Exercise 2.2.5 b)

The set of all strings whose tenth symbol from the right end is a 1.

$A = (Q, \Sigma, \delta, q_0, F)$, 其中 $\Sigma = \{0, 1\}$

$Q = \{\overline{x_1 x_2 \ldots x_n} \mid 1 \leq n \leq 10, x_i \in \{0, 1\}, i \in \{1, \ldots, n\}\}$

$$\delta(\overline{x_1 x_2 \ldots x_n}, y) = \begin{cases} \overline{x_1 x_2 \ldots x_n y} & \text{if } n < 10 \\ \overline{x_2 \ldots x_n y} & \text{if } n = 10 \end{cases}$$

$q_0 = \overline{\varepsilon}$

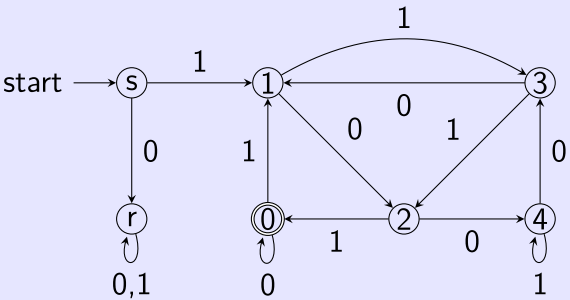
$F = \{\overline{1 x_2 \ldots x_{10}} \mid x_i \in \{0, 1\}, i \in \{2, \ldots, 10\}\}$

Exercise 2.2.6 a)

The set of all strings beginning with a 1 that, when interpreted as binary integer, is a multiple of 5. for example, strings 101(5), 1010(10), and 1111(15) are in the language; 0, 100(4) and 111(7) are not.

Exercise 2.2.6 a)

	0	1
→s	r	q1
*q0	q0	q1
q1	q2	q3
q2	q4	q0
q3	q1	q2
q4	q3	q4
r	r	r



Exercise 2.2.6 b)

The set of all strings that, when interpreted *in reverse* as a binary integer, is divisible by 5. Examples of string in the language are 0, 10011(25), 1001100(25), and 0101(10).

Exercise 2.2.6 b)

Solutions: $A = (Q, \Sigma = \{0,1\}, \delta, q_0, F)$

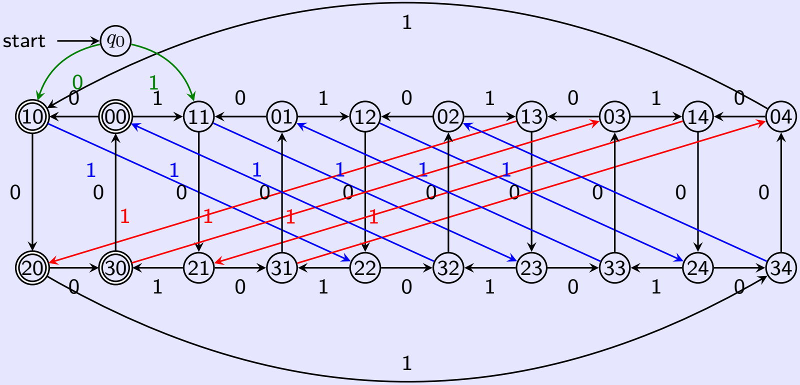
$$Q = \left\{ (x,y) \mid \begin{array}{ll} x \in \{0,1,2,3\}, & \mathrm{len}(w) \mod 4 \\ y \in \{0,1,2,3,4\}, & \mathrm{bin}(\overleftarrow{w}) \mod 5 \end{array} \right\} \cup \{q_0\}$$

$$f(x) \stackrel{def}{=} \left\{ \begin{array}{r|cccc} x & 0 & 1 & 2 & 3 \\ f(x) & 1 & 2 & 4 & 3 \end{array} \right\}$$

$$\delta \begin{cases} \delta((x,y),0) & = & ((x+1) \mod 4, y) \\ \delta((x,y),1) & = & ((x+1) \mod 4, (y + f(x)) \mod 5) \\ \delta(q_0, 0) & = & (1,0) \\ \delta(q_0, 1) & = & (1,1) \end{cases}$$

$F = \{(x,0) \mid x \in \{0,1,2,3\}\}$

Exercise 2.2.6 b)



Exercise 2.2.7

Let A be a DFA and q a particular state of A, such that $\delta(q, a) = q$ for all input symbols a. Show by induction on the length of the input that for all input strings w, $\hat{\delta}(q, w) = q$.

首先，对于 $|w| = 0$ 的 w，显然成立。

假设对所有 $|w| < n$ 的串 w 成立，则当 $|w| = n$ 时，令 $w = xa$，有

$$\begin{aligned}
\hat{\delta}(q, w) &= \hat{\delta}(q, xa) \\
&= \delta(\hat{\delta}(q, x), a) \\
&= \delta(q, a) \\
&= q
\end{aligned}$$

Exercise 2.2.8

Let A be a DFA and a a particular input symbol of A, such that for all states q of A we have $\delta(q, a) = q$.

a) Show by induction on n that for all $n \geq 0$, $\hat{\delta}(q, a^n) = q$, where a^n is the string consisting of n a's.

归纳基础 $\hat{\delta}(q, a^0) = \hat{\delta}(q, \varepsilon) = q$, 归纳递推
$\hat{\delta}(q, a^{n+1}) = \hat{\delta}(q, a^n a) = \delta(\hat{\delta}(q, a^n), a) = \delta(q, a) = q$

b) Show that either $\{a\}^* \subseteq L(A)$ or $\{a\}^* \cap L(A) = \emptyset$.

证 $q_0 \in F \Leftrightarrow \{a\}^* \subseteq L(A)$ 即可.

Exercise 2.2.9

Let $A = (Q, \Sigma, \delta, q_0, \{q_f\})$ be a DFA, and suppose that for all a in Σ we have $\delta(q_0, a) = \delta(q_f, a)$

a) Show that for all $w \neq \varepsilon$ we have $\hat{\delta}(q_0, w) = \hat{\delta}(q_f, w)$.

b) Show that if x is a nonempty string in $L(A)$, then for all $k > 0$, x^k (i.e. x written k times) is also in $L(A)$.

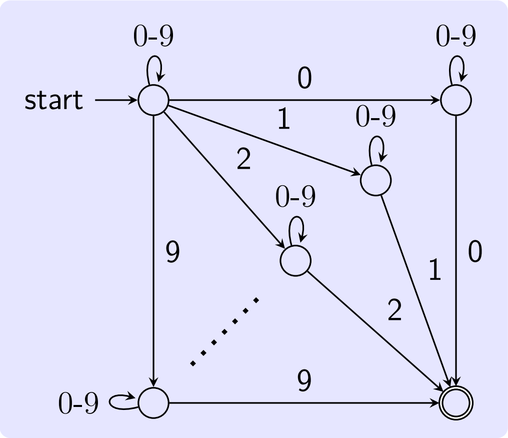
如果 $x \in L(A)$，则有 $\hat{\delta}(q_0, x) = q_f$，即 $k = 1$ 成立；假设 $k = n - 1$ 时，$x^k \in L(A)$ 成立，那么当 $k = n$ 时

$$\hat{\delta}(q_0, x^n) = \hat{\delta}(\hat{\delta}(q_0, x^{n-1}), x) = \hat{\delta}(q_f, x) = \hat{\delta}(q_0, x) = q_f$$

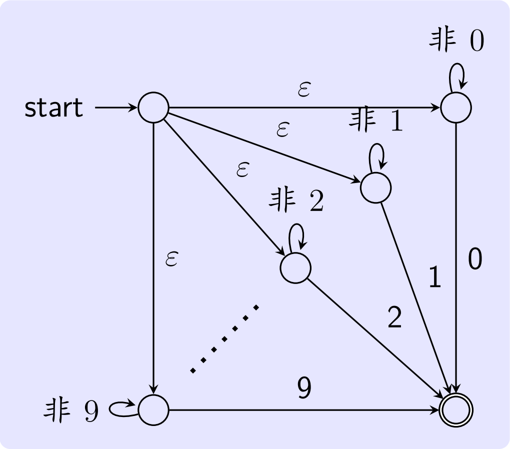
Exercise 2.3.4

Give NFA, try to take advantage of nondeterminism as much as possible.

a) The set of strings over alphabet $\{0, 1, \cdots, 9\}$ such that the final digit has appear before.

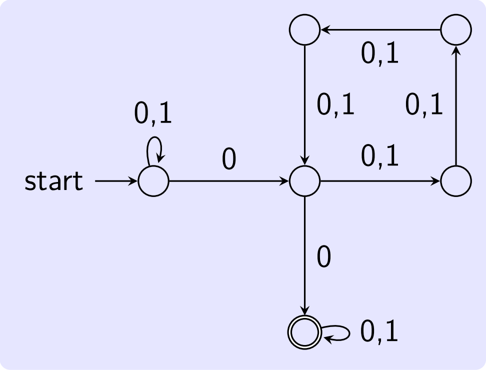


b) The set of strings over alphabet $\{0, 1, \cdots, 9\}$ such that the final digit has *not* appeared before.



Exercise 2.3.4

c) The set of strings of 0's and 1's such that there are two 0's separated by a number of positions that is a mutiple of 4. (Note that 0 is an allowable multiple of 4.)



Exercise 3.1.1

a) The set of strings over alphabet $\{a, b, c\}$ containing at least one a and at least one b.

$$(a+b+c)^*(a(a+b+c)^*b+b(a+b+c)^*a)(a+b+c)^*$$

b) The set of strings of 0's and 1's whose tenth symbol from the right end is 1.

$$(0+1)^*1(0+1)^9$$

c) The set of strings of 0's and 1's with at most one pair of consecutive 1's.

$$(0+10)^*(\varepsilon+1+11)(0+01)^*$$

Exercise 3.1.2

Write regular expressions for the following languages:

a) The set of all strings of 0's and 1's such that every pair of adjacent 0's appears before any pair of adjacent 1's.

$$(0 + 10)^*(01 + 1)^*(\varepsilon + 0)$$

b) The set of strings of 0's and 1's whose number of 0's is divisible by five.

$$(01^*01^*01^*01^*0 + 1)^*$$

Exercise 3.1.3

a) The set of all strings of 0's and 1's not containing 101 as a substring.

$0^*(1+000^*)^*0^*$ or $(0+\varepsilon)(1+000^*)^*(0+\varepsilon)$ or

$(0+\varepsilon)(1+00+000)^*(0+\varepsilon)$

b) The set of all strings with an equal number of 0's and 1's, such that no prefix has two more 0's than 1's, nor two more 1's than 0's.

$(01+10)^*$

c) The set of all strings of 0's and 1's whose number of 0's is divisible by five and whose number of 1's is even.

Exercise 3.1.4

Give English descriptions of the languages of the following regular expressions:

a) $(1+\varepsilon)(00^*1)^*0^*$

没有连续的 1

b) $(0^*1^*)^*000(0+1)^*$

有 3 个连续 0 的串

c) $(0+10)^*1^*$

任何连续 1 以后没有 0

Exercise 4.1.2

Prove that the following are not regular languages.

d) The set of strings of 0's and 1's whose length is a perfect square.

取 $w = 0^{N^2}$

e) The set of strings of 0's and 1's that are of the form ww, that is some string repeated.

取 $w = 0^N 1 0^N 1$

Exercise 4.2.2

If L is a language, and a is a symbol, then L/a, the quotient of L and a, is the set of strings w such that wa is in L. For example, if $L = \{a, aab, baa\}$, then $L/a = \{\varepsilon, ba\}$. Prove that if L is regular, so is L/a. Hint: Start with a DFA for L and consider the set of accepting states.

令 $L = L(M)$, 其中 $M = (Q, \Sigma, \delta, q_0, F)$

构造 $M' = (Q, \Sigma, \delta, q_0, F')$，其中 $F' = \{q \mid \delta(q, a) \in F\}$, $q \in Q$, $a \in \Sigma$. 先证明 $L(M') = L/a$, 再说明 $L(M')$ 正则

$\because \forall w \in L(M')$ 即 $\delta(q_0, w) \in F'$ 即 $\delta(\delta(q_0, w), a) \in F$,
$\therefore w \in L/a$ 又 $\because \forall w \in L/a$ 有 $wa \in L$ 即 $\delta(q_0, wa) \in F$ 即
$\delta(\delta(q_0, w), a) \in F$ 即 $\delta(q_0, w) \in F'$ $\therefore w \in L(M')$

Exercise 4.2.6 a)

Show that the regular languages are closed under the following operations:

min(L) = $\{w \mid w$ is in L, but no proper prefix of w is in L $\}$.

$$\delta'(q,a) = \begin{cases} \delta(q,a) & \text{if } q \notin F \\ \varnothing & \text{if } q \in F \end{cases} \tag{1}$$

证明 $L(M') = \min(L)$

1° $\forall w \in L(M')$ 存在转移序列 $q_0 q_1 \cdots q_n \in F$ 使 M' 接受 w 其中 $q_i \notin F, 0 \le i \le n-1$ $\therefore w \in \min(L)$

2° $\forall w \in \min(L)$ 有 $w \in L$，如果 M 接受 w 的状态序列为 $q_0 q_1 \cdots q_n \in F$ 则显然 $q_i \notin F, 0 \le i \le n-1$(因为否则，$w$ 有 L 可接受的前缀) $\therefore w \in L(M')$

Exercise 4.2.6 a)

$\min(L) = \{ w \mid w$ is in L, but no proper prefix of w is in L $\}$.

用封闭性证明

$$\min(L) = L - L\Sigma^+$$

Exercise 4.2.6 b)

max(L) $= \{\ w \mid w$ is in L and for no x other than ε is wx in $L\}$

由 $M = (Q, \Sigma, \delta, q_0, F)$ 构造 $M' = (Q, \Sigma, \delta, q_0, F')$ 其中

$$F' = \{f \mid f \in F, \forall x \in \Sigma^+, \hat{\delta}(f, x) \notin F\}$$

则 $L(M') = \max(L)$

Exercise 4.2.6 b)

max$(L) = \{\ w \mid w$ is in L and for no x other than ε is wx in $L\}$

利用封闭性。如果 $\Sigma = \{a, b, \cdots\}$，设 $\Gamma = \{a, \hat{a}, b, \hat{b}, \cdots\}$，定义同态 h $(\Gamma \to \Sigma^*)$ 和 g $(\Gamma \to \Sigma^*)$：

$$\begin{array}{ll} h(a) = a & g(a) = a \\ h(\hat{a}) = a & g(\hat{a}) = \varepsilon \\ h(b) = b & g(b) = b \\ h(\hat{b}) = b & g(\hat{b}) = \varepsilon \end{array}$$

那么

$$\max(L) = L - g(h^{-1}(L) \cap (a+b)^*(\hat{a}+\hat{b})^+)$$

Exercise 4.2.6 c)

init(L)={ w | for some x, wx is in L }

用同样的同态 h 和 g，则

$$\text{init}(L) = g(h^{-1}(L) \cap (a+b)^*(\hat{a}+\hat{b})^*)$$

Exercise 4.2.6 c)

init(L)={ w | for some x, wx is in L }

由 $M = (Q, \Sigma, \delta, q_0, F)$ 构造 $M' = (Q, \Sigma, \delta, q_0, Q - Q')$ 其中

$Q' = \{ \ q \mid q \in Q$, 没有从 q 到终态的路径 $\}$.

$$q \in Q - Q' \ \Leftrightarrow \ \exists x, \ \hat{\delta}(q, x) \in F$$

$$\forall w \in \Sigma^*, \hat{\delta}(q_0, w) \in Q - Q' \Leftrightarrow \exists x, \ \hat{\delta}(\hat{\delta}(q_0, w), x) \in F$$

即 $L(M') = \text{init}(L)$.

Exercise 5.1.3

Show that every regular laugnage is a context-free laugnage.
Hint: Construct a CFG by induction on the number of operators in the regular expression.

证明：对正则表达式 R 中运算符的个数 n 进行归纳。(1) 当 $n=0$ 时，R 只能是 ε, $\emptyset$ 或 a $(a \in \Sigma)$，可以构造仅有一条产生式的文法 $S \to \varepsilon$，$S \to \emptyset$ 或 $S \to a$ 得到。(2) 假设当 $n \leq m$ 时成立，当 $n = m+1$ 时，R 的形式只能由表达式 R_1 和 R_2 由连接、并或闭包形成：

(i) 若 $R = R_1 + R_2$，则 R_1 和 R_2 中运算符都不超过 m，所以都存在文法 G_1 和 G_2，分别开始于 S_1 和 S_2，只需构造新产生式和开始符号 $S \to S_1 \mid S_2$，连同 G_1 和 G_2 的产生式，构成 R 的文法；

(ii) 若 $R = R_1R_2$，则同理构造 $S \to S_1S_2$ 即可；

(iii) 若 $R = R_1^*$，则构造 $S \to SS_1 \mid \varepsilon$ 即可。

Exercise 7.1.7

Suppose G is a CFG with p productions, and no production body longer than n. Show that if $A \overset{*}{\underset{G}{\Rightarrow}} \varepsilon$, then there is a derivation of ε from A of no more than $(n^p - 1)/(n - 1)$ steps. How close can you actually come to this bound?

取 $A \overset{*}{\underset{G}{\Rightarrow}} \varepsilon$ 节点数最少的派生树，则任何从根节点到叶子的路径长度不超过 $p - 1$。因为否则会有重复变元，可以将重复变元之间的节点去掉，得到节点数更少的派生树。即树的高度最多为 $p - 1$，且第 k 层的内节点，最多为 n^k 个，因为产生式右部最长为 n。所以整个树的内节点数最多为 $1 + n + n^2 + \cdots + n^{p-1} = (n^p - 1)/(n - 1)$，而内节点数与推导的次数相等。