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ELECTROMAGNETISM

Electrostatics

Electric Forces

In Electrodynamics, there is typically a *source point* $\mathbf{r}'$ where a charge is located and a *field point* $\mathbf{r}$ where a field is calculated at. The *seperation vector* is defined as

$$\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}' \qquad \hat{\boldsymbol{\mathscr{r}}} = \boldsymbol{\mathscr{r}}/\mathscr{r} \tag{1}$$

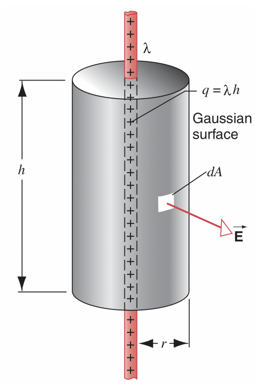

Figure 1: A Gaussian surface as a cylinder.

The force acting on a charge is given by *Coulomb's Law*,

$$\mathbf{F} = k \sum \frac{q_1 q_2}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} = \iiint d\mathbf{F} \tag{2}$$

The *electric field* exerted on a positive test charge $+q_0$ at a point is defined as

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} = k \sum \frac{q}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} \tag{3}$$

Field lines point outwards from $+q$ and towards $-q$, and are parallel or tangential of an electric field. For a continuous charge distribution,

$$\mathbf{E} = k \iiint \frac{dq}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}} = -\nabla V \tag{4}$$

where

$$dq \sim \lambda\, d\ell \sim \sigma\, dA \sim \rho\, dV \sim \lambda R\, d\phi \sim \sigma 2\pi w\, dw \tag{5}$$

The *electric dipole* is a configuration of two equal and opposite charges q separated by a distance d, in which the electric dipole moment $p = qd$ in the direction towards $+q$. If a dipole is in an external field, the torque is

$$\text{ø} = \mathbf{p} \times \mathbf{E} \tag{6}$$

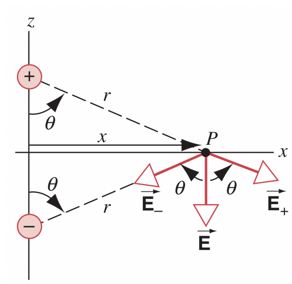

Figure 2: The field at any is the vector sum of the charges.

such that the work done by the field is

$$W = -\int_{\theta_0}^{\theta} \tau\, d\theta = pE(\Delta \cos\theta) \tag{7}$$

Consequently, the potential energy is defined as

$$U = -pE\cos\theta = -\mathbf{p} \cdot \mathbf{E} \tag{8}$$

Potentials

Gauss' Law states the electric *flux* through a field is

$$\Phi_E = \oiint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\varepsilon_0} \tag{9}$$

such that $\|\Phi\| = EA\cos\theta$ where θ is the angle between the field and field lines. The electric field outside a conductor is given by

$$E = \frac{\sigma}{\varepsilon_0} \iff q = \iint \sigma \, dA \tag{10}$$

The change in electric potential is defined as

$$\Delta U = kq_1q_2(\Delta r^{-1}) \tag{11}$$

For a system of charge, the *potential energy* is

$$U = k\sum_i \sum_{j\neq i} \frac{q_iq_j}{\imath} \tag{12}$$

Consequently, the change in *electric potential* is

$$\Delta V = \frac{\Delta U}{q_0} = -\frac{W}{q_0} = -\int \mathbf{E} \cdot d\mathbf{s} \tag{13}$$

The electric potential at a point is thus defined as

$$V = k\iiint_a^b \frac{dq}{\imath} = k\sum \frac{q_i}{\imath} \sim k\frac{p\cos\theta}{\imath^2} \tag{14}$$

where the latter equivalence holds for dipoles. A surface on which the potential has the same value everywhere is *equipotential* such that $\Delta V = W = 0$. The field lines must everywhere be perpendicular to the equipotential surfaces, which implies all conductors are equipotential.

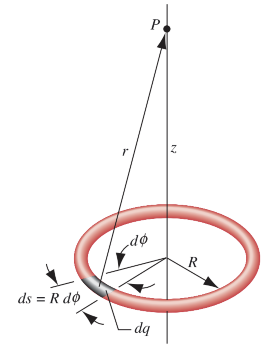

Figure 3: A uniformly charged ring with a potential P and charge element dq.

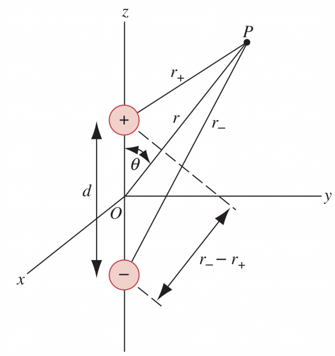

Figure 4: The geometry for calculating V at P for a dipole.

Examples

1. Given $i = 2.5 \times 10^4$ C/s and $t = 20$ μs, calculate the charge q.
2. There is a semicircular ring of charge with radius r and charge Q. Find E at the center of the circle P.
3. Let there be a square of length a with 4 charges on the corners. The charges clockwise are $+q$, $-q$, $-2q$, $+2q$. Find F on the $+2q$ charge.
4. For a square with charges Q, q, Q, q on the corners, if $F_Q = 0$, relate Q and q.
5. Two charges Q are d apart. A charge $-q$ and mass m are midway between them that is slightly displaced and released. Prove $-q$ has a period $(\epsilon_0 m\pi^3 d^3/qQ)^{1/2}$.
6. Prove $\tau = \mathbf{p} \times \mathbf{E}$ and $\Delta U = -\mathbf{p} \cdot \mathbf{E}$.
7. (a) Consider a line charge of length L with λ. The rod is aligned along z and the centre of the rod is the origin. Find E at a point P a distance y from the rod along its perpendicular bisector (the positive y axis). Use the substitution $z = y\sinh t$ and $1 + \sinh 2t = \cosh 2t$. (b) Assume that the line charge extends from $[-\infty, \infty]$ along the z-axis. Calculate E.

Electrodynamics

Current

The electric current is defined as

$$i = \frac{dq}{dt} = \int \mathbf{j} \cdot d\mathbf{A} \tag{15}$$

where j is the current density and is opposite the motion of electrons. The net charge passing through is therefore

$$q = \int i \, dt \tag{16}$$

The current density is defined as

$$\mathbf{j} = q/At = -en\mathbf{v} = \sigma\mathbf{E} \tag{17}$$

where n is the electron density and $\sigma = q/A$ is the conductivity. Furthermore, the net charge passing through the surface $q = enAL$. The resistance of the material is thus $R = L/\sigma A = \Delta V/i$.

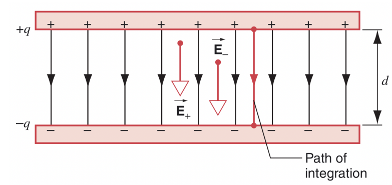

Figure 5: A parallel-plate capacitor.

Capacitance

Capacitance is defined as

$$C = \frac{q}{\Delta V} = \frac{\varepsilon_0 A}{d} \tag{18}$$

In a PPC, SC, and CC, the potential is derived via

$$\Delta V_{ppc} = \frac{qd}{\epsilon A} \qquad \Delta V_{sc} = kq\left(\frac{1}{a} - \frac{1}{b}\right) \qquad \Delta V_{ss} = \frac{q\ln(b-a)}{2\pi\epsilon L} \tag{19}$$

In a parallel combination of capacitors,

$$q = \sum q = C_{eq}\Delta V \iff C_{eq} = \sum C \tag{20}$$

and in a series combination,

$$\Delta V = \sum \Delta V = q/C_{eq} \iff C_{eq}^{-1} = \sum C^{-1} \tag{21}$$

As well, the potential energy in a capacitor is

$$dU = \Delta V \, dq \iff U = \int_0^q dU = \frac{q^2}{2C} \tag{22}$$

with $q = C\Delta V$. Specifically, the energy U is stored in E in the region. Furthermore, the *energy density* u is defined as

$$u = \frac{U}{Ad} = \tfrac{1}{2}\varepsilon_0 E^2 \tag{23}$$

by Eqns 18 & 22. Using *voltage division*, the voltage of any impedance Z including capacitance may be found via $q_0 = \sum q$ by

$$V = \frac{Z_1}{Z_{eq}} V_0 \tag{24}$$

For a *dielectric capacitor*, the charge is defined as

$$q' = q(1 - 1/\kappa) \iff E = E_0/\kappa \tag{25}$$

DC Circuits

The *emf* of a voltage source is defined as

$$\mathcal{E} = dW/dq = iR \tag{26}$$

The electric potential v decreases across a voltage source in a circuit.

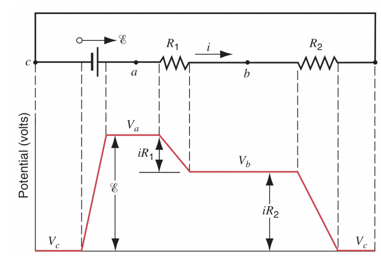

Figure 6: A simple circuit and its potential difference change.

For more information, view the Electric Circuits textbook supplement.

The emf is related to the electric field by

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s} \tag{27}$$

where $\mathbf{E} = \mathbf{F}/q = \rho j$.

Magnetic Forces

Magnetic Field

The magnetic force is defined as

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \|\mathbf{F}\| = |q|vB\sin\phi \tag{28}$$

where B is the field and v the velocity. The combined electric and magnetic fields yield the Lorentz force,

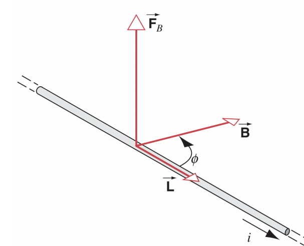

Figure 7: **F** acting on a wire **L** making an angle ϕ with **B**.

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{29}$$

For a velocity selector, then $v = E/B$, and for a particle in a circular path, then:

$$|q|vB = mv^2/r \qquad \omega = v/r = |q|B/m \tag{30}$$

where $\omega = 2\pi f = r'$ is the angular velocity, and the kinetic energy is $K = 0.5mv^2$.

The Hall effect states for a strip of conductor, the electric field is

$$\mathbf{E}_H = -\mathbf{v}_d \times \mathbf{B} \tag{31}$$

which gives $n = iB/et\Delta V_H$. For a straight wire, then

$$\mathbf{F} = i\mathbf{L} \times \mathbf{B} \sim iLB\sin\phi \tag{32}$$

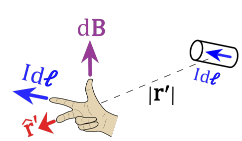

Figure 8: The right hand rule for magnetic field lines.

For a uniform non-straight wire, $dF = iB\,ds$. The total torque on such wire being rotated is given by

$$\tau = NIA\mathbf{n} \times \mathbf{B} \tag{33}$$

Alternate Definition

The magnetic field may also be expressed as

$$\mathbf{B} = K\frac{|q|v\sin\phi}{r^2} = K\int_C \frac{i\,d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \tag{34}$$

where $i\,d\mathbf{s} = dq\,v$, known as the Biot-Savart law. For two parallel wires in a magnetic field, the force of wire 1 on 2 is

$$F_{21} = i_2 L B_1 = \frac{K L i_1 i_2}{d} \tag{35}$$

Note antiparallel currents repel. Ampere's law generalizes the magnetic field to state

$$\oiint \mathbf{B} \cdot d\mathbf{s} = \mu_0 i \tag{36}$$