Some Beautiful and Useful Tricks in Mathematics

Wenbo V. Li
University of Delaware
E-mail: wli@math.udel.edu
Harbin, June, 2006

Abstract: We will present a collection of undergraduate level problems which can be solved by simple and powerful ideas/methods. Of course, this is purely based on my personal taste. This talk is suitable for both undergraduate and graduate students.

Intercepts of an interview of Terence Tao

Collaboration is very important for me, as it allows me to learn about other fields, and, conversely, to share what I learned about my own fields with others. It broadens my experience, not just in a technical mathematical sense but also in being exposed to other philosophies of research, of exposition, and so forth. Also, it is considerably more fun to work in groups than by oneself. Ideally, a collaborator should be close enough to one's own strengths that one can communicate ideas and strategies back and forth with ease, but far enough apart that one's skills complement rather than replicate each other.

It is true that mathematics is more specialized than at any time in its past, but I don't believe that any field of mathematics should ever get so technical and complicated that it could not (at least in principle) be accessible to a general mathematician after some patient work (and with a good exposition by an expert in the field). Even if the rigorous machinery is very complicated, often the ideas and goals of a field are often so simple, elegant, and natural that I feel it is frequently more than one's while to invest the time and effort to learn about other fields.

In fact, I believe that a subfield of mathematics has a better chance of staying dynamic, fruitful, and exciting if people in the area do make an effort to make good surveys and expository articles that try to reach out to other people in neighboring disciplines and invite them to lend their own insights and expertise to attack the problems in the area. The need to develop fearsome and impenetrable machinery in a field is a necessary evil, unfortunately, but as understanding progresses it should not be a permanent evil. If it serves to keep away other skilled mathematicians who might otherwise have useful contributions to make, then that is a loss for mathematics.

- “Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock our efforts. It should be to us a guidepost on the mazy path to hidden truths, and ultimately a reminder of our pleasure in the successful solution.”
—David Hilbert, 1900

- “It is not knowledge, but the act of learning, not possession but the act of getting there which generates the greatest satisfaction.”
—Carl Friedrich Gauss (1777-1855)

• Standard proof of $\sqrt{2}$ is irrational: Assume $\sqrt{2} = m/n$, then $m^2 = 2n^2$ and contradiction.

A different idea is to rewrite $m^2 = 2n^2$ as $(2n-m)^2 = 2(m-n)^2$, i.e.

$$\sqrt{2} = \frac{m}{n} = \frac{2n-m}{m-n}, \quad 2n > m > n$$

but $m - n < n$, a contradiction. This is a useful idea in number theory.

- Can an irrational power of an irrational number be rational?

Geometric Sum: It Can Be Summed

Fermat's idea of integration of $\int_0^1 x^\alpha dx$ before fundamental theorem of calculus is still very useful in many problems. Indeed, Fermat's original argument was not properly justified at the time. Note that the even partition approximation runs into the sum $\sum_{j=1}^n j^\alpha$, which is hard to sum.

Geometric Sum: Representation

It is very useful idea to change sum into integral since there are more tools available, such as substitution, integration by parts, etc. As an example, consider

$$\sum_{j=1}^{n} \frac{1}{j} = \log n - \gamma + o(1)$$

we can use representation

$$\frac{1}{j} = \int_0^1 x^{j-1} dx = \int_0^\infty e^{-jx} dx$$

More general, we have

$$\frac{1}{j^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-jx} dt.$$

More Examples

- The series $\sum_{j=1}^{\infty} \frac{1}{j^{\alpha}}$, $\alpha > 1$. In particular, the value of $\sum_{j=1}^{\infty} \frac{1}{j^3}$ is unknown.
- The series $\sum_{j=1}^{\infty} \frac{1}{(j+a)(j+b)}$.
- For non-integer x,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+x)^2} = \frac{\pi^2}{(\sin \pi x)^2}.$$

- Feynman's formula: For any $v = (v_1, \cdots, v_n) \in \mathbb{R}_+^n$,

$$\int_{S_+^{n-1}} \frac{1}{\langle x, y \rangle^n} \sigma_{n-1}(dy) = \frac{1}{(n-1)!} \cdot \frac{1}{\prod_{1 \leq j \leq n} v_j}.$$

Geometric Sum: Linearity and Representation

The simplest example is

$$\sum_{j=0}^{\infty} j x^j = \frac{x}{(1-x)^2}$$

The following theorem is known to Euler but the ingenious proof is due to Clarkson (1966).

Thm: The series $\sum_{p \text{prime}} \frac{1}{p}$ diverges

Pf: Assume convergence, then there is k such that

$$\sum_{j \geq k} \frac{1}{p_j} \leq \frac{1}{2}$$

where p_j means the j-th prime number. Let $Q = 2 \cdot 3 \cdot 5 \cdots p_k$. Then any prime divisor of $nQ + 1$ must be a prime larger than p_k, and thus

$$\sum_{n=1}^{\infty} \frac{1}{nQ+1} < \sum_{m=1}^{\infty} \left(\sum_{j \geq k} \frac{1}{p_j} \right)^m \leq \sum_{m=1}^{\infty} \left(\frac{1}{p_j} \right)^m = 1.$$

Contradiction!

- More refined result is known: There is a constant C such that

$$\sum_{p \le x} \frac{1}{p} = \ln(\ln x) + C + O\left(\frac{1}{\ln x}\right).$$

Infinite Checker Problem

Integer Length of Rectangles

A rectangle is cut into finite many smaller rectangles. If each smaller piece has at least one side with integer length, then the original rectangle has at least one side with integer length.

All Zeros of Orthogonal Polynomials are Real

Consider orthogonal polynomials $P_n(x)$ on $L^2([a,b],\mu(x))$, i.e.

$$\int_a^b P_n(x)P_m(x)dx\mu(dx) = 0 \quad if m \neq n$$

with degree of $P_n(x) = n$. We can think $\mu(dx) = w(x)dx$, $w(x) \geq 0$.

Representation for $|x|^p$

• Let X and Y be independent and identical distributed. Then for any $0 < p < 2$,

$$\mathbb{E}\,|X+Y|^p \geq \mathbb{E}\,|X-Y|^p.$$

• For i.i.d r.v's $\varepsilon_i = \pm 1$ with $\mathbb{P}(\varepsilon_i = \pm 1) = 1/2$ and i.i.d $\xi_i \sim N(0,1)$, then for $p \geq 2$

$$\mathbb{E}\left|\sum_{j=1}^{n} a_j \varepsilon_j\right|^p \leq \mathbb{E}\left|\sum_{j=1}^{n} a_j \xi_j\right|^p$$

Key representation:

$$|x|^p = c_p \int_0^\infty \frac{1-\cos(xt)}{t^{1+p}} dt$$

which also holds for $x \in \mathbb{R}^n$.

Perturbation

We all know monotonicity way of finding the limit of the sequence x_n defined by

$$x_{n+1} = 1 + \frac{1}{x_n}, \quad x_0 > 0.$$

What about the sequence define by

$$x_{n+1} = a_n + \frac{b_n}{x_n}, \quad x_0 > 0.$$

with $a_n \to 1$ and $b_n \to 1$, $a_n, b_n > 0$?

Inequality via Equality

Every inequality should have an equality behind it and most inequality should have a nice equality associated. Here is an example:

$$\left(\sum a_i b_i\right)^2 \le \sum a_i^2 \cdot \sum b_i^2.$$

The equality behind it is

$$\sum a_i^2 \cdot \sum b_i^2 - \left(\sum a_i b_i\right)^2 = \frac{1}{2}\sum_{i,j}(a_i b_j - a_j b_i)^2.$$

For convex function f, i.e. $f''(x) \ge 0$,

$$\lambda f(x) + (1-\lambda)f(y) = f(\lambda x + (1-\lambda)y) + \int_0^1 f''(\cdots)dt$$

Random Projection

Rectangle Box with fixed sum of three sides.