Chapter 4

Section 1

Exercise 4.1.1

Exercise 4.1.1. Prove Theorem 4.1.6. (Hints: for (a) and (b), use the root test (Theorem 7.5.1). For (c), use the Weierstrass M-test (Theorem 3.5.7). For (d), use Theorem 3.7.1. For (e), use Corollary 3.6.2.

Theorem 4.1.6. *Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence.*

(*a*) *(Divergence outside of the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| > R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is divergent for that value of x.*

(*b*) *(Convergence inside the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| < R$, then the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is absolutely convergent for that value of x.*

For parts (c)-(e) we assume that $R > 0$ (i.e., the series converges at at least one other point than $x = a$). Let $f : (a-R, a+R) \to \mathbf{R}$ be the *function* $f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n$*; this function is guaranteed to exist by (b).*

(c) *(Uniform convergence on compact sets) For any* $0 < r < R$*, the series* $\sum_{n=0}^{\infty} c_n (x-a)^n$ *converges uniformly to* f *on the compact interval* $[a-r, a+r]$*. In particular,* f *is continuous on* $(a-R, a+R)$*.*

(d) *(Differentiation of power series) The function* f *is differentiable on* $(a-R, a+R)$*, and for any* $0 < r < R$*, the series* $\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$ *converges uniformly to* f' *on the interval* $[a-r, a+r]$*.*

(e) *(Integration of power series) For any closed interval* $[y, z]$ *contained in* $(a-R, a+R)$*, we have*

$$\int_{[y,z]} f = \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

(a)
From the root test, define
$\alpha = \lim sup_{n\to\infty} |c_n (x-a)^n|^{1/n} = \lim sup_{n\to\infty} |c_n^{1/n} (x-a)|$
Since $|x-a| > R$ and $R = \dfrac{1}{\lim sup_{n\to\infty} |c_n|^{1/n}}$, it follows that $\alpha > 1$. This implies the series is convergent.

(b)
A similar argument to (a) shows divergence when $|x-a| < R$.

(c)
Notice that $(x-a)^n \leq r^n$. Hence, $|c_n (x-a)^n| \leq c_n r r^n$, and so in particular, $||f^{(n)}||_\infty \leq c_n r^n$ for every n.
From the root test, define
$\alpha = \lim sup_{n\to\infty} |c_n r^n|^{1/n}$. Since $R = \dfrac{1}{\lim sup_{n\to\infty} |c_n|^{1/n}}$, it follows that $\alpha = r/R < 1$, and so the series
$\sum_{n=0}^{\infty} c_n r r^n$ is absolutely convergent, which implies that the series

$\sum_{n=0}^{\infty} ||f^{(n)}||_\infty$ is absolutely convergent from Corollary 7.3.2 (AI). From Theorem 3.5.7, since the $f^{(n)}$ are continuous and bounded, we know that the series
$\sum_{n=0}^{\infty} f^{(n)}$ convergers to some function f.
Hence,

$$f = \sum_{n=0}^{\infty} f^{(n)} = \sum_{n=0}^{\infty} c_n(x-a)^n$$

This equals the desired $f(x)$ for $x \in (x-R, x+R)$ since for every x there is some r such that $0 < r < R$. This also shows that $f(x)$ is continuous.

(d)
Since x^n is differentiable for every n (Exercise 10.1.5 AI), we know from using the binomial formula and Theorem 10.1.13(e) AI that $f_n = c_n(x-a)^n$ is differentiable.
We now show that $f_n' = n(x-a)^{n-1}$.

$$(x-a)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j (-a)^{n-j}$$

Hence,

$$f_n' = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} j x^{j-1} (-a)^{n-j}$$

$$= \sum_{j=1}^{n} \frac{n!}{j!(n-j)!} j x^{j-1} (-a)^{n-j}$$ (the first term equals zero since j=0)

$$= \sum_{j=1}^{n} \frac{n!}{(j-1)!(n-j)!} x^{j-1} (-a)^{n-j}$$

$$= \sum_{j=0}^{n-1} \frac{n!}{j!(n-j-1)!} x^{j} (-a)^{n-j-1}$$ (Lemma 7.1.4(b) AI)

$$= n\sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-j-1)!} x^{j} (-a)^{n-j-1}$$

$$= n(x-a)^{n-1}$$

If we then define $g_n = d_n(x-a)^{n-1}$, such that $d_n = nc_n$, then we can use (c) to assert that the g_n converge uniformly to $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$, which implies that $g(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ from Proposition 7.1.14(d).

Since we already know from (c) that $\sum f_n$ converges uniformly, we know in particular that for some x_0 that $\sum f_n(x_0)$ converges, and hence, using Corollary 7.2.6 AI, we can assert that $\lim_{n\to\infty} f_n(x_0)$ exists.

Next, using Theorem 3.7.1, given all of the above, we can assert that the f_n converge uniformly to a differentiable f, which we already know equals $\sum_{n=0}^{\infty} c_n(x-a)^n$, and hence $f' = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ as desired.

(e)
This follows easily from Corollary 3.6.2, since

$$\int_{[y,z]} f_n = \int_{[y,x]} c_n(x-a)^n = \frac{c_n(z-a)^{n+1}}{n+1} - \frac{c_n(y-a)^{n+1}}{n+1}$$
$$== \frac{c_n(z-a)^{n+1} - c_n(y-a)^{n+1}}{n+1}.$$

Exercise 4.1.2

Exercise 4.1.2. Give examples of a formal power series $\sum_{n=0}^{\infty} c_n x^n$ centered at 0 with radius of convergence 1, which

(a) diverges at both $x = 1$ and $x = -1$;

(b) diverges at $x = 1$ but converges at $x = -1$;

(c) converges at $x = 1$ but diverges at $x = -1$;

(d) converges at both $x = 1$ and $x = -1$.

(e) converges pointwise on $(-1, 1)$, but does not converge uniformly on $(-1, 1)$.

(a)

If c_n, then $\sum_{n=0}^{\infty}(1)^n$ diverges as it approaches $+\infty$, and $\sum_{n=0}^{\infty}(-1)^n$ diverges since from Proposition 7.2.12, c_n does not converge to 0.

(b)

$\sum_{n=0}^{\infty}\frac{1}{n}(1)^n$ diverges but from Proposition 7.2.12 we know that $\sum_{n=0}^{\infty}\frac{1}{n}(-1)^n$ converges.

(c)

$\sum_{n=0}^{\infty}(1-\frac{1}{10^n})(1)^n$ converges to 1, but diverges for $\sum_{n=0}^{\infty}(1-\frac{1}{10^n})(-1)^n$

(d)

$\sum_{n=0}^{\infty}(0.5)^n(-1)^n$ converges (Proposition 7.2.12) and $\sum_{n=0}^{\infty}(0.5)^n(1)^n$ converges (Root test, Theorem 7.5.1)

(e)

Section 2

Exercise 4.2.1

Exercise 4.2.1. Let $n \geq 0$ be an integer, let c, a be real numbers, and let f be the function $f(x) := c(x-a)^n$. Show that f is infinitely differentiable, and that $f^{(k)}(x) = c\frac{n!}{(n-k)!}(x-a)^{n-k}$ for all integers $0 \leq k \leq n$. What happens when $k > n$?

We argue by induction.

Let $k = 1$. Then from Theorem 10.1.15 (AI) and Ex. 10.1.5 (AI), it follows that $f^{(1)}(x)$ is differentiable and

$$f^{(1)}(x) = cn(x-a)^{n-1} = c\frac{n!}{(n-k)!}(x-a)^{n-k}.$$

Assume by induction that $f^{(k)}(x) = c\frac{n!}{(n-k)!}(x-a)^{n-k}$ for $0 \leq k \leq n-1$.
Then,

$$f^{(k+1)}(x) = c\frac{n!}{(n-k)!}(n-k)(x-a)^{n-k-1}$$
$$= c\frac{n!}{(n-(k+1))!}(x-a)^{n-(k+1)}$$

In particular, if $k = n$, then $f^{(k)}(x) = cn!$, which is constant, and hence for every $k > n$, $f^{(k)}(x) = 0$. So f is infinitely differentiable.

Exercise 4.2.2

Exercise 4.2.2. Show that the function f defined in Example 4.2.2 is real analytic on all of $\mathbf{R}\backslash\{1\}$.

Let $a \in \mathbb{R} \setminus 1$.
Then,

$$S = \sum_{n=0}^{\infty}(\frac{1}{1-a})^{n+1}(x-a)^n$$
$$= \frac{1}{1-a}\sum_{n=0}^{\infty}(\frac{x-a}{1-a})^n$$

If we define $r = |1-a|$, then for $x \in (a-r, a+r) \subseteq R \setminus 1$, and using Lemma 7.3.3, we have

$$S = \frac{1}{1-a}\frac{1}{1-\frac{x-a}{1-a}} = \frac{1}{1-x}$$

To calculate the radius of convergence, note that as n approaches infinity, $(\frac{1}{1-a})^{(n+1)/n}$ approaches $1/(1-a)$. Hence, $R = |1-a| = r$.

Exercise 4.2.3

Exercise 4.2.3. Prove Proposition 4.2.6. (Hint: induct on k and use Theorem 4.1.6(d)).

Proposition 4.2.6 (Real analytic functions are k-times differentiable)**.** *Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and and let f be a function which is real analytic at a, thus there is an $r > 0$ for which we have the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$. Then for every $k \geq 0$, the function f is k-times differentiable on $(a-r, a+r)$, and for each $k \geq 0$ the k^{th} derivative is given by

$$\begin{aligned} f^{(k)}(x) &= \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\ldots(n+k)(x-a)^n \\ &= \sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(x-a)^n \end{aligned}$$

for all $x \in (a-r, a+r)$.

Note, by definition, $r \leq R$; hence, $R > 0$. This implies that for the base case, where $k = 1$, we can apply Theorem 4.1.6(d) to conclude that f is differentiable on $(a-R, a+R)$. It immediately follows that f is differentiable on $(a-r, a+r)$. We can also assert that

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$= \sum_{n=0}^{\infty}(n+1)c_{n+1}(x-a)^n$$
$$= \sum_{n=0}^{\infty}\frac{(n+1)!}{n!}c_{n+1}(x-a)^n$$
Which shows the base case.

By induction, we can assume that $f^{(k)}$ is differentiable on $(a-r, a+r)$, and that
$$f^{(k)}(x) = \sum_{n=0}^{\infty}\frac{(n+k)!}{n!}c_{n+k}(x-a)^n$$
Now,
$$R_k = (\lim sup_{n\to\infty}|c_{n+k}\frac{(n+k)!}{n!}|^{\frac{1}{n}})^{-1}$$
$$= (\lim sup_{n\to\infty}|c_{n+k}|^{\frac{1}{n}} \lim sup_{n\to\infty}|\frac{(n+k)!}{n!}|^{\frac{1}{n}})^{-1}$$
$$= (\lim sup_{n\to\infty}|c_{n+k}|^{\frac{1}{n}}$$
$$= R.$$
Hence, $R_k > 0$ and we can apply Theorem 4.1.6(d), and assert that $f^{(k+1)}$ is differentiable on $(a-R, a+R)$, and hence differentiable on $(a-r, a+r)$. We can also assert that
$$f^{(k+1)} = \sum_{n=1}^{\infty}\frac{(n+k)!}{n!}nc_{n+k}(x-a)^{n-1}$$
$$= \sum_{n=0}^{\infty}\frac{(n+k+1)!}{n!}c_{n+k+1}(x-a)^n$$

Exercise 4.2.4

Exercise 4.2.4. Use Proposition 4.2.6 and Exercise 4.2.1 to prove Corollary 4.2.10.

Corollary 4.2.10 (Taylor's formula). *Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and let $f : E \to \mathbf{R}$ be a function which is real analytic at a and has the power series expansion*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$ and some $r > 0$. Then for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! c_k,$$

where $k! := 1 \times 2 \times \ldots \times k$ (and we adopt the convention that $0! = 1$). In particular, we have Taylor's formula

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for all x in $(a-r, a+r)$.

From Proposition 4.2.6, we know that f is k-times differentiable. Hence, for any $k \leq 0$, we know that

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} (x-a)^n$$

Hence,

$$\begin{aligned} f^{(k)}(a) &= \sum_{n=0}^{\infty} c_{n+k} \frac{(n+k)!}{n!} 0^n \\ &= c_k k! + \sum_{n=1}^{\infty} c_{n+k} \frac{(n+k)!}{n!} 0^n \\ &= k! c_k \end{aligned}$$

Note,

$$\frac{f^{(n)}(a)}{n!} = \frac{n! c_n}{n!} = c_n$$

Hence, the second claim follows.

Exercise 4.2.5

Exercise 4.2.5. Let a, b be real numbers, and let $n \geq 0$ be an integer. Prove the identity

$$(x-a)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

for any real number x. (Hint: use the binomial formula, Exercise 7.1.4.) Explain why this identity is consistent with Taylor's theorem and Exercise 4.2.1. (Note however that Taylor's theorem cannot be rigorously applied until one verifies Exercise 4.2.6 below.)

$$(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$$

Since $(x-a)^n = (x-b+b-a)^n$, the equality follows easily.

To show that its consistent with Taylor's theorem (assuming Ex 4.2.6), note that we can write

$$f(x) = \sum_{m=0}^{\infty} \frac{f^m(b)}{m!}(x-b)^m$$

From Ex 4.2.1, we know that

$$f^m(b) = \frac{n!}{(n-m)!}(b-a)^{n-k}$$ for $0 \leq k \leq n$

And

$f^m(b) = 0$ for $k > n$

Hence,

$$f(x) = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m$$

Which is consistent with the result using the binomial formula.

Exercise 4.2.6

Exercise 4.2.6. Using Exercise 4.2.5, show that every polynomial $P(x)$ of one variable is real analytic on $\mathbf{R}$.

Definition 4.2.1 (Real analytic functions)**.** Let E be a subset of $\mathbf{R}$, and let $f : E \to \mathbf{R}$ be a function. If a is an interior point of E, we say that f is *real analytic at* a if there exists an open interval $(a-r, a+r)$ in E for some $r > 0$ such that there exists a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a-r, a+r)$. If E is an open set, and f is real analytic at every point a of E, we say that f is *real analytic on* E.

Let $a \in \mathbb{R}$.

Then

$$P(x) = \sum_{j=0}^{n} \alpha_j x^j$$

$$= \sum_{j=0}^{n} \alpha_j \sum_{m=0}^{j} \frac{j!}{m!(j-m)!} a^{j-m}(x-a)^m$$

$$= \sum_{j=0}^{n} \alpha_j \sum_{m=0}^{n} \mathbb{I}_{m \le j} \frac{j!}{m!(j-m)!} a^{j-m}(x-a)^m$$

$$= \sum_{m=0}^{n} (x-a)^m \sum_{j=0}^{n} \alpha_j \mathbb{I}_{m \le j} \frac{j!}{m!(j-m)!} a^{j-m}$$

$$= \sum_{m=0}^{\infty} c_m (x-a)^m$$

Where

$$c_m = \mathbb{I}_{m \le n} \sum_{j=0}^{n} \alpha_j \mathbb{I}_{m \le j} \frac{j!}{m!(j-m)!} a^{j-m}$$

Since $c_m = 0$ for all $m > n$, it follows that the radius of convergence is given by

$$R = \frac{1}{0} = +\infty$$

Exercise 4.2.7

Exercise 4.2.7. Let $m \geq 0$ be a positive integer, and let $0 < x < r$ be real numbers. Use Lemma 7.3.3 to establish the identity

$$\frac{r}{r-x} = \sum_{n=0}^{\infty} x^n r^{-n}$$

for all $x \in (-r, r)$. Using Proposition 4.2.6, conclude the identity

$$\frac{r}{(r-x)^{m+1}} = \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n}$$

for all integers $m \geq 0$ and $x \in (-r, r)$. Also explain why the series on the right-hand side is absolutely convergent.

Lemma 7.3.3 (Geometric series). *Let x be a real number. If $|x| \geq 1$, then the series $\sum_{n=0}^{\infty} x^n$ is divergent. If however $|x| < 1$, then the series is absolutely convergent and*

$$\sum_{n=0}^{\infty} x^n = 1/(1-x).$$

Proposition 4.2.6 (Real analytic functions are k-times differentiable).

Since $|x/r| < 1$ for $x \in (-r, r)$, we have using Lemma 7.3.3

$$\sum_{n=0}^{\infty} (\frac{x}{r})^n = 1/(1-(x/r)) = r/(1-r)$$

By applying Proposition 4.2.6 to the RHS of the identity, which is a power series with $c_n = r^{-n}$ and $a = 0$,

$$f^m(x) = \sum_{n=0}^{\infty} c_{n+m} \frac{(n+m)!}{n!} (x-a)^n$$

$$= \sum_{n=m}^{\infty} c_n \frac{n!}{(n-m)!} (x-a)^{n-m}$$

It can be shown by induction that for th LHS of the identity

$$f^m(x) = \frac{r(m!)}{(r-m)^{m+1}}$$

And so the result follows easily.

Let $0 \leq x \leq r$, then

$$\sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!} x^{n-m} r^{-n} = \sum_{n=m}^{\infty} |\frac{n!}{m!(n-m)!} x^{n-m} r^{-n}|$$

The RHS converges since the LHS converges, so the series is absolutely convergent.
Now let $-r \leq x \leq 0$, then

$$\sum_{n=m}^{\infty} |\frac{n!}{m!(n-m)!} x^{n-m} r^{-n}| = \sum_{n=m}^{\infty} |\frac{n!}{m!(n-m)!} x_0^{n-m} r^{-n}|$$

Where $x_0 = -x$.
We know the RHS converges and so the LHS does as well, so the series is absolutely convergent.

Exercise 4.2.8

Exercise 4.2.8. Let E be a subset of $\mathbf{R}$, let a be an interior point of E, and let $f : E \to \mathbf{R}$ be a function which is real analytic in a, and has a power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

at a which converges on the interval $(a-r, a+r)$. Let $(b-s, b+s)$ be any sub-interval of $(a-r, a+r)$ for some $s > 0$.

(a) Prove that $|a-b| \leq r-s$, so in particular $|a-b| < r$.

(b) Show that for every $0 < \varepsilon < r$, there exists a $C > 0$ such that $|c_n| \leq C(r-\varepsilon)^{-n}$ for all integers $n \geq 0$. (Hint: what do we know about the radius of convergence of the series $\sum_{n=0}^{\infty} c_n(x-a)^n$?)

(c) Show that the numbers $d_0, d_1, \dots$ given by the formula

$$d_m := \sum_{n=m}^{\infty} \frac{n!}{m!(n-m)!}(b-a)^{n-m} c_n \text{ for all integers } m \geq 0$$

are well-defined, in the sense that the above series is absolutely convergent. (Hint: use (b) and the comparison test, Corollary 7.3.2, followed by Exercise 4.2.7.)

(d) Show that for every $0 < \varepsilon < s$ there exists a $C > 0$ such that

$$|d_m| \leq C(s-\varepsilon)^{-m}$$

for all integers $m \geq 0$. (Hint: use the comparison test, and Exercise 4.2.7.)

(e) Show that the power series $\sum_{m=0}^{\infty} d_m(x-b)^m$ is absolutely convergent for $x \in (b-s, b+s)$ and converges to $f(x)$. (You may need Fubini's theorem for infinite series, Theorem 8.2.2, as well as Exercise 4.2.5).

(f) Conclude that f is real analytic at every point in $(a-r, a+r)$.

(a)

Since $(b-s, b+s) \subseteq (a-r, a+r)$, we know

$a - r \leq b - s$, and so $a - b \leq r - s$

$b + s \leq a + r$, and so $s - r \leq a - b$

Hence,

$$|a-b| \leq r-s$$

The second equality follows since $r-s<r$ and $-r<s-r$.

(b)

Since the series converges, we know that

$$r \leq R = \frac{1}{L}$$

Where

$$L = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

Since the $|c_n|^{\frac{1}{n}}$ are all positive, it follows that

$L \leq |c_n|^{\frac{1}{n}}$ for all n

And consequently

$$L^n \leq |c_n|$$

Note also, for all $0<\epsilon<r$

$$r-\epsilon \leq \frac{1}{L}-\epsilon < \frac{1}{L}$$

Hence,

$$L < \frac{1}{r-\epsilon}$$
$$L^n < (r-\epsilon)^{-n}$$

Let

$$M = |c_n| - L^n \geq 0$$
$$N = (r-\epsilon)^{-n} - L^n > 0$$
$$m = \max(M, N)$$

Then,

$$C = \frac{(r-\epsilon)^{-n}+m}{(r-\epsilon)^{-n}}$$

If $m = M = |c_n| - L^n$, then

$$C(r-\epsilon)^{-n} = |c_n| + \delta \geq |c_n|$$

Where $\delta = (r-\epsilon)^{-n} - L^n$, which is guaranteed to be positive since $N>0$.

If $m=N$, then, $N \geq M$, and

$$\begin{aligned} C(r-\epsilon)^{-n} &= (r-\epsilon)^{-n} + (r-\epsilon)^{-n} - L^n \\ &\geq (r-\epsilon)^{-n} + |c_n| - L^n \end{aligned}$$

And the same logic applies.

c)

Corollary 7.3.2 (Comparison test). *Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact*

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Since $|b-a| < r$, we know that there is some ϵ such that $|a-b| < r-\epsilon$.
We choose this ϵ when applying (b) so that the application of Ex 4.2.7 is valid.
We also assume that $b-a > 0$.

$$\begin{aligned}
|d_m| &= |\sum_{n=m}^{\infty} \frac{n!}{m(n-m)!}(b-a)^{n-m} c_n| \\
&= |\sum_{n=m}^{\infty} \frac{n!}{m(n-m)!}(b-a)^{n-m}||c_n| \\
&\leq |\sum_{n=m}^{\infty} \frac{n!}{m(n-m)!}(b-a)^{n-m}|C(r-\epsilon)^{-n} && \text{(b)} \\
&= C\sum_{n=m}^{\infty} \frac{n!}{m(n-m)!}(b-a)^{n-m}(r-\epsilon)^{-n} \\
&= C\frac{r-\epsilon}{(r-\epsilon-(b-a))^{m+1}} && \text{(Ex 4.2.7)}
\end{aligned}$$

From this, the comparison test shows that d_m is convergent.

d)
Let $0 < \epsilon < s$.
From c), we know that

$$\begin{aligned}
|d_m| &\leq |\sum_{n=m}^{\infty} \frac{n!}{m(n-m)!}(b-a)^{n-m}|C(r-\epsilon)^{-n} \\
&= C\frac{r-\epsilon}{(r-\epsilon-(b-a))^{m+1}}
\end{aligned}$$

If $b-a > 0$, then from a) we know $b-a \leq r-s$ so that $s \leq r-(b-a)$ and $s-\epsilon \leq r-\epsilon-(b-a)$.
If $b-a < 0$, then $r-(b-a) > r \geq s$ and $s-\epsilon \leq r-\epsilon-(b-a)$
Either way, it follows that

$$\frac{1}{(r-\epsilon-(b-a))^{m+1}} \leq \frac{1}{(s-\epsilon)^{m+1}}$$

By defining $D = C(r-\epsilon)(s-\epsilon)$, we have
$|d_m| \leq D(s-\epsilon)^{-m}$
Which shows the result.

e)

Theorem 8.2.2 (Fubini's theorem for infinite sums). *Let $f : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$ be a function such that $\sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m)$ is absolutely convergent.*

Then we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} f(n,m)\right) &= \sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m) \\ &= \sum_{(m,n)\in\mathbf{N}\times\mathbf{N}} f(n,m) \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} f(n,m)\right).\end{aligned}$$

We use the comparison test to show that the series is absolutely convergent.
Since $b - s < x < b + s$ we know that there is some ϵ such that $-s + \epsilon < x < s - \epsilon$, so we choose this one and apply d) to write

$$|d_m(x-b)^m| \leq C(s-\epsilon)^{-m}|(x-b)^m|$$

Since $|x - b| < s - \epsilon$, it follows that

$$\frac{|x-b|}{s-\epsilon} < 1$$

And in particular

$$\sum_{m=0}^{\infty}\left(\frac{|x-b|}{s-\epsilon}\right)^m$$

Is an absolutely convergent geometric series.
Hence, we can apply Fubini's theorem and write

$$\begin{aligned}\sum_{m=0}^{\infty} d_m(x-b)^m &= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{n!}{m!(n-m)!}(b-a)^{n-m}c_n(x-b)^m \\ &= \sum_{n=m}^{\infty} c_n \sum_{m=0}^{\infty}\frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m \\ &= \sum_{n=m}^{\infty} c_n \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}(b-a)^{n-m}(x-b)^m \\ &= \sum_{n=m}^{\infty} c_n(x-a)^n \\ &= \sum_{n=0}^{\infty} c_n(x-a)^n\end{aligned}$$

$= f(x)$

f)

For any point $b \in (a-r, a+r)$, we can choose $s > 0$ such that $(b-s, b+s)$ is a sub-interval, and consequently, we have

$$f(x) = \int_{m=0}^{\infty} d_m (x-b)^m$$

Which shows that a convergent power series exists on the sub-interval. The conclusion follows.

Section 3

Theorem 4.3.1

Theorem 4.3.1 (Abel's theorem). *Let $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ be a power series centered at a with radius of convergence $0 < R < \infty$. If the power series converges at $a+R$, then f is continuous at $a+R$, i.e.*

$$\lim_{x \to a+R: x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

Similarly, if the power series converges at $a-R$, then f is continuous at $a-R$, i.e.

$$\lim_{x \to a-R: x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n (-R)^n.$$

First, we make certain substitutions

$$d_n = c_n R^n$$
$$y = \frac{x-a}{R}$$

Then the above claim can be rewritten as

$$\lim_{y \to 1, y \in (-1,1)} \sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} d_n$$

This follows since

$$(x-a)^n = y^n R^n c_n = d_n y^n$$

And as $x \to a+R$, $y \to 1$.

Now define

$$D = \sum_{n=0}^{\infty} d_n$$

And

$$S_N = (\sum_{n=0}^{N-1} d_n) - D$$ for every $N \geq 0$

Note that

$$S_0 = -D$$

$$\lim_{N \to \infty} S_N = 0$$

Also note

$$S_{n+1} - S_n = (\sum_{m=0}^{n} d_m) - D - (\sum_{m=0}^{n-1} d_m) + D$$
$$= d_n$$

This means that we can write for any $y \in (-1, 1)$

$$\sum_{n=0}^{\infty} d_n y^n = \sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n$$

Here we apply Lemma 4.3.2, the summation by parts formula. In order to do this, we require that the sequence $(S_n)_{n=0}^{\infty}$ converges, that the sequence $(y^n)_{n=0}^{\infty}$ converges, and that the series $\sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n$ also converges.

This follows since

$\lim_{n \to \infty} S_n = 0$, as noted above

$\lim_{n \to \infty} y^n = 0$, since $|y| \leq 1$

$\sum_{n=0}^{\infty} (S_{n+1} - S_n) y^n = \sum_{n=0}^{\infty} d_n y^n = f(x)$ by definition.

Hence,

$$\sum_{n=0}^{\infty} d_n y^n = 0 \cdot 0 - (-D) y^0 - \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n)$$
$$= D - \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n)$$

If we can show that

$$\lim_{y \to 1} : y \in (-1, 1) \sum_{n=0}^{\infty} S_{n+1} (y^{n+1} - y^n) = 0$$

Then the claim follows since we will have

$$\sum_{n=0}^{\infty} d_n y^n = D = \sum_{n=0}^{\infty} d_n$$

Since y converges to 1, we may as well restrict y to $[0,1)$. In particular, we will take y to be positive.

Next, we will apply Proposition 7.2.9, which requires that the series

$$\sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n)$$

Is absolutely convergent.

Note that since the S_{n+1} are partial sums of a convergent series, we know that the $|S_{n+1}|$ are bounded by some $M > 0$.

Since also $y \in [0,1)$, we have

$$|y^{n+1} - y^n| \leq y^n(1-y)$$

Hence, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} |S_{n+1}(y^{n+1} - y^n)| &\leq \sum_{n=0}^{\infty} M y^n (1-y) \\ &= M(1-y)\sum_{n=0}^{\infty} y^n \\ &= \frac{M(1-y)}{1-y} = M \end{aligned}$$

Where we have used the fact that $\sum_{n=0}^{\infty} y^n$ is a geometric series.

Since

$$\sum_{n=0}^{\infty} |S_{n+1}(y^{n+1} - y^n)| \leq M$$

It follows that the series is absolutely convergent, and also conditionally convergent. Since also each S_{n+1} is negative and $y^{n+1} - y^n$ is also negative (for $0 \leq y \leq 1$), it follows that

$$0 \leq \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n)$$

From Proposition 7.2.9, we also know that

$$\begin{aligned} \left|\sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n)\right| &\leq \sum_{n=0}^{\infty} |S_{n+1}(y^{n+1} - y^n)| \\ &= \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \end{aligned}$$

So we can write,

$$0 \leq \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n) \leq \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1})$$

If we can show that the last term converges to 0, then we arrive at the desired result by an application of the squeeze text (Corollary 6.4.14).
Since

$$\sum_{n=1}^{\infty} |S_{n+1}|(y^n - y^{n+1})$$

is non-negative, it will suffice to show

$$\lim \sup_{y \to 1: y \in [0,1)} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) = 0$$

Let $\epsilon > 0$. Since S_n converges to 0, there exists an N such that $|S_n| \leq \epsilon$ for all $n > N$.
Hence,

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n - y^{n+1}) + \sum_{n=N+1}^{\infty} \varepsilon(y^n - y^{n+1})$$

Notice that since y^n converges to 0, the last summation is a telescoping series which sums to ϵy^{N+1} (Lemma 7.2.15).
Hence,

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n - y^{n+1}) + \varepsilon y^{N+1}$$

Note that from Exercise 7.1.5, we have

$$\lim_{n \to \infty} \sum_{x \in X} a_n(x) = \sum_{x \in X} \lim_{n \to \infty} a_n(x)$$

Which allows us to interchange limits and finite sums, assuming that $a_n(x)$ is convergent for all x. This is a slightly confusing equation because in our case the limit is over y and the sum is over n, while in Exercise 7.1.5, the sum is over x and the limit is over n. In any case, this is valid because for every n, $|S_{n+1}|(y^n - y^{n+1}) \to 0$ as $y \to 1$.

We apply this below

$$\begin{aligned}
\limsup_{y \to 1} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) &\leq \limsup_{y \to 1} \sum_{n=0}^{N} |S_{n+1}|(y^n - y^{n+1}) + \limsup_{y \to 1} \epsilon y^{N+1} \\
&= \sum_{n=0}^{N} \limsup_{y \to 1} |S_{n+1}|(y^n - y^{n+1}) + \limsup_{y \to 1} \epsilon y^{N+1} \\
&= \epsilon
\end{aligned}$$

Where we have used the fact that $y^n - y^{n+1} \to 0$ as $y \to 1$.

Since ϵ was arbitrary, it follows that

$$\limsup_{y\to 1} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq 0$$

And the claim follows.

This is a very tricky proof, in the sense that it relies on very little intuition of the problem, but rather a specific set of mathematical manipulations to yield the result. The main flow of the proof starts once it is established that it suffices to show

$$\lim_{y\to 1} : y \in (-1,1) \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n) = 0$$

As this is the final form so to speak. Then, Proposition 7.2.9 is applied to express the problem in terms of an inequality that can be resolved by the squeeze theorem. The right most terms of this inequality is then expressed in terms of the $\limsup$ *rather than the direct* $\lim$*. Why this is done is not clear in the proof; however, it likely makes the application of the squeeze theorem simpler. Further manipulations are done to simplify the RHS, where finally an interchange of the finite sum and limit is performed. In the book, the limit becomes over* $n \to \infty$ *rather than over* $herthanover\$\$$*r than over* $anover\$\$$ *over* $r\$\$y$ *1\$. Whether this is valid from Exercise 7.1.5 is also unclear.*

Exercise 4.3.1

Exercise 4.3.1. Prove Lemma 4.3.2. (Hint: first work out the relationship between the partial sums $\sum_{n=0}^{N}(a_{n+1} - a_n)b_n$ and $\sum_{n=0}^{N} a_{n+1}(b_{n+1} - b_n)$.)

Lemma 4.3.2 (Summation by parts formula). *Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers which converge to limits A and B respectively, i.e., $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose that the sum $\sum_{n=0}^{\infty}(a_{n+1} - a_n)b_n$ is convergent. Then the sum $\sum_{n=0}^{\infty} a_{n+1}(b_{n+1} - b_n)$ is also convergent, and*

$$\sum_{n=0}^{\infty}(a_{n+1} - a_n)b_n = AB - a_0 b_0 - \sum_{n=0}^{\infty} a_{n+1}(b_{n+1} - b_n).$$

Let A_N and B_N be the partial sums

$$A_N = \sum_{n=0}^{N} (a_{n+1} - a_n) b_n$$
$$B_N = \sum_{n=0}^{N} a_{n+1}(b_{n+1} - b_n)$$

Note that

$$A_N = \sum_{n=0}^{N} a_{n+1} b_n - a_n b_n$$
$$= (\sum_{n=0}^{N} a_{n+1} b_n) - a_0 b_0 - \sum_{n=1}^{N} a_n b_n$$

And

$$B_N = \sum_{n=0}^{N} a_{n+1} b_{n+1} - a_{n+1} b_n$$
$$= (\sum_{n=0}^{N-1} a_{n+1} b_{n+1}) + a_{N+1} b_{N+1} - \sum_{n=0}^{N} a_{n+1} b_n$$
$$= (\sum_{n=1}^{N} a_n b_n) + a_{N+1} b_{N+1} - \sum_{n=0}^{N} a_{n+1} b_n$$

It easily follows then that

$A_N + B_N = a_{N+1} b_{N+1} - a_0 b_0$

So that

$B_N = a_{N+1} b_{N+1} - a_0 b_0 - A_N$

Taking limits,

$$\lim_{N \to \infty} B_N = AB - a_0 b_0 - \sum_{n=0}^{\infty} (a_{n+1} - a_n) b_n$$

Where we have applied Theorem 6.1.19(b).

This shows that B_N is convergent, and consequently that $\sum_{n=0}^{\infty} a_{n+1}(b_{n+1} - b_n)$, since all terms on the RHS are convergent (and Theorem 6.1.19(a)).

The second claim easily follows by rearranging the limit of B_N.

This was a straightforward proof, that only required simple manipulations of the finite series and simple applications of limit laws.

Section 4

Theorem 4.4.1

Theorem 4.4.1. *Let $f : (a-r, a+r) \to \mathbf{R}$ and $g : (a-r, a+r) \to \mathbf{R}$ be functions analytic on $(a-r, a+r)$, with power series expansions*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} d_n (x-a)^n$$

respectively. Then $fg : (a-r, a+r) \to \mathbf{R}$ is also analytic on $(a-r, a+r)$, with power series expansion

$$f(x)g(x) = \sum_{n=0}^{\infty} e_n (x-a)^n$$

where $e_n := \sum_{m=0}^{n} c_m d_{n-m}$.

Let $x \in (a-R, a+R)$.
From Theorem 4.1.6(b), we know that the series expansions of $f(x)$ and $g(x)$ are absolutely convergent.
Hence, we can write

$$C = \sum_{n=0}^{\infty} |c_n (x-a)^n|$$

$$D = \sum_{n=0}^{\infty} |d_n (x_a)^n|$$

Note that for every $N \geq 0$, the below partial sum can be written as

$$\begin{aligned}
\sum_{n=0}^{N} \sum_{m=0}^{\infty} |c_m (x-a)^m d_n (x-a)^n| &= \sum_{n=0}^{N} |d_n (x-a)^n| \sum_{m=0}^{\infty} |c_m (x-a)^m| \\
&= \sum_{n=0}^{N} |d_n (x-a)^n| C \\
&\leq DC
\end{aligned}$$

This shows that the series of partial sums is convergent.

In particular,

$$\sum_{n=0}^{\infty} d_n(x-a)^n \sum_{m=0}^{\infty} c_m(x-a)^m$$

Is absolutely convergent.

This means we can write

$$\begin{aligned}
f(x)g(x) &= \sum_{n=0}^{\infty} d_n(x-a)^n \sum_{m=0}^{\infty} c_m(x-a)^m \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_m d_n (x-a)^{n+m} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_m d_n (x-a)^{n+m} && \text{(Theorem 8.2.2)} \\
&= \sum_{m=0}^{\infty} \sum_{n'=m}^{\infty} c_m d_{n'-m} (x-a)^{n'} && \text{(n'=n+m)} \\
&= \sum_{m=0}^{\infty} \sum_{n'=0}^{\infty} c_m d_{n'-m} (x-a)^{n'} && (d_j = 0 \text{ for all } j < 0) \\
&= \sum_{n'=0}^{\infty} \sum_{m=0}^{\infty} c_m d_{n'-m} (x-a)^{n'} && \text{(Theorem 8.2.2)} \\
&= \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{\infty} c_m d_{n'-m} \\
&= \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{n'} c_m d_{n'-m} && (d_j = 0 \text{ for all } j < 0) \\
&= \sum_{n'=0}^{\infty} e_{n'} (x-a)^{n'}
\end{aligned}$$

Note that the application of Theorem 8.2.2 is valid because the series is absolutely convergent. We also assume the convention that $d_j = 0$ for all $j < 0$.

This proof is relatively straightforward. It applies an interesting trick to show the convergence of $$f(x)g(x)$$, as it takes a partial sum whose inner term is an infinite series. Then it applies simple manipulations to get the desired form.

Section 5

Exercise 4.5.1

Exercise 4.5.1. Prove Theorem 4.5.2. (Hints: for part (a), use the ratio test. For parts (bc), use Theorem 4.1.6. For part (d), use Theorem 4.4.1. For part (e), use part (d). For part (f), use part (d), and prove that $\exp(x) > 1$ when x is positive. You may find the binomial formula from Exercise 7.1.4 to be useful.