Chapter 1

Section 1

Exercise 1.1.1

Exercise 1.1.1. Prove Lemma 1.1.1.

Lemma 1.1.1. *Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number. Then $(x_n)_{n=m}^{\infty}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.*

$\Rightarrow$
$(x_n)_{n=m}^{\infty}$ converges to x. (Assumption)
$\forall\epsilon(\epsilon > 0 \to \exists N(N \geq m \wedge \forall n(n \geq N \to |x_n - x| \leq \epsilon)))$ (A1, Def 6.1.5)
$d(x_n - x) = 0 = |x_n - x|$ (Def)
$d(x_n - x) \leq \epsilon$
$|d(x_n - x)| = |d(x_n - x) - 0| \leq \epsilon$
$\forall\epsilon(\epsilon > 0 \to \exists N(N \geq m \wedge \forall n(n \geq N \to |d(x_n - x) - 0| \leq \epsilon)))$ (Substitution)
$(d(x_n, x))_{n=m}^{\infty}$converges to 0.
$\Leftarrow$
Similar argument.

This proof argues pretty much straight from the definitions.

Exercise 1.1.2

Exercise 1.1.2. Show that the real line with the metric $d(x, y) := |x-y|$ is indeed a metric space. (Hint: you may wish to review your proof of Proposition 4.3.3.)

(a)
$x \in \mathbb{R}$ (Assumption)
$x = x$
$d(x, x) = 0$ (A1, P4.3.3 applied to reals)

(b)
$x, y \in \mathbb{R}$ (Assumption)
$d(x, y) \geq 0$ (A1, P4.3.3 applied to reals)

$x \neq y$ (Assumption)
$d(x, y) > 0$ (Deduction)

(c)
(A1, P4.3.3 applied to reals)

(d)
(A1, P4.3.3 applied to reals)

This proof applied properties already proven to the given context.

Exercise 1.1.3

Exercise 1.1.3. Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function.

(a) Give an example of a pair (X, d) which obeys axioms (bcd) of Definition 1.1.2, but not (a). (Hint: modify the discrete metric.)

(b) Give an example of a pair (X, d) which obeys axioms (acd) of Definition 1.1.2, but not (b).

(c) Give an example of a pair (X, d) which obeys axioms (abd) of Definition 1.1.2, but not (c).

(d) Give an example of a pair (X, d) which obeys axioms (abc) of Definition 1.1.2, but not (d). (Hint: try examples where X is a finite set.)

Pass

Exercise 1.1.4

Exercise 1.1.4. Show that the pair $(Y, d|_{Y \times Y})$ defined in Example 1.1.5 is indeed a metric space.

Example 1.1.5 (Induced metric spaces)**.** Let (X,d) be any metric space, and let Y be a subset of X. Then we can restrict the metric function $d : X \times X \to [0,+\infty)$ to the subset $Y \times Y$ of $X \times X$ to create a restricted metric function $d|_{Y\times Y} : Y \times Y \to [0,+\infty)$ of Y; this is known as the metric on Y *induced* by the metric d on X. The pair $(Y, d|_{Y\times Y})$ is a metric space (Exercise 1.1.4) and is known the *subspace* of (X,d) induced by Y.

Each property of the induced metric immediately follows from the fact that X is a metric space, and Y is a subset of X.

Exercise 1.1.5

Exercise 1.1.5. Let $n \geq 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Verify the identity

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right),$$

and conclude the *Cauchy-Schwarz inequality*

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{1.3}$$

Then use the Cauchy-Schwarz inequality to prove the *triangle inequality*

$$\left(\sum_{i=1}^{n}(a_i+b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}.$$

$$(a_ib_j - a_jb_i)^2 = a_i^2b_j^2 - a_ib_ja_jb_i - a_ib_ja_jb_i + a_j^2b_i^2$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2b_j^2 - a_ib_ja_jb_i - a_ib_ja_jb_i + a_j^2b_i^2)$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(2a_i^2b_j^2 - 2a_ib_ja_jb_i) = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i^2b_j^2 - a_ib_ja_jb_i)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_j a_j b_i = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2 b_j^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i b_i \sum_{i=1}^{n}\sum_{j=1}^{n} a_j b_j$$
$$= (\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) - (\sum_{i=1}^{n} a_i b_i)^2$$
Hence,
$$(\sum_{i=1}^{n} a_i b_i)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = (\sum_{i=1}^{n} a_i b_i)^2 + (\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) - (\sum_{i=1}^{n} a_i b_i)^2$$
$$= (\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2)$$
As required.

For the Cauchy-Schwarz inequality
$$|(\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2)| = (\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) \geq 0$$
Since, $a_i^2 > 0, b_i^2 > 0$ for all i.
And, also
$$(\sum_{n=1}^{n} a_i b_i)^2 \geq 0$$
Hence,
$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 \geq 0$$
Given,
$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 = (\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) - (\sum_{i=1}^{n} a_i b_i)^2$$
We have,
$$(\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) - (\sum_{i=1}^{n} a_i b_i)^2 \geq 0$$
Which gives the result, since
$$|(\sum_{i=1}^{n} a_i b_i)^2| = |(\sum_{i=1}^{n} a_i b_i)|^2$$

For the triangle inequality,
$$\sum_{i=1}^{n}(a_i + b_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2$$

And,

$$((\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}})^2 = \sum_{i=1}^{n} a_i^2 + 2(\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}}(\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}} + \sum_{i=1}^{n} b_i^2$$

The only difference between these two equations is the middle term, and, from the above proof, we know that

$$(\sum_{i=1}^{n} a_i^2)(\sum_{j=1}^{n} b_j^2) \geq (\sum_{i=1}^{n} a_i b_i)^2$$

Hence,

$$\sum_{i=1}^{n}(a_i + b_i)^2 \leq ((\sum_{i=1}^{n} a_i^2)^{\frac{1}{2}} + (\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}})^2$$

Which gives the result.

Exercise 1.1.6

Exercise 1.1.6. Show that $(\mathbf{R}^n, d_{l^2})$ in Example 1.1.6 is indeed a metric space. (Hint: use Exercise 1.1.5.)

Example 1.1.6 (Euclidean spaces)**.** Let $n \geq 1$ be a natural number, and let $\mathbf{R}^n$ be the space of n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in \mathbf{R}\}.$$

We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$\begin{aligned} d_{l^2}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &:= \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \\ &= \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}. \end{aligned}$$

(a)

$$d(\vec{x}, \vec{x}) = \sqrt{(x_1 - x_1)^2 + ... + (x_n - x_n)^2} = 0$$

(b)

For any distinct x_i, y_i, $x_i - y_i \neq 0$, which implies $(x_i - y_i)^2 > 0$.

Hence,

$$\sum_{i=1}^{n}(x_i - y_i)^2 > 0$$
Which gives the result.

(c)
For any x_i, y_i, $(x_i - y_i)^2 = (y_i - x_i)^2$, which gives the result.

(d)
$$d(\vec{x},\vec{z}) = (\sum_{i=1}^{n}(x_i - z_i)^2)^{\frac{1}{2}} = (\sum_{i=1}^{n}(x_i - y_i + y_i - z_i)^2)^{\frac{1}{2}}$$
$$\leq (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}} + (\sum_{i=1}^{n}(x_i - z_i)^2)^{\frac{1}{2}} = d(\vec{x},\vec{y}) + d(\vec{y},\vec{z})$$
Where we have used the Cauchy-Schwarz inequality.

Exercise 1.1.7

Exercise 1.1.7. Show that the pair $(\mathbf{R}^n, d_{l^1})$ in Example 1.1.7 is indeed a metric space.

Example 1.1.7 (Taxi-cab metric)**.** Again let $n \geq 1$, and let $\mathbf{R}^n$ be as before. But now we use a different metric d_{l^1}, the so-called *taxicab metric* (or l^1 *metric*), defined by

$$d_{l^1}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) := |x_1 - y_1| + \ldots + |x_n - y_n|$$
$$= \sum_{i=1}^{n} |x_i - y_i|.$$

(a)
$d(\vec{x},\vec{x}) = 0$ since $\forall i |x_i - x_i| = 0$

(b)
This follows since for distance $\vec{x},\vec{y}$, there will be some i, such that $|x_i - y_i| > 0$.

(c)
This follows since $\forall i |x_i - y_i| = |y_i - x_i|$.

(d)
This follows directly from the triangle inequality for distances (A1, P4.3.3 applied to reals).

Exercise 1.1.8

Exercise 1.1.8. Prove the two inequalities in (1.1). (For the first inequality, square both sides. For the second inequality, use Exercise (1.1.5).

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y) \tag{1.1}$$

For the first inequality,
$$d_{l^2}(x,y)^2 = \sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} |x_i - y_i||x_j - y_j| =$$
$d_{l^1}(x,y)^2$
Where have used the fact $(x_i - y_i)^2 = |(x_i - y_i)|^2$

For the second inequality,
$$\sum_{i=1}^{n}|x_i - y_i| \leq |\sum_{i=1}^{n}|x_i - y_i|| \leq (\sum_{i=1}^{n}|x_i - y_i|)^{\frac{1}{2}}(\sum_{i=1}^{n}1^2)^{\frac{1}{2}} = \sqrt{n}d_{l^2}(x,y)$$
Where we have used the Cauchy-Schwarz inequality.

Exercise 1.1.9

Exercise 1.1.9. Show that the pair $(\mathbf{R}^n, d_{l^\infty})$ in Example 1.1.9 is indeed a metric space.

Example 1.1.9 (Sup norm metric)**.** Again let $n \geq 1$, and let $\mathbf{R}^n$ be as before. But now we use a different metric d_{l^∞}, the so-called *sup norm metric* (or l^∞ *metric*), defined by

$$d_{l^\infty}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}.$$

(a)
$d(\vec{x}, \vec{x}) = sup\{0 : 1 \leq i \leq n\} = 0$

(b)

For distinct $\vec{x}, \vec{y}$, there is some j such that $|x_j - y_j| > 0$, in which case $sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_j - y_j| > 0$. This shows the result.

(c)
This follows since $|x_i - y_i| = |y_i - x_i|$.

(d)
From the triangle inequality for distances (A1, P4.3.3 applied to reals),
$|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$
Which gives,
$|x_i - z_i| \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$
Since,
$|x_i - y_i| \leq d_{l^\infty}(x, y)$ and $|y_i - z_i| \leq d_{l^\infty}(y, z)$
The claim follows.

Exercise 1.1.10

Exercise 1.1.10. Prove the two inequalities in (1.2).

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y) \tag{1.2}$$

For the first inequality,
Define $d_{l^\infty}(x, y) = |x_k - y_k|$
$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_k - y_k)^2 = n(x_k - y_k)^2 = n d_{l^\infty}(x, y)^2$$
Where have used the fact $\forall i |x_i - y_i| \leq d_{l^\infty}(x, y)$
Hence,
$$d_{l^2}(x, y) = (\sum_{i=1}^{n} (x_i - y_i)^2)^{\frac{1}{2}} = \sqrt{n} d_{l^\infty}(x, y)$$
And the result follows.

For the second inequality,
Define $d_{l^\infty}(x, y) = |x_k - y_k|$
Then,
$$d_{l^2}(x, y)^2 = (x_k - y_k)^2 + \sum_{i=1, i \neq k} (x_i - y_i)^2 \geq |x_k - y_k|^2 = d_{l^\infty}(x, y)^2$$

Where have used $(x_i - y_i)^2 = |(x_i - y_i)|^2$.
The inequality follows.

Exercise 1.1.11

Exercise 1.1.11. Show that the discrete metric $(\mathbf{R}^n, d_{\text{disc}})$ in Example 1.1.11 is indeed a metric space.

Example 1.1.11 (Discrete metric)**.** Let X be an arbitrary set (finite or infinite), and define the *discrete metric* d_{disc} by setting $d_{\text{disc}}(x, y) := 0$ when $x = y$, and $d_{\text{disc}}(x, y) := 1$ when $x \neq y$. Thus, in this metric, all points are equally far apart. The space (X, d_{disc}) is a metric space (Exercise 1.1.11). Thus every set X has at least one metric on it.

(a)
This follows by the definition of the discrete metric.

(b)
This follows by the definition of the discrete metric.

(c)
This follows easily from (a) and (b).

(d)
Considering the cases where x, y, z are distinct or not, this follows easily from the definition.

Exercise 1.1.12

Exercise 1.1.12. Prove Proposition 1.1.18.

Proposition 1.1.18 (Equivalence of l^1, l^2, l^∞). *Let $\mathbf{R}^n$ be a Euclidean space, and let $(x^{(k)})_{k=m}^\infty$ be a sequence of points in $\mathbf{R}^n$. We write $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$, i.e., for $j = 1, 2, \ldots, n$, $x_j^{(k)} \in \mathbf{R}$ is the j^{th} co-ordinate of $x^{(k)} \in \mathbf{R}^n$. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbf{R}^n$. Then the following four statements are equivalent:*

(a) $(x^{(k)})_{k=m}^\infty$ *converges to x with respect to the Euclidean metric d_{l^2}.*

(b) $(x^{(k)})_{k=m}^\infty$ *converges to x with respect to the taxi-cab metric d_{l^1}.*

(c) $(x^{(k)})_{k=m}^\infty$ *converges to x with respect to the sup norm metric d_{l^∞}.*

(d) *For every $1 \le j \le n$, the sequence $(x_j^{(k)})_{k=m}^\infty$ converges to x_j. (Notice that this is a sequence of real numbers, not of points in $\mathbf{R}^n$.)*

(a) ⇒ (b)
Since $d_{l^1}(x,y) \le \sqrt{n} d_{l^2}(x,y)$, the result follows.

(b) ⇒ (c)
Since $d_{l^\infty}(x,y) \le d_{l^2}(x,y) \le d_{l^1}(x,y)$, we have $d_{l^\infty}(x,y) \le d_{l^1}(x,y)$. And the result follows.

(c) ⇒ (a)
Since $\frac{1}{\sqrt{n}} d_{l^2}(x,y) \le l^\infty(x,y)$, the result follows.

(b) ⇒ (d)
Since, by assumption, $\forall k \sum_{i=1}^{n} |x_i^{(k)} - x_i| \le \epsilon$, we know $\forall i \forall k |x_i^{(k)} - x_i| \le \epsilon$, and hence the claim follows.

(d) ⇒ (b)
This is similar to the preceding argument.

Exercise 1.1.13

Exercise 1.1.13. Prove Proposition 1.1.19.

Proposition 1.1.19 (Convergence in the discrete metric). *Let X be any set, and let d_{disc} be the discrete metric on X. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the discrete metric d_{disc} if and only if there exists an $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.*

⇒

Since $(x^{(n)})_{n=m}^{\infty}$ converges, we know that there exists an $N \geq m$ such that $d_{disc}(x^{(n)}, x) \leq \epsilon \forall n \geq N \forall \epsilon$. Given that the discrete metric only equals 1 or 0, this is only possible if $d_{disc}(x^{(n)}, x) = 0 \forall n \geq N$, which requires $x^{(n)} = x$.

⇐

$x^{(n)} = x$ immediately implies that $d_{disc}(x^{(n)}, x) = 0$, and so $d_{disc}(x^{(n)}, x) \leq \epsilon \forall n \geq N \forall \epsilon$, which implies convergence.

Exercise 1.1.14

Exercise 1.1.14. Prove Proposition 1.1.20. (Hint: modify the proof of Proposition 6.1.7.)

Proposition 1.1.20 (Uniqueness of limits). *Let (X, d) be a metric space, and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in X. Suppose that there are two points $x, x' \in X$ such that $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d, and $(x^{(n)})_{n=m}^{\infty}$ also converges to x' with respect to d. Then we have $x = x'$.*

$\forall \epsilon \exists N \geq m \forall n \geq N (d(x^{(n)}, x) \leq \epsilon)$, and

$\forall \epsilon \exists M \geq m \forall n \geq N (d(x^{(n)}, x') \leq \epsilon)$

Setting $n = max(M, N)$, we have

$d(x^{(n)}, x) \leq \frac{\epsilon}{2}$ and $d(x^{(n)}, x') \leq \frac{\epsilon}{2}$

Using the triangle inequality for metric spaces,

$d(x, x') \leq d(x, x^{(n)}) + d(x^{(n)}, x') \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Hence, $d(x, x')$ converges to 0, but given it doesn't depend on n, this is only possible if $d(x, x') = 0$, which gives the result.

Exercise 1.1.15

Exercise 1.1.15. Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the l^1 and l^∞ metrics on this space by

$$d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|;$$
$$d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n \in \mathbf{N}} |a_n - b_n|.$$

Show that these are both metrics on X, but show that there exist sequences $x^{(1)}, x^{(2)}, \ldots$ of elements of X (i.e., sequences of sequences) which are convergent with respect to the d_{l^∞} metric but not with respect to the d_{l^1} metric. Conversely, show that any sequence which converges in the d_{l^1} metric automatically converges in the d_{l^∞} metric.

$d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})$

(a)

If the sequences are equal (not just equivalent), then each term in the series will equal 0 and the metric will converge to 0.

(b)

Given that it's an absolute sum, each term in the series is greater than 0 (since the sequences are distinct, and so the series will necessarily be greater than 0.

(c)

The results follows because $|a_n - b_n| = |b_n - a_n|$.

(d)

$d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})$

$$\sum_{n=0}^{\infty} |a_n - b_n| \leq \sum_{n=0}^{\infty} |a_n - c_n + c_n - b_n| \leq \sum_{n=0}^{\infty} |a_n - c_n| + \sum_{n=0}^{\infty} |c_n - b_n|$$

Which gives the result.

$d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty})$

(a)

If the sequences are equal (not just equivalent), then each term in the series will equal 0 and the supremum will be 0.

(b)

If the sequences are distinct, then there will be some non-zero $|a_n - b_n|$, in which case the supremum will be greater than or equal to this value. This gives the result.

(c)

The results follows because $|a_n - b_n| = |b_n - a_n|$.

(d)

$|a_n - b_n| \leq |a_n - c_n| + |c_n - b_n| \leq sup|a_n - c_n| + sup|c_n - b_n|$

Hence,

$sup|a_n - b_n| \leq sup|a_n - c_n| + sup|c_n - b_n|$

See solutions manual…

Conversely, by assumption,

$$\forall \epsilon > 0 \exists N \geq m \forall n \geq N (\sum_{k=0}^{\infty} |x_k^{(n)} - x_k| \leq \epsilon)$$

Then, obviously, if $sup_{n \in N}|x_k^{(n)} - x_k| > \epsilon$, then $\sum_{k=0}^{\infty} |x_k^{(n)} - x_k| > \epsilon$, which is a contradiction.

Hence,

$\forall \epsilon > 0 \exists N \geq m \forall n \geq N (sup_{n \in N}|x_k^{(n)} - x_k| \leq \epsilon)$

Which is the desired result.

Exercise 1.1.16

Exercise 1.1.16. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in a metric space (X, d). Suppose that $(x_n)_{n=1}^{\infty}$ converges to a point $x \in X$, and $(y_n)_{n=1}^{\infty}$ converges to a point $y \in X$. Show that $\lim_{n \to \infty} d(x_n, y_n) = d(x, y)$. (Hint: use the triangle inequality several times.)

By assumption,

$\forall \epsilon > 0 \exists N \geq m \forall n \geq N (d(x_n, x) \leq \epsilon)$

$\forall \epsilon > 0 \exists N \geq m \forall n \geq N (d(y_n, y) \leq \epsilon)$

Hence, we can write,

$d(x_n, x) \le \frac{\epsilon}{2}$ and $d(y_n, y) \le \frac{\epsilon}{2}$

From the triangle inequality,

$d(x_n, y_n) \le d(x_n, x) + d(x, y_n)$

$d(x, y_n) \le d(x, y) + d(y, y_n)$

Hence,

$d(x_n, y_n) \le d(x_n, x) + d(x, y_n) \le d(x_n, x) + d(x, y) + d(y, y_n)$

Then,

$d(x_n, y_n) - d(x, y) \le d(x_n, x) + d(y, y_n) \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

This gives the desired result.

Section 2

Exercise 1.2.1

Exercise 1.2.1. Verify the claims in Example 1.2.8.

Example 1.2.8. When we give a set X the discrete metric d_{disc}, and E is any subset of X, then every element of E is an interior point of E, every point not contained in E is an exterior point of E, and there are no boundary points; see Exercise 1.2.1.

(i)

By selecting $r < 1$, the ball centred at any element of E will only contain that centre, which is obviously an element of E. This is enough to show that every element is an interior point.

(ii)

By selecting $r < 1$, the ball centred at any element of X that is not an element of X will only contain that centre, which is obviously not an element of E. This is enough to show that every element not contained in E is an exterior point.

(iii)

Every element of X either is or is not an element of E, and so will either be an interior point or exterior point. As boundary points are neither interior points nor exterior points, no boundary points exist.

Exercise 1.2.2

Exercise 1.2.2. Prove Proposition 1.2.10. (Hint: for some of the implications one will need the axiom of choice, as in Lemma 8.4.5.)

Proposition 1.2.10. *Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are logically equivalent.*

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) There exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d.

(a) ⇒ (b)
If $x_o \in X$, x_o is either an interior point, exterior point, or a boundary point. Given, by assumption
$\forall r > 0 \exists S \neq \emptyset (B(x_o, r) \cap E = S)$
Since a non-empty intersection between the ball and E is given for every r, we see that x_o is not an exterior point. This gives the result.

(b) ⇒ (a)
Assume x_o is an interior point, then for some r, $B(x_o, r) \subseteq E$, which obviously implies $B(x_o, r) \cap E \neq \emptyset$, and so x_o is an adherent point.
Assume x_o is a boundary point, then for every r, the ball has a non-empty intersection with E, since we know it is not an exterior point.
This gives the result.

(a) ⇒ (c)
For each positive natural n, define
$X_n := \{x \in E : x \in B(x_o, \frac{1}{n}) \cap E\}$
We know this is not empty for each n because x_o is an adherent point.

Using the axiom of choice, we can construct a sequence $(x_n)_{n=1}^{\infty}$ such that $0 \leq d(x_n, x_o) \leq \frac{1}{n}$.
From the squeeze test, this shows the sequence converges to x_o.

(c) ⇒ (a)
Let $r > 0$ and $(x_n)_{n=1}^{\infty}$ be some sequence that converges to x_o in E. Let $N \geq 1$ be some natural number such that $\forall n \geq N (d(x_n, x_o) \leq r)$. Then we know $d(x_N, x_o) \leq r$, which implies $x_N \in B(x_o, r)$. Since the sequence converges in E, we know $x_N \in E$ and hence $B(x_o, r) \cap E \neq \emptyset$. Since r was arbitrary, this gives the result.

Exercise 1.2.3

Exercise 1.2.3. Prove Proposition 1.2.15. (Hint: you can use earlier parts of the proposition to prove later ones.)

Proposition 1.2.15 (Basic properties of open and closed sets). *Let (X, d) be a metric space.*

(a) *Let E be a subset of X. Then E is open if and only if $E = \mathrm{int}(E)$. In other words, E is open if and only if for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.*

⇐
Assuming $E = int(E)$, say $x \in \partial E$; this immediately implies that $x \notin int(E)$. Since x was arbitrary, this shows that $\partial E \cap E = \emptyset$, which implies that E is open.

⇒
Assuming E is open, let $x \in E$. x is either an interior point, exterior point, or boundary point. Since $x \in E$, it is not an exterior point. Since E is open, we know $\partial E \cap E = \emptyset$, and hence x is not a boundary point. Hence, x is an interior point. This shows $E \subseteq int(E)$ because x was arbitrary. We already know $int(E) \subseteq E$; hence, $E = int(E)$.

(b) *Let E be a subset of X. Then E is closed if and only if E contains all its adherent points. In other words, E is closed if and only if for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n \to \infty} x_n$ of that sequence also lies in E.*

⇒
Assuming E is closed, say $x \in E$. Because E contains all its boundary points, x is either an interior point or a boundary point, which implies that x is an adherent point from P1.2.10.

⇐
Assuming E contains all its adherent points, say x is a boundary point of E.This implies that x is an adherent point of E, as per P1.2.10. By assumption, this implies that $x \in E$. Since x was arbitrary, this shows E is closed.

(c) *For any $x_0 \in X$ and $r > 0$, then the ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set. (This set is sometimes called the* closed ball *of radius r centered at x_0.)*

Given $B(x_o, r)$, say $x \in B(x_o, r)$. Hence, $d(x_o, x) < r$. Define $m = d(x_o, x)$, and then let $B(x, r-m)$ be the ball centred at x with radius $r - m$ using the same metric. This ball is not empty since $x \in B(x, r-m)$. Given $\hat{x} \in B(x, r-m)$, we can see $d(x, \hat{x}) \leq d(x_o, x) + d(x, \hat{x}) < m + r - m = r$, which shows that $\hat{x} \in B(x_o, r)$. Given that $\hat{x}$ was arbitrary, this shows that $B(x, r-m) \subseteq B(x, r)$. This is enough to show that x is an interior point of $B(x_o, r)$, since we have shown there exists some radius q such that $B(x, q) \subseteq B(x, r)$. Given that x was arbitrary, we have shown that $B(x_o, r) = int(B(x_o, r))$, which from (a) shows that $B(x_o, r)$ is open.

This part of the proof relies on being able to, for every element of $B(x_o, r)$, construct a ball that is necessarily a subset of $B(x_o, r)$. This follows from the metric structure, specifically the triangle inequality.

Define $B_c(x_o, r) := \{x \in X : d(x, x_o) \leq r\}$. Let x be some adherent point of $B_c(x_o, r)$. Then we know $\forall q > 0 B(x, q) \cap B_c(x_o, r) \neq \emptyset$. We know $d(x, x_o) > r \vee d(x, x_o) \leq r$.
We will show $d(x, x_o) \leq r$ by showing that $d(x, x_o) > r$ leads to a contradiction.
Hence, assume $d(x, x_o) > r$, and define $m = d(x, x_o)$. Then define
$B_F(x, \frac{m-r}{2}) := \{\hat{x} \in X : d(\hat{x}, x) < \frac{m-r}{2}\}$.
Say $\hat{x} \in B_F(x, \frac{m-r}{2})$, then we know $d(x, \hat{x}) < \frac{m-r}{2}$, and hence also $-d(x, \hat{x}) > -\frac{m-r}{2}$.
Since $d(x, x_o) \leq d(x, \hat{x}) + d(\hat{x}, x_o)$, we can write $d(x, x_o) - d(x, \hat{x}) \leq d(\hat{x}, x_o)$.
If we assume that $d(x_o, \hat{x}) \leq r$, then we can show
$m - \frac{m-r}{2} < d(x, x_o) - d(x, \hat{x}) \leq d(\hat{x}, x_o) \leq r$, since we have assumed $d(x, x_o) > r$ and $r > m$.
It follows that $\frac{m+r}{2} < r$, which is a contradiction since $m > r$. Henc, $d(x_o, \hat{x}) > r$, and it follows that $\hat{x} \notin B_c(x_o, r)$. Given $\hat{x}$ was arbitrary, we have shown that
$B_F(x, \frac{m-r}{2}) \cap B_c(x_o, r) = \emptyset$, and hence shown that there exists some q such that $B(x, q) \cap B_c(x_o, r) = \emptyset$. This implies that x is an exterior point, which contradicts the fact that it is an adherent point. Hence, we conclude that $d(x, x_o) \leq r$, and $x \in B_c(x_o, r)$. Given x was arbitrary, we have shown that every adherent point of $B_c(x_o, r)$ is an element of $B_c(x_o, r)$, which, from (b), implies that $B_c(x_o, r)$ is closed.

This part of the proof uses the definition that if the set contains all its adherent points, then it is closed. We take an arbitrary adherent point, and assert that its distance from the centre of the

closed ball is either greater than the radius or less than or equal to the radius. If we can show that the adherent point necessarily has a distance less than or equal to the radius, then it automatically follows by definition that it is contained in the closed ball.
So we assume $d(x, x_o) > r$, where x is the adherent point. We show that it is always possible to construct a ball that is centred at x that has an empty intersection with the original closed ball. This shows that the adherent point is an exterior point, which is obviously a contradiction. The result follows.
This type of approach won't work with an open ball, because, although we can show that every adherent point x will have $d(x, x_o) \leq r$, this doesn't imply that x is an element of the open ball.

Let $B_c(x_o, r) := \{x \in X : d(x, x_o) \leq r\}$, and let $(x_n)_{n=m}^{\infty}$ be a convergent sequence in B_c. Since the sequence is in B_c, we know $d(x_n, x_o) \leq r$ for every n. Let the limit $c \notin B_c$. We know $\forall \epsilon \exists N \geq m \forall n \geq N (d(x_n, c) \leq \epsilon)$. Then, let $\epsilon = d(x_n, c) < \epsilon)$. Hence, we can find some N such that $d(x_n, c) < d(x_o, c) - r$ for all $n \geq N$. We can immediately write $r < d(x_o, c) - d(x_n, c)$. Since $d(x_o, c) \leq d(x_o, x_n) + d(x_n, c)$, we can write $r < d(x_o, c) - d(x_n, c) \leq d(x_o, x_n)$. It follows that $r < d(x_o, x_n)$ for all $n \geq N$. This is a contradiction since $d(x_n, x_o) \leq r$ for every n. Hence, $c \in B_c$.

This proof also shows that a closed ball is closed. However, it works from the definition that all convergent sequences in a closed set have their limit contained in the set. If the limit is not contained inside the closed ball, then there must be some point in the sequence such that all x_n after this point are coming arbitrarily close to the limit, and this implies that these x_n are outside the closed ball. Obviously, this is a contradiction.

(d) *Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.*

Since the set is a singleton, the only sequence $(x_n)_{n=m}^{\infty}$ is going to be $(x_o)_{n=m}^{\infty}$, which automatically converges to x_o. Hence, for every convergent sequence in $\{x_o\}$, the limit also lies in $\{x_o\}$. From (b), this implies that $\{x_o\}$ is closed.

(e) *If E is a subset of X, then E is open if and only if the complement $X \backslash E := \{x \in X : x \notin E\}$ is closed.*

Define $\partial E := \{x \in X : \ x \ boundary \ E\}$, and $\partial X/E := \{x \in X : x \ boundary \ point \ of \ E\}$.

Let $x \in \partial E$. Since x is a boundary point, we know $\forall r > 0(B(x,r) \cap E \neq \emptyset)$, and $\forall r > 0(B(x,r) \not\subseteq E)$. Let $r > 0$ be some real number. Since $B(x,r) \not\subseteq E$, we know there exists some $x' \in B(x,r)$ which is also an element of X/E. Hence, given that r was arbitrary, we conclude $\forall r > 0(B(x,r) \cap X/E \neq \emptyset)$.
Again, let $r > 0$ be some real number. Then, $B(x,r) \cap E \neq \emptyset$ which implies there exists some $x' \in B(x,r)$ which is also an element of E. Given that r was arbitrary, we have shown that $\forall r > 0(B(x,r) \not\subseteq X/E)$. From these two facts, we can conclude that $x \in \partial X/E$. Since x was arbitrary, we have shown $\partial E \subset \partial X/E$. By a similar argument, we can show $\partial X/E \subset \partial E$. Hence, $\partial E = \partial X/E$.

⇒

Since E is open, it contains none of its boundary points, which implies that X/E contains all the boundary points of E. However, since X and E have the same boundary points, this implies that X/E contains all of its boundary points, and hence is closed.

⇐

If $x \setminus E$ is closed, it contains all of its boundary points. Since these boundary points are also E's boundary points, this implies that E does not contain any of its boundary points. THis follows because $E \cap X \setminus E = \emptyset$

(f) If $E_1, \ldots, E_n$ are a finite collection of open sets in X, then $E_1 \cap E_2 \cap \ldots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is also closed.

Proof by induction on n:
Base case:
If E_1 is open, then $E_1 \cap E_1 = E_1$ is open.

Inductive case:
Assume: $E_1 \cdots E_n$ are open $\rightarrow E_1 \cap \cdots \cap E_n$.
Define $M := E_1 \cap \cdots \cap E_n$, and let E_{n+1} be open. The case where $M \cap E_{n+1} = \emptyset$ is trivial, so assume $M \cap E_{n+1} \neq \emptyset$.
Let $x \in M \cap E_{n+1}$. Since $x \in M$, and M is open, we know $\exists r > 0 B(x,r) \subseteq M$. Since $x \in E_{n+1}$, and E_{n+1} is open, we know $\exists q > 0 B(x,q) \subseteq E_{n+1}$.
Let $x \in M \cap E_{n+1}$. Assume x is a boundary point of $M \cap E_{n+1}$. W.l.o.g. assume $r < q$. Then $B(x,r) \subseteq E_{n+1}$, which implies $B(x,r) \subseteq M \cap E_{n+1}$, since $B(x,r) \subseteq M$ as well. Since x is a boundary point, we know $\forall s > 0 B(x,s) \not\subseteq M \cap E_{n+1}$. Hence, we can write $B(x,r) \not\subseteq M \cap E_{n+1}$, which is a contradiction.

It follows that x is not a boundary point of $M \cap E_{n+1}$. Since x was arbitrary, we have shown that $M \cap E_{n+1}$ does not contain any boundary points, and is open.
This closes the induction.

Proof by induction on n:
Base case:
If F_1 is closed, then $F_1 \cup F_1 = F_1$ is closed.

Inductive case:
Assume: $F_1 \cdots F_n$ are closed $\to$ $F_1 \cup \cdots \cup F_n$ is closed.
Define $M := F_1 \cup \cdots \cup F_n$, and let $F_{n+1} \neq \emptyset$ be closed.
Let x be a boundary point of $M \cup F_{n+1}$.
Then we know $\forall r > 0 B(x, r) \cap (M \cup F_{n+1}) \neq \emptyset$ (x is an adherent point), which implies $\forall r > 0 B(x, r) \cap M \neq \emptyset \vee B(x, r) \cap F_{n+1} \neq \emptyset$.
We also know $\forall q > 0 B(x, q) \nsubseteq (M \cup F_{n+1})$ (x is not an interior point), which implies both $\forall q > 0 B(x, q) \nsubseteq M$, and $\forall q > 0 B(x, q) \nsubseteq F_{n+1}$.
Hence, we can either assert $\forall r > 0 B(x, r) \cap M \neq \emptyset$ and $\forall q > 0 B(x, q) \nsubseteq M$, which implies x is a boundary point of M and hence $x \in M$, since M is closed. Hence, $x \in M \cup F_{n+1}$.
Or, we can assert $\forall r > 0 B(x, r) \cap F_{n+1} \neq \emptyset$, and $\forall q > 0 B(x, q) \nsubseteq F_{n+1}$, which implies x is a boundary point of F_{n+1}. It follows that $x \in F_{n+1}$, since F_{n+1} is closed. Hence, $x \in M \cup F_{n+1}$.
Since x was arbitrary, we have shown that $M \cup F_{n+1}$ contains all its boundary points, and hence is closed.

(g) If $\{E_\alpha\}_{\alpha \in I}$ is a collection of open sets in X (where the index set I could be finite, countable, or uncountable), then the union $\bigcup_{\alpha \in I} E_\alpha := \{x \in X : x \in E_\alpha \text{ for some } \alpha \in I\}$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in I} F_\alpha := \{x \in X : x \in F_\alpha \text{ for all } \alpha \in I\}$ is also closed.

Let $\{E_\alpha\}_{\alpha \in I}$ be a collection of open sets in X, and $\cup_{\alpha \in I} E_\alpha$ be the union of these sets. Let x be some boundary point of $\cup_{\alpha \in I} E_\alpha$. Then we know $\forall r > 0 (B(x, r) \nsubseteq \cup_{\alpha \in I} E_\alpha)$ (x is not an interior point), and $\forall r > 0 (B(x, r) \cap \cup_{\alpha \in I} E_\alpha) \neq \emptyset)$ (x is an adherent point). Say $\alpha \in I$, and assume $x \in E_\alpha$. Then it immediately follows that $\forall r > 0 (B(x, r) \cap E_\alpha \neq \emptyset)$ (x is an adherent point of E_α since $x \in E_\alpha$). Since $E_\alpha \subseteq \cup_{\alpha \in I} E_\alpha$, it follows that if $\exists r > 0 (B(x, r) \subseteq E_\alpha)$, then $\exists r > 0 (B(x, r) \subseteq \cup_{\alpha \in I} E_\alpha)$. This is a contradiction, since x is a boundary point of $\cup_{\alpha \in I} E_\alpha$.
Hence, we conclude $\forall r > 0 (B(x, r) \nsubseteq E_\alpha)$.

Hence, x is a boundary point of E_α. Since E_α is open by assumption, this a contradiction, and hence $x \notin E_\alpha$. Since x was arbitrary, we have shown that every boundary point of $\cup_{\alpha \in I} E_\alpha$ is not an element of E_α. Since every element of $\cup_{\alpha \in I} E_\alpha$ is necessarily an element of some E_α, we have shown that $\cup_{\alpha \in I} E_\alpha$ does not contain any of its boundary points, and is hence open.

This proof works from the definition that an open set does not contain any of its boundary points. It proves by contradiction. Assuming that x is a boundary point of $\cup_{\alpha \in I} E_\alpha$, we show that it is a boundary point for some E_α, if $x \in E_\alpha$. Since every E_α is open, it follows that x cannot be an element of any of the E_α, which implies that it cannot be an element of $\cup_{\alpha \in I} E_\alpha$.

Let $\{E_\alpha\}_{\alpha \in I}$ be a collection of open sets in X, and $\cup_{\alpha \in I} E_\alpha$ be the union of these sets. Let $x \in \cup_{\alpha \in I} E_\alpha$. It follows that $x \in E_\alpha$ for some E_α. Since E_α is open we know there is some ball such that $B(x, r) \subseteq E_\alpha$. Since $E_\alpha \subseteq \cup_{\alpha \in I} E_\alpha$, it follows that $B(x, r) \subseteq \cup_{\alpha \in I} E_\alpha$. Hence, x is an interior point of $\cup_{\alpha \in I} E_\alpha$. Since x was arbitrary, we have shown that $\cup_{\alpha \in I} E_\alpha$ only contains its interior points, and is open.

This proof also shows that a (possibly infinite) collection of open sets is open. However, it works from the definition that an open set only contains interior ponts, more specifically, for every element of an open set, there exists a ball that is contained in the set. Since every element of $\cup_{\alpha \in I} E_\alpha$ is necessarily an element of some E_α, and E_α is open, the claim follows easily.

Let $\{F_\alpha\}_{\alpha \in I}$ be collection of closed sets in X, and let $\cap_{\alpha \in I} F_\alpha$ be the intersection of these sets. Let x be some boundary point of $\cap_{\alpha \in I} F_\alpha$. Then we know $\forall r > 0 (B(x, r) \nsubseteq \cap_{\alpha \in I} F_\alpha)$ (x is not an interior point), and $\forall r > 0 (B(x, r) \cap \cap_{\alpha \in I} F_\alpha \neq \emptyset)$ (x is an adherent point). Say α is some element in I, then we know x is either an interior point, exterior point, or a boundary point or F_α. Say x is an exterior point. Then $B(x, r) \cap F_\alpha = \emptyset$ for some $r > 0$. However, this would be a contradiction, since we can assert $B(x, r) \cap \cap_{\alpha \in I} F_\alpha \neq \emptyset$, which implies that $B(x, r)$ and F_α have a common element. It follows then that x is either an interior point, or a boundary point or F_α. If x is an interior point, then it is easy to show that $x \in F_\alpha$. If x is a boundary point, then $x \in F_\alpha$ by virtue of F_α being closed. Since α was arbitrary, we have shown that α is an element of every F_α, and hence an element of $\cap_{\alpha \in I} F_\alpha$. Since x was an arbitrary boundary point, we have shown that $\cap_{\alpha \in I} F_\alpha$ contains all its boundary points, and hence is closed.

This proof relies on the fact that every element of $\cap_{\alpha \in I} F_\alpha$ must be an interior point, exterior point or boundary point of every F_α. Assuming then that x is a boundary point of $\cap_{\alpha \in I} F_\alpha$, we show that it cannot be an exterior point of any F_α. It quickly follows that $x \in \cap_{\alpha \in I} F_\alpha$, since all *$F_\alpha$ are closed.*

(h) *If E is any subset of X, then* $\operatorname{int}(E)$ *is the largest open set which is contained in E; in other words,* $\operatorname{int}(E)$ *is open, and given any* *other open set $V \subseteq E$, we have $V \subseteq \text{int}(E)$. Similarly $\overline{E}$ is the smallest closed set which contains E; in other words, $\overline{E}$ is closed, and given any other closed set $K \supset E$, $K \supset \overline{E}$.*

We prove that $int(E)$ is open.
Let x be a boundary point of $int(E)$, then we know $\forall r > 0(B(x,r) \cap int(E) \neq \emptyset)$ (x is an adherent point) and $\forall r > 0(B(x,r) \nsubseteq int(E)$ (x is not an interior point). Since $int(E) \subset E$, we can show $\forall r > 0(B(x,r) \cap E \neq \emptyset)$ (x is an adherent point of E). Assume x is an interior point of E, i.e, $\exists r > 0(B(x,r) \subseteq E$. We can assert $B(x,r) \subseteq E$ and $B(x,r) \nsubseteq int(E)$ for some r. Hence, there is some $\hat{x} \in B(x,r)$, which is a boundary point of E, $\forall r > 0(B(\hat{x},r) \nsubseteq E)$.

Let $d(x,\hat{x}) = d$; we know $d < r$, so let $q = \dfrac{r-d}{2}$, and define $B(\hat{x},q)$ using the same metric. Say $x_o \in B(\hat{x},q)$, then $d(x_o,\hat{x}) < q$. Hence, $d(x_o,x) \leq d(x_o,\hat{x}) + d(\hat{x},x) \leq \dfrac{r-2}{2} + d = \dfrac{r+d}{2} < r$. Hence, $x_o \in B(x,r)$. Since x_o was arbitrary, we have shown that $B(\hat{x},q) \subseteq B(x,r) \subseteq E$. Hence, $\exists r > 0(B(\hat{x},q) \subseteq E$, which is a contradiction, since $\hat{x}$ is a boundary point. Hence, $\forall r > 0(B(x,r) \nsubseteq E$, and x is a boundary point of E, which means it cannot be an element of $int(E)$. Since x was arbitrary, we have shown that $int(E)$ is open.

This part of the proof works from the definition that boundary points cannot be elements of an open set. By showing that every boundary point of $int(E)$ is a boundary point of E, it follows that $int(E)$ cannot contain its boundary points.
The proof uses contradiction. Assuming x is a boundary point of $int(E)$, we then assume that it is an interior point of E (we already know that is not an exterior point), which implies that there is a ball centred at x that is contained in E. We also know that this ball cannot be contained in $int(E)$, since x is a boundary point of $int(E)$. Hence, there is some element of this ball that is an element E and is not an element of $int(E)$. In other words, there is an element of the ball that is a boundary point of E.
We then construct a ball that is centred at this boundary point of E. Using the structure of a metric, we show that this new ball is necessarily contained in the ball centred at x, which implies that it is contained in E. However, this is a contradiction, since this would imply that the boundary point of E is also an interior point of E. Hence, x must be a boundary point of E.

Suppose $V \subseteq E$ is some non-empty open set. Let $x \in V$, which immediately applies that x is an adherent point of V, $\forall r > 0(B(x,r) \cap V \neq \emptyset)$. Assume x is a boundary point of E, $\forall r > 0(B(x,r) \nsubseteq E)$. Assume x is an interior point of V, then we know that for some $s > 0$,

$B(x,s) \subseteq V$. However, since $V \subseteq E$, we can assert $B(x,s) \subseteq E$. This is a contradiction because we assumed x is a boundary point of E. Hence, x is not an interior point of V. Since it is adherent point, we know it is a boundary point of V. However, this is a contradiction, since we assumed V was open. Hence, x is not a boundary point of E. Since x was arbitrary, we have shown that V does not contain any boundary points of E. Since $V \subseteq E$, it necessarily follows that $V \subseteq int(E)$.

This part of the proof works to show that V does not contain any boundary points of E. Since $V \subseteq E$, it immediately follows that $V \subseteq int(E)$. We show that if $x \in V$ is a boundary point of E, then this leads to a contradiction. x cannot be an interior of point of V, since then it would be an interior point of E. And, it cannot be a boundary point of V, since V is open. It is obviously not an exterior point of V. This is a contradiction because it has to be one of these.

$\bar{E}$ is the set of all adherent points of E. Let x be a boundary point of $\bar{E}$. Then, $\forall r > 0(B(x,r) \cap \bar{E} \neq \emptyset)$ (adherent point), $\forall r > 0(B(x,r) \nsubseteq \bar{E})$ (not an interior point). Assume x is an interior point of E, $\exists r > 0(B(x,r) \subseteq E)$. Since $E \subseteq \bar{E}$, we can assert that it is also an interior point of $\bar{E}$. However, this is a contradiction, and hence x is a boundary point of E, $\forall r > 0(B(x,r) \nsubseteq E)$.
Let $r > 0$ be some real number. Let $B(x,r)$ be the ball with radius r. We can also assert $B(x, \frac{r}{2}) \cap \bar{E} \neq \emptyset$, since $\forall r > 0(B(x,r) \cap \bar{E} \neq \emptyset)$, and it follows there is some $\hat{x}$ such that $\hat{x} \in B(x, \frac{r}{2})$ and $\hat{x} \in \bar{E}$. Since $\hat{x} \in \bar{E}$, then $\hat{x}$ is either a boundary point or an interior point of E. Assume $\hat{x}$ is a boundary point of E. Then it follows $\forall s > 0(B(\hat{x},s) \cap E \neq \emptyset)$, and hence we can assert $B(\hat{x}, \frac{r}{2}) \cap E \neq \emptyset$. Hence, there is some $\tilde{x}$ such that $\tilde{x} \in B(\hat{x}, \frac{r}{2})$ and $\tilde{x} \in E$. From this, we can assert $d(\hat{x}, \tilde{x}) < \frac{r}{2}$ and $d(\hat{x}, x) < \frac{r}{2}$. Hence, $d(x, \tilde{x}) \leq d(\hat{x}, \tilde{x}) + d(\hat{x}, x) < \frac{r}{2} + \frac{r}{2} = r$. It follows that $\tilde{x} \in B(x,r)$, and $B(x,r) \cap E \neq \emptyset$. Now assume $\hat{x}$ is an interior point of E. Then it easily follows that $B(x,r) \cap E \neq \emptyset$, since $\hat{x} \in E$, and $d(x, \tilde{x}) < \frac{r}{2} < r$. Hence, either way, $B(x,r) \cap E \neq \emptyset$. Since r was arbitrary, we have shown $\forall r > 0(B(x,r) \cap E \neq \emptyset)$, and hence x is a boundary point of E, and is an element of $\hat{E}$. Since x was arbitrary, we have shown that $\hat{E}$ is closed.

Let $K \supset E$ be some closed set. Let x be some boundary point of E. Then we know $\forall r > 0(B(x,r) \cap E \neq \emptyset)$. We can assert $\forall r > 0(B(x,r) \nsubseteq K) \vee \exists r > 0(B(x,r) \subset E)$. Assume $\forall r > 0(B(x,r) \nsubseteq K)$. Since $\forall r > 0(B(x,r) \cap E \neq \emptyset)$, and $E \subset K$, it follows that

$\forall r > 0(B(x,r) \cap K \neq \emptyset)$. Hence, x is a boundary point of K, and an element of K. Now assume $\exists r > 0(B(x,r) \subset E)$. In that case, it immediately follows that x is an element of K. Either way, x is an element of K. Since x was arbitrary, we have shown that every boundary point of E is an element of K. Hence $\bar{E} \subset K$.

Exercise 1.2.4. Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leq r\}$.

(a) Show that $\overline{B} \subseteq C$.

Let $x \in \bar{B}$, then x is an interior point or a boundary point of B. If x is an interior point of B, then it immediately follows that $x \in C$. If x is a boundary point of B, then it can be shown that $d(x, x_o) = r$, and hence $x \in C$. Either way, since x was arbitrary, we have shown that every element of $\bar{B}$ is an element of C, and hence $\bar{B} \subset C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

Pass

Section 3

Proposition 1.3.4

Proposition 1.3.4. *Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.*

(*a*) *E is relatively open with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.*

(*b*) *E is relatively closed with respect to Y if and only if $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.*

(a)
⇒

Let E be relatively open with respect to Y. Then, from Definition 1.3.3 (AII), we know that E is open in the metric subspace $(Y, d|_{Y\times Y})$. Hence, $\forall x \exists r > 0 (B_{(Y, d_{Y\times Y})}(x, r) \subseteq E)$. Each r depends on x, so we write r_x. Hence, using Proposition 8.4.7 (AI), we can write $\forall x (B_{(Y, d_{Y\times Y})}(x, r_x) \subseteq E$.

Now consider $V := \bigcup_{x \in E} B_{(X,d)}(x, r_x)$. This is a subset of X. And from Proposition 1.2.15 (c) and (g) (AII), V is open. Say x is a point in E. Then it necessarily lies in Y, since $E \subset Y$, and it lies in V because $x \in B_{(X,d)}(x, r_x)$. Say y is a point in $V \cap Y$, then since $y \in V$, we know there exists $x \in E$ such that $y \in B_{(X,d)}(x, r_x)$. This implies $y \in B_{(Y, d_{Y\times Y})}(x, r_x)$, since $y \in Y$. From the definition of r_x, this implies $y \in E$. Hence, we have found an open set V such that $E = V \cap Y$.

Because E is open with respect to Y, every element of E has a ball with respect to Y that is contained in E. If each ball is extended to be with respect to X, then the balls will contain elements of X that are not elements of Y. This is true for every radius, and so each element of E cannot be an interior point. Hence, E is not open. However, if we take all the balls with respect to Y centred at an element of E, and extend them to X, the union of the extended balls, V, will be open because each ball is open. Every element of E will be an element of Y, and it will also be an element of V because it will be the centre of some ball extended to X, which makes part of the union. Every element of the intersection of Y and V will be an element of some extended ball centred at E. Because it's an element of Y, it will also be an element of the ball restricted to Y alone. This restricted ball is a subset of E as already stated. Hence, E is equal to the intersection of Y and V.

⇐

Let $E = V \cap Y$ for some open $V \subseteq X$. We need to show $\forall x \in E \exists r > 0 (B_{Y, d_{Y\times Y}}(x, r) \subset E)$. Let $x \in E$. Then we know $x \in V$ and $x \in Y$. Since V is open, we can write $\exists r > 0 (B_{X,d}(x, r) \subset V)$. Since $x \in Y$, it follows that $\exists r > 0 (B_{Y, d_{Y\times Y}}(x, r) \subset V)$. And since this ball only contains elements of Y, and is a subset of V, then it is necessarily a subset of $Y \cap V$. Hence, $\exists r > 0 (B_{Y, d_{Y\times Y}}(x, r) \subset E)$, as desired.

Exercise 1.3.1

Exercise 1.3.1. Prove Proposition 1.3.4(b).

⇒

Let E be relatively closed with respect to Y. Then we know $Y \setminus E$ is open with respect to Y. Since $Y \setminus E$ is open, we know from (a) that $Y \setminus E = V \cap Y$ for some $V \subseteq X$ that is open with respect to X. Hence, we know $X \setminus V$ is closed with respect to X. Define $K = X \setminus V \cap Y$.

Let $x \in K$. Then $x \in Y$, $x \in X$, and $x \notin V$. Assume $x \notin E$. Then $x \in Y \wedge x \notin E$, so it follows that $x \in Y \setminus E$, which implies $x \in V$. This is a contradiction, so $x \in E$. Now let $x \in E$, then $x \in Y$ and $x \in X$. Assume $x \in V$, then $x \in V \cap Y$, hence, $x \in Y \setminus E$, and $x \notin E$, which is a contradiction. Hence, $x \notin V$. Then $x \in X \setminus V$. Since x was arbitrary, we have shown $E = K \cap Y$, as desired.

$\Leftarrow$

Let $E = K \cap Y$ and $K \subseteq X$, which is closed in X. It follows that $X \setminus K$ is open in X. Define $F = (X \setminus K) \cap Y$, then it follows that F is open in Y. Then $Y \setminus F$ is closed in Y. Note, $Y \setminus F = Y \setminus ((X \setminus K) \cap Y) = Y \setminus (X \setminus K) \cup Y \setminus Y = Y \setminus (X \setminus K)$.

Let $x \in Y \setminus F$. Then $x \in Y$ and $x \notin X \setminus K$. Hence, $x \in X$ and $x \in K$. Then $x \in K \cap Y$, hence $x \in E$.

Now let $x \in E$, then $x \in K$ and $x \in Y$. Assume $x \in X \setminus K$, which is a contradiction, hence, $x \notin X \setminus K$. Then $x \in Y \setminus (X \setminus K)$.

Since x was arbitrary, we have shown $E = F$ and E is closed.

Section 4

Exercise 1.4.1

Exercise 1.4.1. Prove Lemma 1.4.3. (Hint: review your proof of Proposition 6.6.5.)

Lemma 1.4.3. *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then every subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of that sequence also converges to x_0.*

We can write $\lim_{n\to\infty} d(x^{(n)}, x_o) = 0$, hence $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(|d(x^{(n)}, x_o)| < \epsilon)$. Let $(x^{(n_j)})_{n=m}^{\infty}$ be some subsequence of $(x^{(n)})_{n=m}^{\infty}$. Let $\epsilon > 0$. Then we can write $\forall n \geq N(|d(x^{(n)}, x_o)| \leq \epsilon)$ for some $N \geq m$. Let $n_j \geq N$, then it follows $|d(x^{(n_j)}, x_o)| \leq \epsilon$. Since n_j was arbitrary, we have shown $\forall n_j \geq N(|d(x^{(n_j)}, x_o)| \leq \epsilon)$. Since ϵ was arbitrary, we have shown $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(|d(x^{(n_j)}, x_o)| \leq \epsilon)$, as desired.

Exercise 1.4.2

Exercise 1.4.2. Prove Proposition 1.4.5. (Hint: review your proof of Proposition 6.6.6.)

Proposition 1.4.5. *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Then the following are equivalent:*

- *L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.*
- *There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence $(x^{(n)})_{n=m}^{\infty}$ which converges to L.*

⇒

Let L be a limit point of $(x^{(n)})_{n=m}^{\infty}$, then $\forall N \geq m \forall \epsilon \exists n \geq N(d(x^{(n)}, L) \leq \epsilon)$. For each j, define $n_j = min\{n > n_{j-1} : d(x^{(n)}, x_o) \leq \frac{1}{j}\}$, where $n_o := m$. We know this isn't empty because x_o is a limit point, and so there must be some $n > m$ such that $d(x^{(n)}, x_o) \leq \frac{1}{j}$. Now consider $(x^{(n_j)})_{j=1}^{\infty}$. We know $\forall j \in N^+(d(x^{(n_j)}, x_o) \leq \frac{1}{j})$. Let $\epsilon > 0$ be some real. We know from Exercise 5.4.4 (AI), $\exists N(\epsilon > \frac{1}{N})$. Hence, $d(x^{(n_N)}, x_o) \leq \frac{1}{N} < \epsilon$. Since $\forall k > N(\frac{1}{k} < \frac{1}{N})$, it easily follows that $\forall k > N(d(x^{(n_k)}, x_o) < \epsilon)$. Since ϵ was arbitrary, we have shown $\forall \epsilon > 0 \exists N \geq j \forall k \geq N(d(x^{(n_k)}, x_o) < \epsilon)$.

⇐

Let $(x^{(n_j)})_{n=m}^{\infty}$ be a subsequence that converges to x_o. Then we know $\forall \epsilon > 0 \exists l \geq j \forall k \geq l(d(x^{(n_k)}, x_o) < \epsilon)$. Let $N \geq m$ and $\epsilon > 0$. Then we can write $\forall k \geq l(d(x^{(n_k)}, x_o) < \epsilon)$ for some l. There is some $n_k \geq N$, since $(x^{(n_j)})_{n=m}^{\infty}$ is an infinite sequence. Either $N \geq n_l$, in which case we can immediately assert $d(x^{(n_k)}, x_o) < \epsilon$, or $n_l > N$, in which case we can select some other $n_k \leq n_l$ and then assert $d(x^{(n_k)}, x_o) < \epsilon$. Either way, since N and ϵ was arbitrary, we can write $\forall N \geq m \forall \epsilon > 0(d(x^{(n)}, x_o) < \epsilon)$, as desired.

Exercise 1.4.3

Exercise 1.4.3. Prove Lemma 1.4.7. (Hint: review your proof of Proposition 6.1.12.)

Lemma 1.4.7 (Convergent sequences are Cauchy sequences)**.** *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^{\infty}$ is also a Cauchy sequence.*

We can write $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(|d(x^{(n)}, x_o)| < \epsilon)$. Let $\epsilon > 0$. We can write $\forall n \geq N(d(x^{(n)}, x_o) < \frac{\epsilon}{2})$ and for some $N \geq m$. Let $j, k \geq N$. Then we can write $d(x^{(j)}, x_o) < \frac{\epsilon}{2}$ and $d(x^{(k)}, x_o) < \frac{\epsilon}{2}$. It follows that $d(x^{(j)}, x^{(k)}) \leq d(x^{(j)}, x_o) + d(x^{(k)}, x_o) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since ϵ was arbitrary, we have shown $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(|d(x^{(j)} x^{(k)})| < \epsilon)$.

Exercise 1.4.4

Exercise 1.4.4. Prove Lemma 1.4.9.

Lemma 1.4.9. *Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in (X, d). Suppose that there is some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of this sequence which converges to a limit x_0 in X. Then the original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to x_0.*

We can write $\forall \epsilon > 0 \exists N \geq m \forall j, k \geq N(d(x^{(j)}, x^{(k)}) < \epsilon)$ and, for some subsequence, $\forall \epsilon > 0 \exists F \geq 1 \forall l \geq F(d(x^{(n_l)}, x_o) < \epsilon)$.

Let $\epsilon > 0$ be some real. Then we can write $\geq m \forall j, k \geq N(d(x^{(j)}, x^{(k)}) < \frac{\epsilon}{2})$ for some N, and $\geq 1 \forall l \geq F(d(x^{(n_l)}, x_o) < \frac{\epsilon}{2})$ for some F. Let $M := max(N, n_F)$, and let $n \geq M$. Then there is some l such that $n_l \geq n$. Henc, we can write $d(x^{(n)}, x_o) \leq d(x^{(n)}, x^{(n_l)}) + d(x^{(n_l)}, x_o) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Since n was arbitrary, we have shown $\forall n \geq N(d(x^{(n)}, x_o) < \epsilon)$. Since ϵ was arbitrary, we have shown $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(d(x^{(n)}, x_o) < \epsilon)$.

Exercise 1.4.5

Exercise 1.4.5. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d), and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\{x^{(n)} : n \geq m\}$. Is the converse true?

We can write $\forall \epsilon > 0 \forall N \geq m \exists n \geq N (d(x^{(n)}, L) \leq \epsilon)$. Let $r > 0$ be some real. Then we can write $\forall N \geq m \exists n \geq N (d(x^{(n)}, L) \leq r)$. It follows that there exists some $x^{(n)}$ such that $d(x^{(n)}, L) \leq r)$, hence $x^{(n)} \in B(L, r)$. Obviously, $x^{(n)} \in \{x^{(n)} : n \geq m\}$. Since r was arbitrary, we have shown $\forall r > 0 (B(L, r) \cap \{x^{(n)} : n \geq m\} \neq \emptyset)$. Hence, L is an adherent point of $\{x^{(n)} : n \geq m\}$.

The converse is not true since, although the intersection of the ball centred at L and the set $\{x^{(n)} : n \geq m\}$ is non-empty for every $r >$, there is no guarantee that their shared element/s will be an $x^{(n)}$ such that $n \geq N$ for every $N \geq m$, and this is required for L to be a limit point.

Exercise 1.4.7

Exercise 1.4.6. Show that every Cauchy sequence can have at most one limit point.

If a Cauchy sequence had two distinct limit points, then by Lemma 1.4.9 (AII), the sequence would converge to two distinct values. However, by Proposition 1.1.20, this is a contradiction. Hence, every Cauchy sequence cannot have more than one limit point.

Exercise 1.4.8

Exercise 1.4.8. The following construction generalizes the construction of the reals from the rationals in Chapter 5, allowing one to view any metric space as a subspace of a complete metric space. In what follows we let (X, d) be a metric space.

(a) Given any Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, we introduce the *formal limit* $\mathrm{LIM}_{n\to\infty} x_n$. We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n$ and $\mathrm{LIM}_{n\to\infty} y_n$ are equal if $\lim_{n\to\infty} d(x_n, y_n)$ is equal to zero. Show that this equality relation obeys the reflexive, symmetry, and transitive axioms.

Reflexive:

This follows since $d(x_n, x_n) = 0$.

Symmetric:

This follows since $d(x_n, y_n) = d(y_n, x_n)$.

Transitive:

This follows since $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) = 0$, and Theorem 6.1.9 (AI).
$d(x_n, z_n) = 0$ because the metric is strictly non-negative.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, with the above equality relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to \mathbf{R}^+$ by setting

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Show that this function is well-defined (this means not only that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists, but also that the axiom of substitution is obeyed; cf. Lemma 5.3.7), and gives $\overline{X}$ the structure of a metric space.

We can write $\forall \epsilon > 0 \exists N \geq m \forall n \geq N (d(x_j, x_k) \leq \epsilon)$ and $\forall \epsilon > 0 \exists N \geq m \forall n \geq N (d(y_j, y_k) \leq \epsilon)$. Then we can write $\forall n \geq N_1 (d(x_j, x_k) \leq \dfrac{\epsilon}{2})$ for some

$N_1 \geq m$, and $\forall n \geq N_2(d(y_j, y_k) \leq \frac{\epsilon}{2})$ for some $N_2 \geq m$. Letting $M := max(N_1, N_2)$, and $n \geq M$, we can write $d(x_n, y_n) \leq d(x_n, x_M) + d(y_n, x_M)$. Then $d(y_n, x_M) \leq d(y_n, y_M) + d(x_M, y_M)$. Hence, $d(x_n, y_n) \leq d(x_n, x_M) + d(y_n, y_M) + d(x_M, y_M)$. Then, $d(x_n, y_n) - d(x_M, y_M) \leq d(x_n, x_M) + d(y_n, y_M) < \frac{\epsilon}{2}) + \frac{\epsilon}{2}) = \epsilon$. By a similar argument, we can write $d(x_M, y_M) - d(x_n, y_n) < \epsilon$. This allows us to write $-\epsilon < d(x_n, y_n) - d(x_M, y_M) < \epsilon$. Since ϵ was arbitrary, we have shown $\forall \epsilon \exists N \geq m \forall j, k \geq N(|d(x_j, y_j) - d(x_k, y_k)| < \epsilon$. Hence, the series $(d(x_n, y_n))_{n=m}^{\infty}$ is Cauchy, and convergent. So the limit exists.

To show that the axiom of substitution holds, we need to show $d(a_n, b_n,) = d(a'_n, b_n)$ given $d(a_n, a'_n) = 0$. We can immediately write $d(a_n, b_n) \leq d(a_n, a'_n) + d(a'_n, b_n)$. Hence, $d(a_n, b_n) \leq d(a'_n, b_n)$. By way of similar argument, we can write $d(a'_n, b_n) \leq d(a_n, b_n)$. Hence, $d(a_n, b_n) = d(a'_n, b_n)$, as desired.

The metric space structure follows easily from what we have already shown.

(c) Show that the metric space $(\overline{X}, d_{\overline{X}})$ is complete.

Let $(LIM_{k\to\infty} x_k^{(n)})_{n=m'}^{\infty}$ be a Cauchy sequence in $\bar{X}$. This means that every element in the sequence is a formal limit of a Cauchy sequence in X. Hence, $\forall n \geq m'((x_k^{(n)}))_{n=m}^{\infty}\ is\ Cauchy$. Hence, $\forall n \geq m' \forall \epsilon > 0 \exists N \geq m' \forall i, j(d(x_i^{(n)}, x_j^{(n)}) < \epsilon)$. From this, it easily follows that $\forall n \geq m' \exists N \geq m \forall k \geq N(d(x_N^{(n)}, x_k^{(n)}) < 1/n)$. Using the axiom of choice, as in Proposition 8.4.7 (AI), we can write $\forall n \geq m' \forall k \geq N_n(d(x_{N_n}^{(n)}, x_k^{(n)}) < 1/n)$, and hence, we can write a sequence $S := (x_{N_n}^{(n)})_{n=m'}^{\infty}$, which is obviously in X.

We will show that this sequence is Cauchy, and hence its formal limit is an element of $\bar{X}$. First, since $(LIM_{k\to\infty} x_k^{(n)})_{n=m'}^{\infty}$ is Cauchy, we can write $\forall \epsilon > 0 \exists N \geq m' \forall i, j \geq N(d_{\bar{X}}(LIM_{k\to\infty} x_k^{(i)}, LIM_{k\to\infty} x_k^{(j)}) < \epsilon)$, which implies $\forall \epsilon > 0 \exists N \geq m' \forall i, j \geq N(\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon)$.

Now, let > 0 be some real. Then we know $\epsilon > 1/n$ for some n. We can also write $\forall i, j \geq N(\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon/3)$ for some $N \geq m'$. Say $i, j \geq N$, then we know

$\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon/6$, and since $i, j \geq N \geq m'$, we can write $\forall k \geq N_i(d(x_{N_i}^{(i)}, x_k^{(i)}) < 1/i)$ and $\forall k \geq N_j(d(x_{N_j}^{(j)}, x_k^{(j)}) < 1/j)$.

Since $\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon/6$, and we know the limit exists from (b), we can write $\forall \epsilon' > 0 \exists N' \geq 1 \forall k \geq N'(|d(x_k^{(i)}, x_k^{(j)}) - L| < \epsilon')$. Setting $\epsilon' = \epsilon/6$, then for some N', we can write $|d(x_k^{(i)}, x_k^{(j)}) - L| < \epsilon/6$ for every k. It follows easily that $d(x_k^{(i)}, x_k^{(j)}) < \epsilon/3$, since we know $L < \epsilon/6$. Then, setting $M := max(N', N_i, N_j)$, we can write $d(x_M^{(i)}, x_M^{(j)}) < \epsilon/3$, $d(x_{N_i}^{(i)}, x_M^{(j)}) < 1/i$, and $d(x_{N_j}^{(j)}, x_M^{(j)}) < 1/j$.

Then it follows, $d(x_{N_i}^{(i)}, x_{N_j}^{(j)} \leq d(x_{N_i}^{(i)}, x_M^{(i)}) + d(x_{N_j}^{(j)}, x_M^{(i)})$, and also, $d(x_{N_j}^{(j)}, x_M^{(i)}) \leq d(x_{N_j}^{(j)}, x_M^{(j)}) + d(x_M^{(i)}, x_M^{(j)})$. Hence, $d(x_{N_i}^{(i)}, x_{N_j}^{(j)}) < 1/i + 1/j + \epsilon/3 < \epsilon/3 + \epsilon/3 + \epsilon/3 < \epsilon$. Since ϵ was arbitrary, we have shown $\forall \epsilon > 0 \exists N \geq m' \forall i, j \geq N(d(x_{N_i}^{(i)}, x_{N_j}^{(j)}) < \epsilon$, which implies the sequence S is Cauchy.

Now, we will show that our original $(LIM_{k\to\infty} x_k^{(n)})_{n=m'}^{\infty}$ converges to sequence S, and hence converges to an element of $\bar{X}$. This is what we know:

$\forall \epsilon > 0 \exists N \geq m' \forall i, j \geq N(d(x_{N_i}^{(i)}, x_{N_j}^{(j)}) < \epsilon$

$\forall \epsilon > 0 \exists N \geq m' \forall i, j \geq N(\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon)$

$\forall n \geq m' \forall k \geq N_n(d(x_{N_n}^{(n)}, x_k^{(n)}) < 1/n)$

Let $\epsilon > 0$ be some real. Then, we know $\forall i, j \geq N(\lim_{k\to\infty} d(x_k^{(i)}, x_k^{(j)}) < \epsilon)$ for some $N \geq m$, and $\forall i, j \geq N'((x_{N_i}^{(i)}, x_{N_j}^{(j)}) < \epsilon)$ for some $N' \geq m$. Let $M := max(N, N')$. Let $n \geq M$. It follows that $\lim_{k\to\infty} d(x_k^{(n)}, x_k^{(M)}) < \epsilon$.

Given the limit laws, we can write:

$\lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) \leq \lim_{k\to\infty} d(x_k^{(n)}, x_{N_M}^{(M)}) + \lim_{k\to\infty} d(x_{N_k}^{(k)}, x_{N_M}^{(M)})$. Since the sequence $(x_{N_n}^{(n)})_{n=m}^{\infty}$ is Cauchy, it follows that $\lim_{k\to\infty} d(x_{N_k}^{(k)}, x_{N_M}^{(M)}) = 0$. Hence, $\lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) \leq \lim_{k\to\infty} d(x_k^{(n)}, x_{N_M}^{(M)})$.

Next, we can write $\lim_{k\to\infty} d(x_k^{(n)}, x_{N_M}^{(M)}) \leq \lim_{k\to\infty} d(x_k^{(n)}, x_k^{(M)}) + \lim_{k\to\infty} d(x_k^{(M)}, x_{N_M}^{(M)})$. We know from the near the beginning of the proof that $\forall n \geq m' \exists N \geq m \forall k \geq N(d(x_N^{(n)}, x_k^{(n)}) < 1/n)$.

Hence, it easily follows that $\lim_{k\to\infty} d(x_k^{(M)}, x_{N_M}^{(M)}) = 0$.

Hence, $\lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) \leq \epsilon$.

Since ϵ was arbitrary we have shown $\forall\epsilon > 0 \exists N \geq m \forall n \geq N (\lim_{k\to\infty} d(x_k^{(n)}, x_{N_k}^{(k)}) \leq \epsilon$, which implies, $\forall\epsilon > 0 \exists N \geq m \forall n \geq N (d_{\bar{X}}(LIM_{k\to\infty x_k^{(n)}} LIM_{k\to\infty x_{N_k}^{(k)}}) \leq \epsilon$, which is the definition of convergence in $\bar{X}$.

Since $(LIM_{k\to\infty} x_k^{(n)})_{n=m'}^{\infty}$ was arbitrary, we have shown that $(\bar{X}, d_{\bar{X}})$ is complete.

(d) We identify an element $x \in X$ with the corresponding formal limit $\mathrm{LIM}_{n\to\infty} x$ in $\overline{X}$; show that this is legitimate by verifying that $x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$. With this identification, show that $d(x, y) = d_{\overline{X}}(x, y)$, and thus (X, d) can now be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

Let $x = y$. Then $d(x, y) = 0$, and $\lim_{n\to\infty} d(x, y) = 0$. This implies, from (a), that $LIM_{n\to\infty} x = LIM_{n\to\infty} y$.
Now suppose $LIM_{n\to\infty} x = LIM_{n\to\infty} y$. This implies $d_{\bar{X}}(LIM_{n\to\infty} x, LIM_{n\to\infty} y) = 0$. It follows that $\lim_{n\to\infty} d(x, y) = 0$, and consequently $x = y$.

Let $d(x, y) = L$. Then $\lim_{n\to\infty} d(x, y) = L$. Hence, $d_{\bar{X}}(LIM_{n\to\infty} x, LIM_{n\to\infty} y) = L$. Given the identity relation, we can write $d_{\bar{X}}(x, y) = L$. Hence, $d(x, y) = d_{\bar{X}}(x, y)$.

(e) Show that the closure of X in $\overline{X}$ is $\overline{X}$ (which explains the choice of notation $\overline{X}$).

Let x be some adherent point of X in $\bar{X}$. Then we know there is some sequence $(x_n)_{n=1}^{\infty}$ that converges to x with respect to $d_{\bar{X}}$. From each element of the sequence, we can identify it with a corresponding element in $\bar{X}$. This sequence in $\bar{X}$ is Cauchy. Since it is Cauchy, and $\bar{X}$ is complete, we know the sequence is convergent in $\bar{X}$. Since the sequence converges to x, it follows that $x \in \bar{X}$.

Let $LIM_{n\to\infty} x_n \in \bar{X}$. Then we know $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in X. We can immediately write $d_{\bar{X}(LIM_{n\to\infty} x_n, LIM_{n\to\infty} x_n)=0}$. Since $d_X = d_{\bar{X}}$, it follows that

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(f) Show that the formal limit agrees with the actual limit, thus if $(x_n)_{n=1}^{\infty}$ is any Cauchy sequence in X, then we have $\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n$ in $\overline{X}$.

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X. Then we know the formal limit $LIM_{n\to\infty}x_n \in \bar{X}$

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Section 5

Theorem 1.5.8

Theorem 1.5.8. *Let (X, d) be a metric space, and let Y be a compact subset of X. Let $(V_\alpha)_{\alpha\in I}$ be a collection of open sets in X, and suppose that*

$$Y \subseteq \bigcup_{\alpha\in I} V_\alpha.$$

(i.e., the collection $(V_\alpha)_{\alpha\in I}$ covers *Y). Then there exists a* finite *subset F of I such that*

$$Y \subseteq \bigcup_{\alpha\in F} V_\alpha.$$

Assume $\sim(\exists F \subset I, F finite(Y \subseteq \bigcup_{\alpha\in F} \bigvee_\alpha))$. Let $y \in Y$. Then, obviously, there is some α such that $y \in \bigvee_\alpha$. Since $\bigvee_\alpha$ is open, we know $\exists r > 0(B_{X,d}(y, r)_\alpha)$. Now let:
$r(y) := sup\{r \in (0, \infty) : B_{X,d}(y, r) \subseteq V_\alpha \;\; for \;\; some \;\; \alpha \in A\}$
We know the given set is non-empty because of what we have just said. We also know $r(y) > 0$ for every $y \in Y$. Then define:
$r_0 := inf\{r(y) : y \in Y\}$
Since $r(y) > 0$ for every $y \in Y$, it follows that $r_0 \geq 0$. Hence, $r_0 = 0 \vee r_0 > 0$.

Assume $r_0 = 0$. Let $n \geq 1$ be some integer. We know there is some y such that $r(y) < 1/n$, because otherwise 0 would not be the infimum of $\{r(y) : y \in Y\}$. Hence, from the axiom of choice, we can choose $y^{(n)}$ in Y such that $r(y^{(n)}) < 1/n$ for every $n \geq 1$. It follows easily from the squeeze test that $\lim_{n\to\infty} r(y^{(n)}) = 0$. The sequence $(y^{(n)})_{n=1}^{\infty}$ is a sequence in Y, and hence we can find a subsequence $(y^{(n_j)})_{j=1}^{\infty}$ that converges to a point y_0, since Y is compact.

Since $y_0 \in Y$, we know $y_0 \in \bigvee_\alpha$ for some α. Since $\bigvee_\alpha$ is open, we know that $B(y_0, \epsilon) \subseteq \bigvee_\alpha$ for some ϵ. Since $y^{(n_j)}$ converges to y_0, there must be some $N \geq 1$ such that $\forall n \geq N y^{(n_j)} \in B(y_0, \epsilon/2)$. Say $x \in B(y^{(n_j)}, y_0)$. Then $d(x, y_0) \leq d(x, y^{(n_j)}) + d(y^{(n_j)}, y_0) \leq \epsilon/2 + \epsilon/2 = \epsilon$. Hence, $B(y^{(n_j)}, \epsilon/2) \subseteq B(y_0, \epsilon)$, and $B(y^{(n_j)}, \epsilon/2) \subseteq \bigvee_\alpha$.

Since $r(y^{n_j})$ is the supremum of all the $r's$, it follows that $r(y^{n_j}) \geq \epsilon/2$ for every $n \geq N$. This contradicts the fact that $\lim_{n\to\infty} r(y^{(n)}) = 0$.

Assume $r_0 > 0$. Hence, $r(y) > r_0/2$ for every $y \in Y$. Since for every $x \in B(y, r_0/2)$, it immediately follows that $x \in B(y, r(y))$, and hence $B(y, r_0/2) \subseteq \bigvee_\alpha$.

We will recursively construct a sequence using Lemme 3.1.6 (AI). Let $y^{(1)}$ be some element in Y. It follows that $B(y^{(1)}, r_0/2)$ does not cover Y since otherwise this V_α that covers $B(y^{(1)}, r_0/2)$ would be covering Y, and then there would be a finite number of sets that covers Y, a contradiction. Hence, we can choose some $y^{(2)} \in Y \setminus B(y^{(1)}, r_0/2)$. In particular $d(y^{(2)}, y^{(1)}) \geq r_0/2$. It also follows that $B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$, such that $B(y^{(2)}, r_0/2) \subseteq V_\alpha$ for some $alpha$, also does not cover Y. Otherwise it would be a contradiction in the same way. So we can choose some $y^{(3)} \in Y \setminus B(y^{(1)}, r_0/2) \cup B(y^{(2)}, r_0/2)$. Continuing in this way, we construct a sequence $(y^{(n)})_{n=1}^{\infty}$ with the property $d(y^{(k)}, y^{(j)}) \geq r_0/2$ for every $k > j$. It follows that this sequence is not a Cauchy sequence, and no subsequence is Cauchy either. From Lemma 1.4.7 (AII), no subsequence is convergent. This contradicts the fact that Y is compact.

We are supposing that an infinite number of sets cover a company Y in union, and we will assume for the sake of the proof that there is no finite number of V_α that contains Y in union. Let $y \in Y$, then we know it is an element of some V_α. And, since V_α is open, we also know we can construct a ball that fits inside the V_α. We then define $r(y)$ as the supremum of the radii of balls centred at y that fit inside any of the V_α. There may be multiple of these balls, and hence radii, but we have defined $r(Y)$ as the supremum of these. We then consider, of all the largest radii (the supremums technically), which one is the smallest (the infimum of these supremums). By definition, none of the radii are negative, so we have two cases.

If $r_0 = 0$, this means that many of the sets, V_α, have increasingly small radii. We construct a sequence from the radii of the largest balls contained in the V_α. By doing so, we show that the radii converge to 0. If we were to construct a sequence of the $y's$ that produce these radii,

there would be some subsequence that converges to some $y_0 \in Y$, since Y is compact. We know immediately that y_0 is contained in some open V_α, and hence y_0 is contained in some ball in V_α. Because the subsequence converges to y_0, there comes a point where all the remaining elements of the subsequence are contained in this ball. It easily follows that each element of the subsequence is itself the centre of a ball contained in the original ball centred at y_0 and hence contained in V_α. And each of the balls share the same radius. Note, each element of this subsequence is necessarily related to an $r(y)$ because that's how we defined the sequence. Since $r(y)$ is the supremum of all the r's of balls centred at an element of y contained in a V_α, it must be the case that these $r(y)$ are maintaining a constant distance from 0. This prevents the sequence of $r(y)$ from converging to 0.
This case boils down to showing that a certain sequence of $r(y)$ converge to 0 by virtue of r_0 being 0, while simultaneously not converging to 0 due to the compactness of Y. The proof moves between the sequence of radii that converge to 0 and the sequence of the associated y values that produce these radii. The compactness of Y applies to the sequence of y's, rather than the radii directly, but has consequences on the properties of the radii sequence.

If $r_0 > 0$, then there is some limit on how small the V_α can get. Because of this, for any element y, we can always construct a ball that is contained in one of the V_α, and every ball we construct has some fixed radius, $r_0/2$.. If we choose a $y^{(1)}$, we can guarantee that there is some other $y^{(2)}$ that isn't an element of the ball centred at $y^{(1)}$. This is because if we couldn't, that would mean all of Y is contained in this ball and hence contained in one of the V_α. This would be a contradiction since we assumed there is not finite number of V_α that cover Y. If we take the ball centred at $y^{(2)}$, union it with the ball centred at $y^{(1)}$, the resulting union would also not cover Y because of the same contradiction. So we can always produce another element of Y that is not an element of the preceding union of balls. Taking the sequence of these $y^{(i)}$'s, we find that every element necessarily has a distance of at least $r_0/2$ each other. This property of the sequence prevents the sequence from being Cauchy, and it also prevents any subsequence from being Cauchy as well. However, this contradicts Y being compact.
This case involves identifying a sequence that doesn't have a Cauchy subsequence. Constructing this sequence results from $r_0 > 0$, and the assumption that no finite number of V_α's cover Y. If we hadn't assumed this, then we wouldn't be able to guarantee our next element of the sequence. And if $r_0 = 0$, we wouldn't be able to guarantee a fixed distance for each element.

Exercise 1.5.1

Exercise 1.5.1. Show that Definitions 9.1.22 and 1.5.3 match when talking about subsets of the real line with the standard metric.

Definition 9.1.22 (Bounded sets). A subset X of the real line is said to be *bounded* if we have $X \subset [-M, M]$ for some real number $M > 0$.

Definition 1.5.3 (Bounded sets). Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is *bounded* iff there exists a ball $B(x, r)$ in X which contains Y.

Definition 9.1.22 (AI) can be written as $\exists M > 0(X \subset [-M, M])$, given some $X \subset \mathbb{R}$.
Definition 1.5.3 (AII) can be written as, $\exists x \in X \exists r > 0(Y \subseteq B(x, r))$ for some $Y \subseteq X$, where X is a metric space..
Let $\mathbb{R}$, the real line, be the metric space, and let $X \subset \mathbb{R}$. Then the distance function is $d_X(x, y) = |x - y|$. Then we can rewrite 1.5.2 as $\exists y \in \mathbb{R} \exists r > 0 \forall x \in X(|x - y| < r)$.
Similarly, we can write 9.1.22 as $\exists M > 0 \forall x \in X(|x| > M)$. Since M is a dummy variable, and observing that $|x - 0| > M$, we can write $\exists y \in \mathbb{R} \exists r > 0 \forall x \in X(|x - y| < r)$. Hence, the definitions are the same when the metric space is $\mathbb{R}$.

Exercise 1.5.2

Exercise 1.5.2. Prove Proposition 1.5.5. (Hint: prove the completeness and boundedness separately. For both claims, use proof by contradiction. You will need the axiom of choice, as in Lemma 8.4.5.)

Proposition 1.5.5. *Let (X, d) be a compact metric space. Then (X, d) is both complete and bounded.*

Let $(x_n)_{n=m}^{\infty}$ be some Cauchy sequence in (X, d). Because (X, d) is compact, we know that there exists some subsequence $(x_{n_j})_{j=1}^{\infty}$ that converges to some $L \in X$. From Lemma 1.4.9, and the sequence is Cauchy, we know that it also converges to L. Since $(x_n)_{n=m}^{\infty}$ was abitrary, we have shown that every Cauchy sequence converges in (X, d), and hence the metric space is complete.

Assume (X, d) is unbounded. Let $x_0 \in X$. Then for every $n \in \mathbb{N}^+$, define $X_n := \{x \in X : x(x_0, n) \setminus B(x_0, n-1)\}$. We know that for each n, $X_n \neq \emptyset$ because otherwise there would be some $B(x_0, n)$ that covers X, and this contradicts X being unbounded. Hence, using the axiom of choice, we can construct a sequence $(x_n)_{n=m}^{\infty}$ from these sets.
Let $N \geq m$, and define $n := N + 5$. We know $x_N \in B(x_0, N)$, and $x_n \in B(x_0, n) \setminus B(x_0, n-1)$. It follows then that $d(x_0, x_N) < N$, and $n - 1 \leq d(x_0, x_n) < n$

. Hence, we can write $d(x_n, x_0) \leq d(x_n, X_n) + d(x_0, x_N)$. Then, $n - 1 - N < d(x_n, x_N)$. It follows that $4 < d(x_n, x_N)$. Since N was arbitrary we have shown $\exists \epsilon > 0 \forall N \geq 1 \exists i, j \geq N(d(x_i, x_j) > \epsilon)$. This implies that the sequence is not Cauchy. Then it follows that any subsequence is not Cauchy, but this contradicts that X is compact. Hence, (X, d) is bounded.

Exercise 1.5.3

Exercise 1.5.3. Prove Theorem 1.5.7. (Hint: use Proposition 1.1.18 and Theorem 9.1.24.)

Theorem 1.5.7 (Heine-Borel theorem). *Let $(\mathbf{R}^n, d)$ be a Euclidean space with either the Euclidean metric, the taxicab metric, or the sup norm metric. Let E be a subset of $\mathbf{R}^n$. Then E is compact if and only if it is closed and bounded.*

Theorem 9.1.24 (Heine-Borel theorem for the line). *Let X be a subset of $\mathbf{R}$. Then the following two statements are equivalent:*

(a) *X is closed and bounded.*

(b) *Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X (i.e., $a_n \in X$ for all n), there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.*

Proposition 1.1.18 (Equivalence of l^1, l^2, l^∞). *Let $\mathbf{R}^n$ be a Euclidean space, and let $(x^{(k)})_{k=m}^\infty$ be a sequence of points in $\mathbf{R}^n$. We write $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$, i.e., for $j = 1, 2, \ldots, n$, $x_j^{(k)} \in \mathbf{R}$ is the j^{th} co-ordinate of $x^{(k)} \in \mathbf{R}^n$. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbf{R}^n$. Then the following four statements are equivalent:*

(a) *$(x^{(k)})_{k=m}^\infty$ converges to x with respect to the Euclidean metric d_{l^2}.*

(b) *$(x^{(k)})_{k=m}^\infty$ converges to x with respect to the taxi-cab metric d_{l^1}.*

(c) *$(x^{(k)})_{k=m}^\infty$ converges to x with respect to the sup norm metric d_{l^∞}.*

(d) *For every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^\infty$ converges to x_j. (Notice that this is a sequence of real numbers, not of points in $\mathbf{R}^n$.)*

Let $E \subseteq \mathbb{R}^n$, and suppose it is compact. Then we know
$\forall (x^{(k)})_{k=m}^\infty \exists (x^{(k_j)})_{j=m}^\infty \exists L \in E(\lim_{j\to\infty} d(x^{(k_j)}, L) = 0$.

For every $1 \leq i \leq n$, define $X_i := \{x_i^k \in x^{(k)} : x^{(k)} \in E\}$. It is obvious that $X_i \subseteq \mathbb{R}$.
Let i be some integer such that $1 \leq i \leq n$. Let $(x_i^{(k)})_{k=m}^\infty$ be some sequence in X_i. Since for every $x_i^{(k)} \in X_i$, there is some $x^{(k)} \in E$, we can construct a sequence of $(x^{(k)})_{k=m}^\infty$ using the axiom of choice (P8.4.7 AI). Since $(x^{(k)})_{k=m}^\infty$ is in E, and E is compact, we can write $\exists L \in E(\lim_{j\to\infty} d(x^{(k_j)}, L) = 0$. From P1.1.18 (AII), we know $\exists L_i X_i(\lim_{j\to\infty} d(x_i^{k_j}, L_i) = 0)$. Since $x_i^{(k)}$ was arbitrary, we have shown that every sequence in X_i has a convergent subsequence in X_i. From T9.1.24 (AI), this implies that X_i is closed and bounded. Since i was arbitrary, we have shown that every X_i is closed and bounded.
Now let $(x^{(k)})_{k=m}^\infty$ be some convergent sequence in E. Then for every $1 \leq i \leq n$, the sequence $(x_i^{(k)})_{k=m}^\infty$ converges to L_i. Since every X_i is closed, this implies that $L_i \in X_i$, from P1.2.15 (AII). It follows then that $L \in E$ from P1.1.18 (AII). Because $(x^{(k)})_{k=m}^\infty$ was arbitrary, we have shown that every convergent sequence in E has it's limit in E, and hence E is closed as per P1.2.15 (AII).
E is bounded, which immediately follows from E being compact.

Assume E is closed and bounded.
For every $1 \leq i \leq n$, define $X_i := \{x_i^k \in x^{(k)} : x^{(k)} \in E\}$. It is obvious that $X_i \subseteq \mathbb{R}$.

Assume there exists some X_i that is unbounded. Since E is bounded, we can find a point $x \in \mathbb{R}$ and an $r > 0$, such that $E \subseteq B(x, r)$. Since X_i is unbounded, we can find an $x_i^{(k)} \in X_i$ such that $d(x^{(k)}, x) \geq r$, where $x_i^{(k)} \in x^{(k)}$. In the case where the metric is the Euclidean metric, this follows from the fact that the square root function is unbounded. For the taxi-cab metric, it follows from the absolute value function being unbounded. Similarly, for the sup norm metric. Since we can find a $x^{(k)}$ where $d(x^{(k)}, x) \geq r$, this contradicts the fact that $E \subseteq B(x, r)$. Hence, every X_i is bounded.

Let $1 \leq i \leq n$, and let $(x_i^{(k)})_{n=m}^{\infty}$ be some convergent sequence in X_i. Then it follows that the limit of this sequence is in X_i since E is closed. Since i is arbitrary, then we have shown every X_i is closed and bounded.

Let $(x^{(k)})_{n=m}^{\infty}$ be some sequence in E. Let $((x_i^{(k)})_{n=m}^{\infty}$ be the sequence for each i. Then, since each X_i is closed and bounded, we know from T9.1.24 (AI), that there exists some subsequence $(x_i^{(k_j)})_{n=m}^{\infty}$ that converges to $L_i \in X_i$. Then it follows from P1.1.18 (AII) that there is some subsequence $(x^{(k_j)})_{n=m}^{\infty}$ that converges to $L \in E$. This shows that E is compact.

Exercise 1.5.4

Exercise 1.5.4. Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and an open set $V \subseteq \mathbf{R}$, such that the image $f(V) := \{f(x) : x \in V\}$ of V is *not* open.

Let $V = (0, 1)$, which is obviously open. And let $f(x) = 1$ for every $x \in (0, 1)$. Then $f(V) = 1$, which is obviously closed.

Exercise 1.5.5

Exercise 1.5.5. Let $(\mathbf{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbf{R} \to \mathbf{R}$, and a closed set $F \subseteq \mathbf{R}$, such that $f(F)$ is *not* closed.

Let $F = [1, +\infty)$ and $f(x) = 1/x$. Then $f(F) = (0, 1]$.

Exercise 1.5.6

Exercise 1.5.6. Prove Corollary 1.5.9. (Hint: work in the compact metric space $(K_1, d|_{K_1 \times K_1})$, and consider the sets $V_n := K_1 \backslash K_n$, which are open on K_1. Assume for sake of contradiction that $\bigcap_{n=1}^{\infty} K_n = \emptyset$, and then apply Theorem 1.5.8.)

Corollary 1.5.9. *Let (X, d) be a metric space, and let $K_1, K_2, K_3, \ldots$ be a sequence of non-empty compact subsets of X such that*

$$K_1 \supset K_2 \supset K_3 \supset \ldots .$$

Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

The $K_1 \setminus K_n$ are open. Assuming $\cap_{n=1}^{\infty} K_n = \emptyset$, it follows that $\cup_{n=1}^{\infty} K_1 \setminus K_n = K_1 \setminus \cap_{n=1}^{\infty} K_n = K_1$. Using Theorem 1.5.8, we can assert that there are some $n_1 \cdots n_k$ such that $\cup_{i=1}^{k} K_1 \setminus K_{n_i} = K_1$. If we then take $N \geq n_i$ for all i, then we know $K_N \subset K_i$ for all i. Since we can show $K_N \nsubseteq \cup_{i=1}^{k} K_1 \setminus K_{n_i}$, it follows that $\cup_{i=1}^{k} K_1 \setminus K_{n_i} \neq K_1$, a contradiction; hence, $\cap_{n=1}^{\infty} K_n \neq \emptyset$.

Exercise 1.5.7

Exercise 1.5.7. Prove Theorem 1.5.10. (Hint: for part (c), you may wish to use (b), and first prove that every singleton set is compact.)

Theorem 1.5.10. *Let (X, d) be a metric space.*

(a) *If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact if and only if Z is closed.*

(b) *If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then their union $Y_1 \cup \ldots \cup Y_n$ is also compact.*

(c) *Every finite subset of X (including the empty set) is compact.*

(a)
Suppose $Y \subseteq X$ is compact and $Z \subset Y$.
Suppose Z is compact. Then it immediately follows that Z is closed, from Corollary 1.5.6 (AII).
Suppose now Z is instead closed. Let $(z_n)_{n=m}$ be some sequence in Z. Since it is also in Y, we know there is some subsequence $(z_{n_j})_{j=m}$ that converges to some L in Y. However, since

Z is closed and the subsequence is also in Z, then we know $L \in Z$, from P1.2.15(b) (AII). Hence, Z is compact.

(b)
Let $(y_n)_{n=m}^{\infty}$ be some sequence in $Y_1 \cup \cdots \cup Y_n$. Since the sequence is infinite we know there is some Y_n that has an infinite number of elements in $(y_n)_{n=m}^{\infty}$. Let $(y_{n_j}^{Y_n})_{j=1}^{\infty}$ be the subsequence that only contains elements from Y_n. Since Y_n is compact, we know there is some subsequence $(y_{n_{j_k}}^{Y_n})_{k=1}^{\infty}$ that converges to some L in Y_n. This subsequence is also a subsequence of $(y_n)_{n=m}^{\infty}$, and obviously $L \in Y_1 \cup \cdots \cup Y_n$. Hence, $Y_1 \cup \cdots \cup Y_n$ is compact.

(c)
The empty set is vacuously compact.
The singleton set is compact since, given $\{x\}$, the only sequence is $(x)_{n=m}^{\infty}$, which obviously converges to x, and hence every subsequence converges to x as well.
Every finite set can be constructed using $\{x_1\} \cup \cup \{x_n\}$, for each element x_i in the finite set. From (b), and the fact that every singleton set is compact, it follows that the finite set is compact.

Exercise 1.5.8

Exercise 1.5.8. Let (X, d_{l^1}) be the metric space from Exercise 1.1.15. For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^{\infty}$ be the sequence in X such that $e_j^{(n)} := 1$ when $n = j$ and $e_j^{(n)} := 0$ when $n \neq j$. Show that the set $\{e^{(n)} : n \in \mathbf{N}\}$ is a closed and bounded subset of X, but is not compact. (This is despite the fact that (X, d_{l^1}) is even a complete metric space - a fact which we will not prove here. The problem is that not that X is incomplete, but rather that it is "infinite-dimensional", in a sense that we will not discuss here.)

Exercise 1.1.15. Let

$$X := \left\{ (a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty \right\}$$

be the space of absolutely convergent sequences. Define the l^1 and l^∞ metrics on this space by

$$d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|;$$
$$d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n \in \mathbf{N}} |a_n - b_n|.$$

Let $(e^{(n)})_{n=m}^{\infty}$ be some convergent sequence in $\{e^{(n)} : n \in \mathbb{N}\}$. Assume $L \notin \{e^{(n)} : n \in \mathbb{N}\}$. Then L is some sequence with more than one non-zero element or only one non-zero element and it is not equal to 1.

Assume there is more than one non-zero element in L. Then for every $e^{(n)}$, at least one non-zero element of L is at the same position as the 1 in $e^{(n)}$, or none are. In the first case, $d_{l^1}(e^{(n)}, L) > c_n$ for some c_n, where c_n depends on the specific non-zero values, l_i, of L and the position of the 1 in $e^{(n)}$.

c_n will always equal $\sum_{i \neq j} l_i + |l_j - 1|$, where j is the position of the 1 in $e^{(n)}$. For the sequence to converge, the c_n need to be arbitrarily small, which implies that $\sum_{i \neq j} l_i$ is arbitrarily small and $|l_j - 1|$ is arbitrarily small. However, if all the l_i are arbitrarily small, then each $|l_i - 1|$ will be close to 1 and hence c_n will not be arbitrarily small. If the $|l_i - 1|$ are arbitrarily small, then each l_i is close to 1, and the c_n will not be arbitrarily small. Either way, $d_{l^1}(e^{(n)}, L)$ cannot converge to 0. In the second case, $d_{l^1}(e^{(n)}, L) > d$ where d is the sum of the l_i plus 1. This will be constant whenever L does not have any l_i and the same position as a 1 in the $e^{(n)}$. Hence, the sequence $(e^{(n)})_{n=m}^{\infty}$ cannot converge to L.

Assume then that Then for a given $e^{(n)}$, the 1 has the same position as l_i or it does not have the same position. If they have the same position, then $d_{l^1}(e^{(n)}, L) = |1 - l_i|$. If they do not have the same position, then $d_{l^1}(e^{(n)}, L) = 1 + |l_i|$. The sequence then cannot converge to L, since $d_{l^1}(e^{(n)}, L) \geq min(|1 - l_i|, 1 + |l_i|$ for all n.

Hence, $L \in \{e^{(n)} : n \in \mathbb{N}\}$, and $\{e^{(n)} : n \in \mathbb{N}\}$ is closed.

For every n, $d_{l^1}(e^{(1)}, e^{(n)}) = \sum_{j=0}^{\infty} |e_j^{(1)} - e_j^{(n)}|$, which equals 0 when $n = 1$ and 2 otherwise. Hence, $\{e^{(n)} : n \in \mathbb{N}\} \subset B(e^{(1)}, 3)$. Hence, the set is bounded.

The set $\{e^{(n)} : n \in \mathbb{N}\}$ is not compact, since in the sequence $(e^{(n)})_{n=1}^{\infty}$, $d_{l^1}(e^{(k)}, e^{(j)}) = 2$ for every $k > j$. Hence, there is no subsequence that is Cauchy, and hence no subsequence is convergent.

Exercise 1.5.9

Exercise 1.5.9. Show that a metric space (X, d) is compact if and only if every sequence in X has at least one limit point.

Say (X, d) is compact. Suppose S is some sequence in X. There we know it has a subsequence that converges to some $L \in X$. From P1.4.5 (AII), this implies that L is a limit point. Since S was arbitrary, we have shown that every sequence has at least one limit point.
Say for every sequence S in (X, d), it has some limit points. Suppose S is some sequence in X. Then we know it has a limit. From P1.4.5 (AII), we know that it has a subsequence that converges to some $L \in X$. Since S was arbitrary, we have shown every sequence has a convergent subsequence in X. This implies that (X, d) is compact.

Exercise 1.5.10

Exercise 1.5.10. A metric space (X, d) is called *totally bounded* if for every $\varepsilon > 0$, there exists a positive integer n and a finite number of balls $B(x^{(1)}, \varepsilon), \ldots, B(x^{(n)}, \varepsilon)$ which cover X (i.e., $X = \bigcup_{i=1}^{n} B(x^{(i)}, \varepsilon)$.

(a) Show that every totally bounded space is bounded.

For some $r > 0$ and n, let $X = \bigcup_{i=1}^{n} B(x^{(i)}, r)$. Then, define $m := max\{d(x^{(i)}, x^{(j)} : 1 \leq i, j \leq n\}$. Hence, $\forall 1 \leq i, j \leq n(d(x^{(i)}, x^{(j)} \leq m)$. Consider the ball, $B(x^{(1)}, m + r)$. Let $x \in X$. Then we know there is some i, such that $x \in B(x^{(i)}, r)$. Hence, $d(x, x^{(i)}) < r$. Hence, we can write $d(x, x^{(1)}) \leq d(x, x^{(i)}) + d(x^{(i)}, x^{(1)} < r + m$. Hence $x \in B(x^{(1)}, m + r)$. Since x was arbitrary, we have shown that $X \subseteq B(x^{(1)}, m + r)$. Hence, there is some ball that contains X, and X is bounded.

L has only one non-zero element and it is not equal to 1.

(b) Show the following stronger version of Proposition 1.5.5: if (X, d) is compact, then complete and totally bounded. (Hint: if X is not totally bounded, then there is some $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Then use Exercise 8.5.20 to find an infinite sequence of balls $B(x^{(n)}, \varepsilon/2)$ which are disjoint from each other. Use this to then construct a sequence which has no convergent subsequence.)

Assume (X, d) is not totally bounded. Then we can write $X \neq \bigcup_{i=1}^{n} B(x^{(i)}, \epsilon)$ for some $\epsilon > 0$ and for every $n \in \mathbb{N}$. Let 2^X denote all subsets of X. And define $:= \{B(x, \epsilon/2) : B(x, \epsilon) \subset X\}$. Obviously, $\subseteq 2^X$. Then from Exercise 8.5.20, we have $' \subseteq$, such that all $B(x, \epsilon/2) \in'$ are disjoint from each other. Constructing a sequence using the axiom of choice, it easy to show that for every n, m, $d(x^{(n)}, x^{(m)}) \geq \epsilon/2$. Then it follows that there is no convergent subsequence, contradicting the fact that (X, d) is compact.

(c) Conversely, show that if X is complete and totally bounded, then X is compact. (Hint: if $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X, use the total boundedness hypothesis to recursively construct a sequence of subsequences $(x^{(n;j)})_{n=1}^{\infty}$ of $(x^{(n)})_{n=1}^{\infty}$ for each positive integer j, such that for each j, the elements of the sequence $(x^{(n;j)})_{n=1}^{\infty}$ are contained in a single ball of radius $1/j$, and also that each sequence $(x^{(n;j+1)})_{n=1}^{\infty}$ is a subsequence of the previous one $(x^{(n;j)})_{n=1}^{\infty}$. Then show that the "diagonal" sequence $(x^{(n;n)})_{n=1}^{\infty}$ is a Cauchy sequence, and then use the completeness hypothesis.)

Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X. We define the sequence of subsequences, starting with some $j \in \mathbb{N}$. Since X is totally bounded, we can write $\exists n \in \mathbb{N}(X = \bigcup_{i=1}^{n} B(x^{(i)}, 1/j))$. Given that every $x \in X$ is in one of these balls, we can guarantee that there is some subsequence $(x^{(n;j)})_{n=1}^{\infty}$ in $B(x^{(i)}, 1/j)$ for some $x^{(i)}$. Then we define the next subsequence as $(x^{(n;j+1)})_{n=1}^{\infty}$ in $B(x^{(i)}, 1/(j+1))$ for the same $x^{(i)}$, and so on. Then, using the axiom of choice, we can construct the sequence $(x^{n;n})_{n=1}^{\infty}$, where $x^{n;n} \in (x^{n;j})_{n=1}^{\infty}$ where $j = n$.

Let $\epsilon > 0$. We know there is some N such that $\epsilon > 1/N$. Then let $n, m \geq 2$. We can write $d(x^{n:n}, x^{n:m}) \leq d(x^{n:n}, x^{(i)}) + d(x^{n:m}, x^{(i)}) < 1/n + 1/m < 1/2N + 1/2N = 1/N < 1/\epsilon$. Since ϵ was arbitrary, we have shown that the sequence $(x^{n;n})_{n=1}^{\infty}$ is Cauchy. Since X is complete, we know that the limit of $(x^{n;n})_{n=1}^{\infty}$ is in X. Since $(x^{n;n})_{n=1}^{\infty}$ is a subsequence of $(x^{(n)})_{n=1}^{\infty}$, and $(x^{(n)})_{n=1}^{\infty}$ was arbitrary, we have shown that every sequence of X has a convergent subsequence.

Exercise 1.5.11

Exercise 1.5.11. Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact. (Hint: if X is not compact, then by Exercise 1.5.9, there is a sequence $(x^{(n)})_{n=1}^{\infty}$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing x which contains at most finitely many elements of this sequence. Now use the hypothesis.)

Assume X is not compact. Then there is some sequence in X that has no limit points. It follows that for every $x \in X$, there exists a ball $B(x, \epsilon)$ that contains only a finite number of elements of the sequence. Using the axiom of choice, we can choose a ball $B(x, \epsilon_x)$. Since these balls are open and $X \subseteq \cup_{x \in X} B(x, \epsilon_x)$, we can assert that there is some finite subset $Y \subseteq X$ such that $X \subseteq \cup_{x \in Y} B(x, \epsilon_x)$. Since there are only a finite number of these balls, but the sequence is infinite, at least one of them must contain infinitely many elements of the sequence. This is a contradiction. Hence, X is compact.

Exercise 1.5.12

Exercise 1.5.12. Let (X, d_{disc}) be a metric space with the discrete metric d_{disc}.

(a) Show that X is always complete.

Suppose we have some Cauchy sequence in (X, d_{disc}). Since for every n, $d_{disc}(x^{(n)}, L) = 0$ or $d_{disc}(x^{(n)}, L) = 1$, for the sequence to be Cauchy we must have $d_{disc}(x^{(n)}, L) = 0$ for every $n \geq N$ for some N. From the definition of d_{disc}, this implies that $x^{(n)} = L$ for every $n \geq N$. Since $x^{(n)} \in X$, it immediately follows that $L \in X$. Since the sequence was arbitrary, we have shown that X is complete.

(b) When is X compact, and when is X not compact? Prove your claim. (Hint: the Heine-Borel theorem will be useless here since that only applies to Euclidean spaces.)

It is compact when it is finite. If it is not finite, then an infinite sequence of distinct elements will never have a convergent subsequence. This follows since every element will have a constant distance of 1 from each other, preventing the subsequence from being Cauchy.

Exercise 1.5.13

Exercise 1.5.13. Let E and F be two compact subsets of $\mathbf{R}$ (with the standard metric $d(x,y) = |x-y|$). Show that the Cartesian product $E \times F := \{(x,y) : x \in E, y \in F\}$ is a compact subset of $\mathbf{R}^2$ (with the Euclidean metric d_{l^2}).

Let $(x^{(k)})_{k=m}^{\infty}$ be a sequence in $E \times F$. Define $(x_E^{(k)})_{k=m}$ as the projection of the sequence on E, and $(x_F^{(k)})_{k=m}$ as the projection of the sequence on F. Since E and F are compact, then we know there exist subsequences $(x_E^{(k_j)})_{j=1}$ and $(x_F^{(k_j)})_{j=1}$ that converge to $L_E \in E$ and $L_F \in F$, respectively. From Proposition 1.1.18, this implies that the subsequence $(x^{(k_j)})_{j=1}^{\infty}$ converges to $L \in E \times F$.

Exercise 1.5.14

Exercise 1.5.14. Let (X,d) be a metric space, let E be a non-empty compact subset of X, and let x_0 be a point in X. Show that there exists a point $x \in E$ such that

$$d(x_0, x) = \inf\{d(x_0, y) : y \in E\},$$

i.e., x is the closest point in E to x_0. (Hint: let R be the quantity $R := \inf\{d(x_0,y) : y \in E\}$. Construct a sequence $(x^{(n)})_{n=1}^{\infty}$ in E such that $d(x_0, x^{(n)}) \leq R + \frac{1}{n}$, and then use the compactness of E.)

Define $R := inf\{d(x_o, y) : y \in E\}$. Then define $X_n := \{x : d(x, x_o) \leq R + 1/n\}$. If $X_n = \emptyset$, then $R + 1/n = \{d(x_o, y) : y \in E\}$, a contradiction. Hence, using the axiom of choice, we can construct a sequence $(x^{(n)})_{n=m}^{\infty}$ such that $d(x, x_o) \leq R + 1/n$ for every n. Since E is compact we know there is some subsequence, $(x^{(n_j)})_{j=1}^{\infty}$, that converges to some $x \in E$. It follows from the squeeze theorem that the sequence, $d(x^{(n_j)}, x_o)$ converges to R for $x \in E$.

Exercise 1.5.15

Exercise 1.5.15. Let (X, d) be a compact metric space. Suppose that $(K_\alpha)_{\alpha\in I}$ is a collection of closed sets in X with the property that any finite subcollection of these sets necessarily has non-empty intersection, thus $\bigcap_{\alpha\in F} K_\alpha \neq \emptyset$ for all finite $F \subseteq I$. (This property is known as the *finite intersection property*.) Show that the *entire* collection has non-empty intersection, thus $\bigcap_{\alpha\in I} K_\alpha \neq \emptyset$. Show by counterexample that this statement fails if X is not compact.

Assume $\cap_{\alpha\in I} K_\alpha = \emptyset$. Define $X_n := \{x \in \cap_{\alpha\in F} K_\alpha : (F) = n\}$ for every positive integer n. Note, that every K_α is associated with some n where it is added to the intersection $\cap_{\alpha\in F} K_\alpha$ for the first time. We can then construct a sequence using the axiom of choice, $(x^{(n)})_{n=1}^{\infty}$, where $x_n^{(n)}$. Since this subsequence is in X and X is compact, there is some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ that converges to some $L \in X$. Suppose $L \notin \cap_{\alpha\in I} K_\alpha$. Then there is some K_β such that $L \notin K_\beta$. Let m be the associated positive integer where K_β is added to the intersection $\cap_{\alpha\in F} K_\alpha$ for the first time when constructing the X_n. Let k be the first position of the subsequence such that $n_k > m$. It follows that $(x^{(n_j)})_{j=k}^{\infty}$ also converges to L. Since for every $j > k$, $x^{(n_j)} \in K_\beta$, it follows that $(x^{(n_j)})_{j=k}^{\infty}$ is in K_β. Since X is compact and K_β is closed, it follows from Theorem 1.5.10 (AII) that K_β is compact. Since it is compact, it is complete, Proposition 1.5.5 (AII). Hence, $L \in K_\beta$, a contradiction. Hence, $\cap_{\alpha\in I} K_\alpha \emptyset$.