AE 6210: Advanced Dynamics Homework 3

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February 10, 2025

1 Moments of Inertia

The two links are identical, so their moments of inertia about their respective center of masses are identical. Denote the moment of inertia about the z-axis resolved in the body frames as I_{zz}^C. Since the motion is restricted to the xy-plane (only rotations about the z-axis), I_{zz}^C resolved in the inertial frame is the same as I_{zz}^C resolved in the body frame.

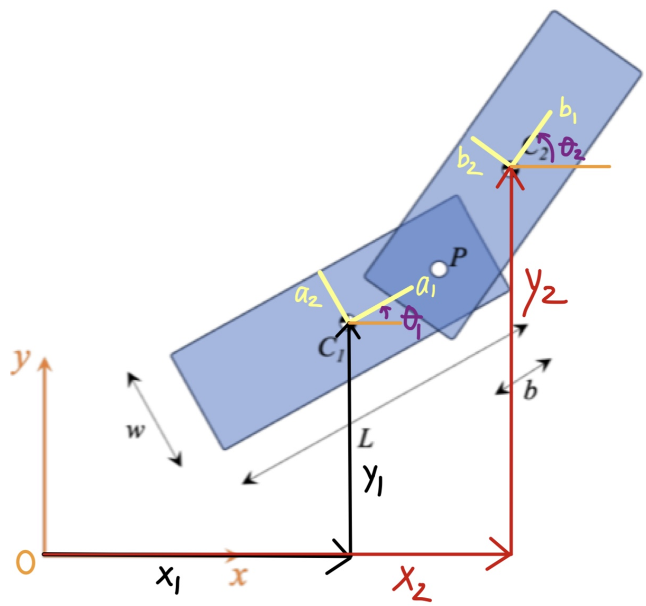

Figure 1: Diagram of the problem. Frame A is fixed to the first link and frame B is fixed to the second link.

The blocks have uniform mass distribution. Denote the mass per unit area of the blocks as $\rho = \frac{dm}{dA} = \frac{m}{wL}$. The substitution $dm = \rho\, dA$ is made.

$$
\begin{aligned}
I_{zz}^{C} &= \int_m (x_A^2 + y_A^2)\, dm \\
&= \int\int_A (x_A^2 + y_A^2)\, \rho\, dA \\
&= \frac{m}{wL} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{-\frac{w}{2}}^{\frac{w}{2}} (x_A^2 + y_A^2)\, dy_A\, dx_A \\
&= \frac{m}{wL} \left[\frac{1}{12} \left(wL^3 + w^3 L \right) \right] \\
\implies & \boxed{I_{zz}^{C} = \frac{m\,(w^2 + L^2)}{12}}
\end{aligned}
$$

The moment of inertia about point P can be calculated using the parallel axis theorem. Similarly, I_{zz}^{P} is the only significant element and is the same in the body and inertial frames.

$$
\begin{aligned}
I_{zz}^{P} &= I_{zz}^{C} + m||\boldsymbol{r^{CP}}||^2 \\
\implies & \boxed{I_{zz}^{P} = I_{zz}^{C} + m\left(\frac{L}{2} - b \right)^2}
\end{aligned}
$$

2 Holonomic Constraints

For convenience, define:

$$
\boxed{D = \left(\frac{L}{2} - b \right)}
$$

Assume the configuration variables to be x_1, y_2, θ_1, x_2, y_2, θ_2. x_2, y_2 are constrained because of the link. x_1 and y_1 are known, so the center of mass of the first link is $\boldsymbol{r^{OC_1}} = x_1\,\hat{\mathbf{i}} + y_1\,\hat{\mathbf{j}}$. The position of the second link's center of mass is:

$$
\begin{aligned}
\boldsymbol{r^{OC_2}} &= \boldsymbol{r^{OC_1}} + \boldsymbol{r^{C_1C_2}} \\
&= \boldsymbol{r^{OC_1}} + D\,\hat{\mathbf{a}}_1 + D\,\hat{\mathbf{b}}_2 \\
&= \left[x_1\,\hat{\mathbf{i}} + y_1\,\hat{\mathbf{j}} \right] + \left[D\left(\cos\theta_1\,\hat{\mathbf{i}} + \sin\theta_1\,\hat{\mathbf{j}} \right) + D\left(\cos\theta_2\,\hat{\mathbf{i}} + \sin\theta_2\,\hat{\mathbf{j}} \right) \right]
\end{aligned}
$$

The two holonomic constraints are:

$$
\boxed{\begin{cases} x_2 &= x_1 + D\left[\cos\theta_1 + \cos\theta_2 \right] \\ y_2 &= y_1 + D\left[\sin\theta_1 + \sin\theta_2 \right] \end{cases}}
$$

Taking the first and second time derivatives of these constraints, we get differential constraints on the velocity and acceleration of the two links.

$$\begin{cases} \dot{x}_2 & = \dot{x}_1 - D\left(\dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2\right) \\ \dot{y}_2 & = \dot{y}_1 + D\left(\dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2\right) \end{cases}$$

$$\begin{cases} \ddot{x}_2 & = \ddot{x}_1 - D\left(\ddot{\theta}_1 \sin\theta_1 + \dot{\theta}_1^2 \cos\theta_1 + \ddot{\theta}_2 \sin\theta_2 + \dot{\theta}_2^2 \cos\theta_2\right) \\ \ddot{y}_2 & = \ddot{y}_1 + D\left(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 \sin\theta_1 + \ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2\right) \end{cases}$$

3 Case 1: Eight DAE's with Six EOM's

Calculate the position of point P relative to C_1 and C_2.

$$\begin{aligned} \boldsymbol{r}^{\boldsymbol{C_1P}} &= D\cos\theta_1\,\hat{\mathbf{i}} + D\sin\theta_1\,\hat{\mathbf{j}} \\ \boldsymbol{r}^{\boldsymbol{C_2P}} &= -D\cos\theta_2\,\hat{\mathbf{i}} - D\sin\theta_2\,\hat{\mathbf{j}} \end{aligned}$$

Breaking the two links into individual bodies, the linkage exerts a constraint force of $F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$ at point P on the first link and a constraint force of $-F_x\,\hat{\mathbf{i}} - F_y\,\hat{\mathbf{j}}$ at point P on the second link. Perform force and moment balance about each of the link's center of masses:

$$\begin{aligned} \Sigma\boldsymbol{F}_1 &= -mg\,\hat{\mathbf{j}} + F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}} \\ &= m\left(\ddot{x}_1\,\hat{\mathbf{i}} + \ddot{y}_1\,\hat{\mathbf{j}}\right) \end{aligned}$$

$$\begin{aligned} \Sigma\boldsymbol{M}_1^C &= \boldsymbol{r}^{\boldsymbol{C_1P}} \times \left(F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}\right) = D\left(F_y\cos\theta_1 - F_x\sin\theta_1\right)\,\hat{\mathbf{k}} \\ &= \left(I_{zz}^C\right) {}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{A}} \\ &= \ddot{\theta}_1\,\hat{\mathbf{k}} \end{aligned}$$

$$\begin{aligned} \Sigma\boldsymbol{F}_2 &= -mg\,\hat{\mathbf{j}} - F_x\,\hat{\mathbf{i}} - F_y\,\hat{\mathbf{j}} \\ &= m\left(\ddot{x}_2\,\hat{\mathbf{i}} + \ddot{y}_2\,\hat{\mathbf{j}}\right) \end{aligned}$$

$$\begin{aligned} \Sigma\boldsymbol{M}_2^C &= \boldsymbol{r}^{\boldsymbol{C_2P}} \times \left(-F_x\,\hat{\mathbf{i}} + -F_y\,\hat{\mathbf{j}}\right) = D\left(F_y\cos\theta_2 - F_x\sin\theta_2\right)\,\hat{\mathbf{k}} \\ &= \left(I_{zz}^C\right) {}^{\boldsymbol{I}}\boldsymbol{\alpha}^{\boldsymbol{B}} \\ &= \ddot{\theta}_2\,\hat{\mathbf{k}} \end{aligned}$$

The force and moment balances give six equations of motion, and combined with the two constraint equations, there are eight equations. The second derivatives of the configuration variables and the constraint forces are unknown, so there ends up being 8 unknowns with 8 linear equations.

$$m\ddot{x}_1 - F_x = 0 \tag{1.1}$$

$$m\ddot{y}_1 - F_y = -mg \tag{2.1}$$

$$I_{zz}^C\ddot{\theta}_1 + D\sin\theta_1 F_x - D\cos\theta_1 F_y = 0 \tag{3.1}$$

$$m\ddot{x}_2 + F_x = 0 \tag{4.1}$$

$$m\ddot{y}_2 + F_y = -mg \tag{5.1}$$

$$I_{zz}^C\ddot{\theta}_2 + D\sin\theta_2 F_x - D\cos\theta_2 F_y = 0 \tag{6.1}$$

$$\ddot{x}_1 - D\sin\theta_1\ddot{\theta}_1 - \ddot{x}_2 - D\sin\theta_2\ddot{\theta}_2 = D\left[\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right] \tag{7.1}$$

$$\ddot{y}_1 + D\cos\theta_1\ddot{\theta}_1 - \ddot{y}_2 + D\cos\theta_2\ddot{\theta}_2 = D\left[\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right] \tag{8.1}$$

This can be written in the linear form $A\{q\} = \{B\}$ and A can be inverted to find $\{q\}$.

$$\begin{bmatrix} m & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & m & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & I_{zz}^C & 0 & 0 & 0 & D\sin\theta_1 & -D\cos\theta_1 \\ 0 & 0 & 0 & m & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & m & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & I_{zz}^C & D\sin\theta_2 & -D\cos\theta_2 \\ 1 & 0 & -D\sin\theta_1 & -1 & 0 & -D\sin\theta_2 & 0 & 0 \\ 0 & 1 & D\cos\theta_1 & 0 & -1 & D\cos\theta_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta}_1 \\ \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\theta}_2 \\ F_x \\ F_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -mg \\ 0 \\ 0 \\ -mg \\ 0 \\ D\left(\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right) \\ D\left(\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right) \end{bmatrix}$$

4 Case 2: Generalized Coordinates With Constraint Forces

The set of general coordinates i.e. the minimal set of coordinates to define this system is $\{x_1, x_2, \theta_1, \theta_2\}$. Looking back as **Case 1**, equation (4.1) can be substituted into equation (7.1) to get rid of $\ddot{x}_2$, and equation (5.1) can be substituted into equation (8.1) to get rid of $\ddot{y}_2$. Note that equations (7.1) and (8.1) are the constraint equations. 6 linear equations with 6 unknowns, which are the second derivatives of the generalized coordinates and the constraint forces, are obtained.

$$m\ddot{x}_1 - F_x = 0 \tag{1.2}$$
$$m\ddot{y}_1 - F_y = -mg \tag{2.2}$$
$$I_{zz}^C\ddot{\theta}_1 + D\sin\theta_1 F_x - D\cos\theta_1 F_y = 0 \tag{3.2}$$
$$I_{zz}^C\ddot{\theta}_2 + D\sin\theta_2 F_x - D\cos\theta_2 F_y = 0 \tag{4.2}$$
$$\ddot{x}_1 - D\sin\theta_1\ddot{\theta}_1 - D\sin\theta_2\ddot{\theta}_2 + \frac{1}{m}F_x = D\left[\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right] \tag{5.2}$$
$$\ddot{y}_1 + D\cos\theta_1\ddot{\theta}_1 - \ddot{y}_2 + D\cos\theta_2\ddot{\theta}_2 + \frac{1}{m}F_y = D\left[\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right] - g \tag{6.2}$$

This can also be written in the linear form $A\{q\} = \{B\}$ and A can be inverted to find $\{q\}$.

$$\begin{bmatrix} m & 0 & 0 & 0 & -1 & 0 \\ 0 & m & 0 & 0 & 0 & -1 \\ 0 & 0 & I_{zz}^C & 0 & D\sin\theta_1 & -D\cos\theta_1 \\ 0 & 0 & 0 & I_{zz}^C & D\sin\theta_2 & -D\cos\theta_2 \\ 1 & 0 & -D\sin\theta_1 & -D\sin\theta_2 & \frac{1}{m} & 0 \\ 0 & 1 & D\cos\theta_1 & D\cos\theta_2 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ F_x \\ F_y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -mg \\ 0 \\ 0 \\ D\left(\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right) \\ D\left(\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right) - g \end{bmatrix}$$

5 Case 3: Generalized Coordinates with No Constraint Forces

Now, consider **Case 2** but without the constraint forces. Equations (1.2) and (2.2) can be substituted into the rest of the equations to get rid of F_x and F_y.

$$mD\sin\theta_1\ddot{x}_1 - mD\cos\theta_1\ddot{y}_1 + I_{zz}^C\ddot{\theta}_1 = mgD\cos\theta_1 \tag{1.3}$$
$$mD\sin\theta_2\ddot{x}_1 - mD\cos\theta_2\ddot{y}_1 + I_{zz}^C\ddot{\theta}_2 = mgD\cos\theta_2 \tag{2.3}$$
$$2\ddot{x}_1 - D\sin\theta_1\ddot{\theta}_1 - D\sin\theta_2\ddot{\theta}_2 = D\left[\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right] \tag{3.3}$$

$$2\ddot{y}_1 + D\cos\theta_1\ddot{\theta}_1 + D\cos\theta_2\ddot{\theta}_2 = D\left[\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right] - 2g \tag{4.3}$$

Like before, this can be written in the linear form $A\{q\} = \{B\}$ and A can be inverted to find $\{q\}$.

$$\begin{bmatrix} mD\sin\theta_1 & -mD\cos\theta_1 & I_{zz}^C & 0 \\ mD\sin\theta_2 & -mD\cos\theta_2 & 0 & I_{zz}^C \\ 2 & 0 & -D\sin\theta_1 & -D\sin\theta_2 \\ 0 & 2 & D\cos\theta_1 & D\cos\theta_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} mgD\cos\theta_1 \\ mgD\cos\theta_2 \\ D\left(\dot{\theta}_1^2\cos\theta_1 + \dot{\theta}_2^2\cos\theta_2\right) \\ D\left(\dot{\theta}_1^2\sin\theta_1 + \dot{\theta}_2^2\sin\theta_2\right) - 2g \end{bmatrix}$$

6 Numerical Example

The following values are given as an example case and implemented in MATLAB. The code file is attached separately.

$$\begin{aligned} L &= 0.3\,\mathrm{m} & w &= 0.1\,\mathrm{m} & b &= 0.04\,\mathrm{m} \\ m &= 0.1\,\mathrm{kg} & g &= 9.81\,\mathrm{m/s^2} \\ x_1 &= 1\,\mathrm{m} & \dot{x}_1 &= 0.15\,\mathrm{m/s} \\ y_1 &= 2\,\mathrm{m} & \dot{y}_1 &= 0.25\,\mathrm{m/s} \\ \theta_1 &= 0.1\,\mathrm{rad} & \dot{\theta}_1 &= 0.35\,\mathrm{rad/s} \\ \theta_2 &= 0.2\,\mathrm{rad} & \dot{\theta}_2 &= 0.45\,\mathrm{rad/s} \end{aligned}$$

Resulting MOI's, Position of P, and Position of Second Link

$$
\begin{aligned}
I_{zz}^{C} &= 8.3333 \times 10^{-4}\,\mathrm{kg\ m^2} & \qquad I_{zz}^{C} &= 2.0433 \times 10^{-4}\,\mathrm{kg\ m^2} \\
J_1 &= 0.5008\,\mathrm{kg\ m^2} & J_2 &= 0.5623\,\mathrm{kg\ m^2} \\
x_2 &= 1.2173\,\mathrm{m} & y_2 &= 2.0328\,\mathrm{m} \\
x_p &= 1.1095\,\mathrm{m} & y_p &= 2.0110\,\mathrm{m}
\end{aligned}
$$

Case 1: Eight DEA's

$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{\theta}_1 \\ \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{\theta}_2 \\ F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0.017575 \\ -9.80725 \\ 0.0129194 \\ -0.017575 \\ -9.81275 \\ -0.0105512 \\ 0.0017575 \\ 0.000274704 \end{bmatrix}
$$

Case 2: Six DEA's

$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ F_x \\ F_y \end{bmatrix} = \begin{bmatrix} 0.017575 \\ -9.80725 \\ 0.0129194 \\ -0.0105512 \\ 0.0017575 \\ 0.000274704 \end{bmatrix}
$$

Case 2: Four DEA's

$$
\begin{bmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0.017575 \\ -9.80725 \\ 0.0129194 \\ -0.0105512 \end{bmatrix}
$$

The results match perfectly with the expected values.