

Lecture 2: Modelling Approaches and Spline Basics

MATH5824 Generalised Linear and Additive Models

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Course notes: Chapter 1, Section 1.3 and Chapter 2, Section 2.1

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The General Model

We observe n pairs (x_i, y_i) at ordered locations $x_1 < x_2 < \dots < x_n$:

$$y_i = f(x_i) + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

Goal: Estimate the unknown function f .

Two broad strategies:

- ❶ **Parametric:** Specify a functional form for f with finitely many parameters
- ❷ **Non-parametric:** Assume f is “smooth” without specifying its form

Parametric Models

Examples:

- Linear: $f(x) = \alpha + \beta x$
- Polynomial: $f(x) = \alpha + \beta x + \gamma x^2 + \dots + \omega x^p$
- Exponential: $f(x) = \alpha e^{-\beta x}$

Strengths:

- Interpretable parameters
- Well-understood inference (confidence intervals, hypothesis tests)

Limitation: Contains only a small number of parameters, restricting flexibility for arbitrary fluctuations in f .

Non-parametric Models

Assumption: f is *smooth*, meaning $f(x_i)$ is close to $f(x_j)$ when x_i is close to x_j .

No fixed functional form — the data determine the shape.

Approaches:

- **Splines:** Piecewise polynomials with smoothness constraints (our focus)
- Kernel smoothing
- Wavelets
- Local polynomial regression

Spline methods fit naturally within the normal linear model and GLM framework.

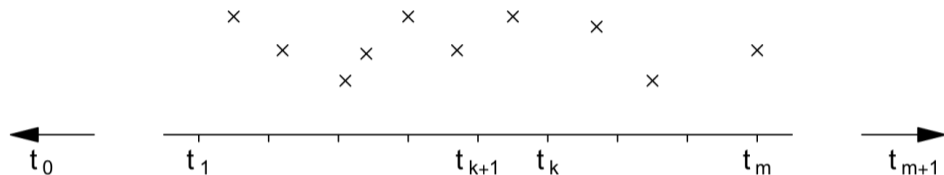
Problem with global polynomials:

- Low-order polynomials fit poorly across the whole range
- High-order polynomials oscillate wildly (Runge's phenomenon)

Idea: Fit *different* low-order polynomials in different regions.

Requiring continuity and differentiability at the join points leads to **splines**.

Knots and Sites



Important distinction:

- **Knots** $\{t_k\}$: Where polynomial pieces join
- **Sites** $\{x_i\}$: Where data are observed

Knots and sites need not coincide, though in many applications we place knots at data locations. The knots create $m + 1$ intervals, each with its own polynomial.

Spline Definition

Definition

A **spline of order p** ($p \geq 1$) with knots $t_1 < \dots < t_m$ is a piecewise polynomial of order p that is $(p - 1)$ times differentiable at each knot.

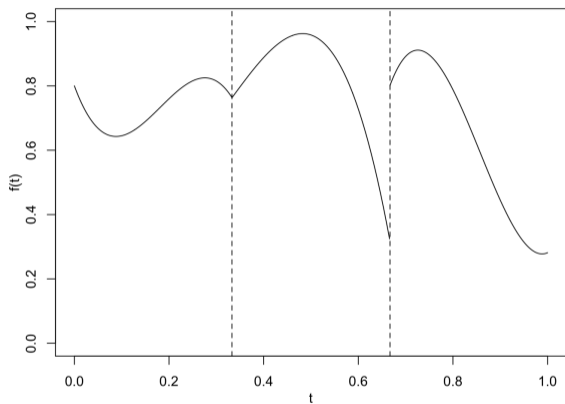
On each interval $[t_k, t_{k+1})$, the spline has the form:

$$f(t) = \sum_{\ell=0}^p a_{k\ell} t^\ell$$

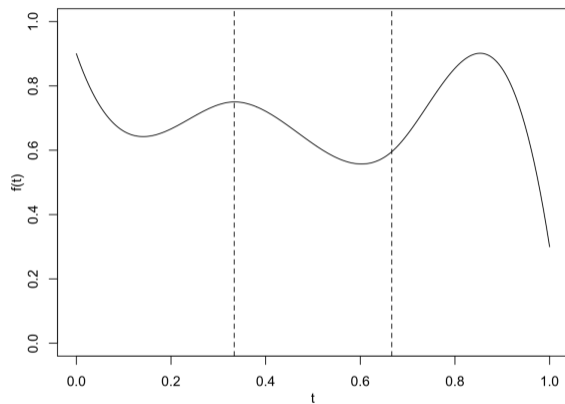
Key property:

- Between knots: infinitely differentiable (it is a polynomial)
- At knots: exactly $(p - 1)$ times differentiable

Piecewise Cubics: Without and With Smoothness



(a) No smoothness constraints



(b) Cubic spline (smooth)

- (a): Piecewise cubic with discontinuities — not a spline
- (b): Piecewise cubic with continuity of f , f' , and f'' at knots — a cubic spline

Common Spline Orders

| Order p | Name | Continuity at knots | Typical use |
|-----------|--------|--|----------------------|
| 1 | Linear | f continuous Slope may jump | Simple interpolation |
| 3 | Cubic | f, f', f'' continuous f''' may jump | Standard choice |

Why cubic?

- Cubic is the lowest order with continuous curvature (f'')
- Knot locations are “invisible” to the eye
- Higher orders are rarely needed in practice

Key points:

- Non-parametric models allow f to be flexible, guided by the data
- Splines are piecewise polynomials joined smoothly at knots
- A spline of order p is $(p - 1)$ times differentiable at each knot
- Cubic splines ($p = 3$) are the standard choice: continuous f , f' , and f''
- Knots and observation sites are conceptually distinct

Next lecture: Smoothness constraints and degrees of freedom for splines.