Pose

$$g_{01} = \left[\begin{array}{c|c} R_{01} & \vec{d}_{01}^{\,0} \\ \hline 0_{3\times1} & 1 \end{array}\right]$$

$$Q_{01} = \begin{Bmatrix} q_{01} \\ (0, \vec{d}_{01}^{\,0}) \end{Bmatrix}$$

Twist

$$\vec{\xi}^{\,0} = \left[\begin{array}{c} \vec{\omega}^{\,0} \\ \hline \vec{v}^{\,0} \end{array}\right]$$

$$\tilde{\xi}^{\,0} = \begin{Bmatrix} (0, \vec{\omega}^{\,0}) \\ (0, \vec{v}^{\,0}) \end{Bmatrix}$$

Twist → Pose

$$\vec{\xi}\theta \rightarrow \begin{cases} \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & \dfrac{[I-e^{\hat{\omega}\theta}]\hat{\omega}\vec{v}+\vec{\omega}\vec{\omega}^T\vec{v}\theta}{\vec{\omega}^T\vec{\omega}} \\ \hline 0_{3\times1} & 1 \end{array}\right] & \text{Revolute} \\ \left[\begin{array}{c|c} I & \vec{v}\theta \\ \hline 0_{3\times1} & 1 \end{array}\right] & \text{Prismatic} \end{cases}$$

$$\tilde{\xi}\theta \rightarrow \begin{cases} \left\{\begin{array}{l} \left(\cos\frac{\theta}{2}, \vec{\omega}\sin\frac{\theta}{2}\right) \\ \left(0, \vec{v}\sin\frac{\theta}{2}\right) \end{array}\right\} & \text{Revolute} \\ \left\{\begin{array}{l} \left(1, \vec{0}\right) \\ \left(0, \vec{v}\frac{\theta}{2}\right) \end{array}\right\} & \text{Prismatic} \end{cases}$$

Twist Transformation

$$\vec{\xi}^{\,0} = Ad_{g_{01}}\vec{\xi}^{\,1}$$

$$\tilde{\xi}^{0} = Q_{01}\tilde{\xi}^{1}Q_{01}^{*}$$

Product of Exponentials

0) Identify home pose of end effector.

1) Identify joint axes: $\vec{\omega}_1^s, \vec{\omega}_2^s, \dots, \vec{\omega}_N^s$

2) Find points on joint axes: $\vec{p}_1^s, \vec{p}_2^s, \dots, \vec{p}_N^s$

3) Calculate velocities: $\vec{v}_i^s = -\vec{\omega}_i^s \times \vec{p}_i^s$

4) Formulate twists: $\vec{\xi}_i^s = \begin{bmatrix} \vec{\omega}_i \\ \hline \vec{v}_i \end{bmatrix}$

SE(3) Method

5) $g_{st}(\theta) = e^{\hat{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_N \theta_N} g_{st}(0)$

Store $g_{chain} = [g_1, g_1 g_2, \dots, g_1 g_2 \dots g_N]$

DQ Method

5) cast twists as DQ: $\tilde{\xi} = \begin{Bmatrix} (0, \vec{\omega}) \\ (0, \vec{v}) \end{Bmatrix}$

6) $Q_{st}(\theta) = e^{\tilde{\xi}_1 \theta_1} e^{\hat{\xi}_2 \theta_2} \dots e^{\hat{\xi}_N \theta_N} Q_{st}(0)$

Store $Q_{chain} = [Q_1, Q_1 Q_2, \dots, Q_1 Q_2 \dots Q_N]$

Manipulator Jacobian

SE(3) Method

6) Calculate transformed twists:

$$\begin{cases} \vec{\xi}_1' = \vec{\xi}_1 \\ \vec{\xi}_2' = Ad_{g_1} \vec{\xi}_2 \\ \vec{\xi}_3' = Ad_{g_1 g_2} \vec{\xi}_3 \\ \quad \vdots \\ \vec{\xi}_N' = Ad_{g_1 g_2 \dots g_{N-1}} \vec{\xi}_N \end{cases}$$

where

$$\begin{cases} g_i = e^{\hat{\xi}_i \theta_i} \\ Ad_{g_i} = \left[\begin{array}{c|c} R & 0_{3\times3} \\ \hline \hat{d}R & R \end{array}\right] \end{cases}$$

7) Formulate Spatial Manipulator Jacobian:

$$J_{st}^{s} = \left[\vec{\xi}_1 \quad \vec{\xi}_2' \quad \dots \quad \vec{\xi}_N'\right]$$

DQ Method

7) Calculate transformed twists:

$$\left\{\begin{aligned}
\tilde{\xi}_1' &= \tilde{\xi}_1 \\
\tilde{\xi}_2' &= Q_1 \, \tilde{\xi}_2 \, Q_2^* \\
\tilde{\xi}_3' &= Q_1 \, Q_2 \, \tilde{\xi}_3 \, Q_2^* \, Q_1^* \\
&\ \ \vdots \\
\tilde{\xi}_N' &= Q_1 \, Q_2 \cdots Q_N \, \tilde{\xi}_N \, Q_N^* \cdots Q_2^* \, Q_1^*
\end{aligned}\right.$$

$$Q_i^* \cdots Q_2^* Q_1^* = (\underbrace{Q_1 Q_2 \cdots Q_i}_{\text{From } Q_{chain}})^*$$

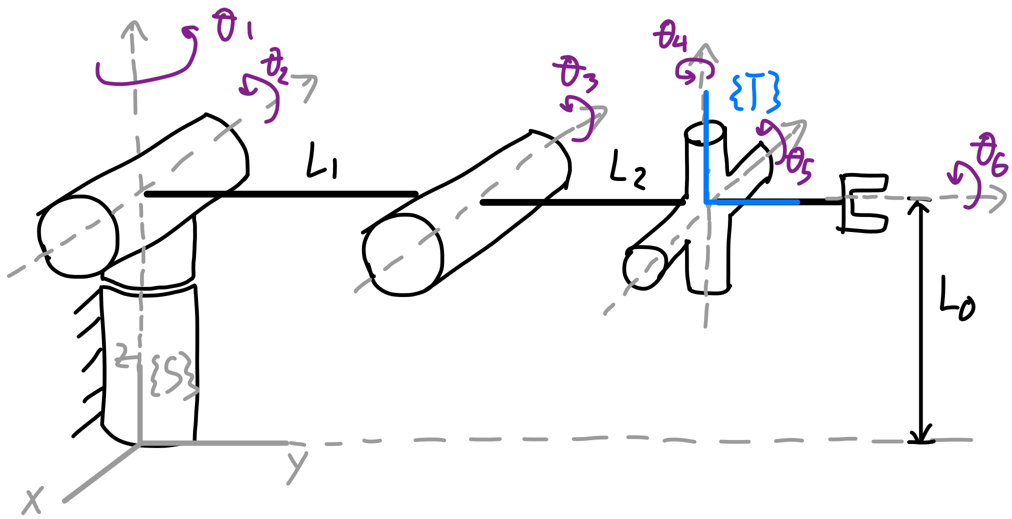
8) Delete padded zeros:

$$\tilde{\xi}_i' = \begin{Bmatrix} (\cancel{0}, \vec{\omega}_i') \\ (\cancel{0}, \vec{v}_i') \end{Bmatrix} \longrightarrow \vec{\xi}_i'$$

9) Formulate Spatial Manipulator Jacobian:

$$J_{st}^s = [\vec{\xi}_1 \ \ \vec{\xi}_2' \ \cdots \ \vec{\xi}_N']$$

Elbow



$$
\begin{cases}
\vec{q}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \vec{q}_2 = \begin{bmatrix} 0 \\ 0 \\ L_0 \end{bmatrix}, \ \vec{q}_3 = \begin{bmatrix} 0 \\ L_1 \\ L_0 \end{bmatrix}, \ \vec{q}_4 = \begin{bmatrix} 0 \\ L_1+L_2 \\ L_0 \end{bmatrix} = \vec{q}_5 = \vec{q}_6 \\
\vec{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{p}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{p}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \vec{p}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \ \vec{p}_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
\end{cases}
$$