A Graphical Characterization of Structurally Controllable Linear Systems with Dependent Parameters

F. Liu and A. S. Morse

Abstract—One version of the concept of structural controllability defined for single-input systems by Lin and subsequently generalized to multi-input systems by others, states that a parameterized matrix pair (A,B) whose nonzero entries are distinct parameters, is structurally controllable if values can be assigned to the parameters which cause the resulting matrix pair to be controllable. In this paper the concept of structural controllability is broadened to allow for the possibility that a parameter may appear in more than one location in the pair (A,B). Subject to a certain condition on the parameterization called the "binary assumption", an explicit graph-theoretic characterization of such matrix pairs is derived.

Index Terms—Linear time-invariant systems, structural controllability, graph theory.

I. Introduction

Over the past few years there has been a resurgence of interest in the question of structural controllability posed by Lin in 1974 [2], which aims to capture the controllability of systems with parameters whose values are not exactly known but only approximately determined. As defined by Lin, a pair of matrices $(A_{n\times n},b_{n\times 1})$ with each entry either a fixed zero or a distinct scalar parameter is structurally controllable if there is a real matrix pair $(\bar{A}_{n\times n}, \bar{b}_{n\times 1})$ with the same pattern of zero entries as (A, b) which is controllable. Thus if (A, b)is structurally controllable, then almost every real matrix pair (\bar{A}, \bar{b}) with the same pattern of zero entries as (A, b) will be controllable. Lin was able to give an explicit graph-theoretic condition for such a matrix pair to be structurally controllable in terms of properties of a suitably defined directed graph determined by the given matrix pair. Lin's result was extended to multi-input matrix pairs $(A_{n\times n}, B_{n\times m})$ in linear algebra terms by Shields and Pearson [3] and reexplained in graph theory terms by Mayeda [4]. Generic properties and design problems of Lin's parameterization of the pair (A, B) were studied in [5]–[12]. Results on structural controllability of linear time-varying systems were presented in [13]-[21]. One line of research deals with the structural controllability of composite systems [22]-[26]. Current interest stems from the realization that structural controllability is a key property of interest in swarming behavior and in the modeling and understanding complex networks [27]–[35]. For example, identification, characterization, and classification of driver vertices or steering vertices in biomedical networks [36]-[43], which tend

This work was supported by National Science Foundation grant n. 1607101.00 and US Air Force grant n. FA9550-16-1-0290.

A short version [1] of this paper which omitted proofs of all lemmas was presented at the 56th IEEE Conference on Decision and Control in December, 2017.

to have strong ability to influence other vertices, may enlighten us on critical underlying relations or mechanisms; the study of robustness of structural controllability to vertex and/or arc failures and disturbances [44]–[49] may give us an insight in the evolution of complex social networks and various issues of network security [50], [51].

In previous work, there was also interest in structural controllability for more general kinds of parameterizations [52]–[55]. In particular, the notion of a "linearly parameterized" matrix pair was introduced in [52] which allowed parameters to appear in multiple locations in the pair (A,B). The engineering motivation for linear parameterization came from physical systems with unknown but dependent design parameters involving the imprecise values of their physical components [56] or measurements [53].

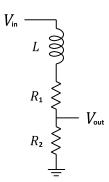


Fig. 1: A voltage divider circuit.

For example, the transfer function of the voltage divider circuit in Figure 1 is

$$T(s) = \frac{V_{\mathrm{out}}(s)}{V_{\mathrm{in}}(s)} = \frac{R_2}{R_1 + R_2 + sL} \label{eq:total_total_total}$$

A one-dimensional realization of the transfer function is

$$\dot{x} = -\frac{R_1 + R_2}{L} \ x + \frac{R_2}{L} \ V_{\rm in}, \qquad V_{\rm out} = x \label{eq:vout}$$

So for this system, $A = -\frac{R_1}{L} - \frac{R_2}{L}$ and $B = \frac{R_2}{L}$. Note that $\frac{R_2}{L}$ appears in both A and B. Suppose the exact values of the physical components R_1 , R_2 , and L are unknown, let $p_1 = \frac{R_1}{L}$ and $p_2 = \frac{R_2}{L}$, then A and B are linear functions of the two independent parameters p_1 and p_2 .

In this paper, we address this same kind of parameterization satisfying a certain condition called the "binary assumption" and show by counterexample that this is the most general class of linear parameterizations for which one can expect a graphical characterization with unweighted graphs. Finally, the structural controllability of this class of linear parameterizations is characterized in strict graph-theoretic terms, which provides a guide to designing and analyzing complex networks with coupled links.

F. Liu and A. S. Morse are with the Department of Electrical Engineering, Yale University, New Haven, CT 06511 USA (e-mail: fengjiao.liu@yale.edu; as.morse@yale.edu).

A. Linear Parameterization

Interesting as the results of Lin's parameterization are, they cannot address many simple but commonly encountered modeling situations such as when A and b are of the forms

$$A = \begin{bmatrix} p_1 & p_1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \tag{1}$$

where at least one parameter, in this example p_1 , appears in more than one location. Recognition of this led to the definition of a "linearly parameterized" matrix pair and to a significant generalization of the concept of structural controllability [52]. The version of a *linearly parameterized* matrix pair to which we are referring is of the form¹

$$A_{n\times n}(p) = \sum_{k\in\mathbf{q}} g_k p_k h_{k1}, \quad B_{n\times m}(p) = \sum_{k\in\mathbf{q}} g_k p_k h_{k2}$$
 (2)

where $p \in \mathbb{R}^q$ is a vector of q > 0 algebraically independent parameters $p_1, p_2, \ldots, p_q, \mathbf{q} \triangleq \{1, 2, \ldots, q\}$, and for each $k \in \mathbf{q}, g_k \in \mathbb{R}^n, h_{k1} \in \mathbb{R}^{1 \times n}, h_{k2} \in \mathbb{R}^{1 \times m}$. In this context, the problem of interest is to find conditions for the existence of a parameter vector $p \in \mathbb{R}^q$ for which (A(p), B(p)) is a controllable matrix pair. If such values exist, the parameterized pair (A(p), B(p)) is structurally controllable. Such pairs are controllable for almost every value of p in the sense that the set of values of p for which (A, B) is controllable is the complement of a proper algebraic set in \mathbb{R}^q .

Necessary and sufficient conditions for such a matrix pair to be structurally controllable in this more generalized sense are developed in [52]. Like the work of Shields and Pearson [3], these conditions are primarily matrix-algebraic. A special form of linearly parameterized matrix pairs can be used to model compartmental systems and corresponds to compartmental graphs, on which graphical conditions for the structural controllability of matrix pairs in this special form have been investigated [53]. Other types of parameterization have also been explored, but either purely algebraically [54], [55], or without equivalent graphical conditions [57]. Since graphical results can reveal important structural properties hidden in matrix representations, there is interest in determining graphical conditions characterizing the generalized concept of structural controllability and this is the specific problem which this paper is addressed.

Before proceeding we point out that not every matrix pair (A,B) with parameters entering "linearly" is a linear parameterization as defined here. For example, while the matrix pair shown in (1) is linearly parameterized, the matrix pair

$$A = \begin{bmatrix} p_1 & p_1 \\ 0 & p_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ p_2 \end{bmatrix} \tag{3}$$

is not. It is claimed that a matrix pair (A, B) whose entries depend linearly on q parameters p_1, p_2, \ldots, p_q will be linearly parameterized if and only if all minors of the partitioned matrix $[A \ B]$ are multilinear functions of the q parameters. It is clear that the matrices in (3) do not have this property. To see why the claim is true, let (A, B) be a linearly parameterized matrix

pair and fix the values of all parameters except for p_k . If p_k appears in only one row or column of a square submatrix of $[A\ B]$, its determinant is a linear function of p_k . If p_k appears in more rows and columns of a square submatrix, it must enters the matrix in a rank-one fashion, as rank $(g_k [h_{k1}\ h_{k2}]) = 1$. So by adding scalar multiples of one row that contains p_k to other rows containing p_k , it is possible to get another square matrix of the same determinant, with p_k appearing in only one row. Therefore all minors of $[A\ B]$ are multilinear functions of the q parameters. The statement in other direction can be easily proved by its contrapositive.

In the sequel it will be convenient to use the $n \times (n+m)$ partitioned matrix $[A(p) \ B(p)]$. In view of (2), this matrix can be expressed as

$$[A(p) \ B(p)] = \sum_{k \in \mathbf{q}} g_k p_k h_k \tag{4}$$

where $h_k \triangleq [h_{k1} \ h_{k2}]$ for $k \in \mathbf{q}$. It will be assumed for simplicity and without loss of generality that the set of matrices $\{g_1h_1,g_2h_2,\ldots,g_qh_q\}$ is linearly independent. To justify this assumption, suppose that the set is not linearly independent and for purposes of illustration that g_qh_q is a linear combination of the remaining matrices $g_1h_1,g_2h_2,\ldots,g_{(q-1)}h_{(q-1)}$. In other words, suppose that

$$g_q h_q = \sum_{k=1}^{q-1} c_k g_k h_k$$

where the c_k are real numbers. Then in view of (4),

$$[A(p) \ B(p)] = \sum_{k=1}^{q-1} g_k(p_k + c_k p_q) h_k$$

Therefore if we define new parameters $\bar{p}_k = p_k + c_k p_q$ for $k \in \{1, 2, \dots, q-1\}$, then the right side of (4) can be written using only the first q-1 matrices in $\{g_1h_1, g_2h_2, \dots, g_qh_q\}$ as

$$\sum_{k=1}^{q-1} g_k \bar{p}_k h_k$$

It is clear from this that the process of defining new parameters and eliminating dependent matrices from $\{g_1h_1,g_2h_2,\ldots,g_qh_q\}$ can be continued until one has a linearly independent subset of matrices. This justifies our claim and accordingly it will henceforth be assumed that $\{g_1h_1,g_2h_2,\ldots,g_qh_q\}$ is a linearly independent set. This implies that $q \leq n(n+m)$.

Since this paper deals with matrix pairs parameterized by p exclusively, it is convenient to drop p in A(p) and B(p), i.e., to write A and B instead of A(p) and B(p), with the understanding that (A,B) is parameterized by p.

B. Graph of (A, B)

It is easy to see that the definition of structural controllability for a linearly parameterized matrix pair (A,B) coincides with Lin's if m=1 and the g_k and h_k are restricted to be unit vectors in \mathbb{R}^n and $\mathbb{R}^{1\times (n+1)}$ respectively. Lin defines the graph of such a matrix pair to be an unweighted directed graph

¹Although written differently, this linear parameterization is in fact the same as the one considered in [52] except that in [52] there are constant matrices A_0 and B_0 also appearing in the sums in (2) for A(p) and B(p) respectively.

on n+1 vertices labeled 1 to n+1 with an arc from vertex j to vertex i if the ijth entry in the matrix $[A \ B]$ is a parameter. For the more general linear parameterization defined by (4), a more elaborate definition of a graph is needed not just because m might be greater than 1, but also because some parameter p_k may appear in multiple locations in $[A \ B]$.

The graph of (A, B), written \mathbb{G} , is defined to be an unweighted directed graph with n+m vertices labeled 1 through n+m and an arc of $\operatorname{color}^2 k$ from vertex j to vertex i if the ijth entry in the matrix $g_k h_k$ is nonzero, i.e., the ijth entry in the partitioned matrix $[A \ B]$ contains p_k . In the sequel, $(j,i)_k$ denotes an arc from vertex j to vertex i with ith color ith. This graph has ith colors.

Figure 2 shows the graph of

$$\left(\begin{bmatrix}
0 & 0 & 0 & p_1 \\
p_5 & 0 & 0 & 0 \\
0 & 0 & p_3 & p_3 \\
0 & 0 & p_4 & 0
\end{bmatrix}, \begin{bmatrix}
0 & p_1 \\
p_2 & p_2 \\
0 & 0 \\
p_4 & 0
\end{bmatrix}\right)$$
(5)

where symbol (k) labels color k for k = 1, 2, ..., 5.

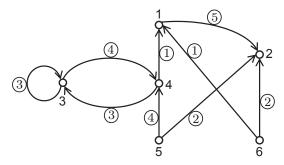


Fig. 2: The graph of the matrix pair in (5).

Note that the graph of (A,B) has three properties: (i) There is no arc pointing toward any of the m vertices with labels n+1 to n+m, since the matrix $[A\ B]$ only has n rows. (ii) There may be more than one arc from one given vertex j to another vertex i, for the ijth entry in the matrix $[A\ B]$ may be a linear combination of more than one parameter. If this is the case, all arcs from vertex j to vertex i will have distinct colors. (iii) If there are two arcs of color $k \in \mathbf{q}$, one leaving vertex j and the other pointing toward vertex i, then there must be an arc $(j,i)_k$. This is because the two given arcs imply that the jth entry in the row vector h_k and the ith entry in the column vector g_k are nonzero, which means the ijth entry in the matrix $g_k h_k$ is nonzero. Any unweighted directed graph on n+m vertices which has these properties is called a $structural\ controllability\ graph$.

C. Binary Assumption

This paper focuses exclusively on linear parameterizations which satisfy a certain "binary assumption". Specifically, the linear parameterization defined by (4) satisfies the *binary assumption* if all of the g_k and h_k appearing in (4) are binary vectors, i.e., vectors of 1's and 0's. Similarly, a linear parameterization satisfies the *unitary assumption* if all of the

 g_k and h_k appearing in (4) are unit vectors. So any linear parameterization satisfying the unitary assumption also satisfies the binary assumption. Lin's parameterization is exactly the linear parameterization satisfying the unitary assumption.

It is quite clear that when the binary assumption holds with n and m specified, the parameterization in (4) is uniquely determined by a structural controllability graph. Because of this, it is possible to characterize the structural controllability of a linearly parameterized matrix pair (A,B) which satisfies the binary assumption, solely in terms of the graph of the pair. On the other hand, without the binary assumption, no such graphical characterization³ is possible. The following example illustrates this.

Note that although the matrix pairs

$$\left(\begin{bmatrix} p_1 & p_1 & p_2 \\ p_1 & p_1 & p_2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix} \right) \text{ and } \left(\begin{bmatrix} p_1 & p_1 & p_2 \\ p_1 & p_1 & 2p_2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix} \right)$$

both have the same graph, only the pair on the right is structurally controllable. Of course the pair on the right does not satisfy the binary assumption.

D. Problem Formulation and Organization

This paper gives necessary and sufficient graph-theoretic conditions for the structural controllability of a linearly parameterized matrix pair $(A_{n\times n}, B_{n\times m})$ which satisfies the binary assumption. To the best of our knowledge, this is the first graph-theoretic result that generalizes the conditions given in [2] and [4]. The rest of the paper is organized as follows. The terminology and concepts used in this paper are defined in Section II. The main result of this paper is presented in Section III and proved in Section IV.

II. PRELIMINARIES

In order to state the main result of this paper, some terminology and a number of graphical and algebraic concepts are needed.

A. Terminology

Let $\mathbb H$ be an unweighted directed graph with a vertex set $\mathcal V$ and an arc set $\mathcal A$. An *induced subgraph* of $\mathbb H$ by a subset of vertices $\mathcal U\subset\mathcal V$ is a subgraph of $\mathbb H$, whose vertex set is $\mathcal U$ and whose arc set is $\{(i,j)|i,j\in\mathcal U,(i,j)\in\mathcal A\}$. For any subset $\mathcal U\subset\mathcal V$, $\mathcal V-\mathcal U$ is the complement of $\mathcal U$ in $\mathcal V$. A *source vertex* in $\mathbb H$ is a vertex with no incoming arc and a *sink vertex* in $\mathbb H$ is a vertex with no outgoing arc. An *isolated vertex* is both a source vertex and a sink vertex. A *partition* π of $\mathcal V$ is a family of nonempty subsets of $\mathcal V$ which are pairwise disjoint and whose union is equal to $\mathcal V$. The *quotient graph* of $\mathbb H$ induced by π , written $\mathbb H/\pi$, is an unweighted directed graph with one vertex for each cell of π , and exactly one arc from vertex i to vertex j whenever $\mathbb H$ has at least one arc from

²In this paper, each color is labeled by a distinct integer.

³If the binary assumption is dropped, one way to proceed is to define the graph of (A,B) as a weighted directed graph, in which the weight of an arc $(j,i)_k$ is the ijth entry in the matrix g_kh_k . The conditions on weighted graphs for the structural controllability of all linearly parameterized matrix pairs will be studied in a sequel of this paper.

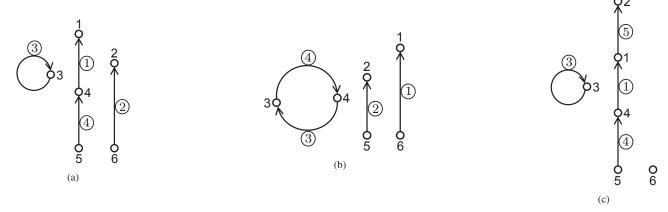


Fig. 3: Multi-colored subgraphs of the graph in Figure 2.

the vertices in the *i*th cell to the vertices in the *j*th cell. \mathbb{H}/π is the *condensation* of \mathbb{H} if π is formed by the collection of strongly connected components of \mathbb{H} .

A directed path graph is a weakly connected [58] graph whose vertices can be labeled in the order 1 to k for some $k \in$ IN such that the arcs are (i, i + 1), where $i = 1, 2, \dots, k - 1$. The *length* of a directed path graph is the number of arcs in it. So in a directed path graph of positive length, the first vertex has exactly one outgoing arc, the last vertex has exactly one incoming arc, and each of the other vertices in between has exactly one incoming arc and one outgoing arc. In this context, a directed path graph of length 0 is an isolated vertex. A directed cycle graph is a strongly connected [58] graph whose vertices can be labeled in the order 1 to k for some $k \in \mathbb{N}$ such that the arcs are (i, i + 1) and (k, 1), where $i=1,2,\ldots,k-1$. So in a directed cycle graph, each vertex has exactly one incoming arc and one outgoing arc. One vertex with a single self-loop is also a directed cycle graph. As this paper is concerned with directed graphs only, a directed path graph and a directed cycle graph will be simply called a path graph and a cycle graph, respectively, in the rest of the paper. The disjoint union of two or more graphs is the union of these graphs whose vertex sets are disjoint. A directed graph is rooted if it contains at least one vertex r called a root with the property that for each remaining vertex v there is a directed path from r to v. Rooted directed graphs arise naturally in the study of consensus problems [59]. A directed rooted tree is a rooted directed graph which is also a weakly connected tree [58]. H has a spanning forest if it has a spanning subgraph [58] which is the disjoint union of directed rooted trees. Let $\mathcal{V}_{\text{root}} \subset \mathcal{V}$ be the set of root vertices of the trees. With a slight abuse of terminology, we will say that $\mathbb H$ has a spanning forest rooted at the vertices in \mathcal{V}_{root} if and only if for each vertex $v \in \mathcal{V} - \mathcal{V}_{\text{root}}$, there is a path to v from one of the root vertices.

B. Graphical Concepts

A multi-colored subgraph of a structural controllability graph \mathbb{G} is a spanning subgraph of \mathbb{G} , which is the disjoint union of m path graphs and any number of cycle graphs with each arc in the union graph of a different color. Clearly, a multi-colored subgraph of \mathbb{G} has n arcs that do not share

colors, start vertices, or end vertices. Figure 3 shows three multi-colored subgraphs of the graph in Figure 2.

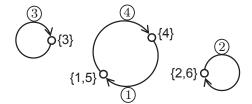
A multi-colored subgraph $\mathbb S$ of a structural controllability graph $\mathbb G$ is obtained by sequentially removing arcs from $\mathbb G$ as follows. First pick any arc $(a,b)_{k_1}$ in $\mathbb G$ and then remove any other arcs with the same color k_1 as well as any arcs other than $(a,b)_{k_1}$ pointing toward vertex b and/or leaving vertex a. Next, from the set of arcs which remain after these removals, pick any arc $(c,d)_{k_2}$ and repeat the process until no further arc picking is possible. If a total of n arcs are left, the graph which remains is $\mathbb S$. Clearly, $\mathbb S$ is not unique. In the sequel, $\mathcal R(\mathbb G)$ denotes the set of all multi-colored subgraphs of $\mathbb G$.

Suppose $\mathbb G$ is the graph of a linearly parameterized matrix pair (A,B). It is possible that $\mathbb G$ does not have any multicolored subgraph, that is, $\mathcal R(\mathbb G)$ is an empty set. As the n arcs in a multi-colored subgraph have n distinct colors, n different start vertices and n different end vertices, $\mathcal R(\mathbb G)=\emptyset$ if and only if there are no n distinct parameters appearing in n different rows and n different columns of the partitioned matrix $[A\ B]$. If so, rank $[A\ B] < n$ for any $p \in \mathbb R^q$, as each parameter enters $[A\ B]$ in a rank-one fashion. Then the pair (A,B) is not structurally controllable.

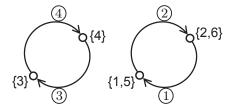
The source vertices (respectively, sink vertices) of a multi-colored subgraph $\mathbb S$ are the m source vertices (respectively, sink vertices) of the path graphs in $\mathbb S$. It is not hard to see that the source vertices of every multi-colored subgraph of $\mathbb G$ are the m vertices with labels n+1 to n+m, since there is no arc pointing toward any of them. But the sink vertices of a multi-colored subgraph may be any m vertices in $\mathbb G$.

Two multi-colored subgraphs $\mathbb{S}_1, \mathbb{S}_2 \in \mathcal{R}(\mathbb{G})$ are called *similar* if \mathbb{S}_1 and \mathbb{S}_2 have the same m sink vertices and the same set of n colors. Graph similarity is an equivalence relation on $\mathcal{R}(\mathbb{G})$. The corresponding equivalence classes induced by this relation are called *similarity classes*.

As an example of this concept, let \mathbb{G} be the graph in Figure 2. Let \mathcal{E}_1 be the similarity class of multi-colored subgraphs with sink vertices 1 and 2, and colors 1, 2, 3, 4. Figure 3a and Figure 3b show the two multi-colored subgraphs in \mathcal{E}_1 . Figure 3c shows a multi-colored subgraph in the similarity class \mathcal{E}_2 with sink vertices 2 and 6, and colors 1, 3, 4, 5. In fact, this graph is the only multi-colored subgraph of \mathbb{G} in \mathcal{E}_2 .



(a) The quotient graph of the graph in Figure 3a.



(b) The quotient graph of the graph in Figure 3b.

Fig. 4

Specific quotient graphs of the multi-colored subgraphs in the same similarity class will be used to define an important property of the class. Let \mathcal{V} be the vertex set of a structural controllability graph \mathbb{G} . For any subset $\mathcal{U} \subset \mathcal{V}$, let $|\mathcal{U}|$ be the number of elements in \mathcal{U} . Let $\mathcal{V}_{\text{source}} \subset \mathcal{V}$ be the set of m source vertices of every multi-colored subgraph of \mathbb{G} . Let $\mathcal{V}_{\text{sink}} \subset \mathcal{V}$ be the set of m sink vertices of a given multicolored subgraph \mathbb{S} of \mathbb{G} . So $\mathcal{V}_{\text{source}} \cap \mathcal{V}_{\text{sink}}$ is the set of isolated vertices in \mathbb{S} and $|\mathcal{V}_{\text{source}} - (\mathcal{V}_{\text{source}} \cap \mathcal{V}_{\text{sink}})| = |\mathcal{V}_{\text{sink}} - (\mathcal{V}_{\text{source}} \cap \mathcal{V}_{\text{source}})|$ $|\mathcal{V}_{\text{sink}}| \leq m$. The desired quotient graph of S is induced by a "matrimonial partition". A partition π of $\mathcal V$ is a matrimonial partition for S if it pairs each vertex in $\mathcal{V}_{\text{source}} - (\mathcal{V}_{\text{source}} \cap$ $\mathcal{V}_{sink})$ with a different vertex in $\mathcal{V}_{sink}-(\mathcal{V}_{source}\cap\mathcal{V}_{sink})$ and assigns each pair to a different cell, then assigns each of the rest vertices in V to a new cell. So there are $|V| - |V_{\text{source}}|$ $(\mathcal{V}_{\text{source}} \cap \mathcal{V}_{\text{sink}})$ cells in π and each of them has at most two vertices. If the pairing is not unique, π is not unique.

An observation made by comparing $\mathbb S$ and the quotient graph $\mathbb S/\pi$ is that the cycle graphs and isolated vertices in $\mathbb S$ remain the same in $\mathbb S/\pi$, while the path graphs with positive lengths in $\mathbb S$ are, roughly speaking, "welded" together to form new cycle graphs in $\mathbb S/\pi$. In the sequel, it is assumed that the quotient graphs of all multi-colored subgraphs in one similarity class are induced by the same matrimonial partition.

For example, $\pi = \{\{1,5\}, \{2,6\}, \{3\}, \{4\}\}\}$ is a matrimonial partition for the two multi-colored subgraphs in Figure 3a and Figure 3b. The quotient graphs of the two graphs induced by π are shown in Figure 4a and Figure 4b, respectively.

A multi-colored subgraph is *odd* (respectively, *even*) if its quotient graph induced by a matrimonial partition has an odd (respectively, even) number of cycle graphs. As will be stated in Lemma 4, the choice of the matrimonial partition does not affect the relative parity of two multi-colored subgraphs in the same similarity class as long as their quotient graphs are induced by the same partition, where parity is the property of being odd or even. A similarity class of multi-colored subgraphs is *balanced* if the numbers of odd and even multi-colored subgraphs in the similarity class are equal. Otherwise, it is *unbalanced*. This important property of a similarity class is regardless of which matrimonial partition is chosen.

From Figure 4, one knows that the graph in Figure 3a is odd and the graph in Figure 3b is even, so the similarity class \mathcal{E}_1 is balanced. The similarity class \mathcal{E}_2 is unbalanced as it only has one multi-colored subgraph.

A "cactus graph" introduced by Lin is a weakly connected graph consisting of one "trunk" and any number of "buds". A *trunk* is a path graph with at least one vertex. A *bud* consists of

one cycle graph with at least one vertex and one additional arc called the bud's stem which is incident to one of the cycle's vertices. A cactus graph is then a weakly connected graph with exactly one trunk and any non-negative number of buds with the understanding that the stem of each bud comes out of either a vertex on the trunk or a vertex on the cycle of another bud in the graph. In this context a path graph is a cactus graph with no bud. Some definition of a cactus graph requires that the stem of a bud cannot come out of the last vertex on the trunk, but the definition in this paper does not, because it does not matter. The graphical condition involving cactus graphs is always that the original graph has a spanning subgraph which is a cactus graph or a disjoint union of cactus graphs. If the original graph has a spanning subgraph that is a cactus graph with one bud whose stem comes out of the last vertex on the trunk, the original graph must also have a spanning subgraph that is a cactus graph with no bud, obtained by removing a specific arc in the cycle of the bud, which points toward the same vertex as the stem of the bud does. So both definitions work. Note that a cactus graph has a unique root vertex. The condensation of a cactus graph which results when all cycles are condensed into vertices is a directed rooted tree.

Figure 5 gives an example of a cactus graph with five buds.

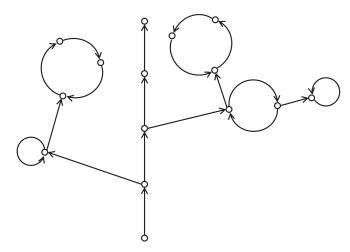


Fig. 5: A cactus graph.

C. Algebraic Concepts

The generic rank of a linearly parameterized matrix

$$M(p) = \sum_{k \in \mathbf{q}} g_k p_k h_k \tag{6}$$

denoted by $\operatorname{grk} M$, is the maximum rank of M that can be achieved as p varies over \mathbb{R}^q . It is generic in the sense that it is achievable by any p in the complement of a proper algebraic set in \mathbb{R}^q . Generalizing the standard notion of irreducibility, a matrix pair (A,B) is said to be *irreducible* if there is no permutation matrix Π bringing (A,B) into the form

$$\Pi A \Pi^{-1} = \begin{bmatrix} A_1 & \mathbf{0} \\ A_2 & A_3 \end{bmatrix}, \quad \Pi B = \begin{bmatrix} \mathbf{0} \\ B_1 \end{bmatrix}$$

where A_1 is an $n_1 \times n_1$ block, B_1 is an $(n - n_1) \times m$ block, $1 \le n_1 < n$.

Proposition 1: [4] A linearly parameterized matrix pair (A,B) is irreducible if and only if the graph of (A,B) has a spanning forest rooted at the m vertices with labels n+1 to n+m.

Although Proposition 1 was initially developed for matrix pairs satisfying the unitary assumption, the same proof applies to all linearly parameterized matrix pairs without change. Therefore a proof of Proposition 1 will not be given here.

III. MAIN RESULT

The following classical result characterizes the structural controllability of linearly parameterized matrix pairs satisfying the unitary assumption.

Proposition 2: [2]–[4] Let (A, B) be a linearly parameterized matrix pair which satisfies the unitary assumption. The following statements are equivalent.

- (i) The pair (A, B) is structurally controllable.
- (ii) $grk[A \ B] = n$ and (A, B) is irreducible.
- (iii) The graph of (A, B) has a spanning subgraph which is a disjoint union of m cactus graphs rooted at the m vertices with labels n+1 to n+m, respectively.

The graphical conditions in Proposition 2 is equivalent to the graphical conditions given in [27] for structural controllability. A "maximum matching" defined in [27] is a maximum-cardinality set of arcs that do not share start vertices or end vertices. It will be called a *nonstandard maximum matching* in the rest of the paper because it differs from the standard definition of maximum matching, i.e., a maximum-cardinality set of arcs that do not share vertices, in the sense that a nonstandard matching allows the start vertex of an arc to be the end vertex of another arc, but a standard matching does not. Let (A,B) be a linearly parameterized matrix pair which satisfies the unitary assumption. The following statements are equivalent.

- (i) $\operatorname{grk}[A \ B] = n$.
- (ii) The graph of (A,B) has a spanning subgraph which is a disjoint union of m path graphs and any number of cycle graphs.
- (iii) The graph of (A,B) has a nonstandard maximum matching of size n.

The equivalence of (i) and (ii) is given by Lemma 2 in [4]. The equivalence of (i) and (iii) is established as follows. It is possible to represent the graph $\mathbb G$ of (A,B) by a bipartite graph $\mathbb B$ such that each vertex i of $\mathbb G$ becomes two vertices i^+ and i^- in $\mathbb B$ and each arc (j,i) of $\mathbb G$ corresponds to an arc (j^+,i^-) in $\mathbb B$. Lemma 1 in [60] implies that $\operatorname{grk} [A\ B]=n$ if and only if $\mathbb B$ has a standard maximum matching of size

n. It is easy to see that a standard maximum matching in \mathbb{B} corresponds to a nonstandard maximum matching in \mathbb{G} . So (i) and (iii) are equivalent. Between the two graphical conditions for generic rank, (ii) is easier to visualize in \mathbb{G} and to combine with the graphical condition for irreducibility.

The following theorem, which is the main result of this paper, shows how the graphical condition in Proposition 2 changes when the unitary assumption is relaxed to the binary assumption.

Theorem 1: Let (A, B) be a linearly parameterized matrix pair which satisfies the binary assumption. The following statements are equivalent.

- (i) The pair (A, B) is structurally controllable.
- (ii) grk $[A \ B] = n$ and (A, B) is irreducible.
- (iii) The graph of (A, B) has an unbalanced similarity class of multi-colored subgraphs and has a spanning subgraph which is a disjoint union of m cactus graphs rooted at the m vertices with labels n+1 to n+m, respectively.
- (iv) The graph of (A, B) has an unbalanced similarity class of multi-colored subgraphs and has a spanning forest rooted at the m vertices with labels n+1 to n+m.

When subject to the unitary assumption, Theorem 1 reduces to Proposition 2. To understand why this is so, let \mathbb{G} be the graph of a matrix pair (A,B) which satisfies the unitary assumption. As no two arcs of \mathbb{G} are of the same color, \mathbb{G} has an unbalanced similarity class of multi-colored subgraphs if and only if \mathbb{G} has a multi-colored subgraph, which can be obtained by removing the stems of all buds in the m cactus graphs. So condition (iii) in Theorem 1 reduces to condition (iii) in Proposition 2.

As an example of Theorem 1, the matrix pair given in (5) is structurally controllable because the graph in Figure 2 satisfies condition (iv).

IV. ANALYSIS

This section focuses on the analysis and proof of Theorem 1, in which the equivalence of statements (i) and (ii) is proved first, followed by the equivalence of statements (ii) and (iv), and then that of statements (iii) and (iv).

A. Proof of Theorem 1, $(i) \iff (ii)$

Apparently, if a linearly parameterized matrix pair (A,B) is structurally controllable, (A,B) is irreducible and $\operatorname{grk}[A\ B]=n$. We will prove the converse. To do that, some concepts and certain result from [52] are summarized as they apply to the proof. It is worth pointing out that the concepts and the result in [52] do not require the binary assumption.

Suppose $S = \{i_1, i_2, \dots, i_k\} \subset \mathbf{q}$ with $i_1 < i_2 < \dots < i_k$. Let matrices G_S , H_S and P_S be

$$G_{\mathcal{S}} \triangleq \begin{bmatrix} g_{i_1} & g_{i_2} & \dots & g_{i_k} \end{bmatrix}, \quad H_{\mathcal{S}} \triangleq \begin{bmatrix} h_{i_1} \\ h_{i_2} \\ \vdots \\ h_{i_k} \end{bmatrix}$$

If $S = \emptyset$, G_S , H_S and P_S are each the 0×0 matrix. The complement of S in \mathbf{q} is denoted by $\mathbf{q} - S$. Note that the linear parameterization $\sum_{i \in \mathbf{q}} g_i p_i h_i$ is exactly $G_{\mathbf{q}} P_{\mathbf{q}} H_{\mathbf{q}}$.

The transfer matrix of $\{\widetilde{G}_{\mathbf{q}}, H_{\mathbf{q}}\}$, denoted by T, is a block matrix with q row partitions and q+1 column partitions defined as

$$T_{i,j} = \begin{cases} h_{i1}g_j, & i, j \in \mathbf{q} \\ h_{i2}, & i \in \mathbf{q}, j = 0 \end{cases}$$

where $T_{i,j}$ is the ijth block of T, $g_j \in \mathbb{R}^n$, $h_{i1} \in \mathbb{R}^{1 \times n}$ and $h_{i2} \in \mathbb{R}^{1 \times m}$. The transfer graph of $\{G_{\mathbf{q}}, H_{\mathbf{q}}\}$, written \mathbb{T} , is the graph of the transfer matrix T and is defined to be an unweighted directed graph with q+1 vertices labeled $0, 1, \ldots, q$ and an arc from vertex j to vertex i whenever $T_{i,j}$ is nonzero. The following proposition is derived from Theorem 1 in [52] with constant matrices $A_0 = \mathbf{0}$ and $B_0 = \mathbf{0}$. It is applicable to any linearly parameterized matrix pair with or without the binary assumption.

Proposition 3: [52] A linearly parameterized matrix pair (A, B) given by (4) is structurally controllable if and only if

$$\min_{\mathcal{S} \subset \mathbf{q}} (\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q} - \mathcal{S}}) = n \tag{7}$$

and the transfer graph of $\{G_{\mathbf{q}}, H_{\mathbf{q}}\}$ has a spanning tree rooted at vertex 0.

In addition to Proposition 3, three lemmas are needed to prove the equivalence of statements (i) and (ii) in Theorem 1. More specifically, Lemma 2 and Lemma 3 draw a connection between Proposition 3 and statement (ii). The following concepts and Lemma 1 are the key ideas for proving Lemma 2. Among the three lemmas, Lemma 1 and Lemma 2 hold without the binary assumption, but Lemma 3 needs the binary assumption.

Suppose we are given two real matrices

$$G_{n_1 \times k} = \begin{bmatrix} g_1 & g_2 & \dots & g_k \end{bmatrix}, \quad H_{k \times n_2} = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{bmatrix}$$

Let $\mathbf{k} \triangleq \{1, 2, \dots, k\}$. Let $\mathcal{P} \triangleq \{(g_i, h_i) \mid i \in \mathbf{k}\}$ be the set of k pairs of vectors. For $\mathcal{I} \subset \mathbf{k}$, a nonempty subset $\{(g_i, h_i) \mid i \in \mathcal{I}\}$ $\mathcal{I} \subset \mathcal{P}$ is jointly independent if $\{g_i \mid i \in \mathcal{I}\}$ and $\{h_i \mid i \in \mathcal{I}\}$ are both linearly independent sets. That is,

$$\operatorname{rank} G_{\mathcal{I}} = \operatorname{rank} H_{\mathcal{I}} = |\mathcal{I}|$$

where $|\mathcal{I}|$ is the cardinality of \mathcal{I} , i.e., the number of elements in \mathcal{I} . Then \mathcal{I} is called a *jointly independent index set* of (G, H). Let $\mathcal{J}(G, H)$ be the set of all jointly independent index sets of (G, H).

Lemma 1: For a linearly parameterized matrix M given by (6),

$$\max_{\mathcal{I} \in \mathcal{J}(G_{\mathbf{q}}, H_{\mathbf{q}})} |\mathcal{I}| = \min_{\mathcal{S} \subset \mathbf{q}} (\mathrm{rank} \; G_{\mathcal{S}} + \mathrm{rank} \; H_{\mathbf{q} - \mathcal{S}})$$

Proof of Lemma 1: Let $\mathcal{M}_1 = \{\mathbf{q}, \mathcal{J}_1\}$ and $\mathcal{M}_2 = \{\mathbf{q}, \mathcal{J}_2\}$ be two finite matroids, where \mathbf{q} is the ground set; \mathcal{J}_1 is the family of the independent sets of \mathbf{q} defined by the linear independence relation of $\{g_i | i \in \mathbf{q}\}$, i.e., $\mathcal{S}_1 \in \mathcal{J}_1$ if and only if $\{g_i | i \in \mathcal{S}_1\}$ is a linearly independent set; \mathcal{J}_2 is the

family of the independent sets of \mathbf{q} defined by the linear independence relation of $\{h_i \mid i \in \mathbf{q}\}$, i.e., $\mathcal{S}_2 \in \mathcal{J}_2$ if and only if $\{h_i \mid i \in \mathcal{S}_2\}$ is a linearly independent set. Let r_1 and r_2 be the rank functions of \mathcal{M}_1 and \mathcal{M}_2 , respectively. Naturally, $\forall \mathcal{S}_1 \subset \mathbf{q}$, $r_1(\mathcal{S}_1) = \operatorname{rank} G_{\mathcal{S}_1}$, and $\forall \mathcal{S}_2 \subset \mathbf{q}$, $r_2(\mathcal{S}_2) = \operatorname{rank} H_{\mathcal{S}_2}$. By the matroid intersection theorem [61],

$$\max_{\mathcal{I} \in \mathcal{J}_1 \cap \mathcal{J}_2} |\mathcal{I}| = \min_{\mathcal{S} \subset \mathbf{q}} (r_1(\mathcal{S}) + r_2(\mathbf{q} - \mathcal{S}))$$

That is,

$$\max_{\mathcal{I} \in \mathcal{J}(G_{\mathbf{q}}, H_{\mathbf{q}})} |\mathcal{I}| = \min_{\mathcal{S} \subset \mathbf{q}} \left(\mathrm{rank} \ G_{\mathcal{S}} + \mathrm{rank} \ H_{\mathbf{q} - \mathcal{S}} \right)$$

Therefore, Lemma 1 is true.

Lemma 2: For a linearly parameterized matrix M given by (6),

$$\operatorname{grk} M = \min_{\mathcal{S} \subset \mathbf{q}} \left(\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q} - \mathcal{S}} \right)$$

Proof of Lemma 2: Let \mathcal{I} be a jointly independent index set of $\{G_{\mathbf{q}}, H_{\mathbf{q}}\}$ with the maximum cardinality. Let $p_i = 1$ if $i \in \mathcal{I}$ and $p_i = 0$ if $i \notin \mathcal{I}$. Then $M = G_{\mathcal{I}}H_{\mathcal{I}}$. As rank $G_{\mathcal{I}} = \operatorname{rank} H_{\mathcal{I}} = |\mathcal{I}|$, rank $G_{\mathcal{I}}H_{\mathcal{I}} = |\mathcal{I}|$. So

$$\operatorname{grk} M \geq \operatorname{rank} G_{\mathcal{I}} H_{\mathcal{I}} = |\mathcal{I}|$$

For any $S \subset \mathbf{q}$, $G_{\mathbf{q}}P_{\mathbf{q}}H_{\mathbf{q}} = G_{\mathcal{S}}P_{\mathcal{S}}H_{\mathcal{S}} + G_{\mathbf{q}-\mathcal{S}}P_{\mathbf{q}-\mathcal{S}}H_{\mathbf{q}-\mathcal{S}}$. So

$$\operatorname{rank} G_{\mathbf{q}} P_{\mathbf{q}} H_{\mathbf{q}} \leq \operatorname{rank} G_{\mathcal{S}} P_{\mathcal{S}} H_{\mathcal{S}} + \operatorname{rank} G_{\mathbf{q}-\mathcal{S}} P_{\mathbf{q}-\mathcal{S}} H_{\mathbf{q}-\mathcal{S}}$$

 $\leq \operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q}-\mathcal{S}}$

holds for all $p \in \mathbb{R}^q$, $S \subset \mathbf{q}$. It follows by varying p over \mathbb{R}^q on the left side of the inequality and by varying S over the power set of \mathbf{q} on the right side of the inequality that

grk
$$M \leq \min_{\mathcal{S} \subset \mathbf{q}} (\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q} - \mathcal{S}})$$

By Lemma 1, $|\mathcal{I}| = \min_{\mathcal{S} \subset \mathbf{q}} (\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q}-\mathcal{S}})$. So

grk
$$M = |\mathcal{I}| = \min_{\mathcal{S} \subset \mathbf{q}} (\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q} - \mathcal{S}})$$

Therefore, Lemma 2 is true.

Corollary 1: For real matrices $G_{n_1 \times k}$ and $H_{k \times n_2}$,

$$\mathrm{rank}\; GH \leq \max_{\mathcal{I} \in \mathcal{J}(G,H)} |\mathcal{I}|$$

Proof of Corallary 1: By Lemma 1 and Lemma 2,

$$\operatorname{grk} GP_{\mathbf{k}}H = \max_{\mathcal{I} \in \mathcal{J}(G,H)} |\mathcal{I}|$$

So

$$\operatorname{rank} GH = \operatorname{rank} GIH \leq \operatorname{grk} GP_{\mathbf{k}}H$$
$$= \max_{\mathcal{I} \in \mathcal{J}(G,H)} |\mathcal{I}|$$

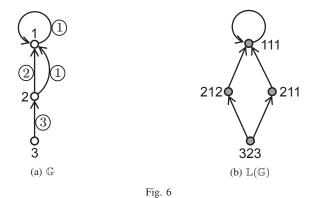
Therefore, Corallary 1 is true.

Corollary 1 gives a tighter upper bound on rank GH than min {rank G, rank H }.

The concept of "line graph" is useful for proving Lemma 3. The *line graph* of a given structural controllability graph \mathbb{G} , written $\mathbb{L}(\mathbb{G})$, is an unweighted directed graph that has one vertex for each arc of \mathbb{G} , for example a vertex ijk for an arc $(i,j)_k$ in \mathbb{G} , and has an arc from vertex abk_1 to vertex bck_2

if \mathbb{G} has arcs $(a,b)_{k_1}$ and $(b,c)_{k_2}$. That is, each arc in $\mathbb{L}(\mathbb{G})$ represents a length-two walk [58] in \mathbb{G} .

Figure 6 gives an example of a structural controllability graph and its line graph.



Lemma 3: Let (A,B) be a linearly parameterized matrix pair given by (4), which satisfies the binary assumption. If (A,B) is irreducible, the transfer graph of $\{G_{\mathbf{q}},H_{\mathbf{q}}\}$ has a spanning tree rooted at vertex 0.

Proof of Lemma 3: For clarity, let v_i denote vertex i in the graph \mathbb{G} of (A,B) and let w_i denote vertex i in the transfer graph \mathbb{T} of $\{G_{\mathbf{q}},H_{\mathbf{q}}\}$. Let $\mathcal{B}\triangleq\{i\in\mathbf{q}\,|\,h_{i2}\neq\mathbf{0}\}$. In other words, p_i appears in B if and only if $i\in\mathcal{B}$. By definition, there is an arc in \mathbb{T} from vertex w_0 to vertex w_i for each $i\in\mathcal{B}$. For $i,j\in\mathbf{q}$, there is an arc in \mathbb{T} from w_i to w_j if $T_{j,i}=h_{j1}g_i\neq0$. As h_{j1} and g_i are binary vectors, $h_{j1}g_i\neq0$ if and only if $\exists\,k\in\{1,2,\ldots,n\}$ such that the kth entry of h_{j1} is one and the kth entry of g_i is also one. So \mathbb{T} has an arc from w_i to w_j if and only if \mathbb{G} has an arc of color i pointing toward v_k and an arc of color j leaving v_k .

Let $\widehat{\mathbb{T}}$ be the subgraph of \mathbb{T} induced by vertices w_1, w_2, \ldots, w_q . Remember that the line graph $\mathbb{L}(\mathbb{G})$ has one vertex for each arc of \mathbb{G} . Let π be the partition of the vertices of $\mathbb{L}(\mathbb{G})$ such that the vertices for the arcs of \mathbb{G} in the same color are in the same cell of the partition. Obviously, the quotient graph $\mathbb{L}(\mathbb{G})/\pi$ has q vertices. For each $i \in \mathbf{q}$, let u_i denote vertex i in $\mathbb{L}(\mathbb{G})/\pi$, which corresponds to the arcs of \mathbb{G} with color i. Then $\mathbb{L}(\mathbb{G})/\pi$ and $\widehat{\mathbb{T}}$ are isomorphic with the bijection that maps vertex u_i in $\mathbb{L}(\mathbb{G})/\pi$ to vertex w_i in $\widehat{\mathbb{T}}$.

If (A,B) is irreducible, by Proposition 1, $\mathbb G$ has a spanning forest rooted at the m vertices v_{n+1}, v_{n+2}, \ldots , and v_{n+m} . So $\mathbb L(\mathbb G)$ has a spanning forest rooted at the vertices for the arcs of $\mathbb G$ leaving v_{n+1}, v_{n+2}, \ldots , or v_{n+m} . The isomorphism of $\mathbb L(\mathbb G)/\pi$ and $\widehat{\mathbb T}$ implies that $\widehat{\mathbb T}$ has a spanning forest rooted at the vertices in the set $\{w_i \mid i \in \mathcal B\}$. Since the transfer graph $\mathbb T$ has an arc from w_0 to w_i for each $i \in \mathcal B$, $\mathbb T$ has a spanning tree rooted at w_0 .

Proof of Theorem 1, (i) \iff (ii): Apparently, (i) \implies (ii). If (ii) is true, by Lemma 2 and Lemma 3,

$$\min_{\mathcal{S} \subset \mathbf{q}} (\operatorname{rank} G_{\mathcal{S}} + \operatorname{rank} H_{\mathbf{q} - \mathcal{S}}) = n$$

and the transfer graph of $\{G_{\mathbf{q}}, H_{\mathbf{q}}\}$ has a spanning tree rooted at vertex 0. By Proposition 3, (i) is true. So (i) \iff (ii).

B. Proof of Theorem 1, (ii) \iff (iv)

Two lemmas are needed to prove the equivalence of statements (ii) and (iv). Lemma 4 implies that the balance or unbalance of a similarity class of multi-colored subgraphs is an intrinsic property, regardless of which matrimonial partition is chosen. It facilitates the understanding of Lemma 5, which converts the generic rank condition into a graphical condition.

For the proofs of Lemma 4 and Lemma 5, some bases of permutation and determinant are needed. Let \mathcal{S}_n be the set of all permutations of the set $\mathbf{n} \triangleq \{1,2,\ldots,n\}$. Let $\sigma \in \mathcal{S}_n$ be one such permutation which maps $i \in \mathbf{n}$ to $\sigma(i) \in \mathbf{n}$. σ is *odd* (respectively, *even*) if $\sigma(1), \sigma(2), \ldots, \sigma(n)$ can be transformed into $1, 2, \ldots, n$ by an odd (respectively, even) number of two-element swaps. The *signature* of σ , denoted by $\operatorname{sgn}(\sigma)$, takes value from $\{1, -1\}$ such that $\operatorname{sgn}(\sigma) = 1$ if σ is even, and $\operatorname{sgn}(\sigma) = -1$ if σ is odd. Each permutation in \mathcal{S}_n can be decomposed into the product of disjoint cycles. Let c be the number of disjoint cycles that σ can be decomposed into, then σ is odd (respectively, even) if n-c is odd (respectively, even). The composition of two permutations with the same parity (respectively, opposite parities) is an even (respectively, odd) permutation.

One definition of the determinant of an $n \times n$ square matrix M is

$$\det M = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i \in \mathbf{n}} m_{i,\sigma(i)}$$
 (8)

where $m_{i,j}$ is the *ij*th entry of M.

If $M_{n\times n}$ is linearly parameterized as given by (6), the graph of M, written \mathbb{G}_M , is an unweighted directed graph with n vertices labeled 1 to n and an arc $(j,i)_k$ if the ijth entry in M contains p_k . So \mathbb{G}_M is exactly the subgraph induced in the graph of the pair $(M,\mathbf{0})$ by the n vertices with labels 1 to n. With the binary assumption, each entry of M is either zero, one parameter or the sum of distinct parameters. After the products of entries in (8) are expanded, each term in $\det M$ is a signed product of n parameters. As no two of the n parameters in a term are taken from the same row or the same column of M, each term in $\det M$ corresponds to a spanning subgraph of \mathbb{G}_M with n arcs, which is a disjoint union of finite number of cycle graphs. The following proposition is derived from Theorem 2 in [62] and will be used to prove Lemma 4.

Proposition 4: [62] Let M be an $n \times n$ linearly parameterized matrix satisfying the binary assumption, whose graph is denoted by \mathbb{G}_M . The sign of a term in $\det M$ is positive if n-c is even, and is negative if n-c is odd, where c is the number of cycle graphs in the corresponding spanning subgraph of \mathbb{G}_M .

For an $n \times n$ linearly parameterized matrix M which satisfies the binary assumption, a term in $\det M$ is valid if it contains n distinct parameters. Since $\det M$ is a multilinear function of p_1, p_2, \ldots, p_q , only valid terms appear in the final expression of $\det M$. That is,

$$\det M = \sum_{\substack{\mathcal{C} \subset \mathbf{q} \\ |\mathcal{C}| = n}} a_{\mathcal{C}} \prod_{k \in \mathcal{C}} p_k \tag{9}$$

where $a_{\mathcal{C}} \in [-n!, n!]$ is the integer coefficient of the product of the n distinct parameters labeled by elements of \mathcal{C} .

Let \mathbb{G} be the graph of a linearly parameterized matrix pair (A, B) which satisfies the binary assumption. By replacing any m columns, such as columns t_1, t_2, \ldots, t_m , of the partitioned matrix [A B] with 0, we get another $n \times (n+m)$ matrix $[\widehat{A} \ \widehat{B}]$. The graph of the pair $(\widehat{A}, \widehat{B})$, denoted by $\widehat{\mathbb{G}}$, is then a spanning subgraph of G which results when all the arcs leaving vertices $t_1, t_2, \ldots,$ or t_m are removed from \mathbb{G} . Let M be the $n \times n$ submatrix obtained by deleting columns t_1 , t_2, \ldots, t_m of [A B]. Each valid term in det M has n distinct parameters and no two of them are taken from the same row or the same column of $[A \ B]$. So each valid term in $\det M$ corresponds to a spanning subgraph of \mathbb{G} with n arcs in ndistinct colors and with no two arcs pointing toward the same vertex or leaving the same vertex, which is a multi-colored subgraph of \mathbb{G} with sink vertices t_1, t_2, \ldots, t_m . Therefore, each valid term in the determinant of an $n \times n$ submatrix of $[A \ B]$ corresponds to a multi-colored subgraph of \mathbb{G} . Valid terms which are in the determinant of the same submatrix and which contain the same n distinct parameters correspond to multi-colored subgraphs of \mathbb{G} in the same similarity class.

Lemma 4: Let $\mathbb G$ be the graph of a linearly parameterized matrix pair (A,B) which satisfies the binary assumption. The relative parity of two multi-colored subgraphs of $\mathbb G$ in the same similarity class remains unchanged regardless of which matrimonial partition induces their quotient graphs.

Proof of Lemma 4: Suppose \mathbb{G} has a similarity class \mathcal{E} of multi-colored subgraphs with sink vertices t_1, t_2, \ldots, t_m . Let

$$\mathcal{L} \triangleq \{1, 2, \dots, n+m\} - \{t_1, t_2, \dots, t_m\}$$

Note that $|\mathcal{L}| = n$. Let M be the $n \times n$ submatrix obtained by deleting columns t_1, t_2, \ldots, t_m of $[A \ B]$. Let

$$f_0 \colon \mathbf{n} \to \mathcal{L}$$

be the bijection such that for each $i \in \mathbf{n}$, the ith column of M is taken from the $f_0(i)$ th column of $[A\ B]$. Let π be the matrimonial partition that induces the quotient graphs of all multi-colored subgraphs in \mathcal{E} . For each $i \in \mathcal{L} \cap \{n+1,n+2,\ldots,n+m\}$, vertex i is a source vertex but not a sink vertex. So vertex i shares a cell of π with a sink vertex, denoted by vertex t_π^i . Let

$$f_{\pi}:\mathcal{L} \to \mathbf{n}$$

be the bijection such that $f_{\pi}(i) = t_{\pi}^{i}$ if $i \in \mathcal{L} \cap \{n+1, n+2, \ldots, n+m\}$ and $f_{\pi}(i) = i$ if $i \in \mathcal{L} \cap \mathbf{n}$.

Rearrange columns of M to get another $n \times n$ matrix \overline{M} such that for each $i \in \mathbf{n}$, the ith column of M is the $f_\pi(f_0(i))$ th column of \overline{M} . Let z_1 be a valid term in $\det M$, which corresponds to a multi-colored subgraph \mathbb{S}_1 in \mathcal{E} . Let $\sigma_1 \in \mathcal{S}_n$ be the permutation associated with term z_1 . That is, each parameter in z_1 is taken from a location in the ith row and the $\sigma_1(i)$ th column of M for some $i \in \mathbf{n}$. So the sign of z_1 is $\mathrm{sgn}\ (\sigma_1)$. Term z_1 naturally pairs with a valid term in $\det \overline{M}$, denoted by \overline{z}_1 . To be precise, if a parameter in z_1 is taken from the location in the ith row and the $\sigma_1(i)$ th column of M, \overline{z}_1 has the same parameter taken from the location in the ith row and the $f_\pi(f_0(\sigma_1(i)))$ th column of \overline{M} . Let $\overline{\sigma}_1 \in \mathcal{S}_n$

be the permutation associated with term \bar{z}_1 . So $\bar{\sigma}_1 = f_{\pi} f_0 \sigma_1$ and the sign of \bar{z}_1 is sgn $(\bar{\sigma}_1)$. As $f_{\pi} f_0 \in \mathcal{S}_n$,

$$\operatorname{sgn}(\bar{\sigma}_1) = \operatorname{sgn}(f_{\pi}f_0) \operatorname{sgn}(\sigma_1)$$

Let c_1 be the number of cycle graphs in the quotient graph \mathbb{S}_1/π . Let \mathbb{Q}_1 be the subgraph of \mathbb{S}_1/π obtained by removing all the isolated vertices, if any, from \mathbb{S}_1/π . So \mathbb{Q}_1 is the disjoint union of c_1 cycle graphs. It can be checked that \mathbb{Q}_1 has n vertices and n arcs in n distinct colors. In fact, \mathbb{Q}_1 is exactly the spanning subgraph of $\mathbb{G}_{\overline{M}}$ that term \bar{z}_1 in det \overline{M} corresponds to. By Proposition 4, $\operatorname{sgn}(\bar{\sigma}_1)=1$ if $n-c_1$ is even, and $\operatorname{sgn}(\bar{\sigma}_1)=-1$ if $n-c_1$ is odd. It means that c_1 is even if $\operatorname{sgn}(\bar{\sigma}_1)=(-1)^n$, and c_1 is odd if $\operatorname{sgn}(\bar{\sigma}_1)=-(-1)^n$. So \mathbb{S}_1 is even if

$$\operatorname{sgn}(f_{\pi}f_0)\operatorname{sgn}(\sigma_1) = (-1)^n$$

and \mathbb{S}_1 is odd if

$$sgn (f_{\pi}f_0) sgn (\sigma_1) = -(-1)^n$$

Let z_2 be another valid term in det M, which corresponds to a multi-colored subgraph \mathbb{S}_2 in \mathcal{E} . Let $\sigma_2 \in \mathcal{S}_n$ be the permutation associated with term z_2 . So the sign of z_2 is $\operatorname{sgn}(\sigma_2)$. Similarly, \mathbb{S}_2 is even if

$$\operatorname{sgn}(f_{\pi}f_0)\operatorname{sgn}(\sigma_2) = (-1)^n$$

and S_2 is odd if

$$\operatorname{sgn}(f_{\pi}f_0)\operatorname{sgn}(\sigma_2) = -(-1)^n$$

Therefore, the relative parity of S_1 and S_2 in \mathcal{E} only depends on the relative sign of z_1 and z_2 . If the two valid terms have the same sign, their corresponding multi-colored subgraphs have the same parity, and vice versa.

Lemma 5: For a linearly parameterized matrix pair (A, B) which satisfies the binary assumption,

$$grk[A \ B] = n$$

if and only if the graph of (A, B) has an unbalanced similarity class of multi-colored subgraphs.

Proof of Lemma 5: When the binary assumption holds, $\operatorname{grk}[A \ B] = n$ if and only if there exists an $n \times n$ submatrix of $[A \ B]$, written M, such that $\operatorname{grk} M = n$. By (9), $\operatorname{grk} M = n$ if and only if $\exists \ \mathcal{C} \subset \mathbf{q}, \ |\mathcal{C}| = n$ such that $a_{\mathcal{C}} \neq 0$. As each valid term in $\det M$ is a signed product of n distinct parameters, $a_{\mathcal{C}} \neq 0$ if and only if the number of positive valid terms $\prod_{k \in \mathcal{C}} p_k$ and the number of negative valid terms $-\prod_{k \in \mathcal{C}} p_k$ are not equal. By the proof of Lemma 4, a positive valid term $\prod_{k \in \mathcal{C}} p_k$ and a negative valid term $-\prod_{k \in \mathcal{C}} p_k$ correspond to two multi-colored subgraphs with opposite parities in the same similarity class. So $a_{\mathcal{C}} \neq 0$ if and only if the similarity class is unbalanced. Therefore, $\operatorname{grk}[A \ B] = n$ if and only if the graph of (A, B) has an unbalanced similarity class of multi-colored subgraphs.

Proof of Theorem 1, (ii) \iff **(iv):** By Lemma 5 and Proposition 1, (ii) \iff (iv).

The following lemma makes the proof of the equivalence of statements (iii) and (iv) fairly straightforward.

Lemma 6: Let \mathbb{G} be a directed graph on n+m vertices. Then \mathbb{G} has a spanning subgraph which is a disjoint union of m cactus graphs rooted at m distinct vertices if and only if \mathbb{G} has two spanning subgraphs: One is a spanning forest rooted at the same m vertices; The other is a disjoint union of m path graphs and a non-negative number of cycle graphs, where the source vertices are the m root vertices of the cactus graphs.

Note that Lemma 6 has no requirement on the color of arcs. **Proof of Lemma 6:** The necessity is obvious. Let us prove the sufficiency. Let $\mathbb U$ be a spanning subgraph of $\mathbb G$, which is the disjoint union of m path graphs and c cycle graphs. If c=0, $\mathbb U$ is already a disjoint union of m cactus graphs with no bud. Now assume c>0.

Let \mathcal{V} be the vertex set of \mathbb{G} . Let $\mathcal{V}_0 \subset \mathcal{V}$ be the set of vertices in the m path graphs of \mathbb{U} . Let $\mathcal{V}_{\text{root}} \subset \mathcal{V}_0$ be the set of source vertices of the m path graphs in \mathbb{U} . For each $i \in \mathbf{c} \triangleq \{1, 2, \dots, c\}$, let $\mathcal{V}_i \subset \mathcal{V}$ be the set of vertices in the ith cycle graph of \mathbb{U} . So

$$\mathcal{V} = \bigcup_{i=0}^{c} \mathcal{V}_i$$

and

$$\mathcal{V}_i \cap \mathcal{V}_j = \emptyset, \ i \neq j, \ i, j \in \{0, 1, \dots, c\}$$

Since $\mathbb G$ has a spanning forest rooted at the m vertices in $\mathcal V_{\rm root}$, there exists an arc in $\mathbb G$ from a vertex in $\mathcal V_0$ to a vertex in $\mathcal V_{i_1}$ for some $i_1 \in \mathbf c$. Otherwise, there is no path to the vertices in $\mathcal V - \mathcal V_0$ from any root vertex. Let $\mathcal V_0^1 \triangleq \mathcal V_0 \cup \mathcal V_{i_1}$. Similarly, there exists another arc in $\mathbb G$ from a vertex in $\mathcal V_0^1$ to a vertex in $\mathcal V_{i_2}$ for some $i_2 \in \mathbf c - \{i_1\}$, otherwise there is no path to the vertices in $\mathcal V - \mathcal V_0^1$ from any root vertex. The process continues until one finds c arcs in $\mathbb G$ that connect $\mathcal V_0, \mathcal V_1, \ldots, \mathcal V_c$. The addition of the c arcs to $\mathbb U$ renders a disjoint union of m cactus graphs rooted at the m vertices in $\mathcal V_{\rm root}$.

Proof of Theorem 1, (iii) \iff (iv): Obviously, (iii) \implies (iv). If the graph $\mathbb G$ of (A,B) has an unbalanced similarity class of multi-colored subgraphs, $\mathbb G$ has at least one multi-colored subgraph. So a spanning subgraph of $\mathbb G$ is the disjoint union of m path graphs and a non-negative number of cycle graphs, where the source vertices are the m vertices with labels n+1 to n+m. By Lemma 6, (iv) \implies (iii). Therefore, (iii) \iff (iv).

V. CONCLUSION

This paper extends the graph-theoretic conditions for structural controllability to the class of linearly parameterized matrix pairs satisfying the binary assumption. As a byproduct of the analysis, Corollary 1 presents a tighter upper bound on the rank of a matrix product than the minimum rank of the matrices in the product. If one wants to further extend the graph-theoretic conditions to all linearly parameterized matrix pairs, weighted graphs of matrix pairs must be introduced. To accommodate this, some graphical concepts will have to be modified accordingly, such as quotient graph, multicolored subgraph, balanced or unbalanced similarity class of

multi-colored subgraphs, and line graph. Some future research problems are: (1) to show that it is NP-hard to determine whether the graph of (A,B) has an unbalanced similarity class of multi-colored subgraphs; (2) to find the minimum number of input required for the structural controllability of a given linearly parameterized matrix $A_{n\times n}$; (3) to study the structural controllability of linearly parameterized linear time-varying systems; (4) to eventually generalize the definition and the corresponding characterizations of structural controllability to nonlinear systems for which there is a good understanding of controllability.

REFERENCES

- F. Liu and A. S. Morse, "Structural controllability of linear systems," in *Proc. IEEE Conf. Decision Control*, Melbourne, Australia, 2017, pp. 3588–3593.
- [2] C.-T. Lin, "Structural controllability," IEEE Trans. Autom. Control, vol. 19, no. 3, pp. 201–208, 1974.
- [3] R. W. Shields and J. B. Pearson, "Structural controllability of multi-input linear systems," *Rice University ECE Technical Report*, no. TR7502, 1975.
- [4] H. Mayeda, "On structural controllability theorem," *IEEE Trans. Autom. Control*, vol. 26, no. 3, pp. 795–798, 1981.
- [5] C.-T. Lin, "System structure and minimal structure controllability," *IEEE Trans. Autom. Control*, vol. 22, no. 5, pp. 855–862, 1977.
- [6] S. Hosoe, "Determination of generic dimensions of controllable subspaces and its application," *IEEE Trans. Autom. Control*, vol. 25, no. 6, pp. 1192–1196, 1980.
- [7] J.-M. Dion, C. Commault, and J. van der Woude, "Generic properties and control of linear structured systems: a survey," *Automatica*, vol. 39, no. 7, pp. 1125–1144, 2003.
- [8] S. Maza, C. Simon, and T. Boukhobza, "Impact of the actuator failures on the structural controllability of linear systems: a graph theoretical approach," *IET Control Theory & Applications*, vol. 6, no. 3, pp. 412– 419, 2012.
- [9] P. J. Zufiria, L. Úbeda-Medina, C. Herrera-Yagüe, and I. Barriales-Valbuena, "Mathematical foundations for efficient structural controllability and observability analysis of complex systems," *Mathematical Problems in Engineering*, vol. 2014, 2014.
- [10] A. Olshevsky, "Minimum input selection for structural controllability," in *Proc. Amer. Control Conf.*, Chicago, IL, USA, 2015, pp. 2218–2223.
- [11] S. Pequito, S. Kar, and A. P. Aguiar, "A framework for structural input/output and control configuration selection in large-scale systems," *IEEE Trans. Autom. Control*, vol. 61, no. 2, pp. 303–318, 2016.
- [12] Y. Zhang and T. Zhou, "On the edge insertion/deletion and controllability distance of linear structural systems," in *Proc. IEEE Conf. Decision Control*, Melbourne, Australia, 2017, pp. 2300–2305.
- [13] S. Poljak, "On the gap between the structural controllability of timevarying and time-invariant systems," *IEEE Trans. Autom. Control*, vol. 37, no. 12, pp. 1961–1965, 1992.
- [14] C. Hartung, G. Reißig, and F. Svaricek, "Necessary conditions for structural and strong structural controllability of linear time-varying systems," in *Proc. Eur. Control Conf.*, Zurich, Switzerland, 2013, pp. 17–19
- [15] X. Liu, H. Lin, and B. M. Chen, "Structural controllability of switched linear systems," *Automatica*, vol. 49, no. 12, pp. 3531–3537, 2013.
- [16] Y. Pan and X. Li, "Structural controllability and controlling centrality of temporal networks," *PloS One*, vol. 9, no. 4, p. e94998, 2014.
- [17] M. Pósfai and P. Hövel, "Structural controllability of temporal networks," New J. Physics, vol. 16, no. 12, p. 123055, 2014.
- [18] M. I. García Planas and M. D. Magret, "Structural controllability and observability of switched linear systems," in *Proc. Int. Conf. Applied Mathematics*, Budapest, Hungary, 2015, pp. 15–21.
- [19] B. Hou, X. Li, and G. Chen, "Structural controllability of temporally switching networks," *IEEE Trans. Circuits Syst. I*, vol. 63, no. 10, pp. 1771–1781, 2016.
- [20] S. Pequito and G. J. Pappas, "Structural minimum controllability problem for switched linear continuous-time systems," *Automatica*, vol. 78, pp. 216–222, 2017.
- [21] P. Yao, B.-Y. Hou, Y.-J. Pan, and X. Li, "Structural controllability of temporal networks with a single switching controller," *PloS One*, vol. 12, no. 1, p. e0170584, 2017.

- [22] C. Rech and R. Perret, "About structural controllability of interconnected dynamical systems," *Automatica*, vol. 27, no. 5, pp. 877–881, 1991.
- [23] K. Li, Y. Xi, and Z. Zhang, "G-cactus and new results on structural controllability of composite systems," *Int. J. Syst. Sci.*, vol. 27, no. 12, pp. 1313–1326, 1996.
- [24] L. Blackhall and D. J. Hill, "On the structural controllability of networks of linear systems," *IFAC Proceedings Volumes*, vol. 43, no. 19, pp. 245– 250, 2010.
- [25] J. F. Carvalho, S. Pequito, A. P. Aguiar, S. Kar, and K. H. Johansson, "Composability and controllability of structural linear time-invariant systems: distributed verification," *Automatica*, vol. 78, pp. 123–134, 2017.
- [26] J. Mu, S. Li, and J. Wu, "On the structural controllability of distributed systems with local structure changes," *Science China Information Sciences*, vol. 61, no. 5, p. 052201, 2018.
- [27] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, "Controllability of complex networks," *Nature*, vol. 473, no. 7346, pp. 167–173, 2011.
- [28] N. J. Cowan, E. J. Chastain, D. A. Vilhena, J. S. Freudenberg, and C. T. Bergstrom, "Nodal dynamics, not degree distributions, determine the structural controllability of complex networks," *PloS One*, vol. 7, no. 6, p. e38398, 2012.
- [29] J. C. Nacher and T. Akutsu, "Structural controllability of unidirectional bipartite networks," *Scientific Reports*, vol. 3, 2013.
- [30] X. Zhang, T. Lv, X. Yang, and B. Zhang, "Structural controllability of complex networks based on preferential matching," *PloS One*, vol. 9, no. 11, p. e112039, 2014.
- [31] H. Yin and S. Zhang, "Minimum structural controllability problems of complex networks," *Physica A: Statistical Mechanics and its Applica*tions, vol. 443, pp. 467–476, 2016.
- [32] S. Sun, Y. Ma, Y. Wu, L. Wang, and C. Xia, "Towards structural controllability of local-world networks," *Physics Lett. A*, vol. 380, no. 22, pp. 1912–1917, 2016.
- [33] E. Tang, C. Giusti, G. L. Baum, S. Gu, E. Pollock, A. E. Kahn, D. R. Roalf, T. M. Moore, K. Ruparel, R. C. Gur, R. E. Gur, T. D. Satterthwaite, and D. S. Bassett, "Developmental increases in white matter network controllability support a growing diversity of brain dynamics," *Nature Communications*, vol. 8, no. 1, p. 1252, 2017.
- [34] M. S. Riasi and L. Yeghiazarian, "Controllability of surface water networks," Water Resources Research, vol. 53, no. 12, pp. 10450– 10464, 2017
- [35] Y. Guan and L. Wang, "Structural controllability of multi-agent systems with absolute protocol under fixed and switching topologies," *Science China Information Sciences*, vol. 60, no. 9, p. 092203, 2017.
- [36] X. Liu and L. Pan, "Detection of driver metabolites in the human liver metabolic network using structural controllability analysis," BMC Syst. Biol., vol. 8, no. 1, p. 1, 2014.
- [37] X. Liu and L. Pan, "Identifying driver nodes in the human signaling network using structural controllability analysis," *IEEE/ACM Trans. Comput. Biol. Bioinformatics*, vol. 12, no. 2, pp. 467–472, 2015.
- [38] L. Wu, M. Li, J. Wang, and F.-X. Wu, "Minimum steering node set of complex networks and its applications to biomolecular networks," *IET Syst. Biol.*, vol. 10, no. 3, pp. 116–123, 2016.
- [39] Y. Chu, Z. Wang, R. Wang, N. Zhang, J. Li, Y. Hu, M. Teng, and Y. Wang, "Wdnfinder: A method for minimum driver node set detection and analysis in directed and weighted biological network," J. Bioinformatics and Comput. Biol., vol. 15, no. 05, p. 1750021, 2017.
- [40] W.-F. Guo, S.-W. Zhang, Z.-G. Wei, T. Zeng, F. Liu, J. Zhang, F.-X. Wu, and L. Chen, "Constrained target controllability of complex networks," J. Statistical Mechanics: Theory and Experiment, vol. 2017, no. 6, p. 063402, 2017.
- [41] P. Yao, C. Li, and X. Li, "The functional regions in structural controllability of human functional brain networks," in *Proc. IEEE Int. Conf.* Syst., Man, and Cybernetics. IEEE, 2017, pp. 1603–1608.
- [42] K. Kanhaiya, E. Czeizler, C. Gratie, and I. Petre, "Controlling directed protein interaction networks in cancer," *Scientific Reports*, vol. 7, no. 1, p. 10327, 2017.
- [43] V. Ravindran, V. Sunitha, and G. Bagler, "Identification of critical regulatory genes in cancer signaling network using controllability analysis," *Physica A: Statistical Mechanics and its Applications*, vol. 474, pp. 134–143, 2017.
- [44] C. Rech, "Robustness of interconnected systems to structural disturbances in structural controllability and observability," *Int. J. Control*, vol. 51, no. 1, pp. 205–217, 1990.
- [45] M. A. Rahimian and A. G. Aghdam, "Structural controllability of multiagent networks: robustness against simultaneous failures," *Automatica*, vol. 49, no. 11, pp. 3149–3157, 2013.

- [46] B. Wang, L. Gao, Y. Gao, and Y. Deng, "Maintain the structural controllability under malicious attacks on directed networks," *Europhysics Lett.*, vol. 101, no. 5, p. 58003, 2013.
- [47] M. A. Rahimian and A. G. Aghdam, "Structural controllability of multiagent networks: importance of individual links," in *Proc. Amer. Control Conf.*, Washington, DC, USA, 2013, pp. 6871–6876.
- [48] S. A. Mengiste, A. Aertsen, and A. Kumar, "Effect of edge pruning on structural controllability and observability of complex networks," *Scientific Reports*, vol. 5, 2015.
- [49] Z. Zhang, Y. Yin, X. Zhang, and L. Liu, "Optimization of robustness of interdependent network controllability by redundant design," *PloS One*, vol. 13, no. 2, p. e0192874, 2018.
- [50] F. Pasqualetti, F. Dörfler, and F. Bullo, "Control-theoretic methods for cyberphysical security: geometric principles for optimal cross-layer resilient control systems," *IEEE Control Syst.*, vol. 35, no. 1, pp. 110– 127, 2015.
- [51] S. Weerakkody, X. Liu, S. H. Son, and B. Sinopoli, "A graph-theoretic characterization of perfect attackability for secure design of distributed control systems," *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 1, pp. 60–70, 2017.
- [52] J. Corfmat and A. S. Morse, "Structurally controllable and structurally canonical systems," *IEEE Trans. Autom. Control*, vol. 21, no. 1, pp. 129–131, 1976.
- [53] Y. Hayakawa, S. Hosoe, M. Hayashi, and M. Ito, "On the structural controllability of compartmental systems," *IEEE Trans. Autom. Control*, vol. 29, no. 1, pp. 17–24, 1984.
- [54] J. L. Willems, "Structural controllability and observability," Syst. & Control Lett., vol. 8, no. 1, pp. 5–12, 1986.
- [55] B. D. O. Anderson and H.-M. Hong, "Structural controllability and matrix nets," *Int. J. Control*, vol. 35, no. 3, pp. 397–416, 1982.
- [56] S. Dasgupta and B. D. O. Anderson, "Physically based parameterizations for designing adaptive algorithms," *Automatica*, vol. 23, no. 4, pp. 469– 477, 1987.
- [57] S. S. Mousavi, M. Haeri, and M. Mesbahi, "On the structural and strong structural controllability of undirected networks," *IEEE Trans. Autom. Control*, (in press).
- [58] C. Godsil and G. Royle, Algebraic graph theory. Springer-Verlag, 2013.
- [59] M. Cao, A. S. Morse, and B. D. O. Anderson, "Reaching a consensus in a dynamically changing environment: a graphical approach," SIAM J. Control and Optimization, vol. 47, no. 2, pp. 575–600, 2008.
- [60] C. Commault, J.-M. Dion, and J. W. van der Woude, "Characterization of generic properties of linear structured systems for efficient computations," *Kybernetika*, vol. 38, no. 5, pp. 503–520, 2002.
- [61] K. Murota, Matrices and Matroids for Systems Analysis. Berlin, Germany: Springer-Verlag, 2010.
- [62] C. Coates, "Flow-graph solutions of linear algebraic equations," *IEEE Trans. Circuit Theory*, vol. 6, no. 2, pp. 170–187, 1959.