Univalent Functions & The Bieberbach Conjecture

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1 Introduction

This project serves as an introduction to geometric function theory, a topic in Complex Analysis. In particular, we focus on the theory of univalent functions. We begin by stating and proving basic mapping properties of univalent functions, in particular their connection with conformal mappings. After this, we focus on the class S, which is the set of normalized functions univalent and holomorphic on the open unit disk, and the closely related class Σ. We mention the *Riemann Mapping Theorem* and prove famous results such as *Gronwall's Area Theorem*, *Bieberbach's Theorem*, *Koebe One-Quarter Theorem* and *Littlewood's Thoerem*. Most of the results in this project, especially sections 3 and 4, can be found in the book by Duren[2], which covers significantly more material than this text. If one is interested in geometric function theory, especially in the work on *Bieberbach's Conjecture*, Duren [2] is a good read. One goal of this project was to "fill in the gaps" found in this book, as it was sometimes difficult to read for an undergraduate.

2 Definitions and Notation

Before we begin exploring the theory of univalent functions we should remind ourselves of the basic definitions in Complex Analysis, so that everyone is on the same page.

Definition 1. *We say a set $\Omega \subset \mathbb{C}$ is a* ***domain*** *if it is open and connected.*

Definition 2. *A continuous function $\gamma : [0,1] \to \mathbb{C}$ is called a* ***path****. We denote the image of γ by γ^*.*

Definition 3. *We say a domain Ω is **simply connected** if it is path connected and if for any two paths $p : [0,1] \to \Omega$, $q : [0,1] \to \Omega$ with $p(0) = q(0)$ and $p(1) = q(1)$, there exists a homotopy $F : [0,1] \times [0,1] \to \Omega$ such that $F(x,0) = p(x)$ and $F(x,1) = q(x)$.*

Definition 4. *The **circle** of radius $r > 0$, centered at $a \in \mathbb{C}$, denoted $\gamma(a,r)$, is the set*

$$\gamma(a,r) := \{z : |z-a| = r\}$$

Definition 5. *The **open disk** of radius $r > 0$, centered at $a \in \mathbb{C}$ denoted $D(a,r)$ is the set*

$$D(a,r) := \{z : |z-a| < r\}$$

Definition 6. *The **closed disk** of radius $r > 0$, centered at $a \in \mathbb{C}$, denoted $\overline{D}(a,r)$ is the set*

$$\overline{D}(a,r) := \{z : |z-a| \leq r\}$$

Definition 7. *The **punctured disk** of radius $r > 0$, centered at $a \in \mathbb{C}$ denoted $D'(a,r)$ is the set*

$$D'(a,r) := \{z : 0 < |z-a| < r\}$$

Definition 8. *We say a function $f : \Omega \to \mathbb{C}$ is **holomorphic** in a domain Ω, denoted $f \in H(\Omega)$, if the limit:*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists $\forall z_0 \in \Omega$.

Holomorphic functions have many beautiful properties. We mention some of them:

Theorem 9 (Liouville's Theorem). *If $f \in H(\mathbb{C})$ and bounded, then it is constant.*

Theorem 10 (Taylor's Theorem). *Let $f \in H(D(z_0,r))$ with $r > 0$. Then:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad z \in D(z_0,r)$$

Theorem 11 (Identity Theorem). *Let $f \in H(\Omega)$. Either all zeros of f are isolated, or $f \equiv 0$.*

Theorem 12 (Rouche's Theorem). *Let $f, g \in H(\Omega)$, and let γ be a positively oriented contour such that $\overline{Int\gamma} \subset \Omega$ and such that f has no zeros on γ and $|g(z)| < |f(z)|$ for all $z \in \gamma$. Then f and $f+g$ have the same number of zeroes inside γ.*

Proofs of these theorems can be found in any introductory book about Complex Analysis, for example in Priestley [1].

Definition 13. *Let $f \in H(\Omega)$. We say f is* ***conformal****, if $\forall z_0 \in \Omega$ we have $f'(z_0) \neq 0$.*

Recall that a special property of conformal mappings is that they preserve angles.

Definition 14. *We say a function $f : \Omega \to \mathbb{C}$ is* ***univalent****, if*

$$f(z_1) = f(z_2) \implies z_1 = z_2$$

Saying a function is univalent is the same as saying it is injective.

3 Elementary properties of univalent functions

Let us now look more closely on functions which are holomorphic and univalent on a domain. Most of the results in this section can be found in Priestley [1], chapter 16, but here we try to explain them in more detail.
Holomorphic, univalent functions have very nice mapping properties (as we will see in the *Riemann Mapping Theorem*), and are of great interest to mathematicians working in Complex Analysis. The first interesting result is that any such function is in fact conformal:

Theorem 15. *Let $f \in H(\Omega)$ be univalent, with Ω a domain. Then f is conformal in Ω.*

Proof. By contradiction. Suppose $\exists a \in \Omega$ such that $f'(a) = 0$. Since $f'(z)$ is holomorphic (every holomorphic function is infinitely differentiable), it is either identically zero or the zero at a is isolated (by Identity Theorem). If $f'(z) \equiv 0$, then $f(z)$ is constant since Ω is a domain. This contradicts the univalence of f. Thus the zero at a is isolated.
Choose $r > 0$ so that $\overline{D}(a, r) \in \Omega$ and $f'(z)$ is never zero in $D'(a, r)$ and on γ^* where $\gamma = \gamma(a, r)$. Let $m := \inf\{|f(z) - f(a)| : z \in \gamma^*\}$. Since f is continuous, m is attained and by univalence of f we have $m > 0$. Let $w \in D'(f(a), m)$. For $z \in \gamma^*$ we have:

$$|f(z) - f(a)| \geq m > |f(a) - w|$$

So by Rouche's Theorem, $f(z)-f(a)$ and $(f(z)-f(a)+f(a)-w) = f(z)-w$ have the same number of zeros (counting multiplicity) in $D(a, r)$. But $f(z) - f(a)$ has a zero of order atleast 2 at a (since $f'(a) = 0$). Since f is univalent, $f(z) - w$ can only have one zero. But this zero cannot be of order two, since $(f(z) - w)' = f'(z)$ and the only zero of f' inside the disk is at a, and clearly $f(a) \neq w$. Thus we have arrived at a contradiction. ∎

It is natural to ask if this holds the other way. It turns out, that this is not necessarily the case, but that conformality implies local univalence:

Theorem 16. *Let f be conformal in Ω where Ω is a domain. Then f is locally univalent.*

Proof. Choose $a \in \Omega$ arbitrarily. Since f is conformal, $f'(a) \neq 0$. Define $g(z) := f(z) - f(a)$. Clearly g is holomorphic and $g(a) = 0$. Since $g'(a) = f'(a) \neq 0$, we have $g \not\equiv 0$ and hence by the Identity Theorem, a is an isolated zero of g.
Thus we can choose $r > 0$ so that $\overline{D}(a, r) \in \Omega$ and $g(z)$ is never zero in $D'(a, r)$ and on γ^* where $\gamma = \gamma(a, r)$. Let $m := \inf\{|g(z)| : z \in \gamma^*\}$. By continuity of g the minimum is attained and by our choice of r we have $m > 0$. By the same argument as in Theorem 1, if $w \in D'(f(a), m)$, then $g(z)$ and $f(z) - w$ have the same number of zeros in $D(a, r)$.
Since f is continuous at a, there exists $\delta > 0$ so that:

$$|z - a| < \delta \implies |f(a) - f(z)| < m \tag{1}$$

Let $\delta_1 = \min(\delta, r)$. Let $z_1 \in D'(a, \delta_1)$ be arbitrary. We have $f(z_1) \in D'(f(a), m)$ and hence $g(z)$ and $f(z) - f(z_1)$ have the same number of zeros in $D(a, r)$. But by our choice of r, $g(z)$ has only one zero in $D(a, r)$. Thus $f(z) - f(z_1)$ has only one zero in $D(a, r)$ so $f(z) = f(z_1)$ has a unique solution. Since z_1 was arbitrary, f is univalent in $D(a, \delta_1)$.
Since a was arbitrary, f is locally univalent. □

These two results combined mean that in a domain, conformality and local univalence are in fact equivalent. The proofs of the above theorems introduce an idea, which we can now use to prove a fundamental property of non-constant, holomorphic functions, which will be used extenstively throughout this text:

Theorem 17 (Open Mapping Theorem)**.** *Let Ω be a domain and let $f \in H(\Omega)$ be univalent. Then $f(\Omega)$ is open.*

Proof. Let $a \in \Omega$ be arbitrary. We need to find $r > 0$ such that $D(f(a), r) \subset f(\Omega)$. Since f is a univalent function in a domain, it is conformal in Ω by Theorem 1. Define:

$$g(z) := f(z) - f(a)$$

Clearly $g(a) = 0$ and since $g'(z) = f'(z) \neq 0$ in Ω, a is an isolated 0 of g (By Identity Theorem). Thus we can choose $r > 0$ such that $g(z) \neq 0$ $\forall z \in D'(a, r)$ and $g(z) \neq 0$ $\forall z \in \gamma^*$ where $\gamma = \gamma(a, r)$. Let

$$m := \inf\{|g(z)| : z \in \gamma^*\}$$

By continuity of g, the infimum is attained and by our choice of r, we have $m > 0$. Let $w \in D'(f(a), m)$. Then by Rouche's Theorem, $g(z)$ and $f(z) - w$ have the same number of zeros in $D(a, r)$. Since $g(a) = 0$ (so g has atleast 1 zero), there exists $\xi \in D(a, r)$ such that $f(\xi) = w$. Hence $w \in f(\Omega)$. Thus $D(f(a), m) \subset f(\Omega)$. Since a was arbitrary, this proves $f(\Omega)$ is open. □

Note that we do not need f to be univalent in order for the Open Mapping Theorem to hold. It suffices if f is holomorphic and not constant, so that $g(z) \not\equiv 0$ and the proof is unchanged.
Since a univalent function has a well defined inverse, we would also like to understand its mapping properties.

Theorem 18 (Inverse function Theorem)**.** *Let Ω be a domain and let $f \in H(\Omega)$ be univalent. Then f^{-1} is holomorphic.*

Proof. First note that since f is univalent the inverse $g := f^{-1} : f(\Omega) \to \Omega$ is well defined. Let $G \subset \Omega$ be an open set. The pre-image of G under g is $f(G)$. By the Open Mapping Theorem, $f(G)$ is open. Thus $g(z)$ is continuous. Let $z_0 \in f(\Omega)$. Consider:

$$g'(z_0) = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{g(z) - g(z_0)}{f(g(z)) - f(g(z_0))} = \frac{1}{f'(g(z_0))}$$

where we used the fact that continuity of g means $z \to z_0 \implies g(z) \to g(z_0)$. Furthermore, since f is holomorphic and univalent in Ω, it is conformal by Theorem 1. Thus $f'(g(z_0)) \neq 0$, so $g'(z)$ exists and thus g is holomorphic. $\square$

Corollary 19. *Let Ω be a domain and let $f \in H(\Omega)$ be univalent. Then f^{-1} is conformal.*

Proof. Note that since f is univalent and Ω is a domain, the set $f(\Omega)$ is open and connected, hence a domain. Now we can apply Theorem 18 and Theorem 15 to f^{-1}. $\square$

These results combined tell us that a univalent function holomorphic in a domain is conformal, has a holomorphic inverse and the inverse is also conformal. It really is the 'perfect' function. One can also notice that a holomorphic univalent function $f : \Omega \to \mathbb{C}$ is in fact a homeomorphism between Ω and $f(\Omega)$, and so it preserves topological properties. In particular, the following property will be very useful:

Theorem 20. *Let Ω be a simply connected domain and let $f \in H(\Omega)$ be univalent. Then $f(\Omega)$ is a simply connected domain.*

Proof. By the Open Mapping Theorem, $f(\Omega)$ is open. Since f is continuous, $f(\Omega)$ is connected and thus it is a domain. It remains to prove $f(\Omega)$ is simply connected.
Let $p : [0,1] \to f(\Omega)$, $q : [0,1] \to f(\Omega)$ be paths with $p(0) = q(0)$ and $p(1) = q(1)$. Since f is univalent, f^{-1} is well defined and holomorphic. Consider the paths $f^{-1}(p)$ and $f^{-1}(q)$ (they are paths since f^{-1} is continuous). Clearly we have $f^{-1}(p(0)) = f^{-1}(q(0))$ and $f^{-1}(p(1)) = f^{-1}(q(1))$. Since Ω is simply connected, there exists a homotopy $F : [0,1] \times [0,1] \to \Omega$ such that $F(x,0) = f^{-1}(p(x))$ and $F(x,1) = f^{-1}(q(x))$. Letting

$$G := f \circ F$$

we can see that G is a homotopy with $G(x, 0) = p(x)$ and $G(x, 1) = q(x)$. Thus $f(\Omega)$ is simply connected. $\square$

We will make use of Theorem 20 quite often in later sections. Finally, we state one of the most important results in geometric function theory:

Theorem 21 (Riemann Mapping Theorem)**.** *Let G be a simply connected domain with $G \neq \mathbb{C}$. Then there exists a holomorphic bijection $f : D(0, 1) \to G$.*

The proof of The Riemann Mapping Theorem is beyond the scope of this project, but it is important to acknowledge its existence, since it is a beautiful result. Note that the condition $G \neq \mathbb{C}$ is necessary because of *Liouville's Theorem* (applied to the inverse). Also note that the function f is guaranteed to be conformal, since it is holomorphic and univalent. An immediate corollary is that given any two simply connected domains (which are not all of $\mathbb{C}$) there must exist a holomorphic bijection (which is also conformal) between these two domains. It is quite shocking that any two simply connected domains (which are not the whole plane) can be mapped conformaly onto each other.

3.1 Univalent functions and paths

In this section we would like to understand how univalent functions transform simple closed paths. Most of the results in this section may seem obvious, but it is important to state them, as we will make use of them when proving *Gronwall's Area Theorem*. This section also includes *Green's Theorem* in its complex form, which is used extensively throughout the text.

Lemma 22. *Let $f : \Omega \to \mathbb{C}$ be univalent and let γ be a simple closed path with $\gamma^* \subset \Omega$. Then $f(\gamma)$ is a simple closed path.*

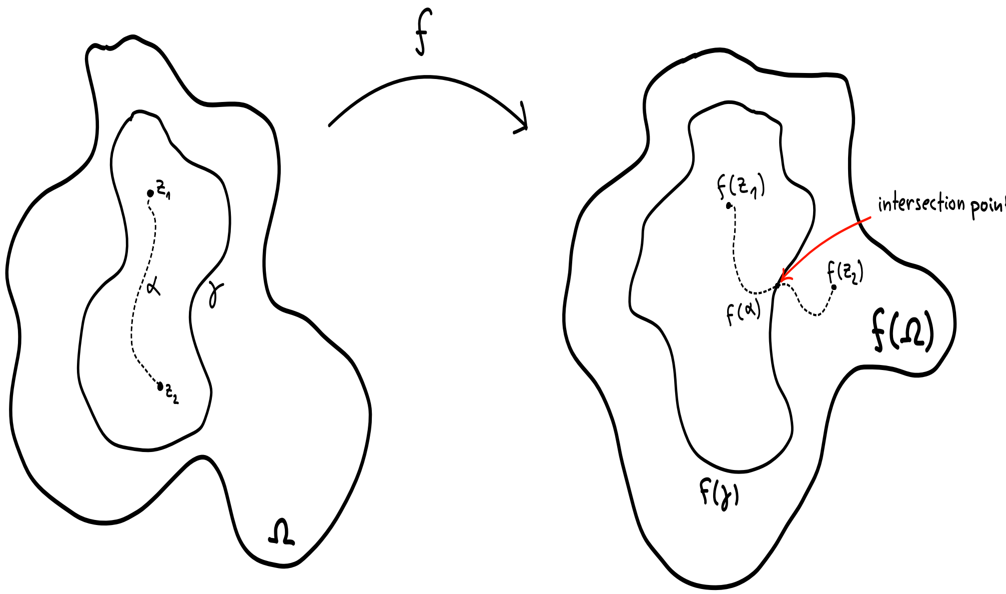
Proof. The fact that $f(\gamma)$ is a path follows from the holomorphicity of f, and it is clearly closed. Since f is univalent, $f(\gamma)$ is simple. $\square$

Recall that if γ is a simple closed path, then by the Jordan Curve Theorem, the complement of γ^* is a union of two disjoint domains, one of which is bounded and called the interior of γ, denoted $Int(\gamma)$, and the other is unbounded and called the exterior of γ, denoted $Ext(\gamma)$. Thus $\mathbb{C} = \gamma^* \dot{\cup} Int(\gamma) \dot{\cup} Ext(\gamma)$. In view of this fact, we would like to understand how univalent functions map the interior/exterior of a simple closed path.

Lemma 23. *Let $f : \Omega \to \mathbb{C}$ be univalent and let γ be a simple closed path, with $\gamma^* \subset \Omega$, and suppose $Int(\gamma) \cap \Omega$ is path connected. Let $z_1 \in Int(\gamma) \cap \Omega$. Then:*

$$\textit{If } f(z_1) \in Int(f(\gamma)) \textit{ then } f(Int(\gamma) \cap \Omega) \subset Int(f(\gamma))$$
$$\textit{If } f(z_1) \in Ext(f(\gamma)) \textit{ then } f(Int(\gamma) \cap \Omega) \subset Ext(f(\gamma))$$

Proof. We will only prove the first statement and note that the second follows by a similar argument.
By contradiction. Suppose that $f(z_1) \in Int(f(\gamma))$ but $f(Int(\gamma) \cap \Omega) \not\subset Int(f(\gamma))$. Then $\exists z_2 \in Int(\gamma) \cap \Omega$ such that $f(z_2) \in Ext(f(\gamma))$ (note that $f(z_2) \in f(\gamma)^*$ is not possible by univalence of f). Since $Int(\gamma) \cap \Omega$ is path connected, let α be a path connecting z_1 and z_2. Since $\alpha^* \subset Int(\gamma)$ the paths α and γ cannot intersect. Clearly $f(\alpha)$ is a path connecting $f(z_1)$ and $f(z_2)$. But since $f(z_1) \in Int(f(\gamma))$ and $f(z_2) \in Ext(f(\gamma))$, it must be the case that the paths $f(\gamma)$ and $f(\alpha)$ intersect.



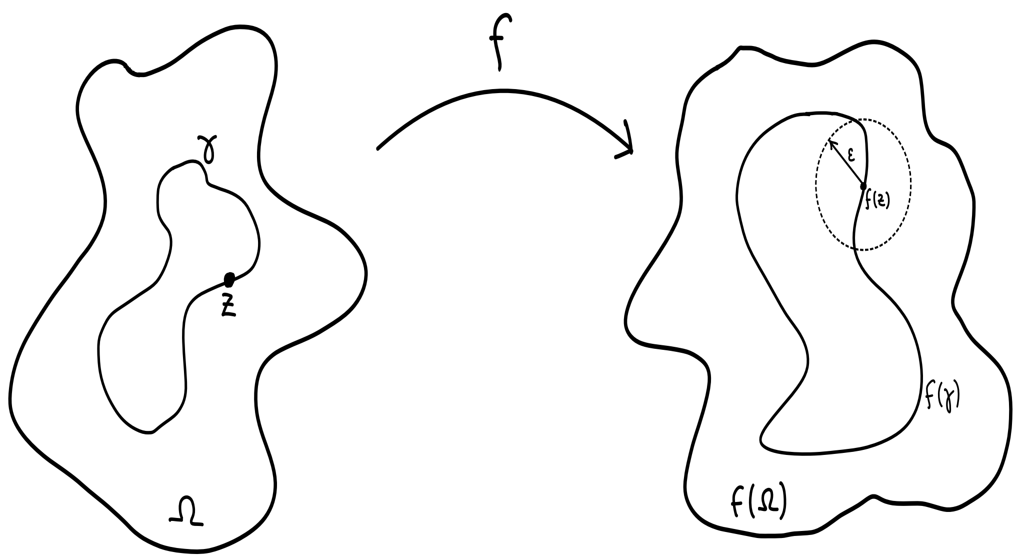
But this contradicts the univalence of f. $\square$

Note that an identical argument can be formed about $Ext(\gamma)$. Thus univalent functions have the property that points on the same "side" of a curve are mapped to the same "side" of the image curve. Another "obvious" observation is summarized in the following.

Lemma 24. *Let $f : \Omega \to \mathbb{C}$ be univalent, with Ω a domain. Let γ be a simple closed curve with $\gamma^* \subset \Omega$. Then:*

$$f(\Omega) \cap Int(f(\gamma)) \neq \emptyset$$
and
$$f(\Omega) \cap Ext(f(\gamma)) \neq \emptyset$$

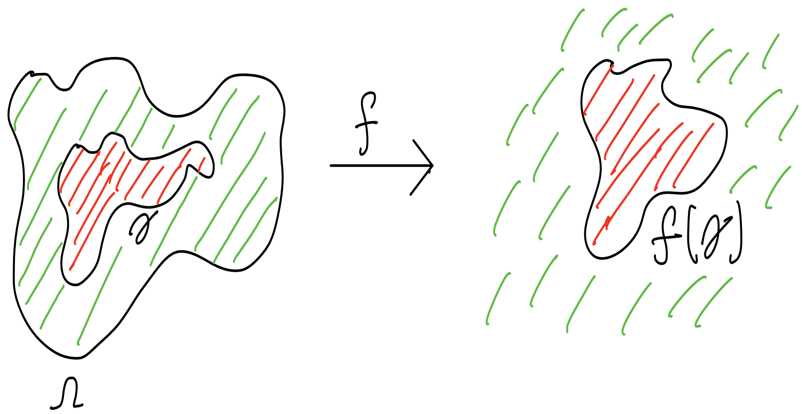
Proof. We only prove the first statement.
By contradiction. Suppose $f(\Omega) \cap Int(f(\gamma)) = \emptyset$. Let $z \in \gamma^*$ be arbitrary, so that $f(z) \in f(\gamma)^*$. Let $\epsilon > 0$ be arbitrary. Clearly the open ball $\{f(w) : |f(z) - f(w)| < \epsilon\}$ contains a point from $Int(f(\gamma))$:



But since $f(\Omega) \cap Int(f(\gamma)) = \emptyset$, this point cannot be in $f(\Omega)$ and thus must be in the complement. Thus this open ball is not contained in the image. Since ϵ was arbitrary, $f(\Omega)$ is not open. This contradicts the Open Mapping Theorem. A similar argument proves the second statement. □

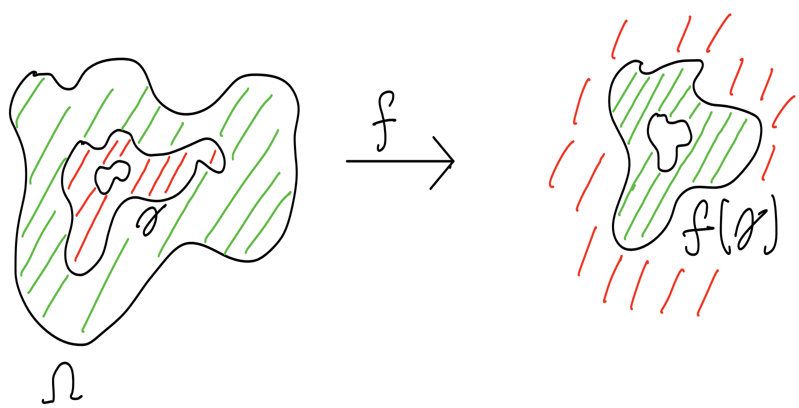
These two lemmas combined allow us to understand the mapping properties in more detail. Lemma 24 tells us that there exists a point $z \in \Omega$ such that $f(z)$ is in the interior of $f(\gamma)$ and another point $w \in \Omega$ such that $f(w)$ is in the exterior of $f(\gamma)$. By Lemma 23, it must be the case that one of these points is in the interior of γ and the other in the exterior of γ (they are not both on the same "side" of γ). If $z \in Int(\gamma)$ and $w \in Ext(\gamma)$. Then by Lemma 23 again, we have:

$$f(Int(\gamma) \cap \Omega) \subset Int(f(\gamma))$$
$$f(Ext(\gamma) \cap \Omega) \subset Ext(f(\gamma)$$



Similarly, if $z \in Ext(\gamma)$ and $w \in Int(\gamma)$, then

$$f(Ext(\gamma) \cap \Omega) \subset Int(f(\gamma))$$
$$f(Int(\gamma) \cap \Omega) \subset Ext(f(\gamma))$$



An example of a univalent function with these mapping properties is $f(z) = 1/z$.

Note that the pictures are only illustrative. We did not specify, whether the image is bounded/unbounded or simply connected.
Lastly, let us state and prove a fundamental result, which will later be essential in our study of univalent functions.

Theorem 25 (Green's Theorem)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a simple closed curve with:*

$$\gamma(t) = A(t) + iB(t) \quad A,B \text{ differentiable}$$

and let Ω be the domain enclosed by γ^*. Then:

$$\textit{Area of } \Omega = \frac{1}{2i}\int_\gamma \overline{z}dz$$

Proof. By definition of the complex line integral we have:

$$\begin{aligned}\int_\gamma \overline{z}dz &= \int_0^1 (A(t) - iB(t))(A'(t) + iB'(t))dt \\ &= \int_0^1 A(t)A'(t) + B(t)B'(t)dt + i\int_0^1 A(t)B'(t) - B(t)A'(t)dt\end{aligned}$$

We recognize that the two integrals can be written as line integrals in $\mathbb{R}^2$. Define the vector fields:

$$\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$$
$$\mathbf{G}(x,y) = -y\mathbf{i} + x\mathbf{j}$$

and let $\mathbf{r}(t) = A(t)\mathbf{i} + B(t)\mathbf{j}$. Then by Green's Theorem in $\mathbb{R}^2$ we have:

$$\begin{aligned}\int_\gamma \overline{z}dz &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt + i\int_0^1 \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt \\ &= \iint_\Omega \frac{\partial x}{\partial y} - \frac{\partial y}{\partial x}d\Omega + i\iint_\Omega \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}d\Omega \\ &= 2i\iint_\Omega d\Omega = 2i * \text{Area of } \Omega\end{aligned}$$

□

Corollary 26. *Let $f : \Omega \to \mathbb{C}$ be a univalent function and let γ be a simple closed curve with $\gamma^* \subset \Omega$. Let A be the area enclosed by $f(\gamma)$. Then:*

$$\textit{Area of } A = \frac{1}{2i}\int_\gamma \overline{f(z)}f'(z)dz$$

Proof. Since f is univalent, $f(\gamma)$ is a simple closed curve. Let $w = f(z)$. By Green's Theorem we have:

$$\text{Area of A } = \frac{1}{2i}\int_{f(\gamma)} \overline{w}dw$$

Substituting in $w = f(z)$ and noting that:

$$\begin{aligned} w &= f(z) \\ \implies \frac{dw}{dz} &= f'(z)\end{aligned}$$

we obtain:

$$\text{Area of A} = \frac{1}{2i}\int_\gamma \overline{f(z)}f'(z)dz$$

□

Corollary 2 will later prove to be very useful. Let us now proceed to study the class S, which is the main topic of this project.

4 Classes S and Σ

4.1 Class S

In this section we would like to study two important classes of univalent functions. The first class is the class S, which is the set of holomorphic, univalent functions $f : D(0,1) \to \mathbb{C}$ normalized by the conditions $f(0) = 0, f'(0) = 1$. By Taylor's Theorem, these functions have an expansion of the form:

$$f(z) = z + a_2 z^2 + \dots \quad |z| < 1$$

Note that if $g : D(0,1) \to \mathbb{C}$ is **any** holomorphic, univalent function, then it is conformal and $\frac{g(z)-g(0)}{g'(0)} \in S$, so g can be transformed into a function from S by adding a constant and rescaling. In view of the Riemann Mapping theorem, understanding the class S is essential to understanding conformal maps between simply connected regions. Let us begin by understanding the mapping properties:

Lemma 27. *Let $f \in S$ and let $f(D(0,1))$ denote the image of f. Then $f(D(0,1)) \neq \mathbb{C}$.*

Proof. Suppose $f(D(0,1)) = \mathbb{C}$. Since f is univalent, it has a holomorphic inverse, $f^{-1} : \mathbb{C} \to D(0,1)$. Thus f^{-1} is entire and bounded and hence constant by Liouville's Theorem. Contradiction. □

Lemma 28. *Let $f \in S$. Then $f(D(0,1))$ is open.*

Proof. Since f is a univalent function on a domain, $f(D(0,1))$ is open by the Open Mapping Theorem. □

If we let E_f denote the set of values omitted by f, meaning $E_f := \mathbb{C} \setminus f(D(0,1))$, we have that E_f is nonempty and closed. The set of omitted values will be important in the next section.

Lemma 29. *Let $f \in S$. Then $f(D(0,1))$ is simply connected.*

Proof. Since $D(0,1)$ is simply connected and f is univalent and holomorphic, $f(D(0,1))$ is simply connected. (By Theorem 20 in Section 3) □

Theorem 30. *Let $f \in S$ and let $\gamma = \gamma(0,r)$ for some $0 < r < 1$. Then the interior of γ, denoted $Int(\gamma)$, is mapped onto $Int(f(\gamma))$.*

Proof. We know $Int(\gamma) = D(0, r)$ is a simply connected domain. Since f is univalent and holomorphic, the image $f(D(0, r))$ must also be simply connected. We know that either:

$$f(D(0, r)) \subset Int(f(\gamma))$$

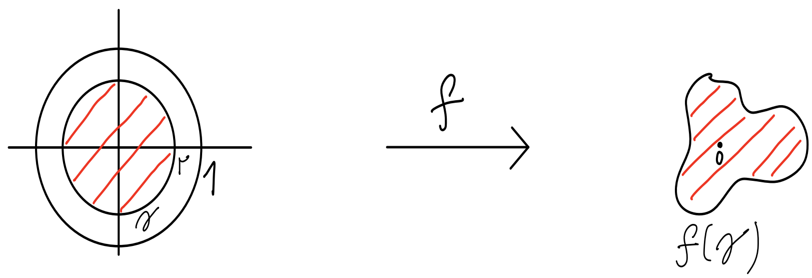
or

$$f(D(0, r)) \subset Ext(f(\gamma))$$

But $f(D(0, r)) \subset Ext(f(\gamma))$ is not possible, since $f(D(0, r))$ would not be simply connected. Thus $f(D(0, r)) \subset Int(f(\gamma))$. Furthermore, since the image of f is simply connected, we have $Int(f(\gamma)) \cap E_f = \emptyset$, so in fact we have:

$$f(D(0, r)) = Int(f(\gamma))$$

□



Note that since $0 \in D(0, r)$ and $f(0) = 0$, this means that $0 \in Int(f(\gamma))$. Also note, there can be no omitted values in the interior of the image of a cicle of radius $r < 1$ and thus all omitted values must lie in the exterior.

To summarize the mapping properties, $f \in S$ maps the unit disk onto an open, simply connected set, with $f(D(0, 1)) \neq \mathbb{C}$. Furthermore, for any curve enclosing 0, f is a bijection between the interior of this curve and the interior of the image curve.

Example 31 (Koebe Function)**.**

An important example of a function from S is the Koebe function, defined as:

$$k(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + \ldots$$

After a little bit of elementary algebra, it can be shown that:

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}$$

Let

$$g(z) := \frac{1+z}{1-z}$$

so that $k(z) = \frac{1}{4}g(z)^2 - \frac{1}{4}$. Observe that g maps $D(0,1)$ conformally onto the right half plane. To see that let's first check $Re(g(z)) > 0$. We have:

$$\begin{aligned} g(z) + \overline{g(z)} &= \frac{1+z}{1-z} + \frac{1+\overline{z}}{1-\overline{z}} \\ &= \frac{2(1-|z|^2)}{|1-z|^2} > 0 \end{aligned}$$

since $|z| < 1$. Next we show g is onto. Let $w \in \mathbb{C}$ with $Re(w) > 0$. We have:

$$\begin{aligned} \frac{1+z}{1-z} &= w \\ \implies \frac{1-w}{1+w} &= z \end{aligned}$$

and since $Re(w) > 0$, we have $|1-w| < |1+w|$, which can be seen geometrically. Thus the inverse function is indeed from the right half plane to the unit disk and so g is onto. Furthermore,

$$g'(z) = \frac{2}{(1-z)^2} \neq 0$$

Thus g maps $D(0,1)$ conformally onto the right half plane. This means $g^2(z)$ maps the unit disk conformally onto the plane with a cut along the negative real axis and hence $k(z)$ maps the unit disk conformally onto the plane with a cut from $(-\infty, -1/4)$. Furthermore, $k(z)$ is univalent as a composition of univalent functions. Note that $k(z)$ omits the value $z = -1/4$, which will be important later.

4.2 Class Σ

Another important class of functions is the class Σ, which is the set of holomorphic, univalent functions mapping the set $\Delta = \{z : |z| > 1\}$ to $\mathbb{C}$, with a Laurent Expansion of the form:

$$g(z) = z + b_0 + \frac{b_1}{z} + \dots \quad |z| > 1$$

We denote by Σ' the set of functions from the class Σ which omit the value 0. Understanding the mapping properties of functions from Σ' will allow us to later formulate *Gronwall's Area Theorem*, which is one of the most important theorems in this document. It turns out there is a one-to-one correspondence between the classes S and Σ', and by proving results about Σ', we will be able to deduce something about S.

Lemma 32 (Inverse Transform)**.** *Let $f \in S$ with*

$$f(z) = z + a_2 z^2 + \dots \quad |z| < 1$$

and define $g(z) := \frac{1}{f(1/z)}$. Then $g(z) \in \Sigma'$ and:

$$g(z) = z - a_2 + (a_2^2 - a_3)z^{-1} + \dots \quad |z| > 1$$

Proof. Since $f(0) = 0, f'(0) = 1$, f has a simple zero at $z = 0$. This means the function $h(z) := \frac{1}{f(z)} \in H(D'(0,1))$ has a simple pole at $z = 0$ and hence its Laurent Series is of the form:

$$h(z) = b_{-1}z^{-1} + b_0 + b_1 z + \dots \quad z \in D'(0,1)$$

Defining $g(z) := h(1/z)$, we have:

$$g(z) = b_{-1}z + b_0 + b_1 z^{-1} + \dots$$

with $g \in H(\Delta)$.
Now let's work out the coefficients in terms of the coefficients a_n. We have:

$$\begin{gathered} g(z)f(1/z) = 1 \\ (b_{-1}z + b_0 + b_1 z{-1} + \dots)(z^{-1} + a_2 z^{-2} + a_3 z^{-3} \dots) = 1 \\ \implies b_{-1} = 1, \quad b_0 + b_{-1}a_2 = 0, \quad b_{-1}a_3 + b_0 a_2 + b_1 = 0 \\ \implies b_0 = -a_2, \quad b_1 = a_2^2 - a_3 \end{gathered}$$

So we have:

$$g(z) = z - a_2 + (a_2^2 - a_3)z^{-1} + \dots$$

It remains to check that g is univalent. But this follows directly from the univalence of f

$$\begin{gathered} g(z_1) = g(z_2) \\ f(1/z_1) = f(1/z_2) \\ z_2 = z_1 \end{gathered}$$

Thus g is univalent. Furthermore, g omits 0 and hence $g \in \Sigma'$. $\square$

We can construct a similar argument the other way around:

Lemma 33. *Let $g \in \Sigma'$. Let $f : D(0,1) \to \mathbb{C}$ be defined as*

$$f(z) = \begin{cases} 0 & z = 0 \\ \frac{1}{g(1/z)} & |z| < 1 \end{cases}$$

Then $f \in S$.

Proof. Since g is never equal to 0, f is well defined. Furthermore since $g \in \Sigma'$, we have:

$$\begin{aligned} g(z) &= z + b_0 + b_1 z^{-1} + \dots \quad z \in \Delta \\ \implies g(1/z) &= 1/z + b_0 + b_1 z + \dots \quad z \in D'(0,1) \end{aligned}$$

and thus:

$$\begin{aligned} f'(0) &= \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} \\ &= \lim_{z \to 0} \frac{1}{g(\frac{1}{z})z} \\ &= \lim_{z \to 0} \frac{1}{z(1/z + b_0 + b_1 z + \dots)} \\ &= \lim_{z \to 0} \frac{1}{1 + b_0 z + b_1 z^2 + \dots} \\ &= 1 \end{aligned}$$

so f is differentiable at 0 with $f'(0) = 1$. If $z \neq 0$, f is differentiable by the chain rule and thus f is a holomorphic function in $D(0,1)$, with $f(0) = 0, f'(0) = 1$. Additionally, f is also univalent, since g is univalent. Thus $f \in S$.

□

This means that any function from Σ' can be written as:

$$g(z) = \frac{1}{z} \circ f \circ \frac{1}{z} \quad |z| > 1$$

where $f \in S$, by defining f as in the previous lemma. Similarly, by defining $g \in \Sigma'$ on the Riemann Sphere with $g(\infty) = \infty$, we could express every $f \in S$ as:

$$f(z) = \frac{1}{z} \circ g \circ \frac{1}{z} \quad |z| > 1$$

Armed with this correspondence between the classes S and Σ', let us now explore some mapping properties of functions from Σ', based on mapping properties of functions from the class S.

Theorem 34. *Let $g \in \Sigma'$ and let $\gamma = \gamma(0, r)$ for some $r > 1$. Then $Int(\gamma) \cap \Delta$ is mapped into $Int(g(\gamma))$.*

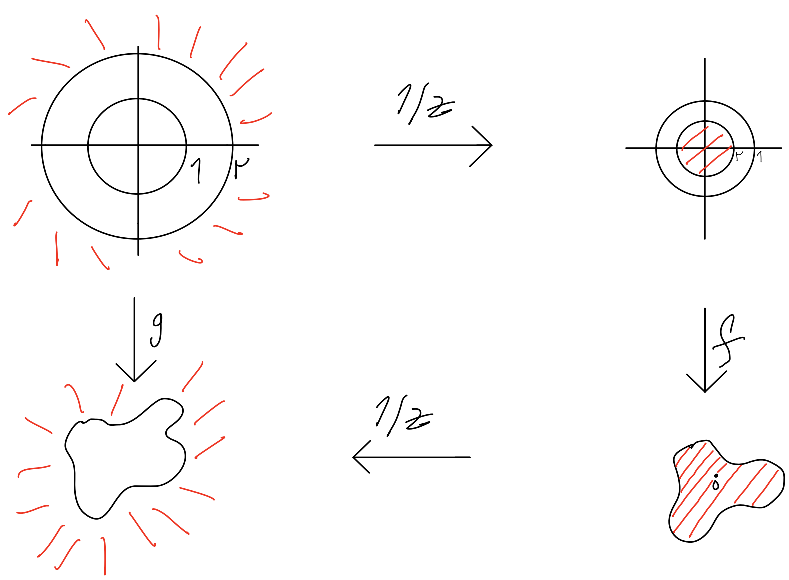
Proof. We will prove $Ext(\gamma)$ is mapped into $Ext(g(\gamma))$ and the result will follow. Define:

$$f(z) = \begin{cases} 0 & z = 0 \\ \frac{1}{g(1/z)} & |z| < 1 \end{cases}$$

We know $f \in S$. We can parametrise γ as $\gamma(t) = re^{it}$, $t \in (-\pi, \pi]$ and note that:

$$g(re^{it}) = \frac{1}{f(\frac{1}{r}e^{-it})}$$

where $1/re^{-it}$ is a circle centered at 0 with radius $1/r$. We denote this circle by $1/\gamma$. The exterior of γ is the set of points z with $|z| > r$, which means $|1/z| < 1/r$. So to figure out where the exterior of γ is mapped under g, it suffices to figure out where $Int(1/\gamma) = D(0, 1/r)$ is mapped under f.



By the mapping properties of $f \in S$, we know $f(D(0, 1/r)) = Int(f(1/\gamma))$, which contains $f(0) = 0$. Since $f(D(0, 1/r))$ is an open set (by Open Mapping Theorem), there exists $\delta > 0$ such that $D(f(0), \delta) \subset f(D(0, 1/r))$. Let $N \in \mathbb{N}$ be such that $1/N < \delta$:

$$n \geq N \implies D(f(0), 1/n) \subset f(D(0, 1/r))$$

This allows us to extract a sequence converging to $f(0)$. More precisely, we can find $\{z_n\} \subset D'(0, 1/r)$ such that:

$$|f(z_n) - f(0)| < 1/n \quad \forall n \geq N$$

Clearly the sequence $\{z_n\}$ satisfies $|z_n| < 1/r$ for all $n \geq N$, which means $|1/z_n| > r$ for all $n \geq N$. Hence the sequence $\{1/z_n\}$ is contained it the exterior of γ. Furthermore, by definition of f:

$$|g(1/z_n)| = \left| \frac{1}{f(z_n)} \right| > n \quad \forall n \geq N$$

Since $Int(g(\gamma))$ is bounded, there exists $N_1 \in \mathbb{N}$, such that:

$$n \geq N_1 \implies g(1/z_n) \notin Int(g(\gamma))$$

In particular, we have $g(1/z_{N_1}) \in Ext(g(\gamma))$. By Lemma 23 in Section 3.1, this means the entire exterior of γ is mapped to the exterior of $g(\gamma)$.
And by the mapping properties of univalent functions (again Section 3.1), we conclude that $\Delta \cap Int(\gamma)$ mapped into $Int(g(\gamma))$. $\square$

The next theorem is absolutely essential in the proof of *Gronwall's Area Theorem* in the following section.

Theorem 35. *Let $g \in \Sigma'$ and let $\gamma = \gamma(0, r)$ for some $r > 1$. Then:*

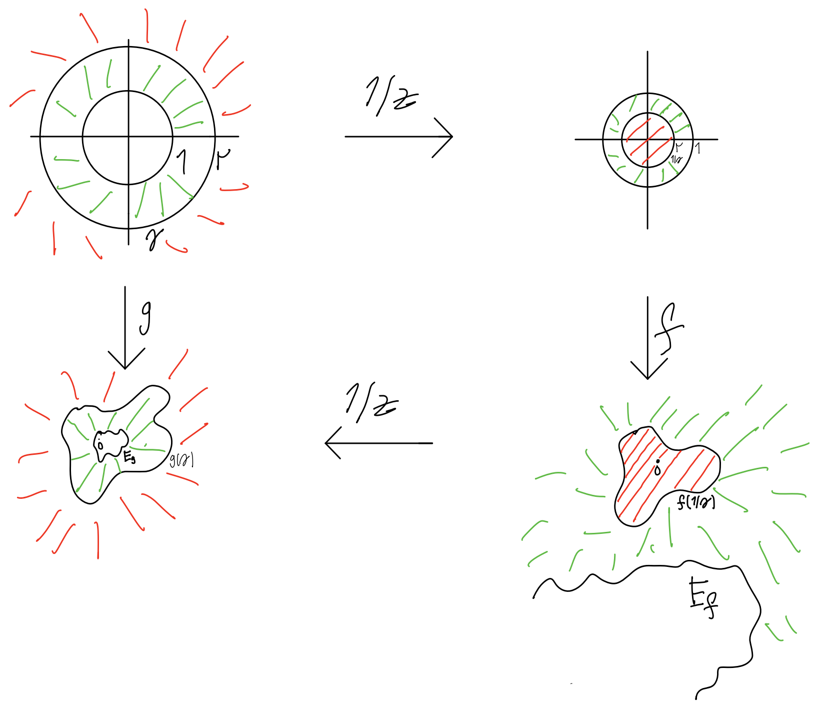
$$E_g \subset Int(g(\gamma))$$

where E_g is the set of values omitted by g.

Proof. Define f by

$$f(z) = \begin{cases} 0 & z = 0 \\ \frac{1}{g(1/z)} & |z| < 1 \end{cases}$$

we know $f \in S$ and clearly $g(z) = \frac{1}{f(1/z)}$. By the mapping properties of functions from the class S, we know that E_f, the set of values omitted by f, must lie in the exterior of the curve $f(1/\gamma)$, with no points in the interior. Applying the transformation $1/z$, this means that E_g must lie inside the curve $g(\gamma)$. $\square$



The reason why this observation is so important, is that now we can approximate the area of E_g by using *Green's Theorem* and enclosing E_g in smaller and smaller curves. Since the interior of every curve is mapped to the interior of the image curve, we can approximate E_g by considering the image of the circle $\gamma_r = \gamma(0, r)$ with $r > 1$ and taking the limit as $r \to 1$. By computing the area of E_g, we will be able to say something about the coefficients in the Laurent Expansion.

The results in this section can be summarized by the following: For any $r > 1$, $g \in \Sigma'$ maps the annulus $1 < |z| < r$ into the interior of a simple closed curve $g(\gamma)$, which must enclose the compact set E_g (E_g is compact since it is closed and bounded). Furthermore, any function from Σ can be transformed into a function from Σ' by adding a constant. Since adding a constant only shifts the image, these mapping properties apply to all functions from the class Σ.

Now that we fully understand the mapping properties of functions from classes S and Σ, we can start proving theorems, which will tell us something about the magnitude of the coefficients in the Laurent Expansions of our functions.

4.3 Estimating the coefficients

Let us begin by refreshing some basic results about power series. We state these without proof as they are all standard results.

Lemma 36. *Let $\sum_{i=0}^{\infty} a_n z^n$ be a power series with radius of convergence R. Then for all z with $|z| \leq R_1 < R$, the power series converges uniformly.*

Lemma 37. *Let $\gamma : [a, b] \to \mathbb{C}$ be a path and let $u_k(z)$ be a family of continuous functions such that $\sum_{k=0}^{\infty} u_k(z)$ converges uniformly on γ^*. Then:*

$$\sum_{k=0}^{\infty} \int_{\gamma} u_k(z) dz = \int_{\gamma} \left(\sum_{k=0}^{\infty} u_k(z) \right) dz$$

Theorem 38. *(Mertens' Theorem) Let $\sum_{k=0}^{\infty} a_k$ be an infinite series converging to A and let $\sum_{k=0}^{\infty} b_k$ converge to B. Suppose one of the series converges absolutely. Then the Cauchy Product:*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

converges to AB.

Lemma 39. *Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence R_1 and let $\sum_{n=0}^{\infty} b_n z^n$ have radius of convergence R_2. Then the Cauchy Product:*

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} a_k b_{n-k}$$

converges uniformly for all z with $|z| \leq R' < \min(R_1, R_2)$

Proof. Since any power series converges absolutely within its radius of convergence, by Mertens' Theorem, the Cauchy Product converges for $|z| < \min(R_1, R_2)$. The Cauchy Product is itself a power series, so using Lemma 36 we get uniform convergence. □

Now we can finally state and prove *Gronwall's Area Theorem.*

Theorem 40 (Area Theorem)**.** *Let $g \in \Sigma$ with:*

$$g(z) = z + b_0 + b_1 z^{-1} + \ldots |z| > 1$$

then we have:

$$\sum_{n=0}^{\infty} n|b_n|^2 \leq 1$$

Proof. Let E_g be the set of values omitted by g. Let $\gamma_r = \gamma(0, r)$ for some $r > 1$ and $C_r = g(\gamma_r)$. We know C_r is a simple closed curve enclosing a domain E_r with $E_g \subset E_r$. By the Corollary of Green's Theorem (Corollary 26), we have:

$$\text{Area of } E_r = \frac{1}{2i} \int_{\gamma_r} \overline{g(z)} g'(z) dz$$

Substituting $z = re^{i\theta}$ and expanding g as a series and differentiating term by term, we obtain:

$$\begin{aligned}
&= \frac{1}{2i} \int_0^{2\pi} \left(re^{-i\theta} + \sum_{n=0}^{\infty} \overline{b_n} r^{-n} e^{ni\theta} \right) \left(1 - \sum_{k=1}^{\infty} k b_k r^{-k-1} e^{-(k+1)i\theta} \right) ire^{i\theta} d\theta \\
&= r^2\pi + \frac{1}{2} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} \overline{b_n} r^{-n+1} e^{(n+1)i\theta} - \sum_{k=1}^{\infty} k b_k r^{-k+1} e^{-(k+1)i\theta} \right) d\theta \\
&- \frac{1}{2} \int_0^{2\pi} \left(\sum_{n=0}^{\infty} \overline{b_n} r^{-n} e^{ni\theta} \sum_{k=1}^{\infty} k b_k r^{-k} e^{-ki\theta} \right) d\theta
\end{aligned}$$

By the Lemmas proved before, the series converge uniformly and hence we can bring the integrals inside the sums. But we have:

$$\int_0^{2\pi} \overline{b_n} r^{-n+1} e^{(n+1)i\theta} d\theta = 0 \quad \forall n \in \mathbb{N}$$

and

$$\int_0^{2\pi} k b_k r^{-k+1} e^{-(k+1)i\theta} d\theta = 0 \quad \forall k \in \mathbb{N}$$

Thus it remains to integrate the product of the two sums. Using the Cauchy Product, we have:

$$\begin{aligned}
&\int_0^{2\pi} \left(\sum_{n=0}^{\infty} \overline{b_n} r^{-n} e^{ni\theta} \sum_{k=1}^{\infty} k b_k r^{-k} e^{-ki\theta} \right) d\theta \\
&= \int_0^{2\pi} \left(\sum_{n=1}^{\infty} \overline{b_n} r^{-n} e^{ni\theta} \sum_{k=1}^{\infty} k b_k r^{-k} e^{-ki\theta} \right) d\theta \\
&= \int_0^{2\pi} \left(\sum_{m=1}^{\infty} \sum_{c=1}^{m} c b_c \overline{b}_{m-c} r^{-m} e^{(m-2c)i\theta} \right) d\theta
\end{aligned}$$

Bringing the integral inside the sums (again we have uniform convergence by the previous Lemma) we find:

$$= \sum_{m=1}^{\infty} \sum_{c=1}^{m} c b_c \overline{b}_{m-c} r^{-m} \int_0^{2\pi} e^{(m-2c)i\theta} d\theta$$

But we have:

$$\int_0^{2\pi} e^{(m-2c)i\theta} d\theta = \begin{cases} 0 & m \neq 2c \\ 2\pi & m = 2c \end{cases}$$

So we get:

$$= 2\pi \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n}$$

Collecting all the terms, we find:

$$\text{Area of } E_r = \pi \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right)$$

We know that the interior of γ_r is mapped into $Int(g(\gamma_r))$ and this is true for every $r > 1$. So as we decrease r, we will approximate the area of E_g. More precisely, we have:

$$\begin{aligned} \text{Area of } E_g &= \lim_{r \to 1^+} \text{ Area of } E_r \\ &= \lim_{r \to 1^+} \pi \left(r^2 - \sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \right) \\ &= \pi \left(1 - \sum_{n=1}^{\infty} n|b_n|^2 \right) \end{aligned}$$

But since the area of E_g (or outer measure) is not negative, we have:

$$\sum_{n=0}^{\infty} n|b_n|^2 \leq 1$$

as claimed. □

One has to be careful when computing limits of infinite series and the computation of this limit deserves an explanation, as it is not completely obvious. It becomes obvious when we consider partial sums. Fix an arbitrary $N \in \mathbb{N}$ and define:

$$S_N(r) := \sum_{n=1}^{N} n|b_n|^2 r^{-2n}$$

we know:

$$\begin{aligned} &\text{Area of } E_r \geq 0 \\ \implies r^2 - &\sum_{n=1}^{\infty} n|b_n|^2 r^{-2n} \geq 0 \quad r > 1 \\ &\implies r^2 - S_N(r) \geq 0 \quad r > 1 \end{aligned}$$

Since $r^2 - S_N(r)$ is continuous for $r \in (0, \infty)$, we have:

$$\begin{aligned} 1 - S_N(1) &\geq 0 \\ \implies 1 &\geq S_N(1) \end{aligned}$$

Since N was arbitrary, the sequence of partial sums is bounded and increasing and thus convergent. Hence $\sum_{n=1}^{\infty} n|b_n|^2$ exists, and:

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$$

as claimed.
From this, we can formulate an obvious result:

Corollary 41. *Let $g \in \Sigma$. Then $b_1 \leq 1$ with equality if and only if g has the form:*

$$g(z) = z + b_0 + b_1/z \quad |b_1| = 1$$

Proof. Follows immediately from the Area Theorem. □

The Area Theorem and its corollary will allow us to obtain bounds for coefficients of functions from the class S. A first estimate for the coefficients of $f \in S$ is obtained in Bieberbach's Theorem, but before we prove it we need a few auxiliary results:

Lemma 42. *Let $f \in H(\Omega)$ where Ω is a simply connected domain. Suppose $0 \notin f(\Omega)$. Then there exists a function $g \in H(\Omega)$ such that $f(z) = \exp(g(z))$.*

Proof. Let $z_0 \in \Omega$ be arbitrary. Since $f(z_0) \neq 0$, we can find $\xi \in \mathbb{C}$ such that $\exp(\xi) = f(z_0)$. Define:

$$g(z) = \xi + \int_{\gamma} \frac{f'(w)}{f(w)} dw$$

where γ is a path from z_0 to z. First we check $g(z)$ is well defined. Since Ω is simply connected and $\frac{f'(w)}{f(w)} \in H(\Omega)$, by Cauchy's Theorem the integral of $\frac{f'(w)}{f(w)}$ along any contour is 0. This implies path independence. Hence $g(z)$ is well defined.
By the Fundamental Theorem of Calculus, $g'(z) = \frac{f'(z)}{f(z)}$, so $g \in H(\Omega)$ and:

$$\frac{d}{dz}[f(z)\exp(-g(z))] = f'(z)\exp(-g(z)) - f(z)\frac{f'(z)}{f(z)}\exp(-g(z)) = 0$$

Since Ω is a domain, this implies $f(z)\exp(-g(z)) = c$ for some constant c. But $g(z_0) = \xi$, so we have $f(z_0)\exp(-\xi) = c$ and hence $c = 1$ by definition of ξ. Thus $f(z) = \exp(g(z))$.

□

Thus for any function, which omits 0 and is holomorphic on a simply connected domain, there exists a branch of the logarithm which is holomorphic on the same domain. This is an extremely useful result, which will also be used later on. We can use it now to prove another lemma:

Lemma 43. *Let $f \in H(\Omega)$ where Ω is a simply connected domain. Suppose $0 \notin f(\Omega)$. Then for any $n \in \mathbb{N}, n \geq 2$, there exists a function $h \in H(\Omega)$ such that $h(z)^n = f(z)\ \forall z \in \Omega$*

Proof. By the previous Lemma, there exists $g \in H(\Omega)$ such that $f(z) = \exp(g(z))$. Now define:

$$h(z) := \exp(g(z)/n)$$

then $h(z)^n = f(z)$ and clearly $h \in H(\Omega)$. □

Even though we only use Lemma 43 for proving *Bieberbach's Theorem* it is a neat result and a nice trick to remember. I imagine this can be used in many ways. Using this Lemma, we can prove a theorem which will be used quite often later on:

Theorem 44 (Koebe Transform)**.** *Let $f \in S$ and $n \in \mathbb{N}$, $n \geq 2$. Then there exists a function $G \in S$ such that $G(z)^n = f(z^n)$ for all $|z| < 1$.*

Proof. Since $f \in S$, we have:

$$\begin{aligned} f(z^n) &= z^n + a_2 z^{2n} + \dots \quad |z| < 1 \\ &= z^n(1 + a_2 z^n + \dots) \end{aligned}$$

Let

$$h(z) := 1 + a_2 z^n + \dots$$

so that $f(z^n) = z^n h(z)$. Note that $h(0) \neq 0$ and thus $0 \notin h(D(0,1))$ (otherwise f would have another 0, contradicting univalence). Thus by the previous Lemma, there exists $g(z) \in H(D(0,1))$ such that $g(z)^n = h(z)$. Furthermore, we can define g so that $g(0) = 1$ (see proof Lemma 43). So clearly we have:

$$f(z^n) = z^n h(z) = (zg(z))^n$$

So let $G(z) = zg(z)$. We need to show $G(z) \in S$. Clearly $G(z) \in H(D(0,1))$ $G(0) = 0, G'(0) = g(0) = 1$ so it remains to check univalence. We have:

$$\begin{aligned} & G(z_1) = G(z_2) \\ \implies & G(z_1)^n = G(z_2)^n \\ \iff & f(z_1^n) = f(z_2^n) \\ \implies & z_1^n = z_2^n \\ \implies & z_1 = z_2 e^{i2\pi k/n} \quad k = 0, 1, \dots n-1 \end{aligned}$$

Since $G(z)^n = f(z^n)$ the Taylor series of G must be of the form:

$$G(z) = z + g_{n+1}z^{n+1} + g_{2n+1}z^{2n+1} + \dots \quad |z| < 1$$

This can be seen by considering the equation $G(z)^n = f(z^n)$ and comparing coefficients. Continuing, we have:

$$\begin{aligned} G(z_2) = G(z_1) &= G(z_2 e^{i2\pi k/n}) \\ &= e^{i2\pi k/n}(z_2 + g_{n+1}z_2^{n+1} + g_{2n+1}z_2^{2n+1} + \dots) \\ &= e^{i2\pi k/n}G(z_2) \end{aligned}$$

So $G(z_2) = e^{i2\pi k/n}G(z_2)$ which means that either $k = 0$, in which case $z_1 = z_2$ or $G(z_2) = 0$. But if $G(z_2) = 0$, we have $0 = G(z_2)^n = f(z_2^n)$ and since f is univalent and $f(0) = 0$, we have $z_2 = 0$ and hence also $z_1 = 0$. In either case, $z_1 = z_2$ and thus G is also univalent. This proves $G \in S$. □

Armed with this the Koebe Transform Theorem, we can proceed to prove the next theorem, which is famous result due to Bieberbach. The proof is quite simple, as we only need to suitably apply the previous lemmas.

Theorem 45 (Bieberbach's Theorem)**.** *Let $f \in S$ with $f(z) = z + a_2z^2 + \dots$. Then $|a_2| \leq 2$, with equality occuring if and only if f is a rotation of the Koebe function.*

Proof. Applying the Koebe Transform Theorem with $n = 2$, we can find $g \in S$ such that $g^2(z) = f(z^2)$. Since $f, g \in S$, we have:

$$\begin{aligned} f(z) &= z + a_2z^2 + \dots \quad |z| < 1 \\ g(z) &= z + b_2z^2 + \dots \quad |z| < 1 \end{aligned}$$

and using $g^2(z) = f(z^2)$ we see that:

$$\begin{aligned} (z + b_2z^2 + b_3z^3 + \dots)(z + b_2z^2 + b_3z^3 + \dots) &= z^2 + a_2z^4 + \dots \quad |z| < 1 \\ \implies b_2 = 0, b_3 = a_2/2 \end{aligned}$$

Furthermore, since $g \in S$ we can apply the Inverse Transform Lemma (Lemma 32) to get $h(z) := \frac{1}{g(1/z)}$ so that $h \in \Sigma'$ and

$$h(z) = z - \frac{a_2}{2}z^{-1} + \dots$$

Since $h \in \Sigma'$, we can apply the Corollary of the Area Theorem (Corollary 41) to get:

$$\begin{aligned} |a_2/2| &\leq 1 \\ |a_2| &\leq 2 \end{aligned}$$

Equality occurs iff $h(z)$ is of the form:

$$h(z) = z - e^{i\theta}z^{-1}$$

for some $\theta \in (-\pi, \pi]$, in which case we have:

$$\begin{aligned}
g(1/z) &= \frac{1}{z - e^{i\theta}z^{-1}} \quad |z| > 1 \\
\implies g(z) &= \frac{z}{1 - e^{i\theta}z^2} \quad |z| < 1 \\
\implies g^2(z) = f(z^2) &= \frac{z^2}{(1 - e^{i\theta}z^2)^2} \\
\implies f(\xi) &= \frac{\xi}{(1 - e^{i\theta}\xi)^2} \\
&= e^{-i\theta}k(e^{i\theta}\xi)
\end{aligned}$$

so f is a rotation of the Koebe function. □

It is important to note that the Koebe function is "special" in this sense. We will observe this quite often, that the Koebe function "maximises" a certain property.
Bieberbach's Theorem will serve as a very useful tool and will allow us to extract information about functions $f \in S$. We can use it to deduce, which values cannot be omitted by functions in S. Before we do that, we need a short lemma:

Lemma 46 (Omitted value transformatiom). *Let $f \in S$. Suppose there exists $w \in \mathbb{C}$ such that $w \notin f(D(0, 1))$. Let:*

$$g(z) := \frac{wf(z)}{w - f(z)}$$

Then $g \in S$.

Proof. Since $w \notin f(D(0, 1))$, g is well defined and $g \in H(D(0, 1))$. Clearly $g(0) = 0$ and:

$$\begin{aligned}
g'(z) &= \frac{wf'(z)(w - f(z)) + wf(z)f'(z)}{(w - f(z))^2} \\
\implies g'(0) &= \frac{w^2}{w^2} = 1
\end{aligned}$$

Univalence follows from the univalence of f:

$$\begin{aligned}
g(z_1) &= g(z_2) \\
\frac{wf(z_1)}{w - f(z_1)} &= \frac{wf(z_2)}{w - f(z_2)} \\
f(z_1)(w - f(z_2)) &= f(z_2)(w - f(z_1) \\
f(z_1) &= f(z_2) \\
z_1 &= z_2
\end{aligned}$$

where we divided by w, which is possible since $w \neq 0$ (because $f(0) = 0$ so 0 is not an omitted value). Thus $g \in S$. $\square$

This Lemma along with Bieberbach's Theorem now allow us to prove a well-known result called the *Koebe One-Quarter Theorem.*

Theorem 47 (Koebe One-Quarter Theorem)**.** *Let $f \in S$ and let $w \in \mathbb{C}$ be a value omitted by f, meaning $w \notin f(D(0,1))$. Then $|w| \geq 1/4$.*

Proof. Let

$$f(z) := z + a_2 z^2 + \dots$$

By Bieberbach's Theorem, we know $|a_2| \leq 2$. Applying the omitted value transformation, we know that if:

$$g(z) := \frac{wf(z)}{w - f(z)}$$

then $g \in S$. Let us now work out the second coefficient in the Taylor Series of $g(z)$. We have:

$$\begin{aligned} g''(z) &= \left(\frac{wf'(w-f) + wff'}{(w-f)^2} \right)' \\ &= \left(\frac{wf'}{w-f} \right)' + \left(\frac{wff'}{(w-f)^2} \right)' \\ \frac{wf''(w-f) + wf'f'}{(w-f)^2} &+ \frac{w(f'f' + ff'')(w-f)^2 + 2wff'(w-f)f'}{(w-f)^4} \\ \implies g''(0) &= \frac{w^2 f''(0) + w}{w^2} + \frac{w^3}{w^4} \\ &= 2(a_2 + 1/w) \end{aligned}$$

which means g is of the form:

$$g(z) = z + (a_2 + 1/w)z + \dots$$

and since $g \in S$, we can apply Bieberbach's Theorem to get:

$$|a_2 + 1/w| \leq 2$$

So we have:

$$\begin{gathered} |1/w| = |1/w + a_2 - a_2| \leq |1/w + a_2| + |a_2| \leq 2 + 2 \\ |1/w| \leq 4 \implies |w| \geq 1/4 \end{gathered}$$

$\square$

The Koebe One-Quarter Theorem tells us, that any value omitted by $f \in S$ cannot lie in in the disk $D(0, 1/4)$. This means that for every $f \in S$, we have $D(0, 1/4) \subset f(D(0, 1))$. Recall that the Koebe function, $k(z)$, omits the value $-1/4$. The proof of the Koebe One-Quarter Theorem actually tells us, that $k(z)$ and its rotations (functions of the form $e^{-i\theta}k(ze^{i\theta})$) are the only functions which omit a value w with $|w| = 1/4$. To see that, let $f \in S$ with:

$$f(z) = z + a_2 z^2 + \dots$$

and suppose f omits a value w with $|w| = 1/4$. By Bieberbach's Theorem, we know $|a_2| \leq 2$. From the proof of the Koebe One-Quarter Theorem, we know:

$$\left| a_2 + \frac{1}{w} \right| \leq 2$$

combining these two facts we have:

$$2 \geq |a_2| = |a_2 + 1/w - 1/w| \geq |1/w| - |a_2 + 1/w| = 4 - |a_2 + 1/w| \geq 2$$
$$\implies |a_2| = 2$$

But we know $|a_2| = 2$ is possible only if f is a rotation of the Koebe function. Thus the Koebe function and its rotations are the only functions in S which omit a value w with $|w| = 1/4$. Notice that once again, it is the Koebe function which "maximizes" this property of functions from S.

5 Littlewood's Theorem & Bieberbach's Conjecture

After proving Bieberbach's Theorem, it is natural to ask whether we can find bounds on other coefficients in the Taylor Expansion of functions from S. It turns out that obtaining these bounds is more complicated and one must develop more sophisticated tools. First, we must find some sort of estimates for $|f'(z)|$ and $|f(z)|$. To do that, we need a few auxiliary results.

Lemma 48. *Let $f \in S$. Let $\xi \in D(0, 1)$. Then the disk automorphism:*

$$h(z) := \frac{f\left(\frac{z+\xi}{1+\bar{\xi}z}\right) - f(\xi)}{(1 - |\xi|^2) f'(\xi)}$$

belongs to S.

Proof. We have:

$$h(0) = \frac{f(\xi) - f(\xi)}{(1 - |\xi|^2) f'(\xi)} = 0$$
$$h'(z) = \frac{f'\left(\frac{z+\xi}{1+\bar{\xi}z}\right)}{f'(\xi)(1 + \bar{\xi}z)^2}$$
$$\implies h'(0) = 1$$

It remains to check univalence:

$$\begin{aligned}
h(z_1) &= h(z_2)\\
f\left(\frac{z_1+\xi}{1+\bar{\xi}z_1}\right) &= f\left(\frac{z_2+\xi}{1+\bar{\xi}z_2}\right)\\
\implies \frac{z_1+\xi}{1+\bar{\xi}z_1} &= \frac{z_2+\xi}{1+\bar{\xi}z_2}\\
\implies z_1(1-|\xi|^2) &= z_2(1-|\xi|^2)\\
z_1 &= z_2
\end{aligned}$$

thus h is univalent and so $h \in S$. $\square$

Corollary 49. *Let $f \in S$ and $\xi \in D(0,1)$. Then:*

$$\left|\frac{\xi f''(\xi)}{f'(\xi)} - \frac{2|\xi|^2}{1-|\xi|^2}\right| \leq \frac{4|\xi|}{1-|\xi|^2}$$

Proof. Define:

$$h(z) := \frac{f\left(\frac{z+\xi}{1+\bar{\xi}z}\right) - f(\xi)}{(1-|\xi|^2)f'(\xi)}$$

by the previous lemma, we know $h \in S$ and hence:

$$h(z) = z + a_2 z^2 + \dots \quad |z| < 1$$

and by Bieberbach's Theorem, we know $|a_2| \leq 2$. Computing a_2 we find:

$$\begin{aligned}
|a_2| &= \left|\frac{h''(0)}{2!}\right|\\
&= \left|\frac{1}{2}\left(\frac{1-|\xi|^2}{(1+\bar{\xi}z)^4}\frac{f''(\frac{z+\xi}{1+\bar{\xi}z})}{f'(\xi)} - 2\frac{\bar{\xi}}{(1+\bar{\xi}z)^3}\frac{f'(\frac{z+\xi}{1+\bar{\xi}z})}{f'(\xi)}\right|_{z=0}\right)\right|\\
&= \left|\frac{1}{2}\left((1-|\xi|^2)\frac{f''(\xi)}{f'(\xi)} - 2\bar{\xi}\right)\right|
\end{aligned}$$

thus

$$\begin{aligned}
&\left|\frac{1}{2}\left((1-|\xi|^2)\frac{f''(\xi)}{f'(\xi)} - 2\bar{\xi}\right)\right| \leq 2\\
\iff &\left|\frac{\xi f''(\xi)}{f'(\xi)} - \frac{2|\xi|^2}{1-|\xi|^2}\right| \leq \frac{4|\xi|}{1-|\xi|^2}
\end{aligned}$$

$\square$

Armed with these results, we can proceed to prove a stronger theorem, which has quite a few applications. The proof is again a little bit technical, but we will later use this theorem to prove *Littlewood's Theorem*, which is a beautiful result, and something to look forward to. Let's just plough on:

Theorem 50 (Distortion Theorem)**.** *Let $f \in S$. Then for all $z \in D'(0,1)$ we have:*

$$\frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq \frac{1+|z|}{(1-|z|)^3}$$

Furthermore, if equality occurs for some $z \in D'(0,1)$ in either the lower or upper estimate, then f must be a rotation of the Koebe function.

Proof. By the previous Corollary, we have:

$$\begin{aligned}
&\left|\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right| \leq \frac{4|z|}{1-|z|^2} \\
\implies & \frac{-4|z|}{1-|z|^2} \leq \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1-|z|^2}\right) \leq \frac{4|z|}{1-|z|^2} \\
\implies & \frac{2|z|^2-4|z|}{1-|z|^2} \leq \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) \leq \frac{2|z|^2+4|z|}{1-|z|^2} \qquad (1)
\end{aligned}$$

for all $z \in D(0,1)$.
Using this estimate, we would somehow like to find an estimate for $|f'(z)|$. To do that, we need to use a small trick. Notice that the ratio $\frac{f''(z)}{f'(z)}$ looks like the derivative of $\log(f'(z))$. However, we need to choose the correct branch of the logarithm, to ensure holomorphicity. But since f is univalent, it is conformal (so $f'(z) \neq 0$) and since $D(0,1)$ is simply connected, we can use Lemma 42 to find a branch of the logarithm so that:

$$w(z) := \log(f'(z))$$

is well defined and holomorphic on $D(0,1)$.
Now, the crucial observation is that:

$$\begin{aligned}
z\frac{dw}{dz} &= z\frac{f''(z)}{f'(z)} \\
\implies \operatorname{Re}\left(z\frac{dw}{dz}\right) &= \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right)
\end{aligned}$$

and now, we can use (1) to extract information about $|f'(z)|$. First, we need to switch to polar coordinates and write $z = re^{i\theta}$ (so $|z| = r$). We have:

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r}$$

and so the identity becomes:

$$\begin{aligned}
\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) = \operatorname{Re}\left(z\frac{dw}{dz}\right) &= \operatorname{Re}\left(r\frac{\partial w}{\partial r}\right) \\
&= r\frac{\partial}{\partial r}\operatorname{Re}(w(z)) \\
&= r\frac{\partial}{\partial r}\log|f'(re^{i\theta})|
\end{aligned}$$

Now (1) becomes:

$$\frac{2r^2-4r}{1-r^2} \le r\frac{\partial}{\partial r}\log|f'(re^{i\theta})| \le \frac{2r^2+4r}{1-r^2}$$

which gives us the identity:

$$\frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log|f'(re^{i\theta})| \le \frac{2r+4}{1-r^2} \tag{2}$$

which holds for **all** $r \in [0,1)$ and $\theta \in [0,2\pi)$.
Now let us fix an arbitrary $R \in (0,1)$. Integrating (2) with respect to r from 0 to R gives:

$$\begin{aligned} &\int_0^R \frac{2r-4}{1-r^2}dr \le \int_0^R \frac{\partial}{\partial r}\log|f'(re^{i\theta})|dr \le \int_0^R \frac{2r+4}{1-r^2}dr \\ \iff &\log\left(\frac{1-R}{(1+R)^3}\right) \le \log|f'(Re^{i\theta})| \le \log\left(\frac{1+R}{(1-R)^3}\right) \end{aligned}$$

since $\log|f'(0)| = \log(1) = 0$.
Exponentiating, we obtain:

$$\frac{1-R}{(1+R)^3} \le |f'(Re^{i\theta})| \le \frac{1+R}{(1-R)^3} \tag{3}$$

Which is exactly what we wanted to prove. Since R was arbitrary, this is true for all $R \in (0,1)$, which proves the theorem.
Let us now prove that if equality occurs for some $z \in D'(0,1)$ in either the lower or upper estimate, then f is a rotation of the Koebe function.
Let us first deal with equality in the upper estimate. Suppose there exists $z = Re^{i\theta}$ such that:

$$|f'(Re^{i\theta})| = \frac{1+R}{(1-R)^3}$$

Taking the log and writing it in integral form, we get:

$$\begin{aligned} &\log|f'(Re^{i\theta})| = \log\frac{1+R}{(1-R)^3} \\ \iff &\int_0^R \frac{\partial}{\partial r}\log|f'(re^{i\theta})|dr = \int_0^R \frac{2r+4}{1-r^2}dr \end{aligned}$$

But by (2) we know $\frac{\partial}{\partial r}\log|f'(re^{i\theta})| \le \frac{2r+4}{1-r^2}$. Furthermore, both functions are continuous and so the equality of the integrals implies [1]:

$$\frac{\partial}{\partial r}\log|f'(re^{i\theta})| = \frac{2r+4}{1-r^2} \quad \forall r \in [0,R]$$

[1] Here we used the fact that if g is continuous on $[a,b]$, with $g(x) \ge 0$ and $\int_a^b g(x)dx = 0$, then $g(x) \equiv 0$.

But we know:

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) = r\frac{\partial}{\partial r}\log|f'(re^{i\theta})|$$

And so we have:

$$\begin{aligned}\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right) &= \frac{2r^2+4r}{1-r^2}\\ \implies \mathrm{Re}\left(e^{i\theta}\frac{f''(z)}{f'(z)}\right) &= \frac{2r+4}{1-r^2}\end{aligned}$$

in particular:

$$\mathrm{Re}\left(\frac{e^{i\theta}f''(0)}{f'(0)}\right) = 4$$

but since $f'(0) = 1$ and $f''(0) = 2a_2$, by Bieberbach's Theorem we have:

$$4 = \mathrm{Re}\left(e^{i\theta}\frac{f''(0)}{f'(0)}\right) \leq \left|\frac{f''(0)}{f'(0)}\right| \leq 4 \implies |f''(0)| = 4$$

which means $|a_2| = 2$ and hence f is a rotation of the Koebe function.
We can construct a similar argument if equality occurs in the lower estimate, but we omit this.

Conversely, let f be a rotation of the Koebe function with:

$$f(z) = e^{-i\theta}k(ze^{i\theta}) = \frac{z}{(1-e^{i\theta}z)^2}$$

for some fixed $\theta \in (-\pi, \pi]$. Then:

$$f'(z) = \frac{1+ze^{i\theta}}{(1-ze^{i\theta})^3}$$

and so if $z = re^{-i\theta}$, for $r < 1$, we have:

$$f'(re^{-i\theta}) = \frac{1+r}{(1-r)^3}$$

so equality in the upper estimate occurs on the line $Arg(z) = -\theta$. Similarly if $z = re^{i(\pi-\theta)}$, for $r < 1$, we have:

$$f'(re^{i(\pi-\theta)}) = \frac{1-r}{(1+r)^3}$$

and so equality in the lower estimate occurs on the line $Arg(z) = \pi - \theta$. $\square$

Notice that once again, the Koebe function "maximizes" a property of functions from the class S.

The upper bound on f' allows us to directly find an upper bound for f, by using the Fundamental Theorem of Calculus:

Corollary 51. *Let $f \in S$ and $z \in D'(0,1)$. Then:*

$$|f(z)| \leq \frac{|z|}{(|z|-1)^2}$$

Proof. Fixing $z \in D'(0,1)$ and integrating f' along the segment connecting 0 to z, one finds:

$$\begin{aligned} f(z) = f(z) - f(0) &= \int_0^z f'(\xi)d\xi \\ &= \int_0^{|z|} f'(re^{i\theta})e^{i\theta}dr \end{aligned}$$

where we used the Fundamental Theorem of Calculus. And thus:

$$\begin{aligned} |f(z)| &= \left| \int_0^{|z|} f'(re^{i\theta})e^{i\theta}dr \right| \\ &\leq \int_0^{|z|} |f'(re^{i\theta})|dr \\ &\leq \int_0^{|z|} \frac{1+r}{(1-r)^3}dr \\ &= \frac{|z|}{(|z|-1)^2} \end{aligned}$$

where we used the upper bound obtained in the Distortion Theorem.

□

This allows us to obtain bounds for $|a_n|$. Using Taylor's Theorem and Cauchy's representation of derivatives, one can obtain the bound $|a_n| \leq n^2e^2/4$. One is encouraged to try this, as it is a good exercise. One can also obtain the bound:

$$|f(z)| \geq \frac{|z|}{(1+|z|)^2}$$

but this is slightly more complicated, as it is done using the *Koebe One-Quarter Theorem*. We refer the interested reader to the *Growth Theorem*, found for example in Duran[2].
Using Corollary 51 and the Koebe Transform, we can obtain the bound $|a_n| < en$, which is proved in *Littlewood's Theorem*. The core idea of this Theorem is proved in the following lemma and the proof has a very neat geometric argument:

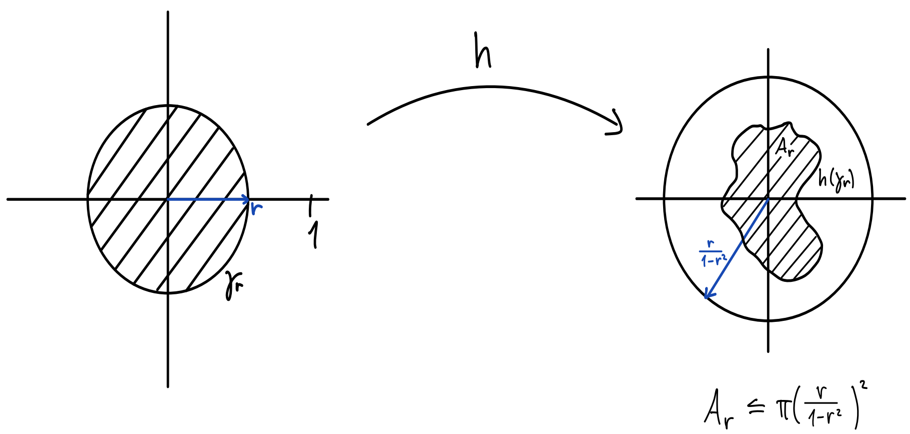
Lemma 52. *Let $f \in S$. Then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|d\theta \leq \frac{r}{1-r}$$

Proof. Applying the Koebe Transform with $n = 2$, we can define $h \in S$ such that $h(z)^2 = f(z^2)$. Let us fix an arbitrary $r \in (0,1)$ and consider the circle $\gamma_r = \gamma(0,r)$. By Corollary 51, we have:

$$|f(z^2)| \leq \frac{r^2}{(1-r^2)^2}$$
$$\iff |h(z)| \leq \frac{r}{1-r^2}$$

for all z with $|z| = r$. Note that this is also true for all z in the interior of the circle γ_r, and hence h maps $D(0,r)$ into $D(0, \frac{r}{1-r^2})$.



Let A_r denote the area enclosed by $h(\gamma_r)$. Clearly A_r cannot be greater than the area of the disk $D(0, \frac{r}{1-r^2})$. We have:

$$A_r \leq \pi\left(\frac{r}{1-r^2}\right)^2$$

by Green's Theorem we have:

$$
\begin{aligned}
A_r &= \frac{1}{2i}\int_{\gamma_r} \overline{h(z)}h'(z)dz \\
&= \frac{1}{2i}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \overline{c_n} r^n e^{-ni\theta}\right)\left(1+\sum_{n=2}^{\infty} nc_n r^{n-1}e^{(n-1)i\theta}\right) ire^{i\theta} \\
&= \frac{1}{2}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} \overline{c_n} r^n e^{-ni\theta}\right)\left(re^{i\theta}+\sum_{n=2}^{\infty} nc_n r^{n}e^{(n)i\theta}\right) \\
&= \pi r^2 + \frac{1}{2}\int_0^{2\pi} \left(\sum_{m=2}^{\infty}\sum_{k=2}^{m} \overline{c_k}(m-k)c_{m-k}r^m e^{(m-2k)i\theta}\right) \\
&= \pi\left(r^2+\sum_{n=2}^{\infty} n|c_n|^2 r^{2n}\right) \\
&= \pi\sum_{n=1}^{\infty} n|c_n|^2 r^{2n}
\end{aligned}
$$

where we used the properties of a uniformly convergent series and exchanged the integral and summation repeatedly.
Thus we have:

$$\sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1} \le \frac{r}{(1-r^2)^2}$$

Since r was arbitrary, this must hold for all $r \in (0,1)$. So we can fix $R \in (0,1)$ and integrate from 0 to R to obtain:

$$\int_0^R \left(\sum_{n=1}^{\infty} n|c_n|^2 r^{2n-1}\right) dr \le \int_0^R \frac{r}{(1-r^2)^2}dr$$

$$\sum_{n=1}^{\infty} |c_n|^2 R^{2n} \le \frac{R^2}{1-R^2}$$

But the left hand side can be expressed in the form of an integral. Observe that:

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |h(Re^{i\theta})|^2 d\theta &= \frac{1}{2\pi}\int_0^{2\pi} h(Re^{i\theta})\overline{h(Re^{i\theta})}d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{n=1}^{\infty} c_n R^n e^{ni\theta}\right)\left(\sum_{n=1}^{\infty} \overline{c_n} R^n e^{-ni\theta}\right) d\theta \\
&= \sum_{n=1}^{\infty} |c_n|^2 R^{2n}
\end{aligned}
$$

and thus:

$$\frac{1}{2\pi}\int_0^{2\pi}|h(Re^{i\theta})|^2 d\theta \leq \frac{R^2}{1-R^2}$$
$$\iff \frac{1}{2\pi}\int_0^{2\pi}|f(R^2e^{2i\theta})|d\theta \leq \frac{R^2}{1-R^2}$$

setting $t = R^2$ and making the substitution $2\theta = \phi$, one can easily check that this implies:

$$\frac{1}{2\pi}\int_0^{2\pi}|f(te^{i\phi})|d\phi \leq \frac{t}{1-t}$$

since R was arbitrary, this is true for all $t \in (0,1)$, as was claimed. $\square$

This Lemma allows us to prove a very elegant result, which now follows trivially.

Theorem 53 (Littlewood's Theorem)**.** *Let $f \in S$ with*

$$f(z) = z + a_2z^2 + a_3z^3 + \dots \quad |z| < 1$$

then $|a_n| < en$.

Proof. By Taylor's Theorem, we have:

$$\begin{aligned}|a_n| &= \left|\frac{f^n(0)}{n!}\right| \\ &= \left|\frac{1}{2\pi}\int_{\gamma_r}\frac{f(\xi)}{\xi^{n+1}}d\xi\right|\end{aligned}$$

where $\gamma_r = \gamma(0,r)$ for some $r \in (0,1)$. Note that we can choose r to be any number in $(0,1)$, since the integrals around any such circle are equal. Continuing, we have:

$$\begin{aligned}&= \frac{1}{2\pi}\left|\int_0^{2\pi}\frac{f(re^{i\theta})}{r^ne^{ni\theta}}id\theta\right| \\ &\leq \frac{r^{-n}}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|d\theta \\ &\leq r^{-n}\frac{r}{1-r}\end{aligned}$$

where we used the previous lemma. Let $g(r) = \frac{r^{-n+1}}{1-r}$. We would like to find the minimum of g on the interval $(0,1)$. Differentiating we find:

$$g'(r) = \frac{(1-n)r^{-n} + nr^{1-n}}{(1-r)^2}$$
$$g'(r) = 0 \iff r = 1 - 1/n$$

One can check that this is indeed a local minimum. Furthermore, we have:

$$\begin{aligned} g(1-1/n) &= n\left(\frac{n-1}{n}\right)^{1-n} \\ &= n\left(1+\frac{1}{n-1}\right)^{n-1} \\ &< en \end{aligned}$$

thus $|a_n| < en$, as claimed. □

In many of the results proved in the previous sections, we saw that the Koebe function 'maximised' a certain property. One would therefore expect that the coefficients in the Taylor Expansion of functions from S are also 'maximised' by the Koebe function. This is precisely the stament of a famous conjecture, posed by Bieberbach in 1916:

Theorem 54 (Bieberbach's Conjecture)**.** *Let $f \in S$ with:*

$$f(z) = z + a_2 z^2 + \ldots \quad |z| < 1$$

then $|a_n| \leq n$.

This conjecture stood as a challenge for mathematicians for almost 70 years. Throughout the years various methods were developed to improve the constant e found in *Littlewood's Theorem.* The conjecture was finally proved in 1985 by Louis De Branges [3]. One thing to note, is that one cannot hope to obtain a sharper bound for the coefficients, since the Koebe function and its rotations satisfy $|a_n| = n$. In view of this conjecture, we can see why Littlewood's Theorem is important, since it bounds the coefficients by order n.

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