Chapter 3

Section 1

Exercise 3.1.1

Exercise 3.1.1. Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f : E \to Y$ be a function, and let x_0 be an element of E. Show that the limit $\lim_{x \to x_0; x \in E} f(x)$ exists if and only if the limit $\lim_{x \to x_0; x \in E \setminus \{x_0\}} f(x)$ exists and is equal to $f(x_0)$. Also, show that if the limit $\lim_{x \to x_0; x \in E} f(x)$ exists at all, then it must equal $f(x_0)$.

Assume $\lim\limits_{x \to x_o; x \in E} f(x)$ exists. Then, we know there is some L such that for every $\epsilon > 0$ there exists some $\delta > 0$ such that for every $x \in E$ $d_Y(f(x), L) < \epsilon$ whenever $d_X(x, x_o) < \delta$. Let $\epsilon > 0$ be arbitrary, and assume by contradiction that for every $\delta > 0$ there is some $x \in E \setminus \{x_o\}$ such that $d_X(x, x_o) < \delta)$ and $\neg d_Y(f(x), L)$. Since $x \in E \setminus \{x_o\}$ implies $x \in E$, the contradiction follows easily. Hence, we have shown that the limit $\lim\limits_{x \to x_o; x \in E \setminus \{x_o\}} f(x)$ exists and is equal to $\lim\limits_{x \to x_o; x \in E} f(x)$.
#contradiction

Assume $\lim\limits_{x \to x_o; x \in E} f(x) = L$. Assume by contradiction that $L \neq f(x_o)$. Then by defining $\epsilon' = d_Y(f(x_o), L)$, we can show that for every $\epsilon \in E$, $d_Y(f(x), L) < d_Y(f(x_o, L)$ whenever $d_X(x, x_o)$. Since $x_o \in E$, and $d_X(x_o, x_o) = 0 < \delta$, it follows that $d_Y(f(x_o), L) < d_Y(f(x_o, L)$, which is a contradiction. Hence, $L = f(x_o)$.
#contradiction #adhocdefinitions

Assume that $\lim_{x \to x_o; x \in E \setminus \{x_o\}} f(x) = f(x_o)$. Let $\epsilon > 0$ be arbitrary. Then assume by contradiction that for every $\delta > 0$ there is some $x \in E$ such that $d_X(x, x_o) < \delta$ and $d_Y(f(x), f(x_o)) \geq \epsilon$. Since $x \in E$, either $x = x_o$ or $x \neq x_o$. If $x = x_o$, then we have $d_X(x_o, x_o) = 0 < \delta$ and $d_Y(f(x_o), f(x_o)) = 0 \geq \epsilon > 0$, which is a contradiction. If $x \neq x_o$, then the contradiction follows in a similar manner to the first part of this proof. Hence, we have shown the desired result.
#contradiction

Exercise 3.1.2

Exercise 3.1.2. Prove Proposition 3.1.5. (Hint: review your proof of Theorem 2.1.4.)

Proposition 3.1.5. *Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following four statements are logically equivalent:*

(a) $\lim_{x \to x_0; x \in E} f(x) = L$.

(b) *For every sequence $(x^{(n)})_{n=1}^\infty$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^\infty$ converges to L with respect to the metric d_Y.*

(c) *For every open set $V \subset Y$ which contains L, there exists an open set $U \subset X$ containing x_0 such that $f(U \cap E) \subseteq V$.*

(d) *If one defines the function $g : E \cup \{x_0\} \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \setminus \{x_0\}$, then g is continuous at x_0. Furthermore, if $x_0 \in E$, then $f(x_0) = L$.*

(a) ⇒ (b)
Let $(x^{(n)})_{n=1}^{\infty}$ converge to x_o. Then we know that for every $\epsilon > 0$ there is some $N \geq m$ such that for every $n \geq N$, $d_X(x^{(n)}, x_o) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to x_o; x \in E} f(x) = L$, we know that for some $\delta > 0$, $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < \epsilon$. Hence, we can write that there is some $N \geq m$ such that for every $n \geq N$, $d_X(x^{(n)}, x_o) < \delta$. It follows easily that for every $n \geq N$, $d_X(f(x^{(n)}), f(x_o)) < \epsilon$. The desired result follows.

(a) ⇒ (c)
Let $V \subset Y$ be open and $L \in V$. Since V is open, we know that $B(L, r) \subseteq V$ for some r. Hence, using (a), we can assert that $\forall x \in E(d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < r$ for some δ. Define $U := B(x_o, \delta) \subset X$. Let $y \in f(U \cap E)$. Then we know $f(x) = y$ for some $x \in U \cap E$. Hence, $d_X(x, x_o) < \delta$ and $d_Y(y, f(x_o)$. Hence, $y \in B(L, r)$, and since y was arbitrary, $f(U \cap E) \subseteq V$. Also, if $x_o \in E$, then… The claim follows.

(a) ⇒ (d)
Let $\epsilon > 0$. Then we know $\forall x \in E(d_X(x, x_o) < \delta \to d_Y(f(x), L) < \epsilon)$ for some δ. Suppose $x \in E \cup \{x_o\}$. Then $x \in E$ or $x \in \{x_o\}$. If $x \in E$, then we can write $f(x) = g(x)$, and $g(x_o) = L$. Hence, since $d_X(x, x_o) < \delta \to d_Y(f(x), L) < \epsilon$, it follows that $d_X(x, x_o) < \delta \to d_Y(g(x), g(x_o) < \epsilon$. If instead $x \in \{x_o\}$, then we can write $g(x) = g(x_o) = L$, and since $d_Y(g(x_o), g(x_o)) = 0 < \epsilon$, it follows immediately that $d_X(x, x_o) < \delta \to d_Y(g(x), g(x_o) < \epsilon$.
Since $x \in E$. Then we know $\forall \epsilon > 0 \exists \delta (d_X(x, x_o) < \delta \to d_Y(f(x_o), L) < \epsilon$. We can write then $\forall \epsilon > 0 (d_X(x, x_o) < \delta_\epsilon \to d_Y(f(x_o), L) < \epsilon$. Letting $\epsilon < 0$ be arbitrary, it follows immediately that $d_X(x_o, x_o) = 0 < \delta_\epsilon$. Hence, $d_Y(f(x_o), L) < \epsilon$. Since ϵ was arbitrary, it follows that $f(x_o) = L$.
The claim follows.

(d) ⇒ (a)
Suppose $\epsilon > 0$. Then from (d) we know that $d_X(x, x_o) < \delta \to d_Y(g(x), g(x_o)) < \epsilon$ for some δ. Let $x \in E$. Then $x = x_o$ or $x \neq x_o$. Suppose $x \neq x_o$. If $d_X(x, x_o) < \delta$, we have $d_Y(g(x), g(x_o)) < \epsilon$. Since $x \neq x_o$, we have $g(x) = f(x)$ and $g(x_o) = L$. It follows that $d_Y(f(x), L) < \epsilon$. Suppose then that $x = x_o$. If $d_X(x, x_o) < \delta$, we can immediately write $d_Y(g(x_o), g(x_o)) < \epsilon$. Since $x = x_o$, from (d), we can write $f(x_o) = L$ and $f(x_o) = g(x_o)$. Hence $d_Y(f(x_o), L)$. The claim follows.

(b) ⇒ (a)
Assume by contradiction that the limit of $f(x) \neq L$. Then there is some $\epsilon > 0$ such that for every $\delta > 0$ there is some $x \in E$ such that $d_X(x, x_o) < \delta$ and $d_Y(f(x), L) \geq \epsilon$. Using the axiom of choice, it easily follows that the sequence $(x^{(n)})_{n=1}^{\infty}$ converges to x_o, since $d_X(x, x_o) < 1 < \delta$ for every δ. Using (b), we can assert that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to L. A contradiction follows easily since we have $d_Y(f(x^{(n)}), L) \geq \epsilon$ for every n.

(c) ⇒ (b)

Let $(x^{(n)})_{n=1}^{\infty}$ is a sequence in E that converges to x_o. Assume by contradiction that $(f(x^{(n)}))_{n=1}^{\infty}$ does not converge to L. Then we know there is some $\epsilon > 0$ such that for every $N \geq m$ there is some $n \geq N$ such that $d_Y(f(x^{(n)}, L) > \epsilon$. Defining $V := B(L, \epsilon)$, which is open and contains L, we can assert, using (c), that there is some open $U \subset X$ that contains x_o such that $f(U \cap E) \subseteq V$. Hence, we can assert that there is some $\delta > 0$ such that $B(x_o, \delta) \subseteq U$, since U is open. Since $(x^{(n)})_{n=1}^{\infty}$ converges to x_o, we can write that there is some $N \geq m$ such that $d_X(x^{(n)}, x_o) \leq \delta$ for every $n \geq N$. It follows easily that there is some $N \geq m$ such that $x^{(n)} \in U \cap E$ for every $n \geq N$. From what we asserted before, we can write there is some $N \geq m$ such that $f(x^{(n)}) \in V$ for every $n \geq N$, which is a contradiction.

Exercise 3.1.3

Exercise 3.1.3. Use Proposition 3.1.5(c) to define a notion of a limiting value of a function $f : X \to Y$ from one topological space $(X, \mathcal{F}_X)$ to another $(Y, \mathcal{F}_Y)$. Then prove the equivalence of Proposition 3.1.5(c) and 3.1.5(d). If in addition Y is a Hausdorff topological space (see Exercise 2.5.4), prove an analogue of Remark 3.1.6. Is the same statement true if Y is not Hausdorff?

(c) *For every open set $V \subset Y$ which contains L, there exists an open set $U \subset X$ containing x_0 such that $f(U \cap E) \subseteq V$.*

(d) *If one defines the function $g : E \cup \{x_0\} \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \backslash \{x_0\}$, then g is continuous at x_0. Furthermore, if $x_0 \in E$, then $f(x_0) = L$.*

Remark 3.1.6. Observe from Proposition 3.1.5(b) and Proposition 1.1.20 that a function $f(x)$ can converge to at most one limit L as x converges to x_0. In other words, if the limit

$$\lim_{x \to x_0; x \in E} f(x)$$

exists at all, then it can only take at most one value.

A topological definition of 3.1.5(c):
Let X and Y be topological spaces. For every neighbourhood V of L in Y, there exists a neighbourhood U of x_o in X such that $f(U \cap E) \subseteq V$.

(c) ⇒ (d)
Let V be a neighbourhood of $g(x_o)$. From the definition of g we know $g(x_o) = L$. Hence, using (c), we know there is some neighbourhood U of x_o such that $f(U \cap E) \subseteq V$. Suppose $g(x) \in g(U)$. If $x = x_o$, then we know $g(x) \in V$ since $g(x) = g(x_o) = L \in V$. If $x \neq x_o$, then $g(x) = f(x)$ and since $x \in U \cap E$, by definition of g, we have $f(x) \in V$. Hence, $g(U) \subseteq V$ and g is continuous from Definition 2.5.8.

(d) ⇒ (c)
Let V be a neighbourhood of L. From (d), we know it is a neighbourhood of $g(x_o)$. Since g is continuous at x_o, we know that there is some neighbourhood U of x_o such that $g(U) \subseteq V$. Let $y \in f(U \cap E)$. Then we have $y = f(x)$ where $x \in U \cap E$. If $x = x_o$, then $f(x) = g(x_o) = L \in V$. If $x \neq x_o$, then $f(x) = g(x) \in V$ since g is continuous and $g(U) \subseteq V$.

Suppose we have L and L' such that $L \neq L'$. Also suppose that for every neighbourhood $V \subset Y$ that contains L, there is some $U_1 \subset X$ that contains x_o such that $f(U \cap E) \subseteq V$. Additionally, suppose that for every neighbourhood $W \subset Y$ that contains L', there is some $U_2 \subset X$ that contains x_o such that $f(U \cap E) \subseteq W$.
Since the topology is Hausdorf we know there are disjoint V and W that are neighbourhoods of L and L', respectively. Hence, we can write $f(U_1 \cap E) \subseteq V$ and $f(U_2 \cap E) \subseteq W$. Since U_1 we know there is some neighbourhood K of x_o such that $K \subseteq U_1$. Similarly, we know there is some neighbourhood J of x_o such that $J \subseteq U_2$, since U_2 is open. Since x_o is an adherent point of E, we know that for every neighbourhood K of x_o, $K \cap E \neq \emptyset$. Hence, from writing $K \subseteq U_1$ and $K \cap E \neq \emptyset$, we can assert $K \subseteq U_1 \cap E$. Similarly, we can write $J \subseteq U_2 \cap E$. Since $x_o \in K \cap J$, it follows that $(U_1 \cap E) \cap (U_2 \cap E) \neq \emptyset$, and consequently $f(U_1 \cap E) \cap f(U_2 \cap E) \neq \emptyset$. However, this is a contradiction since V and W are disjoint and $f(U_1 \cap E) \subseteq V$ and $f(U_2 \cap E) \subseteq W$. Hence, $L = L'$.

Exercise 3.1.4

Exercise 3.1.4. Recall from Exercise 2.5.5 that the extended real line $\mathbf{R}^*$ comes with a standard topology (the order topology). We view the natural numbers $\mathbf{N}$ as a subspace of this topological space, and $+\infty$ as an adherent point of $\mathbf{N}$ in $\mathbf{R}^*$. Let $(a_n)_{n=0}^{\infty}$ be a sequence taking values in a topological space $(Y, \mathcal{F}_Y)$, and let $L \in Y$. Show that $\lim_{n \to +\infty; n \in \mathbf{N}} a_n = L$ (in the sense of Exercise 3.1.3) if and only if $\lim_{n \to \infty} a_n = L$ (in the sense of Definition 2.5.4). This shows that the notions of limiting values of a sequence, and limiting values of a function, are compatible.

⇒

Assume $lim_{n\to+\infty;n\in\mathbb{B}}a_n = L$ in the sense of Exercise 3.1.3. Then we know for every neighbourhood V of L, there is some neighbourhood U of $+\infty$ such that $f(U\cap\mathbb{N})\subseteq V$. Since also $+\infty$ is an adherent point of $\mathbb{B}$, we know that $U\cap\mathbb{N}\neq\emptyset$, and hence there is some $N\in\mathbb{N}$. Let $n\geq N$. Since U is open and contains $+\infty$, given the definition of order topology in Exercise 2.5.5, we see that U is of the form $\{y\in U : a<y\}$ for some $a\in U$. Hence, since $N\in U$ it follows immediately that $n\in U$. Then we can assert that $f(n)=a_n\in V$ since $f(U\cap\mathbb{N})\subseteq V$.

⇐

Assume $lim_{n\to\infty}a_n = L$ in the sense of Exercise 2.5.4. Then we know there is a neighbourhood V of L such that $a_n\in V$ for every $n\geq N$ for some $N\geq m$. Consider then $U:=\{y\in\mathbb{N} : N<y\}$. From the definition of order topology in Exercise 2.5.5, U is open. It follows immediately that $f(U)=f(U\cap\mathbb{N})\subseteq V$ since $a_n\in V$ for every n.

Exercise 3.1.5

Exercise 3.1.5. Let (X,d_X), (Y,d_Y), (Z,d_Z) be metric spaces, and let $x_0\in X$, $y_0\in Y$, $z_0\in Z$. Let $f: X\to Y$ and $g: Y\to Z$ be functions, and let E be a set. If we have $\lim_{x\to x_0;x\in E} f(x)=y_0$ and $\lim_{y\to y_0;y\in f(E)} g(y)=z_0$, conclude that $\lim_{x\in x_0;x\in E} g\circ f(x)=z_0$.

Let $\epsilon>0$. Then by assumption, we know that for every $y\in f(Y)$ there is some $_1$ such that if $d_Y(y,y_o)<\delta_1\to d_Z(g(y),z_o)<\epsilon$. Hence, we can assert that there is some δ_2 such that for every $x\in E$ if $d_X(x,x_o)<\delta_2\to d_Y(f(x),y_o)<\delta_1$. Since $f(x)=y$ for some $y\in Y$, it follows easily that for every $x\in E$ if $d_X(x,x_o)<\delta_2\to d_Z(g(f(x)),z_o)<\delta_1$.

Exercise 3.1.6

Exercise 3.1.6. State and prove an analogue of the limit laws in Proposition 9.3.14 when X is now a metric space rather than a subset of $\mathbf{R}$. (Hint: use Corollary 2.2.3.)

Proposition 9.3.14 (Limit laws for functions). *Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Suppose that f has a limit L at x_0 in E, and g has a limit M at x_0 in E. Then $f + g$ has a limit $L + M$ at x_0 in E, $f - g$ has a limit $L - M$ at x_0 in E, $\max(f, g)$ has a limit $\max(L, M)$ at x_0 in E, $\min(f, g)$ has a limit $\min(L, M)$ at x_0 in E and fg has a limit LM at x_0 in E. If c is a real number, then cf has a limit cL at x_0 in E. Finally, if g is non-zero on E (i.e., $g(x) \neq 0$ for all $x \in E$) and M is non-zero, then f/g has a limit L/M at x_0 in E.*

Corollary 2.2.3. *Let (X, d) be a metric space, let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Let c be a real number.*

(a) *If $x_0 \in X$ and f and g are continuous at x_0, then the functions $f + g : X \to \mathbf{R}$, $f - g : X \to \mathbf{R}$, $fg : X \to \mathbf{R}$, $\max(f, g) : X \to \mathbf{R}$, $\min(f, g) : X \to \mathbf{R}$, and $cf : X \to \mathbf{R}$ (see Definition 9.2.1 for definitions) are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbf{R}$ is also continuous at x_0.*

First, define $f' : E \cup \{x_o\} \to X$ as $f'(x_o) = L$ and $f'(x) = f(x), x \in E \setminus \{x_o\}$. Similarly, define $g' : E \cup \{x_o\} \to X$ as $g'(x_o) = M$ and $g'(x) = g(x), x \in E \setminus \{x_o\}$. From Proposition 3.1.5, it follows that f' and g' are continuous. From Corollary 2.2.3, it follows that $f' + g'$ is continuous. $f' + g'$ also has the property that $(f' + g')(x_o) = L + M$ and $(f' + g')(x) = (f + g)(x)$, hence, using Proposition 3.1.5 again, we can assert that $lim_{x \to x_o; x \in E}(f + g)(x) = L + M$.

The other statements can be proved similarly.

Section 2

Exercise 3.2.1

Exercise 3.2.1. The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence. Let $f : \mathbf{R} \to \mathbf{R}$ be a function. For any $a \in \mathbf{R}$, let $f_a : \mathbf{R} \to \mathbf{R}$ be the shifted function $f_a(x) := f(x-a)$.

(a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

Suppose f is continuous. Then we know for every $\epsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Since $(a_n)^{n=0}$ converges to 0, we can assert that there is some $N \geq 0$ such that for every $n \geq N$, $|a_n| < \delta$. Hence, taking $y = x - a_n$, it follows easily that $|f(x) - f(x-a_n)| < \epsilon$. This shows the result.

Assume by contradiction that f is not continuous. Then we know there is some $x_o \in X$ and some $\epsilon > 0$ such that for every $\delta > 0$ there is some $x \in X$ such that $|x - x_o| < \delta$ and $|f(x) - f(x_o)| \geq \epsilon$. Hence, using the Axiom of Choice, we can write $\forall \delta > 0 |x_\delta - x_o| < \delta$. We then construct a sequence $(a_n)_{n=0}^{\infty}$, where $a_n := x_o - x_{1/n}$. Obviously, $-1/n < a_n < 1/n$, and hence it follows that the sequence converges to 0. However, taking any $N > 0$, it follows that $|f(x_o - a_n) - f(x_o)| = |f(x_{1/n}) - f(x_o)| \geq \epsilon$, which shows that f_{a_n} does not converge pointwise to f, which is a contradiction. Hence, f is continuous.

(b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^{\infty}$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

Assume f is uniformly continuous. Let $\epsilon > 0$. Then we know there is some $\delta > 0$ such that for every $x, x' \in X$, if $d_X(x, x') < \delta$ then $d_Y(f^{(n)}(x), f(x')) < \epsilon$. Assuming $(a_n)^{n=0}$ converges to 0, we know there is some $N \geq 0$ such that for every $n \geq N$, $|a_n| < \delta$. Let $x \in X$ and $n > N$. Define $x' = x - a_n$, then $d_X(x, x') = |x - x + a_n| = |a_n| < \delta$. Hence, $d_Y(f^{(n)}(x), f(x - a_n) < \epsilon$. This shows that f_{a_n} converges uniformly to f.

Assume f_{a_n} converges uniformly to f, whenever $(a_n)_{n=0}^{\infty}$ converges to 0. Assume that f is not uniformly continuous. Then there is some $\epsilon > 0$ such that for every $\delta > 0$ there is some $x, x' \in X$ such that $|x - x'| < \delta$ and $|f(x) - f(x')| \geq \epsilon$. Using the Axiom of Choice, we can write that for every $\delta > 0$, $|x_\delta - x'_\delta| < \delta$ and $|f(x_\delta) - f(x'_\delta)| \geq \epsilon$. Hence, defining

$a_n := x_{1/n} - x'_{1/n}$, it follows that $(a_n)_{n=0}^{\infty}$ converges to 0. However, it also follows that $|f(x_{1/N}) - f(x'_{1/N})| = |f(x_{1/N}) - f(x_{1/N} - a_N)| \geq \epsilon$ for every N, and hence f_{a_n} does not converge uniformly to f. Hence, f is uniformly continuous

Exercise 3.2.2

Exercise 3.2.2. (a) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function from X to Y. Show that if $f^{(n)}$ converges uniformly to f, then $f^{(n)}$ also converges pointwise to f.

Suppose $f^{(n)}$ converges uniformly to f. Take some $x \in X$ and $\epsilon > 0$. Then we know there is some $N > 0$ such that for every $n > N$ and every $x' \in X$, $d_Y(f^{(n)}(x'), f(x')) < \epsilon$, since $f^{(n)}$ converges uniformly to f. Take some $n > N$, then it easily follows that $d_Y(f^{(n)}(x), f(x)) < \epsilon$. This is enough to show that $f^{(n)}$ converges pointwise to f.

(b) For each integer $n \geq 1$, let $f^{(n)} : (-1, 1) \to \mathbf{R}$ be the function $f^{(n)}(x) := x^n$. Prove that $f^{(n)}$ converges pointwise to the zero function 0, but does not converge uniformly to any function $f : (-1, 1) \to \mathbf{R}$.

Let $x \in (-1, 1)$. Then since $|x| < 1$, we know that $\lim_{n\to\infty} x^n = 0$ from Lemma 6.5.2 (AI). Since $f^{(n)}(x) = x^n$, and the zero function equals 0 for every x, it follows that $f^{(n)}$ converges pointwise to the zero function.

Suppose $f^{(n)}$ converges uniformly to some function, f. From (a), we know that it also converges pointwise to this function. Since the limit is unique, we know that this function is the zero function as well. Define $c := (1/2)^{1/n}$, then for every n, $|f^{(n)}(c) - f(c)| = 1/2$, and it follows that $f^{(n)}$ does not converge uniformly to f.

(c) Let $g : (-1, 1) \to \mathbf{R}$ be the function $g(x) := x/(1-x)$. With the notation as in (b), show that the partial sums $\sum_{n=1}^{N} f^{(n)}$ converges pointwise as $N \to \infty$ to g, but does not converge uniformly to g, on the open interval $(-1, 1)$. (Hint: use Lemma 7.3.3.) What would happen if we replaced the open interval $(-1, 1)$ with the closed interval $[-1, 1]$?

From Lemma 7.3.3, it follows that the partial sums $\sum_{n=0}^{N} f^{(n)}$ converge to $1/(1-x)$. Since $\sum_{n=1}^{N} f^{(n)} = \sum_{n=0}^{N} f^{(n)} - 1$, we have $\sum_{n=0}^{N} f^{(n)} = x/(1-x) = g(x)$.

Using the geometric series formula, we have $\sum_{n=1}^{N} f^{(n)} = (x - x^{N+1}/(1-x)$. Hence, $|\sum_{n=1}^{N} f^{(n)} - g(x)| = |(x - x^{N+1}/(1-x) - x/(1-x)| = | - x^{N+1}/(1-x)|$. If we take $c = (1/2)^{1/(N+1)}$, then $| - x^{N+1}/(1-x)| = (1/2)/(1-(1/2)^{1/(N+1)} > 1/2$. Hence, the $\sum_{n=1}^{N} f^{(n)}$ does not converge uniformly to g.

If we use the closed interval $[-1, -1]$, then we can no longer apply Lemma 7.3.3 at the points $x = 1, -1$, and hence the sequence would not converge pointwise.

Exercise 3.2.3

Exercise 3.2.3. Let (X, d_X) a metric space, and for every integer $n \geq 1$, let $f_n : X \to \mathbf{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbf{R}$ on X (in this question we give $\mathbf{R}$ the standard metric $d(x, y) = |x - y|$). Let $h : \mathbf{R} \to \mathbf{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbf{R}$ is the function $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.

Let $x \in X$. Let $\epsilon > 0$. We have $f(x) \in \mathbb{R}$. Since h is continuous, we know that there is some δ such that for every $y \in \mathbb{R}$, if $|f(x) - y| < \delta$ then $|h(f(x)) - h(y)| < \epsilon$. Since f_n converges pointwise to f, we can write that there is some $N > 0$ such that for every $n \geq N$, $|f_n(x) - f(x)| < \delta$. Since $f_n(x) \in \mathbb{R}$, then it follows that $|h(f(x)) - h(f_n(x))| < \epsilon$. This is enough to show the result.

Exercise 3.2.4

Exercise 3.2.4. Let $f_n : X \to Y$ be a sequence of bounded functions from one metric space (X, d_X) to another metric space (Y, d_Y). Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function; i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$. Show that the sequence f_n is *uniformly bounded*; i.e. there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f_n(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$ and all positive integers n.

Since f is bounded, we know there is some y_o and some $R > 0$ such that $f(x) \in B_{(y,d_Y)}(y_o, R)$ for every $x \in X$. Since f_n converges to f, we know that there is some $N > 0$ such that for every $x \in X$ and every $n \geq N$, $d_Y(f(x), f_n(x)) < 1$. It follows that for every $x \in X$ and every $n \geq N$, we can write $d_Y(f_n(x), y_o) \leq d_Y(f_n(x), f(x)) + d_Y(y_o, f(x)) < 1 + R$.

Since all f_n are bounded, we know that for every $1 \leq i < N$, there is some $y_i \in Y$ and some $R_i > 0$ such that $f_i(x) \in B_{(y,d_Y)}(y_i, R_i)$ for every $x \in X$. For each i, we can write $d_Y(f_i(x), y_o) \leq d_Y(f_i(x), y_i) + d_Y(y_o, y_i) < R_i + d_Y(y_o, y_i)$ for every $x \in X$. Hence, we define $r := max_{1 \leq i < N} R_i + max_{1 \leq i < N} d_Y(y_o, y_i)$. Then we can write that for every $x \in X$ and every $n \in N$, $d_Y(f_n(x), y_o) < 1 + R + r$. This shows the result.

#sheercreativity

Section 3

Exercise 3.3.1

Exercise 3.3.1. Prove Theorem 3.3.1. Explain briefly why your proof requires uniform convergence, and why pointwise convergence would not suffice. (Hints: it is easiest to use the "epsilon-delta" definition of continuity from Definition 2.1.1. You may find the triangle inequality

$$\begin{aligned} d_Y(f(x), f(x_0)) \leq & d_Y(f(x), f^{(n)}(x)) + d_Y(f^{(n)}(x), f^{(n)}(x_0)) \\ & + d_Y(f^{(n)}(x_0), f(x_0)) \end{aligned}$$

useful. Also, you may need to divide ε as $\varepsilon = \varepsilon/3 + \varepsilon/3 + \varepsilon/3$. Finally, it is possible to prove Theorem 3.3.1 from Proposition 3.3.3, but you may find it easier conceptually to prove Theorem 3.3.1 first.)

Theorem 3.3.1 (Uniform limits preserve continuity I). *Suppose $(f^{(n)})_{n=1}^{\infty}$ is a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let x_0 be a point in X. If the functions $f^{(n)}$ are continuous at x_0 for each n, then the limiting function f is also continuous at x_0.*

Let $\epsilon > 0$. Since $f^{(n)}$ converges uniformly, we know there is some $N > 0$ such that for every $x \in X$ and for every $n \geq N$, $d_Y(f^{(n)}(x), f(x)) < \epsilon/3$. Since $f^{(N)}$ is continuous, we can write that there is some $\delta > 0$ such that for every $x \in X$, if $d_X(x, x_o) < \delta$, then $d_Y(f^{(N)}(x), f^{(N)}(x_o)) < \epsilon/3$. Let $x \in X$ and let $d_X(x, x_o) < \delta$. Then it follows that

$$d_Y(f(x), f(x_o)) \leq d_Y(f(x), f^{(N)}(x)) + d_Y(f^{(N)}(x), f^{(N)}(x_o)) + d_Y(f^{(N)}(x_o), f(x_o)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

.

This is enough to show the result.
#chainingthemetrics

We require uniform convergence because we need to choose the one N to make f and $f^{(N)}$ $\epsilon/3$-close. See Example 3.2.4 for a case where a sequence of continuous functions does not preserve continuity under pointwise convergence.

Exercise 3.3.2

Exercise 3.3.2. Prove Proposition 3.3.3. (Hint: this is very similar to Theorem 3.3.1. Theorem 3.3.1 cannot be used to prove Proposition 3.3.3, however it is possible to use Proposition 3.3.3 to prove Theorem 3.3.1.)

Proposition 3.3.3 (Interchange of limits and uniform limits). *Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete, and let E be a subset of X. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from E to Y, and suppose that this sequence converges uniformly in E to some function $f : E \to Y$. Let $x_0 \in X$ be an adherent point of E, and suppose that for each n the limit $\lim_{x \to x_0; x \in E} f^{(n)}(x)$ exists. Then the limit $\lim_{x \to x_0; x \in E} f(x)$ also exists, and is equal to the limit of the sequence $(\lim_{x \to x_0; x \in E} f^{(n)}(x))_{n=1}^{\infty}$; in other words we have the interchange of limits*

$$\lim_{n \to \infty} \lim_{x \to x_0; x \in E} f^{(n)}(x) = \lim_{x \to x_0; x \in E} \lim_{n \to \infty} f^{(n)}(x).$$

Let $\epsilon > 0$. Since $f^{(n)}$ converge uniformly to f, we can write that there is some $N > 0$ such that for every $x \in E$ and every $n \geq N$, $d_Y(f^{(n)}(x), f(x)) < \epsilon/4$. Let $n, m > N$. For every n, define limit $L_n := \lim_{x \to x_o; x \in E} f^{(n)}(x)$. Hence, We can write for some $\delta_1 > 0$ and some $\delta_2 > 0$, for every $x \in E$, $d_X(x, x_o) < \delta_1 \to d_Y(f^{(n)}(x), L_n) < \epsilon/4$, and $d_X(x, x_o) < \delta_2 \to d_Y(f^{(m)}(x), L_n) < \epsilon/4$. By choosing some x such that $d(x, x_o) < min(\delta_1, \delta 2)$, it follows that

$d_Y(L_n, L_m) \leq d_Y(L_n, f^{(n)}(x)) + d_Y(f^{(n)}(x), f(x)) + d_Y(f(x), f^{(m)}(x)) + d_Y(L_m, f^{(m)}(x)) \leq \epsilon/4 + \epsilon\ 4 + \epsilon/4 + \epsilon/4$. Since this does not depend on x or the δ_1 and δ_2, it follows that $(L_n)_{n=1}^{\infty}$ is Cauchy. Since Y is complete, we know that the sequence converges to some L.
#closingtheexistential

Let $\epsilon > 0$. Since the L_n converge, we can write that there is some $N_1 > 0$ such that for every $n \geq N$, $d_Y(L_n, L) \leq \epsilon/3$. Since the $f^{(n)}$ converge uniformly, we know there is some N_2 such that for every $x \in E$ and every $n \geq N_2$, $d_Y(f^{(n)}(x), f(x)) < \epsilon/3$. Taking $M := max(N_1, N_2)$, we know there is some $\delta > 0$ such that for every $x \in E$, if $d_X(x, x_o) < \delta$ then $d_Y(f^(M), L_M) < \epsilon/3$. Hence, taking $x \in E$ and $d_X(x, x_o) < \delta$, we can write

$d_Y(f(x), L) \leq d_Y(f^{(M)}(x), L_M) + d_Y(L_M, L) + d_Y(f^{(M)}(x), f(x)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

Hence, the result follows.
#chainingthemetrics

Exercise 3.3.3

Exercise 3.3.3. Compare Proposition 3.3.3 with Example 1.2.8. Can you now explain why the interchange of limits in Example 1.2.8 led to a false statement, whereas the interchange of limits in Proposition 3.3.3 is justified?

Example 1.2.8 (Interchanging limits, again)**.** Consider the plausible looking statement

$$\lim_{x \to 1^-} \lim_{n \to \infty} x^n = \lim_{n \to \infty} \lim_{x \to 1^-} x^n$$

where the notation $x \to 1^-$ means that x is approaching 1 from the left. When x is to the left of 1, then $\lim_{n\to\infty} x^n = 0$, and hence the left-hand side is zero. But we also have $\lim_{x\to 1^-} x^n = 1$ for all n, and so the right-hand side limit is 1. Does this demonstrate that this type of limit interchange is always untrustworthy? (See Proposition 11.15.3 for an answer.)

The problem is that x^n does not converge uniformly on $[0, 1]$, so the interchange of limits does not apply.

Exercise 3.3.4

Exercise 3.3.4. Prove Proposition 3.3.4. (Hint: again, this is similar to Theorem 3.3.1 and Proposition 3.3.3, although the statements are slightly different, and one cannot deduce this directly from the other two results.)

Proposition 3.3.4. *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of continuous functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let $x^{(n)}$ be a sequence of points in X which converge to some limit x. Then $f^{(n)}(x^{(n)})$ converges (in Y) to $f(x)$.*

From Corollary 3.3.2, we know that f is continuous. Let $\epsilon > 0$. Since f is continuous we know there is some $\delta > 0$ such that for every $x \in X$, if $d_X(x, x') < \delta$, then $d_Y(f(x), f(x')) < \epsilon/2$. Since the $f^{(n)}$ converge uniformly to f, we know that there is some N_1 such that for every $x \in X$ and every $n \geq N_1$, $d_Y(f^{(n)}(x), f(x)) < \epsilon/2$. Since the $x^{(n)}$ converge to x, we know that there is some $N_2 > 0$ such that for every $n \geq N_2$, $d_X(x^{(n)}, x') < \delta$. Hence, taking $M := max(N_1, N_2)$, let $n \geq M$. Then we know $d_X(x^{(n)}, x') < \delta$, which implies $d_Y(f^{(n)}(x), f(x')) < \epsilon$. We know also know that $d_Y(f(x^{(n)}), f(x')) < \epsilon$. Hence, it follows that

$$d_Y(f^{(n)}(x^{(n)}), f(x)) \leq d_Y(f^{(n)}(x), f(x')) + d_Y(f(x^{(n)}), f(x')) < \epsilon/2 + \epsilon/2 = \epsilon.$$

The result follows.

Exercise 3.3.5

Exercise 3.3.5. Give an example to show that Proposition 3.3.4 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given earlier will already work here.)

Let $f^{(n)} : [0, 1] \to \mathbb{R}$ be defined as $f^{(n)} = x^n$. It converges pointwise to $f(x) = 1$ when $x = 1$ and $f(x) = 0$ when $0 \leq x < 1$. Take the sequence $x^{(n)} = (1/2)^{1/n}$, which converges to 1. Since $f(1) = 1$ but $f^{(n)}(x^{(n)}) = 1/2$ for all n, it follows that Proposition 3.3.4 fails.

Exercise 3.3.6

Exercise 3.3.6. Prove Proposition 3.3.6. Discuss how this proposition differs from Exercise 3.2.4.

Proposition 3.3.6 (Uniform limits preserve boundedness). *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are bounded on X for each n, then the limiting function f is also bounded on X.*

Since the $f^{(n)}$ converge uniformly to f, we know that for every $\epsilon > 0$ there is some $N \geq 0$ such that for every $x \in X$ and every $n \geq N$, $d_Y(f^{(n)}, f(x)) < \epsilon$. Hence, choose $\epsilon = 1$. Since the $f^{(n)}$ are bounded, we know that there is some $y_o \in Y$ and some $R > 0$ such that for every $x \in X$, $d_Y(f^{(N)}(x), y_o) < R$. Hence, for every $x \in X$, we can write $d_Y(f(x), y_o) \leq d_Y(f(x), f^{(N)}(x)) + d_Y(f^{(N)}(x), y_o) < 1 + R$. The claim follows.

In Exercise 3.2.4, we assume that the limiting function is bounded, and we show that the $f^{(n)}$ are uniformly bounded; however, in this exercise, we assume that the $f^{(n)}$ are bounded, and we show that the limiting function is bounded. Note, although we can show uniform boundedness from the limiting function, we do not require it to show that the limiting function is bounded.

Exercise 3.3.7

Exercise 3.3.7. Give an example to show that Proposition 3.3.6 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given earlier will already work here.)

Define $(f^{(n)}(x))_{n=1}^{\infty}$, where $f^{(n)} : (-1,1) \to \mathbb{R}, f^{(n)}(x) = x^n$. Then define $F^N(x) = \sum_{n=1}^{N} f^{(n)}$.
We know that the $F^N(x)$ pointwise converge to $g : (-1,1) \to \mathbb{R}, g(x) := x/(1-x)$ Each $F^N(x)$ is bounded; however $g(x)$ is bounded, giving the example.

Exercise 3.3.8

Exercise 3.3.8. Let (X, d) be a metric space, and for every positive integer n, let $f_n : X \to \mathbf{R}$ and $g_n : X \to \mathbf{R}$ be functions. Suppose that $(f_n)_{n=1}^{\infty}$ converges uniformly to another function $f : X \to \mathbf{R}$, and that $(g_n)_{n=1}^{\infty}$ converges uniformly to another function $g : X \to \mathbf{R}$. Suppose also that the functions $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are uniformly bounded, i.e., there exists an $M > 0$ such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all $n \geq 1$ and $x \in X$. Prove that the functions $f_n g_n : X \to \mathbf{R}$ converge uniformly to $fg : X \to \mathbf{R}$.

Since the g_n converge uniformly and are bounded, we know that g is bounded. Let M' be this bound, and write $|g(x)| \leq M'$ for every $x \in X$. Let $\epsilon > 0$. From the uniform convergence of the f_n and the g_n, we can that there is some $N_1 > 0$ and some $N_2 > 0$ such that for every $x \in X$, $|f_n(x) - f(x)| \leq \epsilon/2M$ and $|g_n(x) - g(x)| \leq \epsilon/2M'$. Defining $N := max(N_1, N_2)$. Let $n \leq N$. Then
$|f_n(x)g_n(x) - f(x)g(x)| = |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)|$
$\leq |f_n(x)g(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)|$
$= |f_n(x)||g_n(x) - g(x)| + |g(x)||f_n(x) - f(x)|$
$\leq |f_n(x)|\epsilon/2M + |g(x)|\epsilon/2M' \leq |\epsilon/2 + \epsilon/2 = \epsilon$
This shows the result.
#sheercreativity

Section 4

Exercise 3.4.1

Exercise 3.4.1. Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the space $B(X \to Y)$ defined in Definition 3.4.2, with the metric $d_{B(X\to Y)}$, is indeed a metric space.

This follows easily from the fact that Y is a metric space.

Exercise 3.4.2

Exercise 3.4.2. Prove Proposition 3.4.4.

Proposition 3.4.4. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions in $B(X \to Y)$, and let f be another function in $B(X \to Y)$. Then $(f^{(n)})_{n=1}^{\infty}$ converges to f in the metric $d_{B(X\to Y)}$ if and only if $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f.*

Suppose the $f^{(n)}$ converge in the metric $d_{B(X\to Y}$. Let $\epsilon > 0$. Since the $f^{(n)}$ converge in the metric, then we know there is some $N > 0$ such that for every $n \geq N$, $d_{B(X\to Y}(f^{(n)}, f) < \epsilon$. Let $n \geq N$. Let $x \in X$. Since $d_{B(X\to Y}(f^{(n)}, f) = sup_{x\in X}\{d_Y(f^{(n)}(x), f(x))\} < \epsilon$, it follows that for every $x \in X$, $d_Y(f^{(n)}(x), f(x) \leq sup_{x\in X}\{d_Y(f^{(n)}(x), f(x))\} < \epsilon$, otherwise, it would not be the supremum. This is enough to show that the $f^{(n)}$ converge uniformly to f.

Suppose the $f^{(n)}$ converge uniformly to f. Let $\epsilon > 0$. Since they converge uniformly, we know there is some $N > 0$ such that for every $x \in X$ and every $n \geq N$, $d_Y(f^{(n)}(x), f(x)) < \epsilon/2$. Let $n \geq N$. It follows easily that $sup_{x\in X}\{d_Y(f^{(n)}(x), f(x))\} \leq \epsilon/2 < \epsilon$. The result follows.

Exercise 3.4.3

Exercise 3.4.3. Prove Theorem 3.4.5. (Hint: this is similar, but not identical, to the proof of Theorem 3.3.1).

Theorem 3.4.5 (The space of continuous functions is complete). *Let (X, d_X) be a metric space, and let (Y, d_Y) be a* complete *metric space. The space $(C(X \to Y), d_{B(X\to Y)}|_{C(X\to Y)\times C(X\to Y)})$ is a complete subspace of $(B(X \to Y), d_{B(X\to Y)})$. In other words, every Cauchy sequence of functions in $C(X \to Y)$ converges to a function in $C(X \to Y)$.*

Let $(f^{(n)})_{n=1}^{\infty}$ be Cauchy in $C(X \to Y)$. Let $x \in X$, and consider $(f^{(n)}(x))_{n=1}^{\infty}$. Let $\epsilon > 0$. Then we know there is some $N > 0$ such that for every $n, m \geq N$, $d_\infty(f^{(n)}, f^{(m)}) < \epsilon$. It easily follows that $d_Y(f^{(n)}, f^{(m)}) \leq sup\{d_Y(f^{(n)}, f^{(m)}) : x \in X\} < \epsilon$. Hence, $(f^{(n)}(x))_{n=1}^{\infty}$ is Cauchy. Since Y is complete, we know this sequence converges for every $x \in X$.

Define $f : X \to Y, f(x) = \lim_{n\to\infty} f^{(n)}$. Hence, for every $x \in X$, for every $\epsilon > 0$ there is some $N > 0$ such that for every $n \geq N$, $d_Y(f^{(n)}(x), f(x)) < \epsilon$.

Let $\epsilon > 0$. Then we know there is some $N > 0$ such that for every $n, m \geq N$, $d_\infty(f^{(n)}, f^{(m)}) < \epsilon/2$. Let $x \in X$ and let $n \geq N$. We can write $d_\infty(f^{(n)}, f^{(m)}) < \epsilon/2$ for every $m \geq N$. We can also write that there is some $N' > 0$ such that for every $m \geq N'$, $d_Y(f^{(m)}(x), f(x)) < \epsilon/2$. Hence, taking some $m > max(N, N')$, it follows

$$d_Y(f^{(n)}(x), f(x)) \leq d_Y(f^{(n)}(x), f^{(m)}(x)) + d_Y(f^{(m)}(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon$$.

This is enough to show that the $f^{(n)}$ converge uniformly to f. Since the $f^{(n)}$ are continuous, it follows that f is continuous from Corollary 3.3.2, and hence is an element of $C(X \to Y)$. From Proposition 3.4.4, we know that $(f^{(n)})_{n=1}^{\infty}$ converges to f in $d_{C(X\to Y}$. Hence, $C(X \to Y)$ is complete.

#closingtheexistential

Exercise 3.4.4

Exercise 3.4.4. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $Y^X := \{f \big| f : X \to Y\}$ be the space of all functions from X to Y (cf. Axiom 3.10). If $x_0 \in X$ and V is an open set in Y, let $V^{(x_0)} \subseteq Y^X$ be the set

$$V^{(x_0)} := \{f \in Y^X : f(x_0) \in V\}.$$

If E is a subset of Y^X, we say that E is *open* if for every $f \in E$, there exists a finite number of points $x_1, \ldots, x_n \in X$ and open sets $V_1, \ldots, V_n \subseteq Y$ such that

$$f \in V_1^{(x_1)} \cap \ldots \cap V_n^{(x_n)} \subseteq E.$$

- Show that if $\mathcal{F}$ is the collection of open sets in Y^X, then $(Y^X, \mathcal{F})$ is a topological space.
- For each natural number n, let $f^{(n)} : X \to Y$ be a function from X to Y, and let $f : X \to Y$ be another function from X to Y. Show that $f^{(n)}$ converges to f in the topology $\mathcal{F}$ (in the sense of Definition 2.5.4) if and only if $f^{(n)}$ converges to f pointwise (in the sense of Definition 3.2.1).

$\varnothing$ is trivially an open set in the topology.
Let $f \in Y^X$. Choose any $x \in X$ and define $V := B_{(Y,d_Y)}(f(x), 1)$, which is open in Y. It follows that $f \in V^{(x)}$ and hence $Y^X \in \mathbb{F}$.

Consider $W_1, \cdots W_m$ open in $\mathbb{F}$. Let $f \in W_1 \cap \cdots \cap W_m$. Then we know that for each $1 \leq i \leq m$, there are finite $x_{1i}, \cdots, x_{ni}$ and $V_{1i}, \cdots, V_{ni}$ such that $f \in V_{x_{ni}}^{(x_{1i})} \cap \cdots \cap V_{x_{ni}}^{(x_{ni})} \subseteq W_i$. Taking all these points and open sets together, we can assert $f \in V_{x_{ni}}^{(x_{1i})} \cap \cdots \cap V_{x_{nn}}^{(x_{nn})} \subseteq W_1 \cap \cdots \cap W_m$, which shows the property.

Let f be an element of some arbitrary union $\cup_\alpha W_\alpha \in \mathbb{F}$. Then we know that $f \in W_\beta$ for some β. Then there are some points $x_1, \cdots, x_n$ and open sets $V_1, \cdots, V_n$ such that $f \in V_1^{x_1} \cap \cdots \cap V_n^{x_n} \subseteq W_\beta$. Since $W_\beta \subseteq \cup_\alpha W_\alpha$ the property follows.

$\Rightarrow$
Let $x \in X$. Let $\epsilon > 0$. Define $V := B_{(Y,d_Y)}(f(x), \epsilon)$ is open in Y. Then, $V^{(x)} \in \mathbb{F}$ since for every $g \in V^{(x)}$, $g^{(x)} \subseteq V^{(x)}$. Hence, $V^{(x)}$ is a neighbourhood of f, since obviously, $f \in V^{(x)}$. Since the $f^{(n)}$ converge topologically to f, we know there is some $N > 0$ such that for every $n \geq N$, $f^{(n)} \in V^{(x)}$. Hence, $f^{(n)}(x) \in B_{(Y,d_Y)}(f(x), \epsilon)$, and $d_Y(f(x), f^{(n)}(x)) < \epsilon$. The result follows

#constructingaball

⇐

Let V be some neighbourhood of f. Since V is open in the topology and obviously $f \in V$, we can write that are some $x_1, \cdots, x_m$ and some $V_1, \cdots, V_m$ such that
$f \in V_1^{(x_1)} \cap \cdots \cap V_n^{(x_n)} \subseteq V$.
Let $1 \leq i \leq m$. We know V_i is open in Y, and since, by definition, $f(x_i) \in V_i$, it follows that there is some ball $B(f(x_i), r) \subseteq V_i$. Hence, since the $f^{(n)}$ converge pointwise to f, we can write that there is some $N_i > 0$ such that for every $n \geq N_i$, $d_Y(f^{(n)}(x_i), f(x_i)) < r$. Hence, $f^{(n)} \in V_i$ for $n \geq N_i$. Taking $N := max_{1 \leq i \leq m} N_i$, it follows that for every $n \geq N$, $f^{(n)} \in V_i$, and hence $f^{(n)} \in V$. This shows topological convergence.

Section 5

Exercise 3.5.1

Exercise 3.5.1. Let $f^{(1)}, \ldots, f^{(N)}$ be a finite sequence of bounded functions from a metric space (X, d_X) to $\mathbf{R}$. Show that $\sum_{i=1}^N f^{(i)}$ is also bounded. Prove a similar claim when "bounded" is replaced by "continuous". What if "continuous" was replaced by "uniformly continuous"?

Bounded

Since for every $1 \leq i \leq N$, $|f^{(i)}(x)| \leq M_i$, we define $M := \sum_{i=1}^{N} M_i$. Then, it follows that
$|\sum_{i=1}^{N} f^{(i)}| \leq \sum_{i=1}^{N} |f^{(i)}| \leq \sum_{i=1}^{N} M_i$. Hence, the finite summation is bounded.

Continuous
Let $x' \in X$. Let $\epsilon > 0$. Then for every $1 \leq i \leq N$, there is some $\delta_i > 0$ such that for every $x \in X$, if $d_X(x, x') < \delta_i$ then $|f^{(i)}(x) - f^{(i)}(x')| < \epsilon/N$. Define $\delta := min_{1 \leq i \leq N} \delta_i$. Let $x \in X$ and let $d_X(x, x') < \delta$. Hence, for every $1 \leq i \leq N$, $|f^{(i)}(x) - f^{(i)}(x')| < \epsilon/N$. It follows that
$$|\sum_{i=1}^{N} f^{(i)}(x) - \sum_{i=1}^{N} f^{(i)}(x')| = |\sum_{i=1}^{N} f^{(i)}(x) - f^{(i)}(x')|$$

$$\leq \sum_{i=1}^{N} |f^{(i)}(x) - f^{(i)}(x')| \leq \sum_{i=1}^{N} \epsilon/N = \epsilon$$
.

Hence, the finite summation is continuous.

Uniformly continuous

Uniform continuity can be shown in the same way since opening up the existentials on the δ_i does not depend on $x' \in X$.

Exercise 3.5.2

Exercise 3.5.2. Prove Theorem 3.5.7. (Hint: first show that the sequence $\sum_{i=1}^{N} f^{(i)}$ is a Cauchy sequence in $C(X \to \mathbf{R})$. Then use Theorem 3.4.5.)

Theorem 3.5.7 (Weierstrass M-test). *Let (X, d) be a metric space, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series $\sum_{n=1}^{\infty} \|f^{(n)}\|_{\infty}$ is convergent. (Note that this is a series of plain old real numbers, not of functions.) Then the series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to some function f on X, and that function f is also continuous.*

Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} ||f^{(n)}||_{\infty}$ is convergent, we know that from Proposition 7.2.5 (AI), that for every $\epsilon > 0$ there is some $N' > 0$ such that $|\sum_{n=p}^{q} ||f^{(n)}||_{\infty}| \leq \epsilon$ for every $p, q \geq N'$. Taking $N, M \geq N'$, and, w.l.o.g, assuming $M \geq N$, we can show

$$|\sum_{i=1}^{N} f^{(i)} - \sum_{i=1}^{M} f^{(i)}| = |\sum_{i=1}^{N} f^{(i)} - \sum_{i=1}^{N} f^{(i)} + \sum_{i=N+1}^{M} f^{(i)}|$$

$$= |\sum_{i=N+1}^{M} f^{(i)}| \leq |\sum_{i=N+1}^{M} ||f^{(i)}||_{\infty}| \leq \epsilon$$
.

This shows that the sequence $\sum_{i=1}^{N} f^{(i)}$ is Cauchy, and from Exercise 3.5.1 we know it is also continuous, which, from Theorem 3.4.5, shows that it is convergent to some continuous function in $C(X \to \mathbb{R})$. From Proposition 3.4.4, this shows uniform convergence for the sequence of partial sums, and consequently uniform convergence of the infinite series.

Section 6

Theorem 3.6.1

Theorem 3.6.1. *Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f^{(n)} : [a, b] \to \mathbf{R}$ be a Riemann-integrable function. Suppose $f^{(n)}$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \mathbf{R}$. Then f is also Riemann integrable, and*

$$\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f.$$

Let $\epsilon > 0$. Since the $f^{(n)}$ converge uniformly, we know there is some $N > 0$ such that for every $x \in X$ and every $n \geq N$, $|f(x) - f^{(n)}| \leq \epsilon$. Hence, we can write $f^{(n)} - \epsilon \leq f(x) \leq f^{(n)}(x) + \epsilon$. Using Lemma 11.3.3 (AI) and the definition of upper and lower Riemann integrals, it follows that

$$\underline{\int}_{[a,b]} (f^{(n)} - \varepsilon) \leq \underline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} (f^{(n)} + \varepsilon).$$

Since the $f^{(n)}$ are Riemann integrable, we have

$$\left(\int_{[a,b]} f^{(n)}\right) - \varepsilon(b-a) \leq \underline{\int}_{[a,b]} f \leq \overline{\int}_{[a,b]} f \leq \left(\int_{[a,b]} f^{(n)}\right) + \varepsilon(b-a)$$

And

$$0 \leq \overline{\int}_{[a,b]} f - \underline{\int}_{[a,b]} f \leq 2\varepsilon(b-a)$$

Since ϵ was arbitrary, it follows that $\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$, which shows that f is Riemann integrable. By way of similar argument, we can show that for every ϵ there is some $N > 0$ such that

$$\left|\int_{[a,b]} f^{(n)} - \int_{[a,b]} f\right| \leq 2\varepsilon(b-a)$$

For every $n \geq N$, which implies uniform convergence.

Exercise 3.6.1

Exercise 3.6.1. Use Theorem 3.6.1 to prove Corollary 3.6.2.

Corollary 3.6.2. *Let $[a,b]$ be an interval, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a,b]$ such that the series $\sum_{n=1}^{\infty} f^{(n)}$ is uniformly convergent. Then we have*

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}.$$

Since each $f^{(n)}$ is Riemann integrable, it follows that the partial sum $sum_{n=1}^{N} f^{(n)}$ is Riemann integrable. Letting $f = \sum_{n=1}^{\infty} f^{(n)}$, we can apply Theorem 3.6.1 and write

$$\lim_{n\to\infty} \int_{[a,b]} \sum_{n=1}^{N} f^{(n)} = \int_{[a,b]} f$$

$$\lim_{n\to\infty} \sum_{n=1}^{N} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}$$

$$\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}.$$

Section 7

Theorem 3.7.1

Theorem 3.7.1. *Let $[a,b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a,b] \to \mathbf{R}$ be a differentiable function whose derivative $f_n' : [a,b] \to \mathbf{R}$ is continuous. Suppose that the derivatives f_n' converge uniformly to a function $g : [a,b] \to \mathbf{R}$. Suppose also that there exists a point x_0 such that the limit $\lim_{n\to\infty} f_n(x_0)$ exists. Then the functions f_n converge uniformly to a differentiable function f, and the derivative of f equals g.*

Since the f_n' is continuous, and hence Riemann integrable, it follows that we can apply the fundamental theorem of calculus, Theorem 11.9.4 (AI), and arrive at

$f_n(x) - f_n(x_0) = \int_{[x_0,x]} f_n'$, when $x \in [x_0, b]$ and

$f_n(x) - f_n(x_0) = -\int_{[x,x_0]} f_n'$, when $x \in [a, x_0]$.

Let $L := \lim_{n\to\infty} f_n(x_0)$, which exists by assumption. From Corollary 3.3.2, we know that g is continuous since the f_n' are continuous and uniformly converge to g.

Define $f(x) := L - \int_{[a,x_o]} g + \int_{[a,x]} g$.

Exercise 3.7.1

Exercise 3.7.1. Complete the proof of Theorem 3.7.1. Compare this theorem with Example 1.2.10, and explain why this example does not contradict the theorem.

Since the f_n' are Riemann integrable and uniformly converge to g, using Theorem 3.6.1, we can write

$f(x) := L - \lim_{n\to\infty} \int_{[a,x_o]} f_n' + \lim_{n\to\infty} \int_{[a,x]} f_n'$.

Using Theorem 11.9.4 (AI), we can write

$f(x) := L - \lim_{n\to\infty} (f_n(x_0) - f_n(a)) + \lim_{n\to\infty} (f_n(x) - f_n(a))$

$= L + \lim_{n\to\infty} (f_n(x) - f_n(x_0)) = \lim_{n\to\infty} f_n(x_0) + \lim_{n\to\infty} (f_n(x) - f_n(x_0))$

$\lim_{n\to\infty} f_n(x)$.

Which shows that the f_n converge uniformly to f.

To show that f is differentiable, take the derivative of f at some point x'. Hence,

$\lim_{x\to x'} f(x) - f(x')/x - x' = \lim_{x\to x'} \lim_{n\to\infty} (f_n(x) - f_n(x'))/x - x'$

$= \lim_{n\to\infty} \lim_{x\to x'} (f_n(x) - f_n(x'))/x - x = \lim_{n\to\infty} f_n' = g$.

since $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$. We can interchange these limits using Proposition 3.3.3 since the f_n converge uniformly.

Exercise 3.7.2

Exercise 3.7.2. Prove Theorem 3.7.1 without assuming that f'_n is continuous. (This means that you cannot use the fundamental theorem of calculus. However, the mean value theorem (Corollary 10.2.9) is still available. Use this to show that if $d_\infty(f'_n, f'_m) \leq \varepsilon$, then $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| \leq \varepsilon|x - x_0|$ for all $x \in [a, b]$, and then use this to complete the proof of Theorem 3.7.1.)

Suppose $n, m \in \mathbb{N}$, $\epsilon > 0$ and $d_\infty(f'n, f'm) < \epsilon$. Let $x \in [a, b]$. Since $f'n$ and $f'm$ are differentiable, we know that they are continuous, and it follows that $f'n - f'm$ is continuous well. Hence, taking the interval $[x, x_0]$ (or $[x_0, x]$), we can apply Corollary 10.2.9, to assert that there is some x' between x and x_0, such that $(f'n - f'm)(x') = (f_n - f_m)(x) - (f_n - f_m)(x_0)/(x - x_0)$. Since $d_\infty(f'n, f'm) < \epsilon$, it follows that $|(f_n - f_m)(x) - (f_n - f_m)(x_0)| < \epsilon|(x - x_0)|$, and $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| < \epsilon|(x - x_0)|$.

Let $\epsilon > 0$. Since the f'_n converge uniformly to g, we know from Proposition 3.4.4 that they converge in $d_{B(X \to Y)}$, which implies they are Cauchy in this metric as well. Hence, let $N_1 > 0$ be such that for every $n, m \geq N_1$, $d_\infty(f'n, f'm) < \epsilon$. And let $N_2 > 0$ be such that for every $n \geq N_2$ and every $x \in [a, b]$, $f_n(x_0) - L| < \epsilon/2$, which we can do since f_n converges. Let $n, m \geq max(N_1, n_2)$ and $x \in [a, b]$. Then can write $|f_n(x_0) - L| < \epsilon/2$ and $|f_m(x_0) - L| < \epsilon/2$, and it follows that $|f_n(x_0) - f_m(x_0)| < \epsilon$. Hence, when we write $|(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| < \epsilon|(x - x_0)|$ and $|(f_n(x) - f_m(x))| + |(f_m(x_0) - f_n(x_0))| < \epsilon|(x - x_0)|$, it follows that $|(f_n(x) - f_m(x))| < \epsilon|(x - x_0)| - \epsilon < \epsilon(b - a - 1)$.
Hence, f_n is a Cauchy sequence of continuous functions, which from Theorem 3.4.5 implies that the sequence uniformly converges to some function f.

The rest of the proof is similar to Exercsise 3.7.1.

Exercise 3.7.3

Exercise 3.7.3. Prove Corollary 3.7.3.

Corollary 3.7.3. *Let $[a,b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a,b] \to \mathbf{R}$ be a differentiable function whose derivative $f_n' : [a,b] \to \mathbf{R}$ is continuous. Suppose that the series $\sum_{n=1}^{\infty} \|f_n'\|_\infty$ is absolutely convergent, where*

$$\|f_n'\|_\infty := \sup_{x \in [a,b]} |f_n'(x)|$$

is the sup norm of f_n', as defined in Definition 3.5.5. Suppose also that the series $\sum_{n=1}^{\infty} f_n(x_0)$ is convergent for some $x_0 \in [a,b]$. Then the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a,b]$ to a differentiable function, and in fact

$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

for all $x \in [a,b]$.

Since the f_n' are continuous (and bounded real-value) functions, and $\sum_{n=1}^{\infty} ||f_n'||_\infty$ is (absolutely) convergent, we can apply Theorem 3.5.7, and assert that $\sum_{n=1}^{\infty} f_n'$ uniformly converges to some g. Hence, $\sum_{n=1}^{N} f_n'$ converges uniformly to g. Similarly, by assumption, we can write that $\sum_{n=1}^{N} f_n(x_0)$ converges to some L. Hence, applying Theorem 3.7.1, we can assert that $\sum_{n=1}^{N} f_n(x)$ converges uniformly to some differentiable f and $f' = g$. The result follows.

Section 8

Lemma 3.8.13

Lemma 3.8.13. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$, and let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[-1,1]$ which is a polynomial on $[-1,1]$. Then $f * g$ is a polynomial on $[0,1]$. (Note however that it may be non-polynomial outside of $[0,1]$.)*

We know that $g(x) = \sum_{j=0}^{n} c_j x^j$ for every $x \in [-1,1]$.

We also know that

$f * g(x) = \int_{[0,1]} f(y) g(x-y) dy$ for every $x \in [0,1]$.

Since the integration variable is also over $[0,1]$, it follows that $x - y \in [-1,1]$.

Hence,

$$f * g(x) = \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j (x-y)^j dy$$

Using the binomial formula from Exercise 7.1.4, we have

$$= \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} x^k (-y)^{j-k} j dy$$

From Corollary 7.1.14 and the fact that x^k is independent of y, we have

$$= x^k \sum_{k=0}^{j} \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k} j dy$$

Hence, if we define,

$$C_k := \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k} j dy$$

Then it follows that

$f * g(x) = x^k \sum_{k=0}^{j} C_k$ for all $x \in [0,1]$.

Which is a polynomial.

This proof is fairly straightforward. It mostly relies on the fact that g is a polynomial and the binomial formula. Initially the polynomial is enclosed inside the integral, and so we cannot immediately assert the result. However, by expanding the binomial formula, we are able to bring one of the summations outside of the integral, leaving us with a polynomial, where each constant is dependent on an integral.

Corollary 3.8.19

Corollary 3.8.19 (Weierstrass approximation theorem III). *Let $f : [0,1] \to \mathbf{R}$ be a continuous function supported on $[0,1]$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0,1] \to \mathbf{R}$ such that $|P(x) - f(x)| \le \varepsilon$ for all $x \in [0,1]$.*

Define $F : [0,1] \to \mathbb{R}$

$$F(x) := f(x) - f(0) - x(f(1) - f(0))$$

Note, $F(0) = F(1) = 0$.

We know that F is continuous since f and x are continous, and Proposition 9.49.

Hence, we can apply Corollary 3.8.18 and assert that for every $\epsilon > 0$ there is some polynomial $Q : [0,1] \to \mathbb{R}$ such that for every $x \in [0,1]$, $|Q(x) - F(x)| \le \epsilon$.

However, $Q(x) - F(x) = Q(x) - f(x) - f(0) + x(f(1) - f(0)$. So if we define $P(x) := Q(x) - f(0) + x(f(1) - f(0)$, then we have found our P.

Theorem 3.8.3

Theorem 3.8.3 (Weierstrass approximation theorem). *If $[a,b]$ is an interval, $f : [a,b] \to \mathbf{R}$ is a continuous function, and $\varepsilon > 0$, then there exists a polynomial P on $[a,b]$ such that $d_\infty(P,f) \le \varepsilon$ (i.e., $|P(x) - f(x)| \le \varepsilon$ for all $x \in [a,b]$).*

Define $g(x) = f(a + (b-a)x)$ for every $x \in [0,1]$.

Then it follows that $f(y) = g((y-a)/(b-a))$ for every $y \in [a,b]$. We know that $g \text{ is continuous because}$ f$is continuous,$a+(b-a)x $is continuous, and composition preserves continuity($.4.13).

Hence, we can apply Corollary 3.8.19, and assert that for every $\epsilon > 0$ there is some polynomial $Q : [0,1] \to \mathbb{R}$ such that $|Q(x) - g(x)| \le \epsilon$ for every $x \in [0,1]$.

Hence, we have

$|Q((y-a)/(b-a)) - g((y-a)/(b-a))| \le \epsilon$ for every $y \in [a,b]$.

If we define

$P(y) = Q((y-a)/(b-a))$, which is also a polynomial, then the result follows since $f(y) = g((y-a)/(b-a))$.

Exercise 3.8.1

Exercise 3.8.1. Prove Lemma 3.8.5.

Lemma 3.8.5. *If $f : \mathbf{R} \to \mathbf{R}$ is continuous and supported on an interval $[a, b]$, and is also supported on another interval $[c, d]$, then $\int_{[a,b]} f = \int_{[c,d]} f$.*

For every $x \in \mathbb{R}$, if $f(x) \neq 0$, then $x \in [a, b]$ and $x \in [c, d]$, otherwise, f would not be supported on both intervals.

Define $m := max(\{x \in \mathbb{R} : f(x) \neq 0\})$ and $n := min(\{x \in \mathbb{R} : f(x) \neq 0\})$. Obviously, $[n, m] \subset [a, b]$ and $[n, m] \subseteq [c, d]$. Then, applying Thoerem 11.4.1 (AI), we can write

$$\int_{[a,b]} f = \int_{[n,m]} f = \int_{[c,d]} f$$.

Exercise 3.8.2

Exercise 3.8.2. (a) Prove that for any real number $0 \leq y \leq 1$ and any natural number $n \geq 0$, that $(1-y)^n \geq 1 - ny$. (Hint: induct on n. Alternatively, differentiate with respect to y.)

Induction

Let $n = 0$. Then $(1-y)^n = 1$ and $1 - ny = 1; \text{hence,}$ (1-y)^n\geq1-ny.

Assume by indunction that $(1-y)^n \geq 1 - ny$. Then

$$(1-y)^{n+1} = (1-y)(1-y)^n \geq (1-y)(1-ny) = 1-(n+1)y + ny^2 \geq 1-(n+1)y$$

since $ny^2 \geq 0$.

(b) Show that $\int_{-1}^{1}(1-x^2)^n \, dx \geq \frac{1}{\sqrt{n}}$. (Hint: for $|x| \leq 1/\sqrt{n}$, use part (a); for $|x| \geq 1/\sqrt{n}$, just use the fact that $(1-x^2)$ is positive. It is also possible to proceed via trigonometric substitution, but I would not recommend this unless you know what you are doing.)

We can write,

$$\int_{-1}^{1}(1-x^2)^n dx$$

$$= \int_{-1}^{1/\sqrt{(n)}}(1-x^2)^n dx + \int_{-1/\sqrt{(n)}}^{1/\sqrt{(n)}}(1-x^2)^n dx + \int_{1/\sqrt{(n)}}^{1}(1-x^2)^n dx$$.

From the hint, and using Theorem 11.4.1(g) and (e) (AI), it follows that

$$\int_{-1}^{1}(1-x^2)^n dx \geq \int_{-1/\sqrt(n)}^{1/\sqrt(n)} 1-nx^2 dx = 2/\sqrt(n) - 2/3\sqrt(n) \geq 1/\sqrt(n)$$
.

(c) Prove Lemma 3.8.8. (Hint: choose $f(x)$ to equal $c(1-x^2)^N$ for $x \in [-1,1]$ and to equal zero for $x \notin [-1,1]$, where N is a large number N, where c is chosen so that f has integral 1, and use (b).)

Lemma 3.8.8 (Polynomials can approximate the identity). *For every $\varepsilon > 0$ and $0 < \delta < 1$ there exists an (ε, δ)-approximation to the identity which is a polynomial P on $[-1,1]$.*

Let $\epsilon > 0$ and $0 < \delta < 1$. Choose $f(x)$ as defined by the hint, where N is chosen such that $\sqrt(N)(1-\delta^2)^N < \epsilon$. Then, obviously f is supported on $[-1,1]$, and by choosing $c < 0$, we guarantee that $f(x) \geq 0$ for all $x \in [-1.1]$. Since $\int_{-\infty}^{\infty} f \geq c/\sqrt(N)$ (which we know from (b)), it follows that by choosing $0 < c \leq \sqrt(N)$, we have $\int_{-\infty}^{\infty} f = 1$. The choice of N guarantees that $\sqrt(N)(1-x^2)^N < \epsilon$ for all $\delta \leq |x| \leq 1$.

Exercise 3.8.3

Exercise 3.8.3. Let $f : \mathbf{R} \to \mathbf{R}$ be a compactly supported, continuous function. Show that f is bounded and uniformly continuous. (Hint: the idea is to use Proposition 2.3.2 and Theorem 2.3.5, but one must first deal with the issue that the domain $\mathbf{R}$ of f is non-compact.)

Since f is compactly supported, we know there is some a and some b such that for every $x \in [a,b]$, $f(x) = 0$. From Theorem 1.5.7 we know that $[a,b]$ is compact, since it is closed and bounded. It follows easily that $f_{|[a,b]}$ is continuous since $[a,b] \subset \mathbb{R}$. Hence, from Proposition 2.3.2, $f_{|[a,b]}$ is bounded. Then we know there is some $M \geq 0$ such that $|f_{|[a,b]}(x)| \leq M$ for every $x \in [a,b]$. Since $f(x) = 0$ for every $x \notin [a,b]$, it follows that $|f(x)| \leq M$ for every $x \in \mathbb{R}$, i.e., f is bounded.

Since $f_{|[a,b]}$ is continuous, from Theorem 2.3.5, we know that it is uniformly continuous. Let $\epsilon > 0$. Then we know there is some $\delta_1 > 0$ such that for every $x, x' \in [a,b]$, if $|x - x'| < \delta_1$ then $|f(x) - f(x')| < \epsilon$. Let $x, x' \in \mathbb{R}$. Then either $x \in [a,b] \vee x \notin [a,b]$ and $x' \in [a,b] \vee x' \notin [a,b]$. Since f is continuous, we know there is some $\delta_2 > 0$ such that for every $x* \in [a,b]$, if $|x * -x'| < \delta_2$ then $|f(x*) - f(x')| < \epsilon$. Define $\delta := min(\delta_1, \delta_2)$. If

$x \in [a, b]$ and $x' \in [a, b]$, then it follows easily that if $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$. If $x \in [a, b]$ and $x'[a, b]$, then we can assert if $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$. Similarly if $x' \in [a, b]$ and $x[a, b]$. If $x \notin [a, b]$ and $x'[a, b]$, then since $f(x) = 0$ for all $x \in [a, b]$, then $|f(x) - f(x')| = 0 < \epsilon$. This shows that f is uniformly continuous.

Exercise 3.8.4

Exercise 3.8.4. Prove Proposition 3.8.11. (Hint: to show that $f*g$ is continuous, use Exercise 3.8.3.)

Proposition 3.8.11 (Basic properties of convolution). *Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$, and $h : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. Then the following statements are true.*

(a) *The convolution $f * g$ is also a continuous, compactly supported function.*

(b) *(Convolution is commutative) We have $f*g = g*f$; in other words*

$$\begin{aligned} f * g(x) &= \int_{-\infty}^{\infty} f(y)g(x - y)\ dy \\ &= \int_{-\infty}^{\infty} g(y)f(x - y)\ dy \\ &= g * f(x). \end{aligned}$$

(c) *(Convolution is linear) We have $f * (g + h) = f * g + f * h$. Also, for any real number c, we have $f * (cg) = (cf) * g = c(f * g)$.*

Suppose f is supported on $[a, b]$ and g on $[c, d]$.

(a)
Consider $h(y) := f(y)g(x - y)$ for some x. Then for $h(y) \neq 0$, $y \in [a, b]$ and $x - y \in [c, d]$. Hence, $y \in [a, b] \cap [x - d, x - c]$. It follows that if $x > b + d$ or $x < a + c$, then $[a, b] \cap [x - d, x - c] = \emptyset$, in which case $f * g = 0$. Hence, the convolution is supported on the interval $[a + c, b + d]$.
Since g is continuous, we know that for every $x' \in \mathbb{R}$ and every $\epsilon > 0$, there is some $\delta > 0$ such that for every $x \in \mathbb{R}$, if $|x - x'| < \delta$ then $|g(x) - g(x')| < \epsilon$. Suppose $|x - x'| < \delta$, then

$$|f * g(x) - f * g(x')| = |\int_{-\infty}^{(} y)[g(x-y) - g(x'-y)]dy|$$

$$= \int_{[a,b]} f(y)[g(x-y) - g(x'-y)]dy \leq \int_{[a,b]} f(y)dy(b-a)\epsilon$$
.

Since $= \int_{[a,b]} f(y)dy$ is constant, the result follows.

(b)

The argument is done through substitution. First, note that

$$\int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{[a,b]\cap[x-d,x-c]} f(y)g(x-y)dy$$
, and

$$\int_{-\infty}^{\infty} g(y)f(x-y)dy = \int_{[c,d]\cap[x-b,x-a]} g(y)f(x-y)dy$$
.

Consider $u := F(y) := x - y$. Then we can write,

$$\int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{max(a,x-d)}^{min(b,x-c)} f(y)g(x-y)dy$$

$$= \int_{h(max(a,x-d))}^{h(min(b,x-c))} f(x-u)g(u)h'(y)dy$$

$$= \int_{min(x-a,d)}^{max(x-b,c)} f(x-u)g(u)(-1)du$$

$$= \int_{max(x-b,c)}^{min(x-a,d)} f(x-u)g(u)du$$

$$= \int_{[c,d]\cap[x-b,x-a]} f(x-y)g(y)dy$$

(c)

These follow very easily.

Exercise 3.8.5

Exercise 3.8.5. Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. Suppose that f is supported on the interval $[0,1]$, and g is constant on the interval $[0,2]$ (i.e., there is a real number c such that $g(x) = c$ for all $x \in [0,2]$). Show that the convolution $f * g$ is constant on the interval $[1,2]$.

If $x \in [1,2]$ and $y \in [0,1]$, then it follows that $x - y \in [0,2]$. Hence,

$$\int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{[0,1]} f(y)g(x-y)dy = c\int_{[0,1]} f(y)dy$$

Which is constant.

Exercise 3.8.6

Exercise 3.8.6. (a) Let g be an (ε, δ) approximation to the identity. Show that $1 - 2\varepsilon \le \int_{[-\delta,\delta]} g \le 1$.

$$\int_{-\infty}^{\infty} g = \int_{-1}^{1} g = \int_{-1}^{-\delta} g + \int_{-\delta}^{\delta} g + \int_{\delta}^{1} g.$$

Since $g(x) \le \epsilon$ for all $\delta \ge |x| \ge 1$, it follows that

$$\int_{-1}^{1} g \le \int_{-1}^{-\delta} \epsilon + \int_{-\delta}^{\delta} g + \int_{\delta}^{1} \epsilon$$

$$= 2\epsilon(1-\delta) + \int_{-\delta}^{\delta} g.$$

Since $\int_{-1}^{1} g = 1$, and $1 - \delta < 1$, it follows that

$$1 - 2\epsilon \le \int_{-\delta}^{\delta} g \le 1.$$

(b) Prove Lemma 3.8.14. (Hint: begin with the identity

$$f * g(x) = \int f(x-y)g(y)\ dy = \int_{[-\delta,\delta]} f(x-y)g(y)\ dy$$

$$+ \int_{[\delta,1]} f(x-y)g(y)\ dy + \int_{[-1,-\delta]} f(x-y)g(y)\ dy.$$

Lemma 3.8.14. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$, which is bounded by some $M > 0$ (i.e., $|f(x)| \le M$ for all $x \in \mathbf{R}$), and let $\varepsilon > 0$ and $0 < \delta < 1$ be such that one has $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbf{R}$ and $|x - y| < \delta$. Let g be any (ε, δ)-approximation to the identity. Then we have*

$$|f * g(x) - f(x)| \le (1 + 4M)\varepsilon$$

If $y \in (-\delta, \delta)$, then $|x-(x-y)| < \delta$ and hence $|f(x)-f(x-y)| < \epsilon$. From this it follows that $f(x)-\epsilon < f(x-y) < f(x)+\epsilon$. Hence,

$$(f(x)-\epsilon)\int_{[-\delta,\delta]} g(y)dy \leq \int_{[-\delta,\delta]} f(x-y)g(y) \leq (f(x)+\epsilon)\int_{[-\delta,\delta]} g(y)dy$$

Applying (a), we have

$$(f(x)-\epsilon)(1-2\epsilon) \leq \int_{[-\delta,\delta]} f(x-y)g(y) \leq f(x)+\epsilon$$

Since $|g(y)| \leq \epsilon$ for $\delta \leq |x| \leq 1$ and $f(x) \leq M$ for all x, it follows that

$$-\epsilon M\int_{[\delta,1]} g(y)dy \leq \int_{[\delta,1]} f(x-y)g(y)dy \leq \epsilon M\int_{[\delta,1]} g(y)dy$$

And we can write

$$-\epsilon M \leq -\epsilon M(1-\delta) \leq \int_{[\delta,1]} f(x-y)g(y)dy \leq \epsilon M(1-\delta) \leq \epsilon M$$

Similarly, we can write

$$-\epsilon M \leq \int_{[-1,-\delta]} f(x-y)g(y)dy \leq \epsilon M$$

Hence,

$$-2M\epsilon + (f(x)-\epsilon)(1-2\epsilon) \leq f*g(x) \leq 2\epsilon M + f(x)+\epsilon$$
$$-2M\epsilon + f(x) - 2f(x)\epsilon - \epsilon + 2\epsilon^2 \leq f*g(x) \leq 2\epsilon M + f(x)+\epsilon$$
$$-2M\epsilon - 2f(x)\epsilon - \epsilon + 2\epsilon^2 \leq f*g(x) - f(x) \leq 2\epsilon M + \epsilon$$
$$-2M\epsilon - 2M\epsilon - \epsilon \leq f*g(x) - f(x) \leq 2\epsilon M + \epsilon$$
$$-4M\epsilon - \epsilon \leq f*g(x) - f(x) \leq 4M\epsilon + \epsilon$$

And the result folllows.

Exercise 3.8.7

Exercise 3.8.7. Prove Corollary 3.8.15. (Hint: combine Exercise 3.8.3, Lemma 3.8.8, Lemma 3.8.13, and Lemma 3.8.14.)

Corollary 3.8.15 (Weierstrass approximation theorem I). *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$. Then for every $\varepsilon > 0$, there exists a function $P : \mathbf{R} \to \mathbf{R}$ which is polynomial on $[0,1]$ and such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0,1]$.*

Since f is continuously supported on $[0,1]$, it follows from Exercise 3.8.3 that f is bounded and uniformly continuous. Let $\epsilon > 0$. Since f is uniformly continuous, we know there is some $\delta > 0$ such that for every $x, y \in [0,1]$, if $|x-y| < \delta$ then $|f(x) - f(y)| < \epsilon$. If $\delta \geq 1$, then we can just take some $\delta * < 1$ and the result still applies.

From Lemma 3.8.8, we know there is some $(\epsilon/(1+4M), \delta)$-approximation to the identity, g, which is a polynomial on $[-1,1]$. From Lemma 3.8.13, we know that $f * g$ is a polynomial on $[0,1]$. Finally, using Lemma 3.8.14, we can write that

$|f * g - f(x)| \leq \dfrac{1+4M}{1+4M}\epsilon$ and the result follows.

Exercise 3.8.8

Exercise 3.8.8. Let $f : [0,1] \to \mathbf{R}$ be a continuous function, and suppose that $\int_{[0,1]} f(x)x^n \, dx = 0$ for all non-negative integers $n = 0, 1, 2, \ldots$. Show that f must be the zero function $f \equiv 0$. (Hint: first show that $\int_{[0,1]} f(x)P(x) \, dx = 0$ for all polynomials P. Then, using the Weierstrass approximation theorem, show that $\int_{[0,1]} f(x)f(x) \, dx = 0$.)

Let P be some polynomial.

Then $P(x) = \sum_{j=0}^{n} c_j x^j$ for $n \geq 0$.

Hence, $f(x)P(x) = \sum_{j=0}^{n} c_j f(x) x^j$ and

$$\int_{[0,1]} f(x)P(x)dx = \int_{[0,1]} \sum_{j=0}^{n} c_j f(x) x^j$$

$$= \sum_{j=0}^{n} c_j f(x) x^j c_j \int_{[0,1]} f(x) x^j = 0$$

Hence, $\int_{[0,1]} f(x)P(x)dx = 0$ for every polynomial P.

Let $\epsilon > 0$.

Applying the Weierstrass approximation theorem, Theorem 3.8.3, we know that there is some P on $[0,1]$ such that for every $x \in [0,1]$, we have $|P(x) - f(x)| < \epsilon$. It follows that $f(x) - \epsilon \leq P(x) \leq f(x) + \epsilon$. Hence, we can write

$$\int_{[0,1]} f(x)(f(x) - \epsilon)dx \leq \int_{[0,1]} f(x)P(x)dx \leq \int_{[0,1]} f(x)(f(x)+\epsilon)dx$$

.

Since $\int_{[0,1]} f(x)P(x)dx = 0$, it follows that

$$-\epsilon \int_{[0,1]} f(x)dx \leq \int_{[0,1]} f(x)f(x)dx \leq \epsilon \int_{[0,1]} f(x)dx$$

Which implies that

$$|\int_{[0,1]} f(x)f(x)dx| \leq 0$$

Since

$$\int_{[0,1]} f(x)dx = \int_{[0,1]} f(x)x^0dx = 0$$

.

The result follows since f is continuous and $f^2 = 0$.

Exercise 3.8.9

Exercise 3.8.9. Prove Lemma 3.8.16.

Lemma 3.8.16. *Let $f : [0,1] \to \mathbf{R}$ be a continuous function which equals 0 on the boundary of $[0,1]$, i.e., $f(0) = f(1) = 0$. Let $F : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $F(x) := f(x)$ for $x \in [0,1]$ and $F(x) := 0$ for $x \notin [0,1]$. Then F is also continuous.*

There are three cases:

(1) $x, y \in [0,1]$ in which case the result follows immediately.

(2) $x, y \notin [0,1]$, in which case the result follows immediately.

(3) W.l.o.g. $x \in [0,1]$ and $y \notin [0,1]$.

Let $\epsilon > 0$. From the continuity of f we know that there is some $\delta > 0$ such that for every $y \in [0,1]$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$,

Let $|x - y| < \delta$.

It follows that $y - \delta < x < y + \delta$.

Since $y \notin [0,1]$, we know that $F(y) = 0$. We also know that $F(x) = f(x)$.

Since $y \notin [0,1]$, we know that $y < 0 \vee y > 1$.

If $y < 0$, then we know $y + \delta < \delta$, hence $x < \delta$ and since $x > 0$ it follows that $-\delta < x$. Hence, $|x| < \delta$ and so we can write that $|f(x)| < \epsilon$, which implies that $|F(x) - F(y)| < \epsilon$, since $F(y) = 0$ and $f(x) = F(x)$.

If $y > 1$, then we know that $1 - \delta < y - \delta < x$. Since $x < 1$ it follows that $|x - 1| < \delta$, which implies that $|f(x) - f(1)| < \epsilon$. Since $f(1) = 0 = F(y)$, we can write that $|F(x) - F(y)| < \epsilon$.

The result follows.