AE 6210: Advanced Dynamics Homework 2

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January 26, 2025

1 Problem Statement

The inertial frame is $\boldsymbol{I}$ (gray). A helicopter is flying in a maneuver, and its body is modeled as a rigid body with frame $\boldsymbol{A}$ (blue) attached to it. Frame $\boldsymbol{A}$ has an angular acceleration ${}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{A}}$ relative to frame $\boldsymbol{I}$. There is a point C on the rigid body that is fixed in the $\boldsymbol{A}$ frame with a velocity ${}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{C}}$ and acceleration ${}^{\boldsymbol{I}}\boldsymbol{a}^{\boldsymbol{C}}$ viewed in the inertial frame.

At a displacement $\boldsymbol{r}^{\boldsymbol{CO}}$ from point C is the rotor hub which is modeled as point O, where point O is also fixed in the $\boldsymbol{A}$ frame. Point O is also the origin of frame $\boldsymbol{H}$ (red), and frame $\boldsymbol{H}$ is rotating at a constant angular speed ω^H relative to a $\boldsymbol{A}$ about the $\hat{\mathbf{h}}_3 = \hat{\mathbf{a}}_3$ axis. The rotor blade mounting point, modeled as point P, is fixed in the $\boldsymbol{H}$ frame at a displacement $\boldsymbol{r}^{\boldsymbol{OP}}$ along the $\hat{\mathbf{h}}_2$ axis.

The rotor blade is modeled as a rigid body with an attached frame $\boldsymbol{B}$ (green). The blade flaps about the $\hat{\mathbf{b}}_1 = \hat{\mathbf{h}}_1$ axis where the flap angle varies as $\beta = \beta_0 + \beta_1 \sin \omega_b t$. The tip of the rotor blade is modeled as point T, and it is at a displacement $\boldsymbol{r}^{\boldsymbol{PT}}$ from the blade root along the $\hat{\mathbf{b}}_2$ axis.

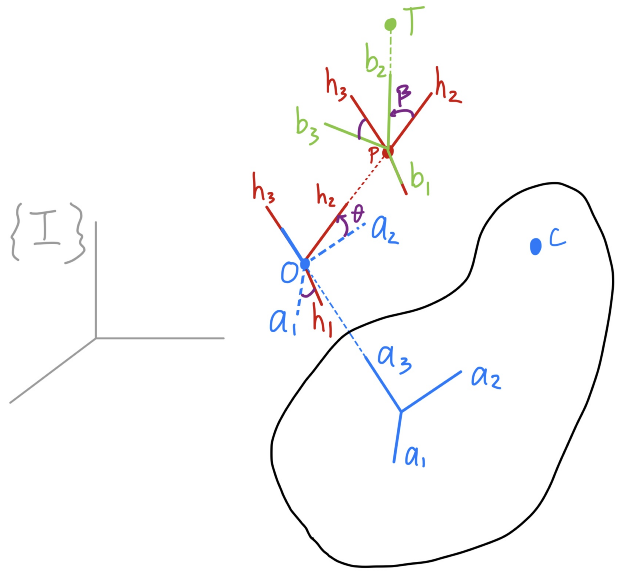

Figure 1: Diagram of the problem. A point that is the same color as a frame means that that point is fixed in that frame.

2 Rotor Blade Angular Velocity ${}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}}$

The angular velocity of the rotor blade relative to the inertial frame ${}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}}$ is the angular velocity of frame $\boldsymbol{B}$ relative to frame $\boldsymbol{I}$. ${}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}}$ can be expressed as the known values.

$$
\begin{aligned}
&{}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}} = {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} + {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}} + {}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}} \\
&\text{where } \begin{cases} {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}} = \omega^H\,\hat{\mathbf{a}}_3 \\ {}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}} = \dot{\beta}\,\hat{\mathbf{b}}_1 = \omega_b\beta_1\cos\omega_b t\,\hat{\mathbf{b}}_1 = \omega_b\beta_1\cos\omega_b t\,\hat{\mathbf{h}}_1 \end{cases}
\end{aligned}
$$

3 Rotor Blade Angular Acceleration ${}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{B}}$

The angular acceleration of the rotor blade relative to the inertial frame ${}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{B}}$ can be computed by taking the inertial frame derivative of ${}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}}$ and using the transport theorem.

$$
\begin{aligned}
{}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{B}} &= \frac{{}^{I}d}{dt}[{}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}}] \\
&= \frac{{}^{I}d}{dt}[{}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} + {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}} + {}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}] \\
&= {}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{A}} + \left(\frac{{}^{A}d}{dt}[{}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}}] + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} \times {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}}\right) + \left(\frac{{}^{H}d}{dt}[{}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}] + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{H}} \times {}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}\right)
\end{aligned}
$$

Noting that $\frac{{}^{A}d}{dt}[{}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}}] = \mathbf{0}$, the equation simplifies.

$$
\begin{aligned}
&{}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{B}} = {}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{A}} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} \times {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}} + \frac{{}^{H}d}{dt}[{}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}] + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{H}} \times {}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}} \\
&\text{where } \begin{cases} \frac{{}^{H}d}{dt}[{}^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}] = -\omega_b^2\beta_1\sin\omega_b t\,\hat{\mathbf{h}}_1 \\ {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{H}} = {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} + {}^{\boldsymbol{A}}\boldsymbol{\omega}^{\boldsymbol{H}} \end{cases}
\end{aligned}
$$

4 Rotor Blade Tip Velocity ${}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{T}}$

Begin by computing the position of the rotor blade tip relative to the inertial frame's origin $\boldsymbol{r}^{\boldsymbol{Tip}}$. Differentiating $\boldsymbol{r}^{\boldsymbol{Tip}}$ in the inertial frame gives the velocity of the rotor blade tip in the inertial frame.

$$
\boldsymbol{r}^{\boldsymbol{Tip}} = \boldsymbol{r}^{\boldsymbol{C}} + \boldsymbol{r}^{\boldsymbol{CO}} + \boldsymbol{r}^{\boldsymbol{OP}} + \boldsymbol{r}^{\boldsymbol{PT}}
$$

$$
\begin{aligned}
{}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{T}} &= \frac{{}^{I}d}{dt}[\boldsymbol{r}^{\boldsymbol{Tip}}] \\
&= \frac{{}^{I}d}{dt}[\boldsymbol{r}^{\boldsymbol{C}} + \boldsymbol{r}^{\boldsymbol{CO}} + \boldsymbol{r}^{\boldsymbol{OP}} + \boldsymbol{r}^{\boldsymbol{PT}}] \\
&= {}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{C}} + \left(\cancelto{\mathbf{0}}{\frac{{}^{A}d}{dt}[\boldsymbol{r}^{\boldsymbol{CO}}]} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} \times \boldsymbol{r}^{\boldsymbol{CO}}\right) + \left(\cancelto{\mathbf{0}}{\frac{{}^{H}d}{dt}[\boldsymbol{r}^{\boldsymbol{OP}}]} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{H}} \times \boldsymbol{r}^{\boldsymbol{OP}}\right) + \left(\cancelto{\mathbf{0}}{\frac{{}^{B}d}{dt}[\boldsymbol{r}^{\boldsymbol{PT}}]} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}} \times \boldsymbol{r}^{\boldsymbol{PT}}\right)
\end{aligned}
$$

The final equation simplifies in terms of the known values.

$$
{}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{T}} = {}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{C}} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} \times \boldsymbol{r}^{\boldsymbol{CO}} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{H}} \times \boldsymbol{r}^{\boldsymbol{OP}} + {}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{B}} \times \boldsymbol{r}^{\boldsymbol{PT}}
$$

5 Rotor Blade Tip Acceleration ${}^{I}\boldsymbol{a}^{T}$

Differentiating ${}^{I}\boldsymbol{v}^{T}$ in the inertial frame gives the acceleration of the rotor blade tip in the inertial frame. For a more organized computation, ${}^{I}\boldsymbol{v}^{T}$ is differentiated in four segments.

First Term

$$\frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{v}^{C}] = {}^{I}\boldsymbol{a}^{C}$$

Second Term

$$\begin{aligned}
\frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{A} \times \boldsymbol{r}^{CO}] &= \frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{A}] \times \boldsymbol{r}^{CO} + {}^{I}\boldsymbol{\omega}^{A} \times \frac{{}^{I}d}{dt}[\boldsymbol{r}^{CO}] \\
&= {}^{I}\boldsymbol{\alpha}^{A} \times \boldsymbol{r}^{CO} + {}^{I}\boldsymbol{\omega}^{A} \times \left(\cancelto{\mathbf{0}}{\frac{{}^{A}d}{dt}[\boldsymbol{r}^{CO}]} + {}^{I}\boldsymbol{\omega}^{A} \times \boldsymbol{r}^{CO} \right)
\end{aligned}$$

Third Term

$$\begin{aligned}
\frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{H} \times \boldsymbol{r}^{OP}] &= \frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{H}] \times \boldsymbol{r}^{OP} + {}^{I}\boldsymbol{\omega}^{H} \times \frac{{}^{I}d}{dt}[\boldsymbol{r}^{OP}] \\
&= \frac{{}^{I}d}{dt}[({}^{I}\boldsymbol{\omega}^{A} + {}^{A}\boldsymbol{\omega}^{H}) \times \boldsymbol{r}^{OP}] + {}^{I}\boldsymbol{\omega}^{H} \times \frac{{}^{I}d}{dt}[\boldsymbol{r}^{OP}] \\
&= \frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{A}] \times \boldsymbol{r}^{OP} + \frac{{}^{I}d}{dt}[{}^{A}\boldsymbol{\omega}^{H}] \times \boldsymbol{r}^{OP} + {}^{I}\boldsymbol{\omega}^{H} \times \left(\cancelto{\mathbf{0}}{\frac{{}^{H}d}{dt}[\boldsymbol{r}^{OP}]} + {}^{I}\boldsymbol{\omega}^{H} \times \boldsymbol{r}^{OP} \right) \\
&= {}^{I}\boldsymbol{\alpha}^{A} \times \boldsymbol{r}^{OP} + \left(\cancelto{\mathbf{0}}{\frac{{}^{A}d}{dt}[{}^{A}\boldsymbol{\omega}^{H}]} + {}^{I}\boldsymbol{\omega}^{A} \times {}^{A}\boldsymbol{\omega}^{H} \right) \times \boldsymbol{r}^{OP} + {}^{I}\boldsymbol{\omega}^{H} \times \left({}^{I}\boldsymbol{\omega}^{H} \times \boldsymbol{r}^{OP}\right)
\end{aligned}$$

Fourth Term

$$\begin{aligned}
\frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{B} \times \boldsymbol{r}^{PT}] &= \frac{{}^{I}d}{dt}[{}^{I}\boldsymbol{\omega}^{B}] \times \boldsymbol{r}^{PT} + {}^{I}\boldsymbol{\omega}^{B} \times \frac{{}^{I}d}{dt}[\boldsymbol{r}^{PT}] \\
&= {}^{I}\boldsymbol{\alpha}^{B} \times \boldsymbol{r}^{PT} + {}^{I}\boldsymbol{\omega}^{B} \times \left({}^{I}\boldsymbol{\omega}^{B} \times \boldsymbol{r}^{PT}\right)
\end{aligned}$$

Sum all of the terms for the final expression.

$$\begin{aligned}
{}^{I}\boldsymbol{a}^{T} &= {}^{I}\boldsymbol{a}^{C} + {}^{I}\boldsymbol{\alpha}^{A} \times \boldsymbol{r}^{CO} + {}^{I}\boldsymbol{\omega}^{A} \times \left({}^{I}\boldsymbol{\omega}^{A} \times \boldsymbol{r}^{CO}\right) \\
&+ {}^{I}\boldsymbol{\alpha}^{A} \times \boldsymbol{r}^{OP} + \left({}^{I}\boldsymbol{\omega}^{A} \times {}^{A}\boldsymbol{\omega}^{H}\right) \times \boldsymbol{r}^{OP} + {}^{I}\boldsymbol{\omega}^{H} \times \left({}^{I}\boldsymbol{\omega}^{H} \times \boldsymbol{r}^{OP}\right) \\
&+ {}^{I}\boldsymbol{\alpha}^{B} \times \boldsymbol{r}^{PT} + {}^{I}\boldsymbol{\omega}^{B} \times \left({}^{I}\boldsymbol{\omega}^{B} \times \boldsymbol{r}^{PT}\right)
\end{aligned}$$

6 Numerical Example

Some numerical values were given for example implementation in MATLAB.

$$
\begin{aligned}
t &= 25 \text{ s} \\
{}^{\boldsymbol{I}}\boldsymbol{\omega}^{\boldsymbol{A}} &= 1.0\,\hat{\mathbf{a}}_1 + 2.0\,\hat{\mathbf{a}}_2 + 3.0\,\hat{\mathbf{a}}_3 \text{ (rad/s)} \\
{}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{A}} &= 0.1\,\hat{\mathbf{a}}_1 + 0.2\,\hat{\mathbf{a}}_2 + 0.3\,\hat{\mathbf{a}}_3 \text{ (rad/s}^2\text{)} \\
{}^{\boldsymbol{I}}\boldsymbol{v}^{\boldsymbol{C}} &= 4.0\,\hat{\mathbf{a}}_1 + 5.0\,\hat{\mathbf{a}}_2 + 6.0\,\hat{\mathbf{a}}_3 \text{ (m/s)} \\
{}^{\boldsymbol{I}}\boldsymbol{a}^{\boldsymbol{C}} &= 0.4\,\hat{\mathbf{a}}_1 + 0.5\,\hat{\mathbf{a}}_2 + 0.6\,\hat{\mathbf{a}}_3 \text{ (m/s}^2\text{)} \\
x_0 &= 1.5 \text{ m} \\
y_0 &= -2.5 \text{ m} \\
\omega^H &= -10 \text{ rad/s} \\
r_h &= 0.5 \text{ m} \\
\beta_0 &= -0.1 \text{ rad} \\
\beta_1 &= 0.1 \text{ rad} \\
\omega_b &= 20 \text{ rad/s} \\
L &= 10 \text{ m}
\end{aligned}
$$

Compute the rotation matrices where $\left[{}^{B}C^{A}\right]$ sends a vector resolved in the $\boldsymbol{B}$ basis to the same vector resolved in the $\boldsymbol{A}$ basis i.e. $\boldsymbol{v_A} = \left[{}^{B}C^{A}\right]\boldsymbol{v_B}$ where $\boldsymbol{v_A} = u_a\,\hat{\mathbf{a}}_1 + v_a\,\hat{\mathbf{a}}_2 + w_a\,\hat{\mathbf{a}}_3$ and $\boldsymbol{v_B} = u_b\,\hat{\mathbf{b}}_1 + v_b\,\hat{\mathbf{b}}_2 + w_b\,\hat{\mathbf{b}}_3$

$$
\left[{}^{H}C^{A}\right] = \begin{bmatrix} \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{h}}_1 & \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{h}}_2 & \hat{\mathbf{a}}_1 \cdot \hat{\mathbf{h}}_3 \\ \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{h}}_1 & \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{h}}_2 & \hat{\mathbf{a}}_2 \cdot \hat{\mathbf{h}}_3 \\ \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{h}}_1 & \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{h}}_2 & \hat{\mathbf{a}}_3 \cdot \hat{\mathbf{h}}_3 \end{bmatrix} = \begin{bmatrix} \cos\omega_b t & -\sin\omega_b t & 0 \\ \sin\omega_b t & \cos\omega_b t & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
\left[{}^{B}C^{H}\right] = \begin{bmatrix} \hat{\mathbf{h}}_1 \cdot \hat{\mathbf{b}}_1 & \hat{\mathbf{h}}_1 \cdot \hat{\mathbf{b}}_2 & \hat{\mathbf{h}}_1 \cdot \hat{\mathbf{b}}_3 \\ \hat{\mathbf{h}}_2 \cdot \hat{\mathbf{b}}_1 & \hat{\mathbf{h}}_2 \cdot \hat{\mathbf{b}}_2 & \hat{\mathbf{h}}_2 \cdot \hat{\mathbf{b}}_3 \\ \hat{\mathbf{h}}_3 \cdot \hat{\mathbf{b}}_1 & \hat{\mathbf{h}}_3 \cdot \hat{\mathbf{b}}_2 & \hat{\mathbf{h}}_3 \cdot \hat{\mathbf{b}}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{bmatrix}
$$

$$
\left[{}^{B}C^{A}\right] = \left[{}^{H}C^{A}\right]\left[{}^{B}C^{H}\right]
$$

Now, every term can be resolved in the $\boldsymbol{A}$ basis.

$$^{\boldsymbol{H}}\boldsymbol{\omega_A}^{\boldsymbol{B}} = \left[{}^{B}C^{A}\right] \begin{bmatrix} \omega_b \beta_1 \cos \omega_b t \\ 0 \\ 0 \end{bmatrix}_B$$

$$\left(\frac{^{H}d}{dt}[^{\boldsymbol{H}}\boldsymbol{\omega}^{\boldsymbol{B}}]\right)_A = \left[{}^{H}C^{A}\right] \begin{bmatrix} -\omega_b^2 \beta_1 \sin \omega_b t \\ 0 \\ 0 \end{bmatrix}_H$$

$$\boldsymbol{r_A}^{\boldsymbol{OP}} = \left[{}^{H}C^{A}\right] \begin{bmatrix} 0 \\ r_h \\ 0 \end{bmatrix}_H$$

$$\boldsymbol{r_A}^{\boldsymbol{PT}} = \left[{}^{B}C^{A}\right] \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix}_B$$

Substituting the example values into the derived equations, the following values are obtained.

$$^{\boldsymbol{I}}\boldsymbol{\omega_A}^{\boldsymbol{B}} = \begin{bmatrix} 0.5740, & 0.2844, & -7.0000 \end{bmatrix}_A \text{ rad/s}$$

$$^{\boldsymbol{I}}\boldsymbol{\omega_B}^{\boldsymbol{B}} = \left[{}^{B}C^{A}\right]^{\intercal} {}^{\boldsymbol{I}}\boldsymbol{\omega_A}^{\boldsymbol{B}} = \begin{bmatrix} 0.4143, & 0.5405, & -6.996 \end{bmatrix}_B \text{ rad/s}$$

$$^{\boldsymbol{I}}\boldsymbol{\alpha_A}^{\boldsymbol{B}} = \begin{bmatrix} -27.4001, & 31.3414, & -0.5636 \end{bmatrix}_A \text{ rad/s}^2$$

$$^{\boldsymbol{I}}\boldsymbol{v_A}^{\boldsymbol{T}} = \begin{bmatrix} 16.1153, & 83.4428, & 8.1899 \end{bmatrix}_A \text{ m/s}$$

$$^{\boldsymbol{I}}\boldsymbol{a_A}^{\boldsymbol{T}} = \begin{bmatrix} 431.8179, & -168.8583, & 291.5781 \end{bmatrix}_A \text{ m/s}^2$$

These values match perfectly with the expected values. The MATLAB code is given in a separate file.