb{J}}, \mathcal{T})$,
- Let $\mathrm{W} = \bigcup_{\mathcal{T}\in\mathbb{T}} \mathrm{L}[G_\mathcal{T}]$.



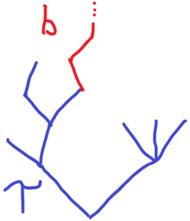
Theorem (Gitman, M.)

1. $\mathrm{W} \models \mathrm{ZFC}^-$ *(+ every cardinal in* L *remains a cardinal in* W*)*,
2. $\mathrm{W} \models \neg\mathrm{DC}_\omega$*-scheme*,
3. *Every proper class over* W *is big.*

DC_ω fails in $\mathrm{W} = \bigcup_{\mathcal{T}\in\mathbb{T}} \mathrm{L}[G_{\mathcal{T}}]$

Idea:

- Consider the definable class tree with domain $\{\vec{r} \mid \vec{r}$ is L-generic for $\mathbb{J}_n$ for some $n\}$ ordered by extension in W,
- This is a tree relation with no terminal nodes.



- Suppose that $b \in \mathrm{W}$ were an infinite branch,
- Then $b \in \mathrm{L}[G_{\mathcal{T}}]$ for some tree $\mathcal{T} \in \mathbb{T}$.
- But $\mathcal{T}$ does not have an infinite branch of Jensen reals. So, for sufficiently large n, $b \restriction n$ is L-generic for $\mathbb{J}_n$ but not in $\mathcal{T}$,
- Contradicting uniqueness of generics for Jensen forcing.

W models ZFC^-

Idea for Collection:

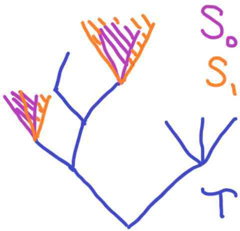
- Suffices to consider instances where we suppose that

$$\mathrm{W} \models \forall \xi \in \delta \, \exists y \, \psi(\xi, y, a).$$

- Let $a \in \mathrm{L}[G_{\mathcal{T}}]$ for some $\mathcal{T} \in \mathbb{T}$.
- For each $\xi < \delta$ there is some tree $\mathcal{S}_\xi \in \mathbb{T}$ such that $\mathrm{W} \models \exists y \in \mathrm{L}[G_{\mathcal{S}_\xi}] \psi(\xi, y, a)$.
- The aim is to find some tree $\mathcal{R} \in \mathbb{T}$ such that for each $\xi < \delta$, $\mathrm{W} \models \exists y \in \mathrm{L}[G_{\mathcal{R}}] \psi(\xi, y, a)$.
- But $\bigcup_{\xi < \delta} \mathcal{S}_\xi$ may contain a cofinal branch.

An automorphism π of $\mathbb{P}(\vec{\mathbb{J}}, \mathrm{ORD}^{<\omega})$ is said to be tree-switching if it is generated by an automorphism of $\mathrm{ORD}^{<\omega}$.

- Since $\bigcup_{\xi<\delta} \mathcal{S}_\xi$ may contain a cofinal branch, we want to find a sequence of tree-switching isomorphisms π_ξ such that
  - $\mathrm{W} \models \exists y \in \mathrm{L}[G_{\pi_\xi``\mathcal{S}_\xi}]\psi(\xi, y, a)$ and
  - $\pi_\xi``\mathcal{S}_\xi$ and $\pi_\eta``\mathcal{S}_\eta$ are disjoint modulo $\mathcal{T}$ for any $\xi, \eta < \delta$.
- However, we can't determine these automorphisms in the ground model and therefore it need not be the case that $\bigcup_{\xi<\delta} \pi_\xi``\mathcal{S}_\xi \in \mathbb{T}$ (in L).
- Instead we use the ccc to construct in L countable sequences of trees $\langle \mathcal{S}_\xi^{(\alpha)} \mid \alpha < \beta_\xi \rangle$ which do the job.



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- However, we can't determine these automorphisms in the ground model and therefore it need not be the case that $\bigcup_{\xi<\delta} \pi_\xi``\mathcal{S}_\xi \in \mathbb{T}$ (in L).
- Instead we use the ccc to construct in L countable sequences of trees $\langle \mathcal{S}_\xi^{(\alpha)} \mid \alpha < \beta_\xi \rangle$ which do the job.
- Namely, taking $\mathcal{T}_\xi = \bigcup_{\alpha<\beta_\xi} \mathcal{S}_\xi^{(\alpha)}$,
    - $\mathrm{W} \models \forall \xi < \delta \, \exists y \in \mathrm{L}[G_{\mathcal{T}_\xi}]\psi(\xi, y, a)$,
    - For any $\xi, \eta < \delta$, $\mathcal{T}_\xi$ and $\mathcal{T}_\eta$ are disjoint modulo $\mathcal{T}$,
    - $\mathcal{R} := \bigcup_{\xi<\delta} \mathcal{T}_\xi \in \mathbb{T} \cap \mathrm{L}$.
- Then for each $\xi < \delta$, $\mathrm{W} \models \exists y \in \mathrm{L}[G_{\mathcal{R}}]\psi(\xi, y, a)$ so we can use collection over this (set) forcing extension.

Closing Remarks

- Instead of doing a class forcing we could instead take λ regular, consider the forcing $\mathbb{P}(\vec{\mathbb{J}}, \lambda^{<\omega})$ and work over $\mathrm{H}_\lambda^{\mathrm{L}}$.
- By combining the model with a generalisation of Zarach's union model, we can also get models of ZFC^- in which the DC_ω-scheme fails and in which there are proper classes that are not big.

Open Question

Is the following situation consistent: There is a non-trivial, cofinal, elementary embedding $j\colon \mathrm{W} \to \mathrm{W}$ such that $\mathrm{W} \models \mathrm{ZFC}_j^-$?

A note on injections

Back

Theorem (Monro)

Let $\mathrm{ZF}(\mathrm{K})$ *be the theory with the language of* ZF *plus a one-place predicate* K *and let* M *be a countable transitive model of* ZF. *Then there is a model* N *such that* N *is a transitive model of* $\mathrm{ZF}(\mathrm{K})$ *and*

$$\begin{aligned}\mathrm{N} \quad \models \quad & \mathrm{K} \textit{ is a proper class which is Dedekind-finite} \\ & \textit{and can be mapped onto the universe.}\end{aligned}$$