§ 2.3 卷积、算子

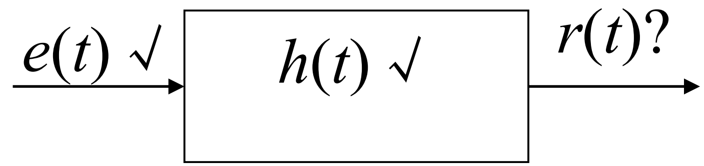
一、卷积及其性质

1. 定义与物理意义

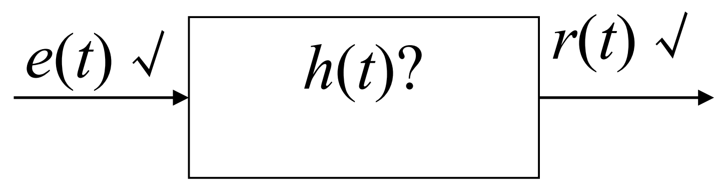
①历史：19世纪，欧拉，泊松，杜阿美尔

②卷积与反卷积互逆

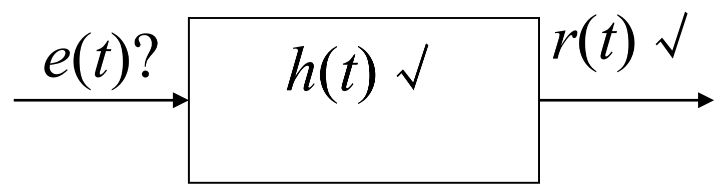
i)卷积



ii)反卷积1：系统辨识



iii)反卷积2：信号检测



§2.3 卷积、算子

③定义：

$$f(t)=f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

④物理意义：将信号分解成冲激信号之和，借助系统的冲激响应$h(t)$，求出系统对任意激励信号的零状态响应，即：

$$e(t)=\int_{-\infty}^{+\infty}e(\tau)\delta(t-\tau)d\tau$$

$$\Rightarrow r(t)=\int_{-\infty}^{+\infty}e(\tau)h(t-\tau)d\tau$$

§ 2.3 卷积、算子

2. 卷积性质

①代数性质

i)交换律：$f_1(t)*f_2(t)=f_2(t)*f_1(t)$

证明：

$$f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

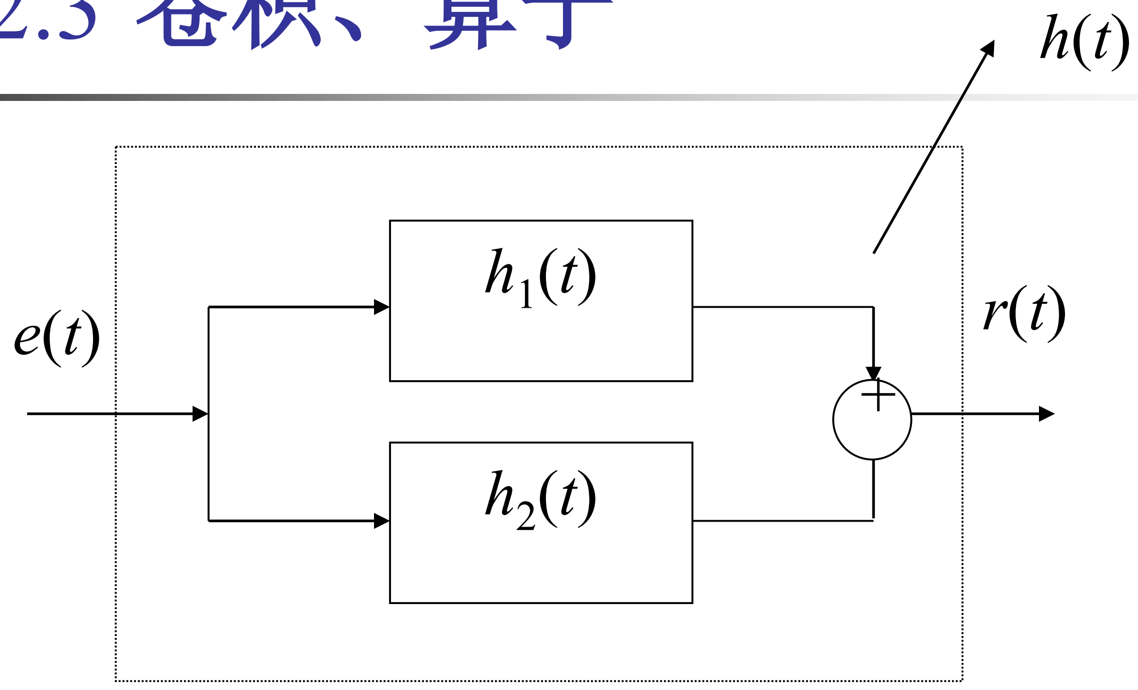
$$\underline{\underline{\tau=t-\lambda}}\int_{-\infty}^{+\infty}f_2(\lambda)f_1(t-\lambda)d\lambda=f_2(t)*f_1(t)$$

ii)分配律：$f_1(t)*\left[f_2(t)+f_3(t)\right]=f_1(t)*f_2(t)+f_1(t)*f_3(t)$

定律成立条件：$f_1(t)*f_2(t)$ $f_1(t)*f_3(t)$ 均存在

物理含义：并联系统的冲激响应，等于各子系统冲激响应之和。

§ 2.3 卷积、算子



$$r(t) = e(t) * [h_1(t) + h_2(t)]$$

$$\Rightarrow h(t) = h_1(t) + h_2(t)$$

§2.3 卷积、算子

iii)结合律：$\left[f_1(t)*f_2(t)\right]*f_3(t)=f_1(t)*\left[f_2(t)*f_3(t)\right]$

定律成立条件：$f_1(t)*f_2(t)$ $f_2(t)*f_3(t)$均存在

物理含义：串联系统的冲激响应=子系统冲激响应卷积

证明：

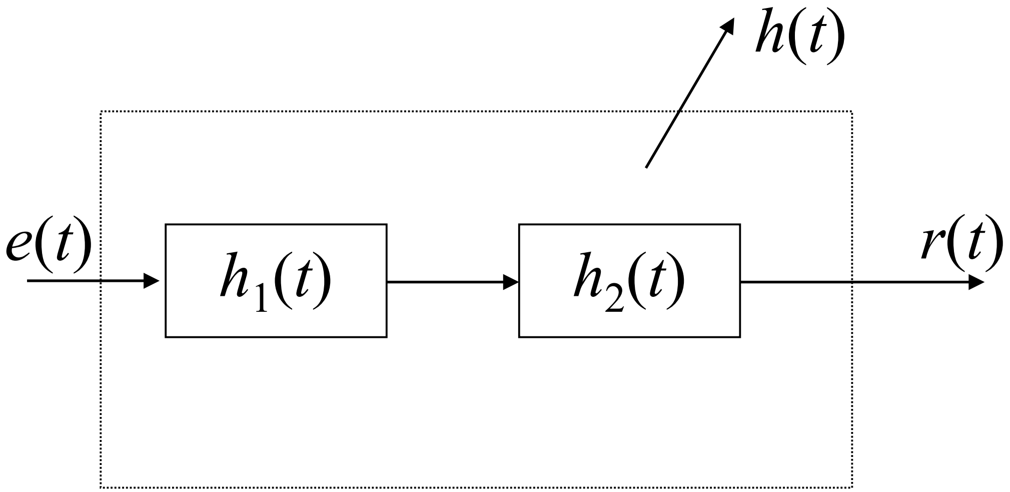
$$\left[f_1(t)*f_2(t)\right]*f_3(t)=\int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty}f_1(\lambda)f_2(\tau-\lambda)d\lambda\right]f_3(t-\tau)d\tau$$

$$=\int_{-\infty}^{+\infty}f_1(\lambda)\left[\int_{-\infty}^{+\infty}f_2(\tau-\lambda)f_3(t-\tau)d\tau\right]d\lambda$$

$$=\int_{-\infty}^{+\infty}f_1(\lambda)\left[\int_{-\infty}^{+\infty}f_2(\tau)f_3(t-\tau-\lambda)d\tau\right]d\lambda$$

$$=f_1(t)*\left[f_2(t)*f_3(t)\right]$$

§ 2.3 卷积、算子



$$r(t) = e(t) * [h_1(t) * h_2(t)]$$

$$\Rightarrow h(t) = h_1(t) * h_2(t)$$

§ 2.3 卷积、算子

[例1]：证明：

$$\left[e^{-t}\mu(t)*e^{-t}\right]*\left[\delta(t)-\frac{1}{e}\delta(t-1)\right]\neq e^{-t}\mu(t)*\left[e^{-t}*\left[\delta(t)-\frac{1}{e}\delta(t-1)\right]\right]$$

证明：因为：

$$\left[e^{-t}u(t)*e^{-t}\right]*\left[\delta(t)-\frac{1}{e}\delta(t-1)\right]=\text{不存在}$$

而

$$e^{-t}u(t)*\left[e^{-t}*\left[\delta(t)-\frac{1}{e}\delta(t-1)\right]\right]$$

$$=e^{-t}u(t)*\left[e^{-t}-\frac{1}{e}e^{1-t}\right]=0$$

§2.3 卷积、算子

②微分积分性质

i)微分性质：$$\frac{d}{dt}\left[f_1(t)*f_2(t)\right]=f_1(t)*\frac{df_2(t)}{dt}=\frac{df_1(t)}{dt}*f_2(t)$$

证明： $$\frac{d}{dt}\left[f_1(t)*f_2(t)\right]=\frac{d}{dt}\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

$$=\int_{-\infty}^{+\infty}f_1(\tau)\frac{df_2(t-\tau)}{dt}d\tau=f_1(t)*\frac{df_2(t)}{dt}=\frac{df_1(t)}{dt}*f_2(t)$$

ii)积分性质：

$$\int_{-\infty}^{t}\left[f_1(\lambda)*f_2(\lambda)\right]d\lambda=f_1(t)*\int_{-\infty}^{t}f_2(\lambda)d\lambda=f_2(t)*\int_{-\infty}^{t}f_1(\lambda)d\lambda$$

证明： $$\int_{-\infty}^{t}\left[f_1(\lambda)*f_2(\lambda)\right]d\lambda=\int_{-\infty}^{t}\left[\int_{-\infty}^{+\infty}f_1(\tau)f_2(\lambda-\tau)d\tau\right]d\lambda$$

$$=\int_{-\infty}^{+\infty}f_1(\tau)\left[\int_{-\infty}^{t}f_2(\lambda-\tau)d\lambda\right]d\tau=f_1(t)*\int_{-\infty}^{t}f_2(\lambda)d\lambda$$

§ 2.3 卷积、算子

iii)推广：设 $s(t)=\left[f_1(t)*f_2(t)\right]$，则

$$s^{(i)}(t)=f_1^{(j)}(t)*f_2^{(i-j)}(t)$$

例：$\dfrac{df_1(t)}{dt}*\int_{-\infty}^{t}f_2(\lambda)d\lambda=f_1(\lambda)*f_2(\lambda)$

§ 2.3 卷积、算子

③ $\delta(t)$ ，u(t)的卷积性质

i) $f(t) * \delta(t) = f(t)$

ii) $f(t) * \delta(t - t_0) = f(t - t_0)$

iii) $f(t) * \delta'(t) = f'(t)$

iv) $f(t) * u(t) = \int_{-\infty}^{t} f(\lambda) d\lambda$

v) $f(t) * \delta^{(k)}(t) = f^{(k)}(t)$

vi) $f(t) * \delta^{(k)}(t - t_0) = f^{(k)}(t - t_0)$

§ 2.3 卷积、算子

3．卷积求法

①图解法，设 $e(t)*h(t)$

i)变量替换： $t \to \tau$

ii)信号反褶： $h(\tau) \to h(-\tau)$

iii)信号移位： $h(-\tau) \to h(t-\tau)$

iv)信号相乘： $e(\tau)h(t-\tau)$

v)求积分： $\int_{-\infty}^{+\infty} e(\tau)h(t-\tau)$

②直接法

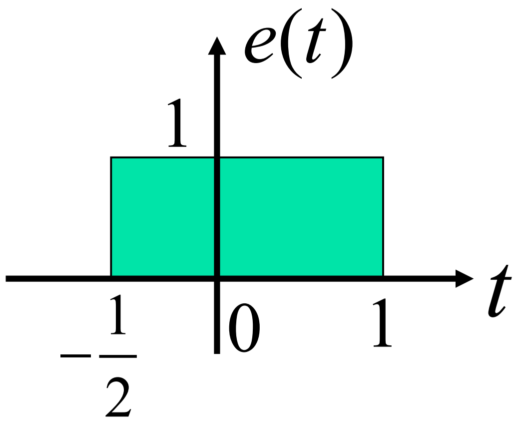
③利用卷积性质

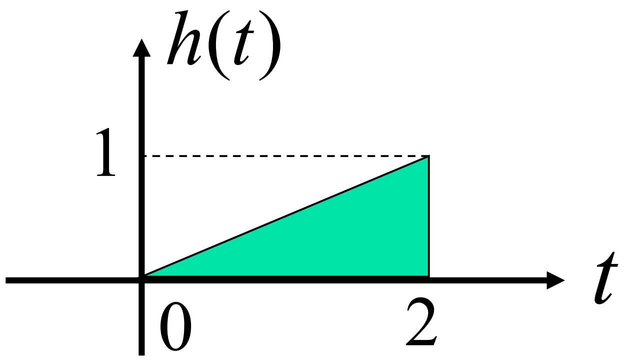
④数值法（积分复杂时采用此法）

§ 2.3 卷积、算子

[例2]: $e(t)=u\left(t+\frac{1}{2}\right)-u(t-1)$　　$h(t)=\frac{1}{2}t\left[u(t)-u(t-2)\right]$

求： $r_{zs}(t)$

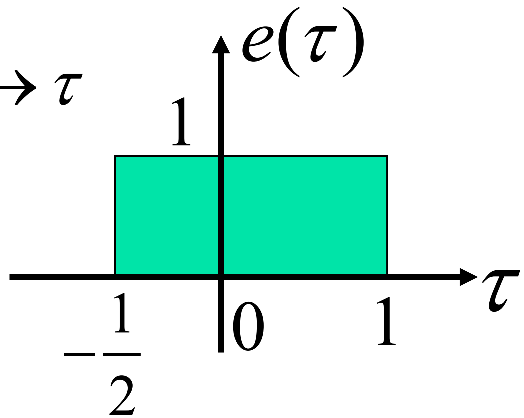


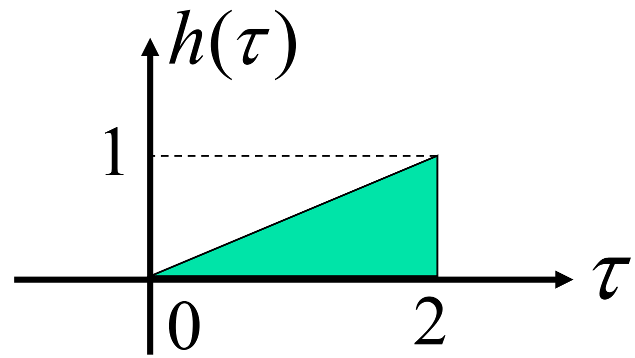


§ 2.3 卷积、算子

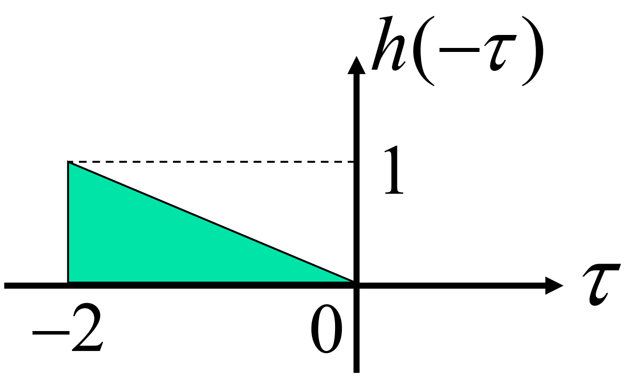
解：①方法一：图解法

i) $t \to \tau$

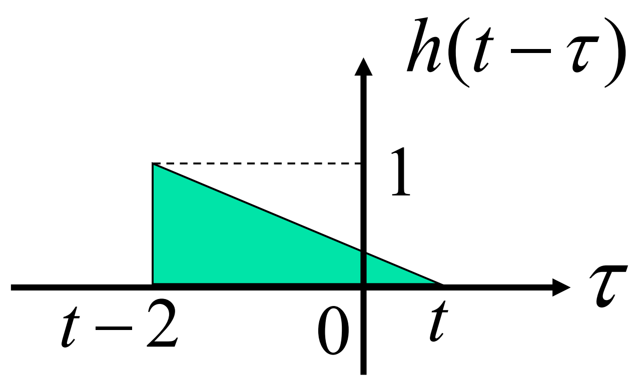




ii) $h(\tau) \to h(-\tau)$

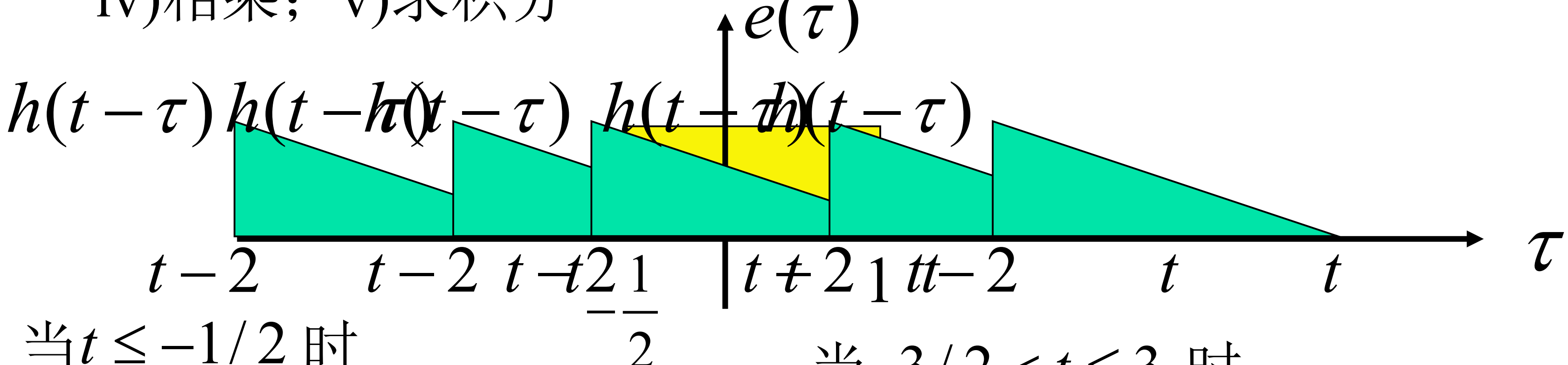


iii) $h(-\tau) \to h(t-\tau)$



§ 2.3 卷积、算子

iv)相乘；v)求积分



当$t \le -1/2$ 时

$$r_{zs}(t) = 0$$

当 $-1/2 < t \le 1$ 时

$$r_{zs}(t) = \int_{-\frac{1}{2}}^{t} \frac{1}{2}(t-\tau)\,d\tau = \frac{t^2}{4} + \frac{t}{4} + \frac{1}{16}$$

当 $1 < t \le 3/2$ 时

$$r_{zs}(t) = \int_{-\frac{1}{2}}^{1} \frac{1}{2}(t-\tau)\,d\tau = \frac{3t}{4} - \frac{3}{16}$$

当 $3/2 < t \le 3$ 时

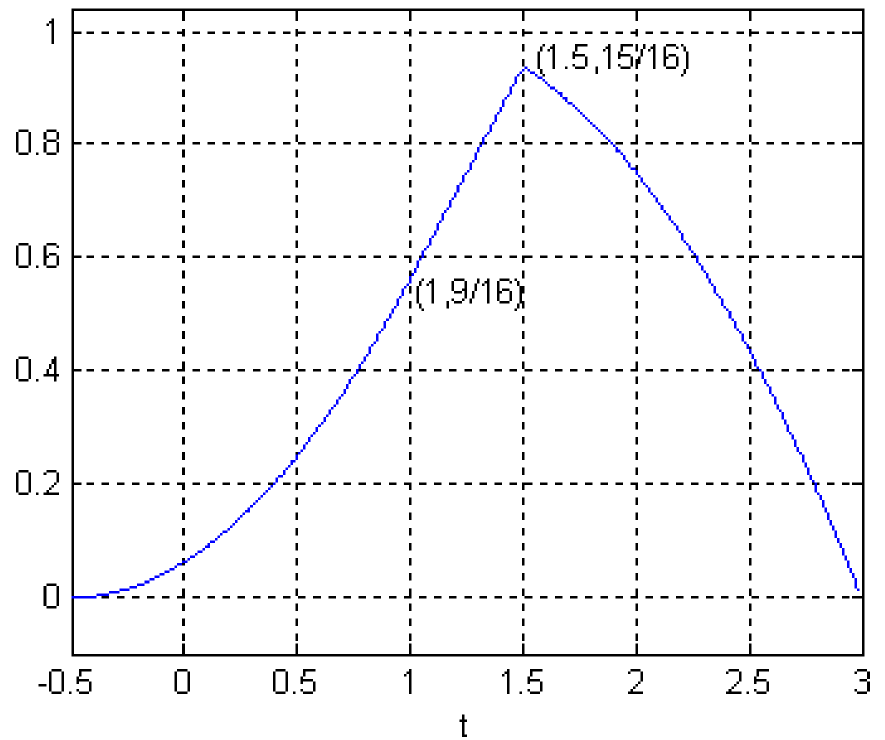
$$r_{zs}(t) = \int_{t-2}^{1} \frac{1}{2}(t-\tau)\,d\tau = -\frac{t^2}{4} + \frac{t}{2} + \frac{3}{4}$$

当 $t > 3$ 时

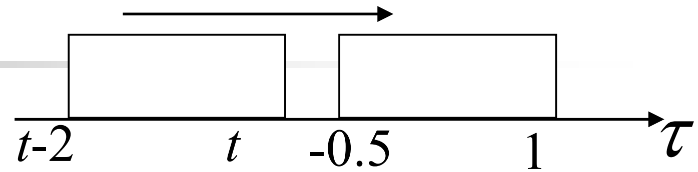
$$r_{zs}(t) = 0$$

§ 2.3 卷积、算子

$$r_{zs}(t)=\begin{cases}0 & -\infty<t\le-\dfrac{1}{2}\\ \dfrac{t^2}{4}+\dfrac{t}{4}+\dfrac{1}{16} & -\dfrac{1}{2}<t\le 1\\ \dfrac{3t}{4}-\dfrac{3}{16} & 1<t\le\dfrac{3}{2}\\ -\dfrac{t^2}{4}+\dfrac{t}{2}+\dfrac{3}{4} & \dfrac{3}{2}<t\le 3\\ 0 & t>3\end{cases}$$



§ 2.3 卷积、算子



解：②方法二：直接法

$$r_{zs}(t)=e(t)*h(t)=\int_{-\infty}^{+\infty}e(\tau)h(t-\tau)d\tau$$

$$=\int_{-\frac{1}{2}}^{1}h(t-\tau)d\tau=\int_{-\frac{1}{2}}^{1}\frac{1}{2}(t-\tau)\left[u(t-\tau)-u(t-\tau-2)\right]d\tau$$

$$=\begin{cases}0 & -\infty<t\le-\frac{1}{2}\\ \int_{-\frac{1}{2}}^{t}\frac{1}{2}(t-\tau)d\tau=\frac{t^2}{4}+\frac{t}{4}+\frac{1}{16} & -\frac{1}{2}<t\le1\\ \int_{-\frac{1}{2}}^{1}\frac{1}{2}(t-\tau)d\tau=\frac{3t}{4}-\frac{3}{16} & 1<t\le\frac{3}{2}\\ \int_{t-2}^{1}\frac{1}{2}(t-\tau)d\tau=-\frac{t^2}{4}+\frac{t}{2}+\frac{3}{4} & \frac{3}{2}<t\le3\\ 0 & t>3\end{cases}$$

§ 2.3 卷积、算子

$$e(t)=u\left(t+\frac{1}{2}\right)-u(t-1)$$

$$h(t)=\frac{1}{2}t\left[u(t)-u(t-2)\right]$$

解：③方法三：利用卷积性质求卷积

$$e(t)*h(t)=e'(t)*\int_{-\infty}^{t}h(\tau)d\tau$$

$$=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*\left[\int_{-\infty}^{t}\frac{1}{2}\tau\left[u(\tau)-u(\tau-2)\right]d\tau\right]$$

$$=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*\left[\int_{-\infty}^{t}\frac{1}{2}\tau u(\tau)d\tau-\int_{-\infty}^{t}\frac{1}{2}\tau u(\tau-2)d\tau\right]$$

$$=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*\left\{\left[\int_{0}^{t}\frac{1}{2}\tau d\tau\right]u(t)-\left[\left(\int_{2}^{t}\frac{1}{2}\tau d\tau\right)u(t-2)\right]\right\}$$

$$=\left[\delta\left(t+\frac{1}{2}\right)-\delta(t-1)\right]*\left[\frac{1}{4}t^2u(t)-\frac{1}{4}\left(t^2-4\right)u(t-2)\right]$$

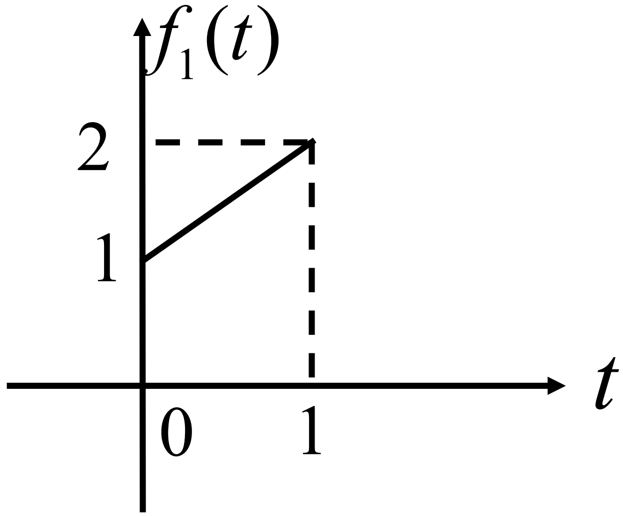
$$=\frac{1}{4}\left(t+\frac{1}{2}\right)^2u\left(t+\frac{1}{2}\right)-\frac{1}{4}\left[\left(t+\frac{1}{2}\right)^2-4\right]u\left(t-\frac{3}{2}\right)-\frac{1}{4}(t+1)^2u(t-1)$$

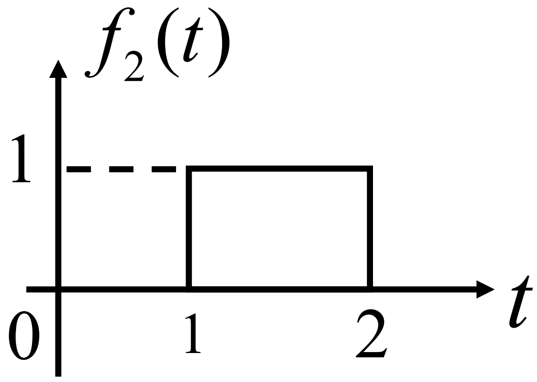
$$+\frac{1}{4}\left[(t-1)^2-4\right]u(t-3)$$

§ 2.3 卷积、算子

[例3]: $f_1(t)=(1+t)\left[u(t)-u(t-1)\right],f_2(t)=u(t-1)-u(t-2)$

求: $f_1(t)*f_2(t)$





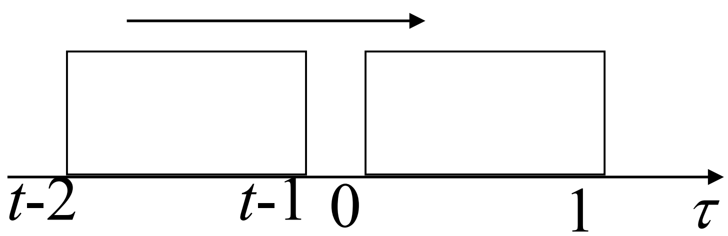
§ 2.3 卷积、算子

解： 用直接法

$$f(t)=f_1(t)*f_2(t)=\int_{-\infty}^{+\infty}f_1(\tau)f_2(t-\tau)d\tau$$

$$=\int_0^1(1+\tau)f_2(t-\tau)d\tau=\int_0^1(1+\tau)[u(t-\tau-1)-u(t-\tau-2)]d\tau$$

$$=\begin{cases}0 & -\infty<t\le 1\\ \int_0^{t-1}(1+\tau)d\tau & 1<t\le 2\\ \int_{t-2}^{1}(1+\tau)d\tau & 2<t\le 3\\ 0 & t>3\end{cases}=\begin{cases}0 & -\infty<t\le 1\\ \frac{1}{2}t^2-\frac{1}{2} & 1<t\le 2\\ -\frac{1}{2}t^2+t+\frac{3}{2} & 2<t\le 3\\ 0 & t>3\end{cases}$$

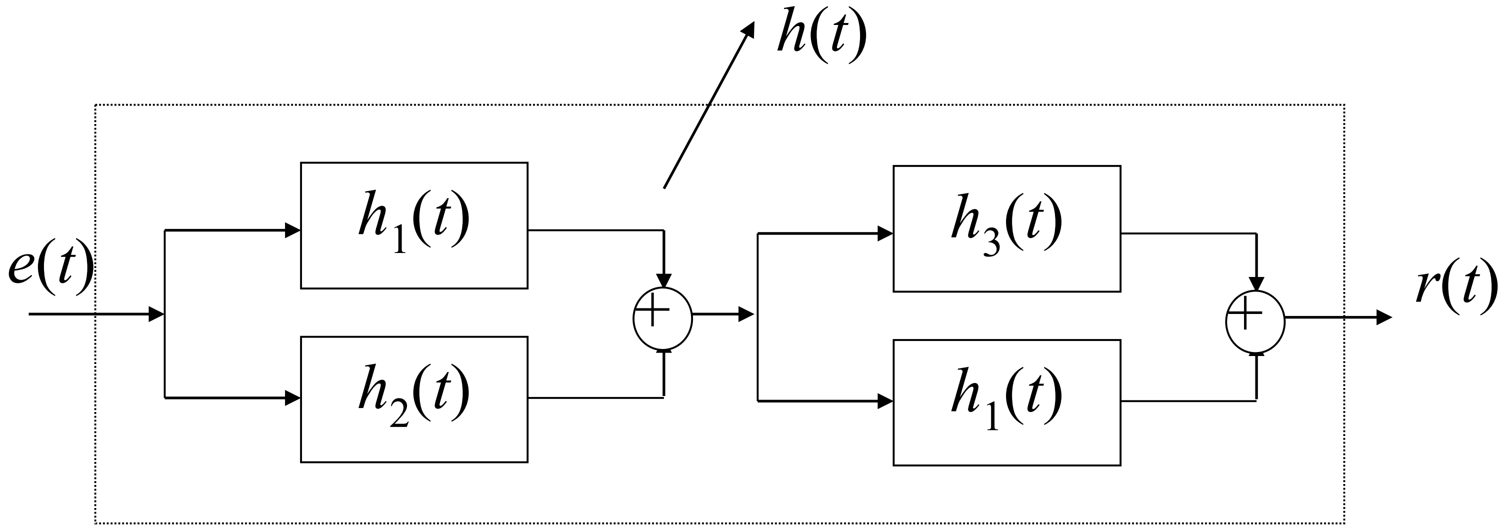


§ 2.3 卷积、算子

[例4]：已知：

$$h_1(t)=u(t-1) \qquad h_2(t)=\delta(t)-\delta(t-1) \qquad h_3(t)=\delta(t)$$

求：$h(t)$



§ 2.3 卷积、算子

解：

$$h(t) = [h_1(t) + h_2(t)] * [h_1(t) + h_3(t)]$$

$$= h_1(t) * h_1(t) + h_1(t) * h_3(t) + h_2(t) * h_1(t) + h_2(t) * h_3(t)$$

$$= (t-2)u(t-2) + u(t-1) + u(t-1) - u(t-2) + \delta(t) - \delta(t-1)$$