Check List For Math104

- **DEF**:
    - operation
    - field
        - 9 axioms $(4+4+1)$
    - ordered field
        - 6 properties
- prove: 1>0
- **DEF**: completeness of ordered field
    - $(\mathbb{Q}, +, \cdot, >)$ is not complete
- **Def**: bounded, bounded above, bounded below
- **DEF**: Sup_A
- **Consequences of completeness of** $(\mathfrak{R}, +, \cdot, >)$
    - Archimedian property (zoom in & out)
    - Existence of integer part of every real & proof
    - **proof**: Denseness of $\mathbb{Q}$
    - **proof**: Denseness of $\mathfrak{R} \setminus \mathbb{Q}$
- **Triangular inequality**
    - with its two corollaries
- **Bernoulli's**
- **Binomial expansion**
- **Cauchy inequality**
- **A-G-H mean**:
    - harmonic mean
    - geometric mean
    - arithmetic mean
- **Def**:
    - **neighborhood** of a for $a \in \mathfrak{R}$
    - **Sequnce**
    - $(a_n)_n \to a, a \in \mathbb{R}$ (2 ways)
- **Proof**: Uniqueness of limits
- **Every convergent Sequence is bdd & proof**
- **Algebra of Limits**
    - $a_n \to a \iff a_n - a \to 0$
    - $a_n \to 0 \iff |a_n| \to 0$
    - $a_n \to 0, b_n$ bounded $\implies a_n \cdot b_n \to 0$

        - & proof
    - $a_n \to a, b_n \to b \implies a_n + b_n \to a + b$
        - & proof

          (*i.e. we can multiply ϵ by any constant in $\mathfrak{R}$*)
    - $a_n \to a, b_n \to b \implies a_n \cdot b_n \to a \cdot b$
        - & proof
        - $(-a \cdot b_n + a \cdot b_n)$
    - $k \in \mathbb{N}, a_n \to a \implies a_n^k \to a^k$
    - $b_n \neq 0, b_n \to b \implies \frac{1}{b_n} \to \frac{1}{b}$
        - & proof (*note: hard)
    - $(a_n)_n \geq 0, k \geq 0, a_n \to a \implies \sqrt[k]{a_n} \to \sqrt[k]{a}$
    - $a > 0$, when $n \to +\infty$, $\sqrt[n]{a} \to 1$
    - when $n \to +\infty$, $\sqrt[n]{n} \to 1$
- **Squeeze Thereom & Proof**
- **Monotone Sequence** $(a_n)_n$ **converges** $\iff$
    - increasing & bdd above *or*
    - decreasing & bdd below
    - & proof
- $(a_n)_n = (1 + \frac{1}{n})^n$ **converges**
    - proof
    - definition of e
- **DEF**: $+\infty, -\infty$ as limit
    - i.e. as $n \to +\infty, (a_n)_n \to +\infty$
    - 2 ways
- **ratio test, root test**
    - & proof
- **bolzano-Weierstrasss** (or whatever it's called)
    - 2 ways of proof
        - peak point
        - nested interval
- **DEF**: Cauchy Sequence
- **Converge** $\implies$ **Cauchy**
    - converse does not hold unless in $\mathfrak{R}$
- **In any metric space,** $(b_n)_n$ **cauchy with a subsequence:**

  $(b_{k_n})_n \to b \in \mathfrak{R} \implies b_n \to b$
    - & proof

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11. Metric Space

- **DEF**:

- topology
- open / closed set
- neighborhood
- basis
- metric (distance)
    - 3 properties
    - prove the triangular eqaulity in d_p
    - Minkowski's inequality

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12. Topology Induced by a Metric

- topology induced by a metric T_d ($\{\emptyset$, union of ... $\}$)
- characterisation
- open ball / open set w.r.t d
- **proof**: finite intersections of sets in T_d are also in T_d
- closed ball / closed set
- discrete topology (all subsets are open and closed)

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13. Open Sets and Closed Sets

- Properties of unions and intersections:
    - **finite intersections** of **open** sets (def)
    - **arbitrary unions** of **open** sets (def)

      $\Longrightarrow$ (due to de morgan's law)
    - **finite unions** of **closed** sets
    - **arbitrary intersections** of **closed** set

      $\Longrightarrow$
    - all **finite** sets are **closed**

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14. Closure

- **DEF**: closure
- $\overline{F} = F \iff$ F closed
- proof: $\overline{F}$ is closed (w.r.t d)
- $\overline{F}$ is the smallest closed set containing F
    - proof

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15. Normed Space

- **DEF**: norm($|| \cdot ||$)

- nonnegativity
- sacalar multiplication
- triangle inequality

- $||\cdot||_p$ for $p \geq 1$, Minkawski's Inequality
- **def**: memtric induced by norm
- Restrictions of normed space
  - X must be vector space
  - translation invariant
  - $d_{||\cdot||}$ takes all non-negative values in $\mathfrak{R}$
    - a result of λ multiplication

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16. Sequence in Metric Space

- **def**:
  - Sequence in a metric space
  - Convergence: $(x_n)_n \to x_0 \in X$, w.r.t d (3 ways)
- 3 properties for convergent Sequence (say, $\to x_o$)in metric spaces
  - unique limit
  - every subsequence converge
  - bounded w.r.t d(meaning ..?)
- Extra 3 properties in normed spaces (normed space gives us addition, zero, etc.)
- **DEF**: characterisation of closures / closed sets via sequences
- 2 techniques to prove $F \subset X$ is closed / not closed

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17. Accumulation Points

- **Def**: accumulation point
- **Proof**: $\overline{A} = A \cup A'$,wher $A' :=$ accumulation points of A,
  - prove: $A \cup A' \subset \overline{A}$
  - prove: $\overline{A} \subset A \cup A'$
- ex: $\overline{(a,b) \cup \{c\}} = ((a,b) \cup \{c\}) \cup ((a,b) \cup \{c\})' = (a,b) \cup \{\} \cup [a,b] = [ab] \cup \{c\}$

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18. Cauchy Sequences In Metric Space

- **Def**: Cauchy sequence in metric space
- cauchy and convergence
  - the key is to note the difference: cauchy only concerns d and all the elements in the squence, while convergence will also need one extra point: the limit. i.e. we need to think about the ambient space

- **Def**: Complete metric space (now we include the extra point)
- **proof**: Complete $\implies$ closed

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19. Limits of Functions

- **Def**: accumulation point x_0 of A, for $x_0 \in \mathbb{R}, A \subset \mathbb{R}$
  - 3 ways
- **Def**: x_0 isolated point of A, for $A \subset R, x_0 \in A$
- **Def**: $\lim_{x \to x_0} f(x) = l$, for $f : A \to \mathbb{R}$, and x_0 a.c point of A.
  - 3 ways
- $+\infty, -\infty$ as limits
- $+\infty, -\infty$ as A.C. points

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20. Characterisation of Accumulation Points

- The following three are equivalent
  - $x_0$is an A.C points of A
  - Every neighbourhood of x_0 contains infinitely many elements of A, different from x_0
  - there exits a sequence $(x_n)_{n \in \mathbb{N}}$in A, all of whose terms are pairwise distinct, such that $x_n \to x_0$

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21. Characterisation of Limits via Limits of Sequences

- $\lim_{x \to x_0} f(x) = l$ vs $f(x_n) \to l$, when $x_n \to x_0$
- Corollary: Algebra of limits

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22. Continuity

- **Def**: f is continuous at x_0
  - 3 ways
  - Compare to limits, we now really care about $f(x_0)$
  - consider x_0 to be an isolated point
- Proof continuity
  - by ϵ definiton
  - by limit of sequence (i.e. when not an isolation point)
- **Def**: Continuity via limits of Sequence
- **Thm**: Continuity of algebra of functions ($f + g, \lambda \cdot f, f \cdot g, \frac{f}{g}$)
- **Thm**: Composition of functions, $g \circ f$, peserves continuity

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23. Continuity as a Local Property

- Continuity is a local property

- if f is cont. at x_0, then f is bounded in a neighbourhood of x_0
- Local preservation of sign and points of continuity

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25. Continuous functions on closed intervals

- Every continuous function $f : [a, b] \to \mathbb{R}$ is bounded.
- (Stronger) Every continuous function $f : [a, b] \to \mathbb{R}$ has a maximal value and minimal value.

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26. IVT

- IVT
  - **Def**: IVT
  - **proof**: IVT
    - 2 ways
  - 3 other forms

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27. Applications of IVT

- Every positive number has a unique n-th root
- Each polynomial of odd degree has a root in $\mathbb{R}$
- Fix point theorem
- **Def**: Interval
- Continuous images of intervals are intervals
- Continuous on I + 1 to 1 $\implies$ strictly monotone
- $f : I \to \mathbb{R}$ continuous and 1-1

  $\implies f^{-1} : f(I) \to I$ is continuous, and has the same kind of monotonicity

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28. Continuous functions on metric spaces

- **Def**: Let (X, d_x), (Y, d_y) be metric spaces, let $f : X \to Y$, f is continuous at $x_0 \in X$
  - 2 ways
- **Def**: $f^{-1}(A), A \subset Y$
  - $f^{-1}(A)$ is well defined $\forall A \subset Y$,whether the function $f^{-1} : Y \to X$ is well defined or not
  - $B \subset f^{-1}(A) \iff f(B) \subset A$
- f continuous $\iff$ f inverts open sets to open sets (i.e. $f^{-1}(U)$ open in (X, d_x), for every $U \subset Y$, open in (Y, d_y)) .
  - proof

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29. Compact Sets

- **Def**: open cover of K, $K \subset X$ in (X, d)
    - **Def**: subcover $A' \subset A$
- **Def**: (X, d)compact
    - **Def**: compact subset
- $K \subset X, K$ compact $\implies$ K closed and bounded in (X, d)
    - the converse is not always true
    - proof
- Thm: Continuous Functions send compact sets to compact sets.

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30. Sequentially Compactness

- **Def**: (X, d) Sequentially compact
- Sequentially compact $\implies$ closed
    - not closed $\implies$ not sequentially compact

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31. Differentiability

- **Def**: f dfferentiable at x_0
    - **def**: derivative of f at x_0
- **Proof**: differentiable $\implies$ continuous
- Rules of differentiation
    - 4
- **Proof**: Chain rule

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32. Local Extrema

- Thm: $f \uparrow \implies f'(x) \geq 0, \forall x \in (a, b)$; .....
    - $f'(x)$ can be zero even strictly increasing, consider $x^3, x = 0$
        - however, $f'(x) > 0 \implies f$ strictly increasing...
- **def**: local extremum
- Fermat's Prop: local extremum at x_0 and $f'(x_0)$ exists $\implies f'(x_0) = 0$
- **def**: critical point

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33. Rolle's, MVT, L' Hopital

- Rolle's Theorem
    - proof
- Mean Value Theorem
    - proof
- L ' Hopistal
    - **Def**: Cauchy's Generalised Mean Value Theorem

34. Uniform Continuity

- **Def**: Uniform Continuity
- $f:(X,d_x)\to(Y,d_y)$ continuous, (X,d_x) compact $\implies$ f uniformly continuous

35. Riemann Integration

- X_I characteristic function
- $\int X_I :=$ length of I
- **Def**: step function ϕ
  - equivalence: $\phi(x)=\sum c_i X_{(x_{i-1},x_i)}$
  - observation:
    - bounded support
    - continuity
  - $\int \phi$
  - observation: adding subset $\{y_0,y_1,\dots y_m\}$ to $\{x_o,x_1,..x_n\}$ of ϕ
  - Linearity of integral for step function
  - $\phi(x)\geq\phi(x),\forall x\in\mathbb{R}$
    $\iff \int\phi\geq\int\psi$
- **def**: Riemann Integrable (R.I.)
  - prop: step functions are R.I.
  - 2 properties of R.I f:
    - f bounded
    - f has bounded support
  - Theorem: R.I. f with bounded support $\iff$
  - $\int f$

36. R.I. Proof & Corollary

- 2 criterions
- 2 corollary
- 4 basic properties
- Thm: Any monotone function $f:[a,b]\to\mathbb{R}$ is Riemann Integrable.
- Thm: Any continuous function $f:[a,b]\to\mathbb{R}$ is Riemann Integrable.