Lecture 3: Spline Smoothness and Degrees of Freedom

MATH5824 Generalised Linear and Additive Models

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Reading

Course notes: Chapter 2, Sections 2.2–2.3

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Recap: Piecewise Polynomials

With m knots, we have $m + 1$ intervals, each with a polynomial of order p:

$$f(t) = \sum_{\ell=0}^{p} a_{k\ell}\, t^{\ell} \quad \text{on interval } k$$

Total unconstrained parameters: $(p + 1)(m + 1)$

Without constraints, the pieces may not join smoothly.

Imposing Smoothness

Smoothness constraints: At each knot t_k, require continuity of the function and its first $(p-1)$ derivatives:

$$\lim_{\varepsilon \to 0} f^{(\ell)}(t_k - \varepsilon) = \lim_{\varepsilon \to 0} f^{(\ell)}(t_k + \varepsilon)$$

for $k = 1, \ldots, m$ and $\ell = 0, 1, \ldots, p-1$.

Number of constraints: p constraints at each of m knots $= pm$.

Resulting degrees of freedom:

$$\mathrm{df}_{\mathrm{spline}} = (p+1)(m+1) - pm = m + p + 1$$

Degrees of Freedom: Examples

Linear spline $(p = 1)$ with m knots:

- Unconstrained parameters: $2(m + 1)$
- Continuity constraints: m (one per knot)
- Degrees of freedom: $m + 2$

Cubic spline $(p = 3)$ with m knots:

- Unconstrained parameters: $4(m + 1)$
- Smoothness constraints: $3m$ (continuity of f, f', f'' at each knot)
- Degrees of freedom: $m + 4$

Note: Degrees of freedom grow linearly with the number of knots, regardless of order p.

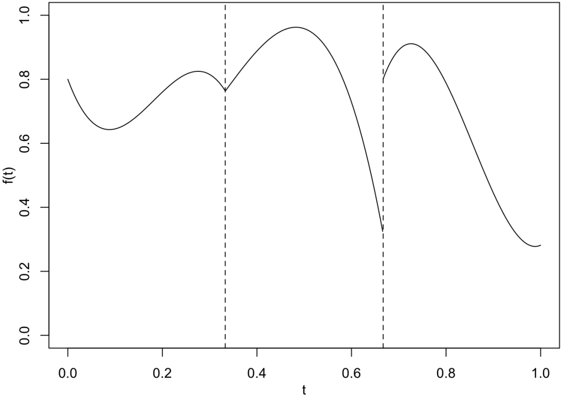
Cubic Spline: Detailed Accounting

Example: Cubic spline with $m = 2$ knots (as in Figure 2.2).

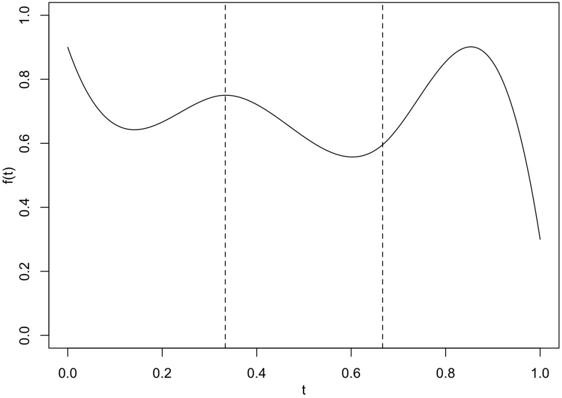
Component	Count
Intervals	3
Coefficients per interval	4
Total coefficients	12
Continuity of f at knots	-2
Continuity of f' at knots	-2
Continuity of f'' at knots	-2
Degrees of freedom	**6**

Check: $m + p + 1 = 2 + 3 + 1 = 6$. ✓

Constrained vs. Unconstrained



(a) No constraints: 12 free parameters



(b) Spline: 6 free parameters

Smoothness constraints reduce the piecewise cubic from 12 to 6 degrees of freedom, producing a visually smooth curve with no visible knot locations.

What Smoothness Means in Practice

Property	Between knots	At knots
f continuous	✓	✓
f' continuous	✓	✓(if $p \geq 2$)
f'' continuous	✓	✓(if $p \geq 3$)
$f^{(p)}$ continuous	✓	Not required
Infinitely differentiable	✓	No

Between knots, the spline is a polynomial and therefore infinitely differentiable. The constraints only matter at the knots themselves.

Looking Ahead: From Definitions to Fitting

We now know:

- What a spline is (piecewise polynomial with smoothness)
- How many degrees of freedom it has $(m + p + 1)$

Next questions:

- How do we choose the coefficients to fit data?
- What happens at the boundaries (beyond the outermost knots)?
- Is there a “best” interpolating spline?

$\Rightarrow$ These lead us to **natural splines** and **roughness penalties**.

Summary

Key points:

- Smoothness constraints require continuity of derivatives at knots
- A spline of order p with m knots has $m + p + 1$ degrees of freedom
- Each smoothness constraint removes one degree of freedom
- Cubic splines have continuous f, f', and f''
- The third derivative f''' may be discontinuous at knots
- Between knots, the spline is a standard polynomial

Next lecture: Natural splines and the interpolation problem.