AE 6210: Advanced Dynamics Homework 1

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1 Equations of Motion for Circular Orbit

Define the center of the Earth as point O, and model the spacecraft as a point mass P. Fix an inertial frame $\{I\}$ with an origin coinciding with point O and a basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$. Also, define a rotating frame $\{B\}$ with an origin also coinciding with point O and a basis $\hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2$, where $\hat{\mathbf{b}}_1$ is always parallel to the position vector of P relative to O, $\mathbf{r_{OP}}$.

$$\mathbf{r_{OP}} = x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2 \implies {}^I\mathbf{v_{OP}} = \dot{x}\hat{\mathbf{e}}_1 + \dot{y}\hat{\mathbf{e}}_2 \implies {}^I\mathbf{a_{OP}} = \ddot{x}\hat{\mathbf{e}}_1 + \ddot{y}\hat{\mathbf{e}}_2$$

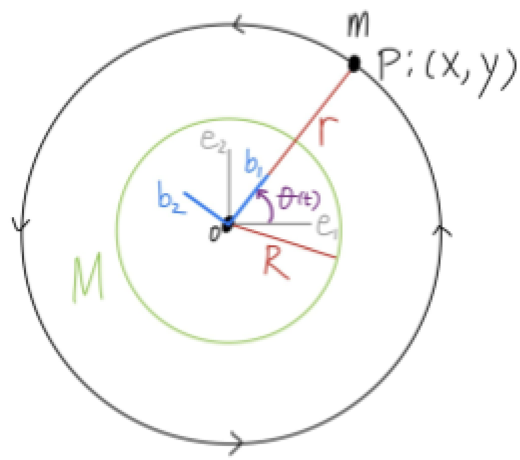

Figure 1: Diagram of the orbit where P represents the spacecraft and O denotes the Earth's center.

In a circular orbit, the velocity vector is always tangent to the path. Using the relation $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$, the following relation between velocity and position is derived.

$${}^I\mathbf{v_{OP}} = V\hat{\mathbf{b}}_2 = V[(\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_1)\hat{\mathbf{e}}_1 + (\hat{\mathbf{b}}_2 \cdot \hat{\mathbf{e}}_2)\hat{\mathbf{e}}_2] = -V\sin\theta\hat{\mathbf{e}}_1 + V\cos\theta\hat{\mathbf{e}}_2 \implies \begin{cases} \dot{x} = -\dfrac{yV}{r} \\ \dot{y} = \dfrac{xV}{r} \end{cases}$$

Additionally, since the orbit is circular, the spacecraft maintains a constant speed. The only acceleration present is the centripetal acceleration.

$$^I\mathbf{a_{OP}} = \frac{||^I\mathbf{v_{OP}}||^2}{||\mathbf{r_{OP}}||}(-\hat{\mathbf{b}}_1) = -\frac{V^2}{r}(\hat{\mathbf{b}}_1)$$

If the orbit is at a high enough altitude, the only significant force acting on the spacecraft is the gravitational force of the Earth. By applying Newton's second law, the constant orbital speed is found.

$$\mathbf{F_P} = m^I\mathbf{a_{OP}} \implies \frac{GMm}{r^2}(-\hat{\mathbf{b}}_1) = -\frac{mV^2}{r}\hat{\mathbf{b}}_1 \implies V = \sqrt{\frac{\mu}{r}} \text{ where } \mu = GM$$

Rewrite the gravitational force in Cartesian Coordinates.

$$\mathbf{F_P} = -\frac{\mu m}{(x^2+y^2)}\left(\frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_1 + \frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{e}}_2\right) = m(\ddot{x}\hat{\mathbf{e}}_1 + \ddot{y}\hat{\mathbf{e}}_2)$$

After simplifying, the Equations of Motion are obtained.

$$\begin{cases} \ddot{x} = -\dfrac{\mu x}{(x^2+y^2)^{3/2}} & \text{where } \mu = GM \\ \ddot{y} = -\dfrac{\mu y}{(x^2+y^2)^{3/2}} & \end{cases}$$

with the constraints:

$$x^2 + y^2 = r^2, \quad V = \sqrt{\frac{\mu}{r}}, \quad \begin{cases} \dot{x} = -V\sin\theta \\ \dot{y} = V\cos\theta \end{cases} \quad \text{where } r,\ V, \text{ and } \mu \text{ are constants}$$

2 Numerical Simulation of an Example Case

Defining a state vector $q = [q_1, q_2, q_3, q_4]^T = [x, \dot{x}, y, \dot{y}]^T$, the system is written in first-order form.

$$\dot{q} = \left[q_2\ , \quad -\frac{\mu q_1}{({q_1}^2 + {q_3}^2)^{3/2}}\ , \quad q_4\ , \quad -\frac{\mu q_3}{({q_1}^2 + {q_3}^2)^{3/2}}\right]^T$$

The example case study is a satellite in low-Earth orbit. The orbital radius r can be calculated using the initial conditions $x(0)$ and $y(0)$, and the orbital speed is then calculated using the gravitational parameter μ and the orbital radius r.

$$\begin{cases} \mu = GM_{Earth} = (6.6743 \times 10^{-11}\dfrac{m^3}{kg\ s^2})(5.97219 \times 10^{24}\ kg)(\dfrac{km^3}{1000^3\ m^3}) = 3.986 \times 10^5\ \dfrac{km^3}{s^2} \\ x(0) = 6500\ km, \quad y(0) = 2700\ km, \end{cases}$$

$$\Longrightarrow\ r = 7039\ km, \quad V = 7.53\ \frac{km}{s}, \quad \dot{x}(0) = -2.89\ \frac{km}{s}, \quad \dot{y}(0) = 6.95\ \frac{km}{s}$$

The dynamics were solved using MATLAB's ode45 function with a relative tolerance of 1×10^{-5}, absolute tolerance of 1×10^{-6}, and time interval of 17 hours $\approx$ 10 orbits.

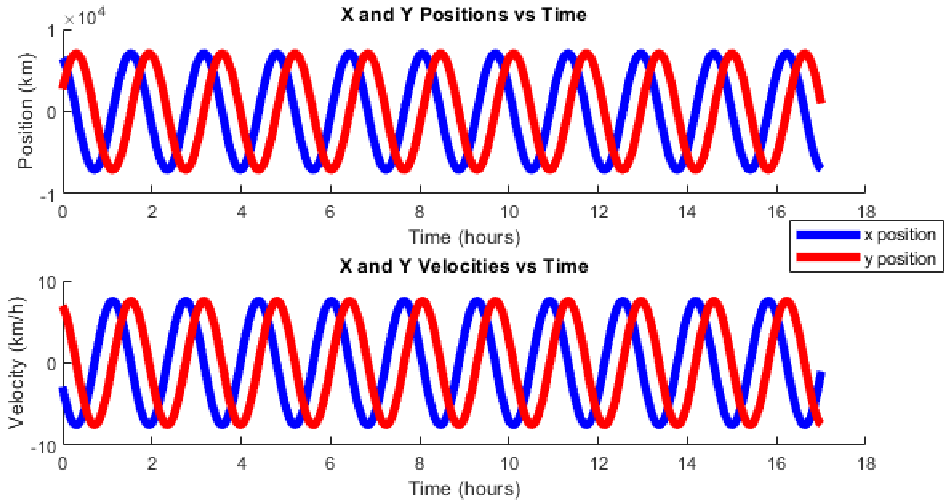

Figure 2: Position and Velocity over a time period of 17 hours $\approx$ 10 orbits.

3 Error Analysis Using Invariants

The error of the numerical solver was quantified by computing the invariants: the orbital radius, angular momentum, and kinetic energy. Since each of these invariants is defined by the initial conditions, the exact error between the truth and the simulation can be computed.

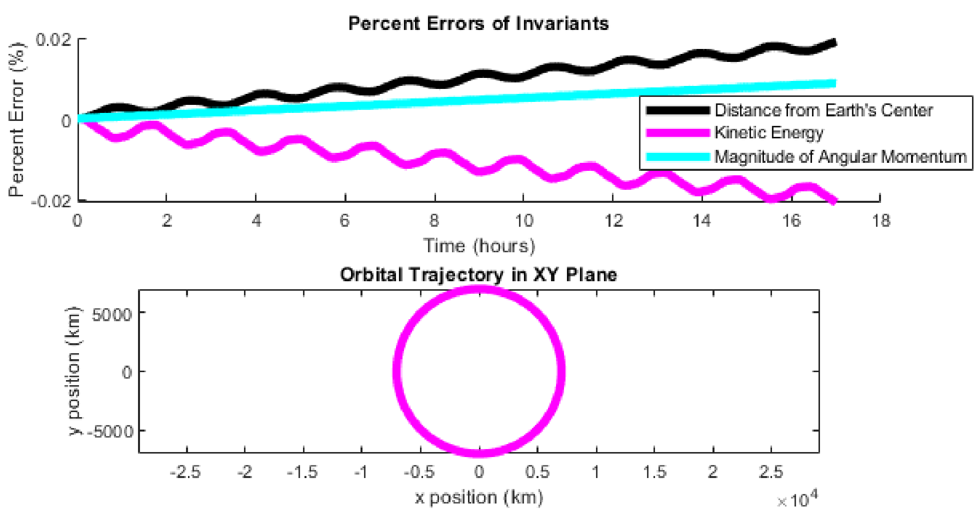

Figure 3: Relative error between the known invariants and simulated invariants. The trajectory is a nice circle as expected.

The error can be reduced by decreasing the relative and absolute tolerances. However, for this system, further reduction in the error of the invariants becomes marginal if the tolerances are decreased below the values mentioned above. Initially, using MATLAB's default relative tolerance of 1×10^{-3} and absolute tolerance of 1×10^{-6}, the errors were significantly larger, up to 20 % for this case, and divergent.

4 Thruster Misfire Case Study

Suppose that there is a catastrophic failure and one of the thrusters becomes stuck, constantly propelling the spacecraft in the tangential direction $\hat{\mathbf{b}}_2$ with a thrust of $\mathbf{T} = T\hat{\mathbf{b}}_2$. The net force on the spacecraft becomes:

$$\mathbf{F_P} = -\frac{\mu m}{(x^2+y^2)^{3/2}}\,(x\hat{\mathbf{e}}_1 + y\hat{\mathbf{e}}_2) + \frac{T}{\sqrt{x^2+y^2}}(-y\hat{\mathbf{e}}_1 + x\hat{\mathbf{e}}_2) = m(\ddot{x}\hat{\mathbf{e}}_1 + \ddot{y}\hat{\mathbf{e}}_2)$$

$$\implies \dot{q} = \left[q_2\ ,\quad -\frac{\mu q_1}{({q_1}^2+{q_3}^2)^{3/2}} - \frac{\frac{T}{m}q_3}{\sqrt{q_1^2+q_3^2}}\ ,\quad q_4\ ,\quad -\frac{\mu q_3}{({q_1}^2+{q_3}^2)^{3/2}} + \frac{\frac{T}{m}q_1}{\sqrt{q_1^2+q_3^2}}\right]^T$$

In this example, a 1700 kg satellite with the same initial circular orbit as the previous example is simulated for various thrust values. Low-Earth orbit satellites typically have very little thrust, which is primarily used for attitude adjustment. Some have only a few milli-Newtons of thrust, while others have up to 5 Newtons. Using MATLAB's ode45 solver with a relative tolerance of 1×10^{-5}, an absolute tolerance of 1×10^{-6}, and time span of 2 hours before the satellite runs out of fuel, the satellite's trajectory is computed.

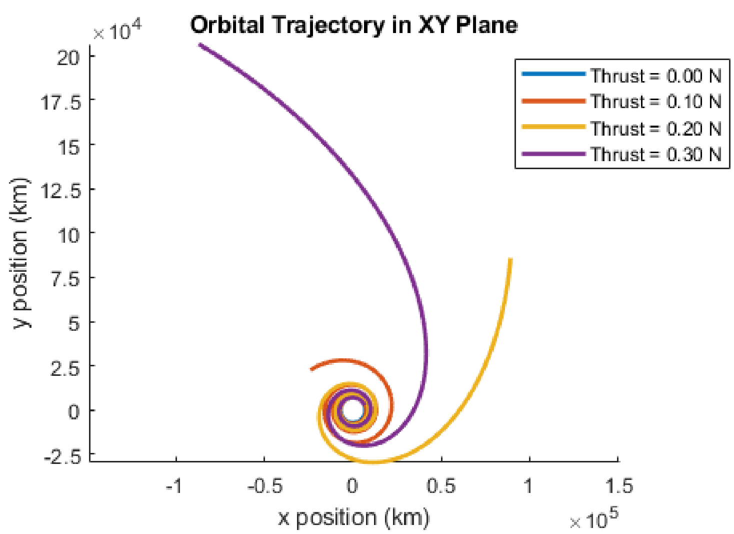

Figure 4: Trajectories for when the thrusters are stuck at different thrust values.

At the satellite's initial orbital altitude of approximately 660 km, the gravitational force of the Earth is the only significant force acting on the satellite when there are no thrusters.

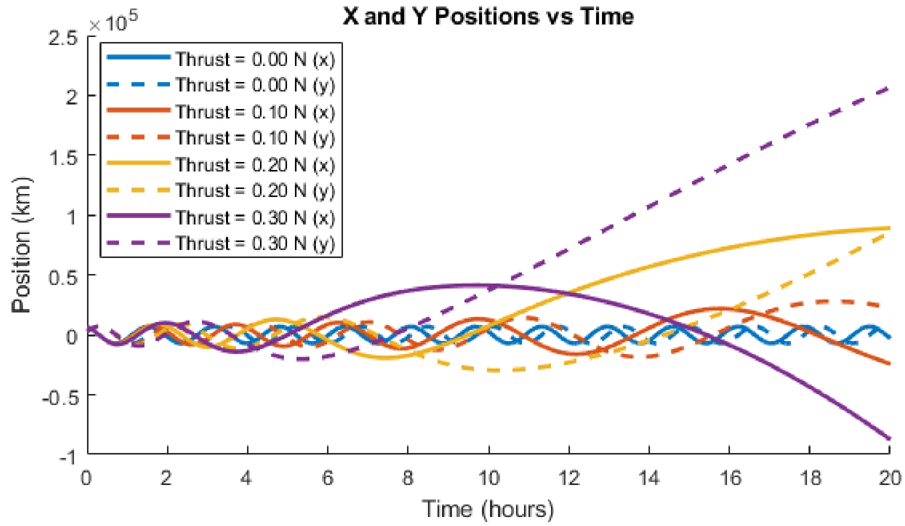

Figure 5: Diagram of the trajectory when the thrusters are stuck at different values.

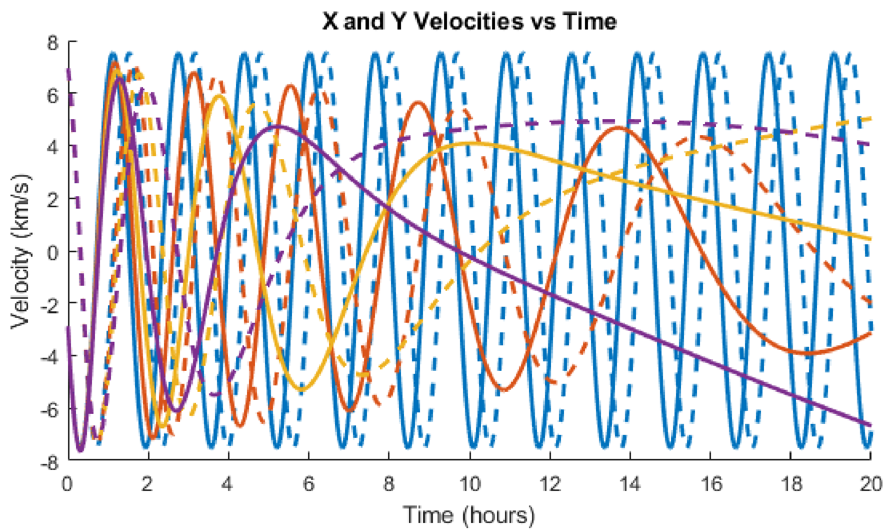

Figure 6: Diagram of the trajectory when the thrusters are stuck at different values.

However, even a small constant thrust value causing tangential acceleration will make the satellite deviate from its original circular orbit. The thrust increases the satellite's kinetic energy, which, in turn, causes it to move to a higher orbit. Once the satellite runs out of fuel, its velocity vector will most likely no longer be tangent to a circular path around the Earth. As a result, it will follow an elliptical orbit around the Earth until the end of time (or an astroid hits it).