Chapter 2

Section 1

Exercise 2.1.1

Exercise 2.1.1. Prove Theorem 2.1.4. (Hint: review your proof of Proposition 9.4.7.)

Theorem 2.1.4 (Continuity preserves convergence). *Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function,*

and let $x_0 \in X$ be a point in X. Then the following three statements are logically equivalent:

(*a*) *f is continuous at x_0.*

(*b*) *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*

(*c*) *For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$.*

(a) ⇒ (b)
Let $\epsilon > 0$ be some real. Since f is continuous, we know that there is some δ such that $d_X(x^{(n)}, x_o) < \delta \to d_Y(f(x^n), f(x_o)) < \epsilon$. Let $(x^{(n)})_{n=1}^{\infty}$ be some sequence in X that converges to x_o. We know that there is some $N \geq m$ such that $\forall n \geq N(d_X(x^{(n)}, x_o)) < \delta$. It then immediately follows that $\forall n \geq N d_Y(f(x^n), f(x_o)) < \delta$ for this N. Since ϵ was arbitrary, we have shown that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_o)$.

(a) ⇒ (c)

Let $V \subset Y$ be an open set, and $f(x_o) \in V$. Since it is open, we know $B(f(x_o), r) \subseteq V$ for some r. From (a), we can write $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < r$ for some $\delta > 0$. We the construct the ball $U := B(x_o, \delta) \subset X$, which is obviously open. We know that for every element, $x \in U$, it follows that $d_Y(f(x), f(x_o)) < r$, and hence $f(x) \in V$ since $B(f(x_o), r) \subseteq V$. Hence, $f(U) \subseteq V$. Since V was arbitrary, the claim follows.

This proof uses the definition that every element of an open set is an interior point. It takes the ball centred at $f(x_o)$ in V, then constructs a corresponding ball, which is a subset of X. This ball in X is our U, and since all balls are open, we only need to show that the image of the ball is a subset of V. This follows from the continuity at x_o, and the fact that we chose the radius of U wisely.

(c) ⇒ (b)
Let $(x^{(n)})_{n=m}^{\infty}$ converge to x_o in X. We can write $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(d(x^{(n)}, x_o) < \epsilon)$. Let $\epsilon > 0$ be some real. We know $B(f(x_o), \epsilon) \subset Y$ is open. From (c) this implies that there is some open $U \subset X$ that contains x_o such that $f(U) \subseteq B(f(x_o), \epsilon)$. Since U is open, we know that $B(x_o, r) \subseteq U$ for some r. Now, we can write $\forall n \geq N(d(x^{(n)}, x_o) < r)$ for some $N \geq m$, since the sequence converges to x_o. It follows then that $x^{(n)} \in B(x_o, r) \subseteq U$ for every $n \geq N$. Since $f(U) \subseteq B(f(x_o), \epsilon)$, it follows that $\forall n \geq N d_Y(f(x^{(n)}), f(x_o))$. Since ϵ was arbitrary, we can write $\forall \epsilon > 0 \exists N \geq m \forall n \geq N(d_Y(f(x^{(n)}), f(x_o)) < \epsilon)$, which is the desired result.

Let some sequence converge to x_o. We construct a ball centred at $f(x_o)$ with radius ϵ. By assumption, since the ball is open, we can assert that there is some open U in X that contains x_o and whose image is contained in the ball. We then construct another ball, which is fully contained in U, since U is open. This allows us to assert that there is some N after which all $x^{(n)}$ are contained in this second ball. Since the image of U is contained in the first ball, it follows that all $f(x^{(n)})$ after N have a distance with $f(x_o)$ less than ϵ. Since ϵ was arbitrary, this shows that the $f(x^{(n)})$ converge to $f(x_o)$.

(b) ⇒ (a)
Assume that f is not continuous. Then, there is some $\epsilon > 0$, for every $\delta > 0$ for some x such that $d_X(x_o, x^{(n)}) < \delta$ and $d_Y(f(x^n), f(x_o)) \geq \epsilon$. We can assert that $(x^{(n)})_{n=1}^{\infty}$ converges to x_o, using the axiom of choice since $d_X(x_o, x^{(n)}) < \delta$ for every $\delta = 1$. We can also show that $(f(x^{(n)}))_{n=1}^{\infty}$ does not converge to $f(x_o)$, since $d_Y(f(x^n), f(x_o)) \geq \epsilon$ for every n. However, this contradicts (b), which we assumed, hence, f is continuous.

This part of the proof works by contradiction. Without using contradiction, finding a δ *for the definition of continuity is required. However, once* f *is assumed to not be continuous, the contradiction of* (b) *naturally follows.*

Exercise 2.1.2

Exercise 2.1.2. Prove Theorem 2.1.5. (Hint: Theorem 2.1.4 already shows that (a) and (b) are equivalent.)

Theorem 2.1.5. *Let* (X, d_X) *be a metric space, and let* (Y, d_Y) *be another metric space. Let* $f : X \to Y$ *be a function. Then the following four statements are equivalent:*

(*a*) f *is continuous.*

(*b*) *Whenever* $(x^{(n)})_{n=1}^{\infty}$ *is a sequence in* X *which converges to some point* $x_0 \in X$ *with respect to the metric* d_X, *the sequence* $(f(x^{(n)}))_{n=1}^{\infty}$ *converges to* $f(x_0)$ *with respect to the metric* d_Y.

(*c*) *Whenever* V *is an open set in* Y, *the set* $f^{-1}(V) := \{x \in X : f(x) \in V\}$ *is an open set in* X.

(*d*) *Whenever* F *is a closed set in* Y, *the set* $f^{-1}(F) := \{x \in X : f(x) \in F\}$ *is a closed set in* X.

(a) $\Longleftrightarrow$ (b)
This follows easily from Theorem 2.1.4.

(a) $\Rightarrow$ (c)
Let $V \subseteq Y$ be an open set. Since f is continuous at every $x_o \in X$, we can write, using Theorem 2.1.4, $\forall x_o \in X \exists U, open, U \subset X(f(U) \subseteq V)$. Using the axiom of choice, we can rewrite this as $\forall x_o \in X(U_{x_o} \subset X, open, f(U_{x_o}) \subseteq V)$. We then define $Z := \cup_{x_o \in X} U_{x_o}$. If $x \in Z$, then we know $x \in U_x$. Since $f(U_x) \subseteq V$, it follows that $f(x) \in V$ and hence $x \in f^{-1}(V)$. If, on the other hand, $x \in f^{-1}(V)$, then we know $x \in X$, and hence, $x \in U_x$, so $x \in Z$. Hence, it follows that $Z = f^{-1}(V)$. From Proposition 1.2.15(g), we know that Z is open. Hence, $f^{-1}(V)$ is open.

Here, since f *is continuous, we are constructing, at every point* $x_o \in X$, *an open set* U *whose image is a subset of* V. *Then, using the axiom of choice, we are taking the union of these open* *sets, which itself is an open set. We then show that every element of this union is an element of $f^{-1}(V)$, and vice versa, Hence, $f^{-1}(V)$ is open.*

(c) ⇒ (a)
Let $x_o \in X$. Let $W \subset Y$ be open and contain $f(x_o)$. By assumption, we know that $f^{-1}(W)$ is open in X. It follows that $x_o \in f^{-1}(W)$ and $f(f^{-1}(W)) \subseteq W$. Since W was arbitrary, and using Theorem 2.1.4, we can assert that f is continuous as x_o. Since x_o was arbitrary, we have shown that f is continuous.

This proof is rather simple, as it is following straight from the definitions. We construct the required definition using the assumption for every x_o. It follows that f is continuous.

(c) ⇒ (d)
Let $F \subset Y$ be closed. Then $Y \setminus F$ is open. Hence, from (c), we know that $f^{-1}(Y \setminus F) := \{x \in X : f(x) \in Y \setminus F\}$ is open. Then $X \setminus f^{-1}(Y \setminus F)$ is closed. Define $f^{-1}(F) := \{x \in X : f(x) \in F\}$. Let $x \in f^{-1}(F)$. Hence, $f(x) \in F$, and it follows that $f(x) \in Y \setminus F$, so $x \in f^{-1}(Y \setminus F)$, and $x \in X \setminus f^{-1}(Y \setminus F)$. Let $x \in X \setminus f^{-1}(Y \setminus F)$. Then $x \notin f^{-1}(Y \setminus F)$, and $f(x) \notin Y \setminus F$. Since $f(x) \in Y$, it follows that $f(x) \in F$. Hence, $x \in f^{-1}(F)$. Hence, $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$. It follows that $f^{-1}(F)$ is closed, since $X \setminus f^{-1}(Y \setminus F)$ is closed.

This proof works by taking complements on open and closed sets. The two propositions are mirrors of each other, so the result follows easily.

(d) ⇒ (c)
By way of a similar argument as (c) ⇒ (d), we can show the desired result.

Exercise 2.1.3

Exercise 2.1.3. Use Theorem 2.1.4 and Theorem 2.1.5 to prove Corollary 2.1.7.

Corollary 2.1.7 (Continuity preserved by composition). *Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces.*

(a) *If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $g \circ f(x) := g(f(x))$, is continuous at x_0.*

(b) *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.*

(a)
Let $\epsilon > 0$ be some real. Then we know there is some δ such that $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < \epsilon$. We can also write $d_Y(f(x), f(x_o)) < \epsilon \to d_Z(g(f(x)), g(f(x_o))) < \epsilon$. Hence, it follows that $d_X(x, x_o) < \delta \to d_Z(g(f(x)), g(f(x_o))) < \epsilon$. Since ϵ was arbitrary, we have shown that $g(f(x))$ is continuous at x_o.

This argument follows from the definition. It connects the two definitions very naturally.

(b)
This easily follows from (a).

Exercise 2.1.4

Exercise 2.1.4. Give an example of functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ such that

(a) f is not continuous, but g and $g \circ f$ are continuous;

(b) g is not continuous, but f and $g \circ f$ are continuous;

(c) f and g are not continuous, but $g \circ f$ is continuous.

Explain briefly why these examples do not contradict Corollary 2.1.7.

(a)

$$f(x) = \begin{cases} 3, & x > 0 \\ 5, & x \leq 0 \end{cases}$$

$g(y) = 4$.

$g(f(x)) = 4$.

(b)
$g(y) = 1/(x-2)$ and $f(x) = 4$. $g(f(x)) = 1/2$.

(c)
$$g(y) = \begin{cases} 1/(y-2), & y < 5 \\ 1/(y-4), & y \geq 5 \end{cases}$$
$$f(x) = \begin{cases} 3, & x > 0 \\ 5, & x \leq 0 \end{cases}$$
$$g(f(x)) = \begin{cases} 1/(3-2), & x > 0 \\ 1/(5-4), & x \leq 0 \end{cases} = 1$$

This doesn't contradict Corollary 2.1.7 because the corollary requires that both g and f are continuous.

Exercise 2.1.5

Exercise 2.1.5. Let (X, d) be a metric space, and let $(E, d|_{E\times E})$ be a subspace of (X, d). Let $\iota_{E\to X} : E \to X$ be the inclusion map, defined by setting $\iota_{E\to X}(x) := x$ for all $x \in E$. Show that $\iota_{E\to X}$ is continuous.

Let $x_o \in E$. Let $\epsilon > 0$ be some real. Let $d_E(x, x_o) < \epsilon$. Then it immediately follows that $d_X(x, x_o) < \epsilon$. Since $\iota_{E\to X}(x) = x$ and $\iota_{E\to X}(x_o) = x_o$, we can write $d_X(\iota_{E\to X}(x), \iota_{E\to X}(x_o)) < \epsilon$. Since ϵ was arbitrary, we have shown that $\iota_{E\to X}$ is continuous at x_o. Since x_o was arbitrary, we have shown that $\iota_{E\to X}$ is continuous.

Exercise 2.1.6

Exercise 2.1.6. Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E\times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus $f|_E(x) := f(x)$ when $x \in E$. If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.) Conclude that if f is continuous, then $f|_E$ is continuous. Thus restriction of the domain of a function does not destroy continuity. (Hint: use Exercise 2.1.5.)

Let $\epsilon >$ be some real. Then we know $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < \epsilon$ for some δ. Assume $d_E(x, x_o) < \delta$. Then it immediately follows that $d_X(x, x_o)$, and hence $d_Y(f(x), f(x_o)) < \epsilon$. Hence, $d_E(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < \epsilon$. Since ϵ was arbitrary, we have shown that $f|_E$ is continuous at x_o.
It follows easily that $f|_E$ is continuous if f is continuous.

The converse is not true because we can have a discontinuous function that is continuous over certain subsets of the domain. In this case, restricting the function to the domain subset, we have a continuous function. However, we could not then infer that the unrestricted function is continuous.

Exercise 2.1.7

Exercise 2.1.7. Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Suppose that the image $f(X)$ of X is contained in some subset $E \subset Y$ of Y. Let $g : X \to E$ be the function which is the same as f but with the range restricted from Y to E, thus $g(x) = f(x)$ for all $x \in X$. We give E the metric $d_Y|_{E \times E}$ induced from Y. Show that for any $x_0 \in X$, that f is continuous at x_0 if and only if g is continuous at x_0. Conclude that f is continuous if and only if g is continuous. (Thus the notion of continuity is not affected if one restricts the range of the function.)

Assume f is continuous at x_o. Let $\epsilon > 0$. Then we know $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < \epsilon$ for some δ. Then assume $d_X(x, x_o) < \delta$. Hence, $d_Y(f(x), f(x_o)) < \epsilon$. It immediately follows that $d_E(g(x), g(x_o)) < \epsilon$. Since ϵ was arbitrary, we have shown that g is continuous as x_o.
From a similar argument, we can shown that g is continuous as x_o implies f is continuous at x_o.
It follows easily that f is continuous if and only if g is continuous.

Section 2

Corollary 2.2.3

Corollary 2.2.3. *Let (X, d) be a metric space, let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Let c be a real number.*

(a) *If $x_0 \in X$ and f and g are continuous at x_0, then the functions $f+g : X \to \mathbf{R}$, $f-g : X \to \mathbf{R}$, $fg : X \to \mathbf{R}$, $\max(f,g) : X \to \mathbf{R}$, $\min(f,g) : X \to \mathbf{R}$, and $cf : X \to \mathbf{R}$ (see Definition 9.2.1 for definitions) are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbf{R}$ is also continuous at x_0.*

(b) *If f and g are continuous, then the functions $f+g : X \to \mathbf{R}$, $f-g : X \to \mathbf{R}$, $fg : X \to \mathbf{R}$, $\max(f,g) : X \to \mathbf{R}$, $\min(f,g) : X \to \mathbf{R}$, and $cf : X \to \mathbf{R}$ are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \mathbf{R}$ is also continuous at x_0.*

(a)
Since f and g are continuous at x_o, it follows from Lemma 2.2.1 that $f \oplus g$ is continuous at x_o. Since the function $(x, y) \mapsto x + y$ is continuous, it follows that it is continuous at $(fg)(x_o)$. Since $(x, y)((f \bigoplus g)(x_o)) = (x, y)(f(x_o), g(x_o)) = f(x_o) + g(x_o) = (f + g)(x_o)$, it follows that $f + g$ is continuous at x_o because continuity is preserved by composition, Corollary 2.1.7. A similar argument follows for the other functions, except that the range for f/g needs to be restricted to $\mathbb{R} \setminus \{0\}$, and we need the result in Exercise 2.1.7.

(b) immediately follows from (a).

This proof relies on Lemma 2.2.2. However, the problem is that the functions in Lemma 2.2.2 require a tuple, while we are proving a property for a function that takes in only one value. This is resolved by using Lemma 2.2.1, which takes in a single value and returns a tuple. Since the functions in the lemmas are continuous and continuity is preserved by composition, we have our result.

Exercise 2.2.1

Exercise 2.2.1. Prove Lemma 2.2.1. (Hint: use Proposition 1.1.18 and Theorem 2.1.4.)

Lemma 2.2.1. *Let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions, and let $f \oplus g : X \to \mathbf{R}^2$ be their direct sum. We give $\mathbf{R}^2$ the Euclidean metric.*

(*a*) *If $x_0 \in X$, then f and g are both continuous at x_0 if and only if $f \oplus g$ is continuous at x_0.*

(*b*) *f and g are both continuous if and only if $f \oplus g$ is continuous.*

Let $(x_1^{(n)}, x_2^{(n)})_{n=1}^{\infty}$ converges to (x_o, x_o). From Proposition 1.1.18 (AII), we know that $(x_1^{(n)})_{n=1}^{\infty}$ converges to x_o, and $(x_2^{(n)})_{n=1}^{\infty}$ converges to x_o. Since f and g are continuous at x_o, we can assert that $(f(x_1^{(n)}))_{n=1}^{\infty}$ converges to $f(x_o)$, and $(g(x_2^{(n)}))_{n=1}^{\infty}$ converges to $g(x_o)$. From Proposition 1.1.18 again, we can assert that $((f(x_1^{(n)}))_{n=1}^{\infty}, (g(x_2^{(n)}))_{n=1}^{\infty})$ converges to $(f(x_o), g(x_o))$. Since the original sequence was arbitrary, we have shown the result.'

A similar argument shows the converse.

(b) follows easily from (a).

Exercise 2.2.2

Exercise 2.2.2. Prove Lemma 2.2.2. (Hint: use Theorem 2.1.5 and limit laws (Theorem 6.1.19).)

Lemma 2.2.2. *The addition function $(x, y) \mapsto x + y$, the subtraction function $(x, y) \mapsto x - y$, the multiplication function $(x, y) \mapsto xy$, the maximum function $(x, y) \mapsto \max(x, y)$, and the minimum function $(x, y) \mapsto \min(x, y)$, are all continuous functions from $\mathbf{R}^2$ to $\mathbf{R}$. The division function $(x, y) \mapsto x/y$ is a continuous function from $\mathbf{R} \times (\mathbf{R}\backslash\{0\}) = \{(x, y) \in \mathbf{R}^2 : y \neq 0\}$ to $\mathbf{R}$. For any real number c, the function $x \mapsto cx$ is a continuous function from $\mathbf{R}$ to $\mathbf{R}$.*

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, f(x, y) = x + y$ be the addition function. Let $((x^{(n)}, y^{(n)}))_{n=m}^{\infty}$ be a sequence that converges to (x_o, y_o) from Proposition 1.1.18. Hence, we know that $x^{(n)}$ converges to x_o and $y^{(n)}$ converges to y_o. From Theorem 6.1.19(a) (AI), we know that $x^{(n)} + y^{(n)}$ converges to $x_o + y_o$. Hence, $f(x^{(n)}, y^{(n)})$ converges to $f(x_o, y_o)$, as desired.

From a similar argument, using Theorem 6.1.19(d) (AI), we can show that $x^{(n)} - y^{(n)}$ converges to $x_o - y_o$.

From a similar argument, using Theorem 6.1.19(b) (AI), we can show that $x^{(n)} \times y^{(n)}$ converges to $x_o \times y_o$.

From a similar argument, using Theorem 6.1.19(g) (AI), we can show that $max(x^{(n)}, y^{(n)})$ converges to $max(x_o, y_o)$.

From a similar argument, using Theorem 6.1.19(h) (AI), we can show that $min(x^{(n)}, y^{(n)})$ converges to $min(x_o, y_o)$.

From a similar argument, using Theorem 6.1.19(f) (AI), we can show that $x^{(n)}/y^{(n)}$ converges to x_o/y_o, assuming $y_o 0$.

From a similar argument, using Theorem 6.1.19(c) (AI), we can show that $cx^{(n)}$ converges to cx_o.

From Theorem 6.1.19(c), we know that $(-f(x^n))_{n=m}^{\infty}$ converges to $-f(x_o)$.

Exercise 2.2.3

Exercise 2.2.3. Show that if $f : X \to \mathbf{R}$ is a continuous function, so is the function $|f| : X \to \mathbf{R}$ defined by $|f|(x) := |f(x)|$.

The function $|f|$ can be rewritten as $|f|(x) = |f(x)| = max(f(x), -f(x))$. Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence that converges to $x_o \in X$. Since f is continuous, we know that $(f(x^n))_{n=m}^{\infty}$ converges to $f(x_o)$. From Lemma 2.2.2, we know that $(-f(x^n))_{n=m}^{\infty}$ converges to $-f(x_o)$. Then we know that $max(f(x^n), -f(x^n))_{n=m}^{\infty}$ converges to $max(f(x_o), -f(x_o))$. Hence, $|f(x^{(n)})|$ converges to $|f(x_o)|$.

Exercise 2.2.4

Exercise 2.2.4. Let $\pi_1 : \mathbf{R}^2 \to \mathbf{R}$ and $\pi_2 : \mathbf{R}^2 \to \mathbf{R}$ be the functions $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$ (these two functions are sometimes called the *co-ordinate functions* on $\mathbf{R}^2$). Show that π_1 and π_2 are continuous. Conclude that if $f : \mathbf{R} \to X$ is any continuous function into a metric space (X, d), then the functions $g_1 : \mathbf{R}^2 \to X$ and $g_2 : \mathbf{R}^2 \to X$ defined by $g_1(x, y) := f(x)$ and $g_2(x, y) := f(y)$ are also continuous.

Let $(x^{(n)}, y^{(n)})_{n=m}^{\infty}$ be a sequence in $\mathbb{R}^2$ that converges to (x_o, y_o). From Proposition 1.1.18 (AII), we know that $(x^{(n)})_{n=m}^{\infty}$ converges to x_o. Since $\pi_1(x^{(n)}, y^{(n)}) = x^{(n)}$, and $\pi_1(x_o, y_o) = x_o$, it follows that $(\pi_1(x^{(n)}, y^{(n)}))_{n=m}^{\infty}$ converges to $\pi_1(x_o, y_o)$. By way of a similar argument, we can show that $(\pi_2(x^{(n)}, y^{(n)}))_{n=m}^{\infty}$ converges to $\pi_2(x_o, y_o)$. The claim follows.

Let $(x^{(n)}, y^{(n)})_{n=m}^{\infty}$ be a sequence in $\mathbb{R}^2$ that converges to (x_o, y_o). From Proposition 1.1.18 (AII), we know that $(x^{(n)})_{n=m}^{\infty}$ converges to x_o. Because f is continuous, we know that $(f(x^{(n)}))_{n=m}^{\infty}$ converges to $f(x_o)$. Since $g_1(x^{(n)}, y^{(n)}) = f(x^{(n)})$ and $g_1(x_o, y_o) = f(x_o)$, it follows that $(g_1(x^{(n)}, y^{(n)}))_{n=m}^{\infty}$ converges to $g_1(x_o, y_o)$. By way of a similar argument, we can show that $(g_2(x^{(n)}, y^{(n)}))_{n=m}^{\infty}$ converges to $g_2(x_o, y_o)$. The claim follows.

Exercise 2.2.5

Exercise 2.2.5. Let $n, m \geq 0$ be integers. Suppose that for every $0 \leq i \leq n$ and $0 \leq j \leq m$ we have a real number c_{ij}. Form the function $P : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$P(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} x^i y^j .$$

(Such a function is known as a *polynomial of two variables*; a typical example of such a polynomial is $P(x, y) = x^3 + 2xy^2 - x^2 + 3y + 6$.) Show that P is continuous. (Hint: use Exercise 2.2.4 and Corollary 2.2.3.) Conclude that if $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are continuous functions, then the function $P(f, g) : X \to \mathbf{R}$ defined by $P(f, g)(x) := P(f(x), g(x))$ is also continuous.

Using Exercise 2.2.4, we can write the polynomial function as

$$P(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} \pi_1(x, y)^i \pi_2(x, y)^j$$

Since π_1 and π_2 are continuous functions, using Corollary 2.2.3, applying fg, cf, and $f+g$, it follows that $P(x,y)$ is continuous.

$P(f,g)$ is continuous. This follows Lemma 2.1.1 and Corollary 2.1.7.

Exercise 2.2.6

Exercise 2.2.6. Let $\mathbf{R}^m$ and $\mathbf{R}^n$ be Euclidean spaces. If $f : X \to \mathbf{R}^m$ and $g : X \to \mathbf{R}^n$ are continuous functions, show that $f \oplus g : X \to \mathbf{R}^{m+n}$ is also continuous, where we have identified $\mathbf{R}^m \times \mathbf{R}^n$ with $\mathbf{R}^{m+n}$ in the obvious manner. Is the converse statement true?

The converse is true.

The argument follows in a similar manner to Lemma 2.2.1.

Exercise 2.2.7

Exercise 2.2.7. Let $k \geq 1$, let I be a finite subset of $\mathbf{N}^k$, and let $c : I \to \mathbf{R}$ be a function. Form the function $P : \mathbf{R}^k \to \mathbf{R}$ defined by

$$P(x_1, \ldots, x_k) := \sum_{(i_1,\ldots,i_k)\in I} c(i_1, \ldots, i_k) x_1^{i_1} \ldots x_k^{i_k}.$$

(Such a function is known as a *polynomial of k variables*; a typical example of such a polynomial is $P(x_1, x_2, x_3) = 3x_1^3 x_2 x_3^2 - x_2 x_3^2 + x_1 + 5$.) Show that P is continuous. (Hint: use induction on k, Exercise 2.2.6, and either Exercise 2.2.5 or Lemma 2.2.2.)

$$P(x_1, \cdots, x_k) = \sum_{(i_1,\cdots,i_k)\in I} c(i_1, \cdots, i_k) x_1^{i_1} \cdots x_k^{i_k}$$

$$= \sum_{j=0}^{n} x_k^j \sum_{(i_1,\cdots,i_{k-1})\in I} c(i_1, \cdots, i_{k-1}, j) x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}$$

Then, we define

$$P(x_1, \cdots, x_{k-1})_j = \sum_{(i_1,\cdots,i_{k-1})\in I} c(i_1, \cdots, i_{k-1}, j) x_1^{i_1} \cdots x_{k-1}^{i_{k-1}}$$

Which is a continuous function by the inductive assumption.

Hence,

$$P(x_1, \cdots, x_k) = \sum_{j=0}^{n} P(x_1, \cdots, x_{k-1})_j$$

.

It follows easily from Lemma 2.2.2 that $P(x_1, \cdots, x_k)$ is continuous.

Exercise 2.2.8

Exercise 2.2.8. Let (X, d_X) and (Y, d_Y) be metric spaces. Define the metric $d_{X\times Y} : (X \times Y) \times (X \times Y) \to [0, \infty)$ by the formula

$$d_{X\times Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y').$$

Show that $(X \times Y, d_{X\times Y})$ is a metric space, and deduce an analogue of Proposition 1.1.18 and Lemma 2.2.1.

(a)
$d_{X\times Y}((x,y),(x,y)) = d_X(x,x) + d_Y(y,y) = 0$.

(b)
$d_{X\times Y}((x,y),(x',y')) := d_X(x,x') + d_Y(y,y')$.
Either $d_X(x,x') > 0$ or $d_Y(y,y') > 0$ since $(x,y)(x',y')$. Hence, $d_{X\times Y}((x,y),(x',y')) > 0$.

(c)
$d_{X\times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y') = d_X(x',x) + d_Y(y',y) = d_{X\times Y}((x',y'),(x,y))$.

(d)
$(x,y),(x',y'),(x'',y'') \in (X \times Y)$.
$d_{X\times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$
$\leq d_X(x,x'') + d_X(x',x'') + d_Y(y,y'') + d_Y(y',y'') = d_X(x,x'') + d_Y(y,y'') + d_X(x',x'') + d_Y(y',y'')$
$d_{X\times Y}((x,y),(x'',y'')) + d_{X\times Y}((x',y'),(x'',y''))$.

Proposition 1.1.18 analogue.
Let $(v^{(k)})_{n=m}^{\infty}$ be a sequence in $\mathbb{R}^{2n}$, where $v^{(k)} = ((x,y)_1^{(k)}, \cdots, (x,y)_n^{(k)})$. Then, $v^{(k)}$ converges if and only if each of the $(x,y)_{(k)}$ converges with respect to the metric $d_{X\times Y}$.

Lemma 2.2.1 analogue.
Let $f : X \times Y \to \mathbb{R}^2$, and $g : X \times Y \to \mathbb{R}^2$ be functions. Define $f \oplus g : X \times Y \to \mathbb{R}^4$. Then given $(x_o, y_o) \in X \times Y$, then $f \oplus g$ is continuous at (x_o, y_o) if and only if f and g are continuous at (x_o, y_o).

Exercise 2.2.9

Exercise 2.2.9. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function from $\mathbf{R}^2$ to $\mathbf{R}$. Let (x_0, y_0) be a point in $\mathbf{R}^2$. If f is continuous at (x_0, y_0), show that

$$\lim_{x\to x_0} \limsup_{y\to y_0} f(x,y) = \lim_{y\to y_0} \limsup_{x\to x_0} f(x,y) = f(x_0,y_0)$$

and

$$\lim_{x\to x_0} \liminf_{y\to y_0} f(x,y) = \lim_{y\to y_0} \liminf_{x\to x_0} f(x,y) = f(x_0,y_0).$$

(Recall that $\limsup_{x\to x_0} f(x) := \inf_{r>0} \sup_{|x-x_0|<r} f(x)$ and $\liminf_{x\to x_0} f(x) := \sup_{r>0} \inf_{|x-x_0|<r} f(x)$.) In particular, we have

$$\lim_{x\to x_0} \lim_{y\to y_0} f(x,y) = \lim_{y\to y_0} \lim_{x\to x_0} f(x,y)$$

whenever the limits on both sides exist. (Note that the limits do not necessarily exist in general; consider for instance the function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $f(x,y) = y \sin\frac{1}{x}$ when $xy \neq 0$ and $f(x,y) = 0$ otherwise.) Discuss the comparison between this result and Example 1.2.7.

Let $x' \in X$. Note, $limsup_{y\to y_o} f(x',y) = inf_{r>0} sup_{|y-y_o|<r} f(x',y)$.
Since $|y_o - y_o| = 0 < r$ for every r, it follows that $f(x',y_o) \le sup_{|y-y_o|<r} f(x',y)$. So, $f(x',y_o)$ is a lower bound. Hence, it cannot be the case that $inf_{r>0} < f(x',y_o)$, because otherwise $inf_{r>0}$ would not be the greatest lowest bound.
Assume that $f(x',y_o) < inf_{r>0}$. Define $g : \mathbb{R} \to \mathbb{R}, g(y) = f(x',y)$. Since $f(x',y)$ is continuous at (x',y_o), it follows that $g(y)$ is continuous at y_o.
Define $\epsilon := inf_{r>0} - f(x',y_o)$. Since $g(y)$ is continuous at y_o, we know that there is some δ such that $|y - y_o| < \delta \to |f(x',y) - f(x',y_o)| < \epsilon$. Hence, it follows that for every y such that $|y-y_o| < \delta$, $f(x',y) < \inf_{r<0}$, a contradiction since it would follow that $sup_{|y-y_o|<\delta} f(x',y) < inf_{r<0}$ and $inf_{r<0}$ is a lower bound.
Hence, $inf_{r>0} sup_{|y-y_o|<r} f(x',y) = f(x',y_o)$. Since x' was arbitrary, we have shown that $limsup_{y\to y_o} f(x,y) = f(x,y)$ for every x.
Hence, $\lim_{x\to x_o} limsup_{y\to y_o} f(x,y) = \lim_{x\to x_o} f(x,y_o)$. Since $f(x,y)$ is continuous at (x_o,y_o), it follows that $\lim_{x\to x_o} limsup_{y\to y_o} f(x,y) = f(x_o,y_o)$.
A similar argument shows that $\lim_{y\to y_o} limsup_{x\to x_o} f(x,y) = f(x_o,y_o)$.

Let $x' < X$. Note, $liminf_{y \to y_o} f(x', y) = sup_{r>0} inf_{|y-y_o|<r} f(x', y)$. Since $|y - y_o| = 0 < r$ for every r, it follows that $inf_{|y-y_o|<r} f(x', y) \leq f(x', y_o)$. Hence, $f(x', y_o)$ is an upper bound for $inf_{|y-y_o|<r} f(x', y)$. Hence, $sup_{r>0} \leq f(x', y_o)$, since $sup_{r>0}$ is the least upper bound. Assume $sup_{r>0} < f(x', y_o)$. And, define $g : \mathbb{R} \to \mathbb{R}, g(y) = f(x', y)$. Since $f(x', y)$ is continuous, $g(y)$ is continuous. Now define $\epsilon := f(x', y_o) - sup_{r>0}$. Since $g(y)$ is continuous, it follows that $|y - y_o| < \delta \to |f(x', y) - f(x', y_o)| < \epsilon$ for some δ. Hence, we can show that for every y such that $|y - y_o|$, $sup_{r>0} < f(x', y)$, a contradiction since it would follow that $sup_{r>0} < inf_{|y-y_o|<\delta}$, and $sup_{r>0}$ is an upper bound. Hence, $sup_{r>0} inf_{|y-y_o|<r} = f(x', y_o)$.
Since x' was arbitrary, we have shown that $liminf_{y \to y_o} f(x, y) = f(x, y_o)$ for every x.
Hence, $\lim_{x \to x_o} liminf_{y \to y_o} f(x, y) = f(x_o, y_o)$, since $f(x, y)$ is continuous at (x_o, y_o)
A similar argument shows that $\lim_{y \to y_o} liminf_{x \to x_o} f(x, y) = f(x_o, y_o)$.

Exercise 2.2.10

Exercise 2.2.10. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuous function. Show that for each $x \in \mathbf{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbf{R}$, and for each $y \in \mathbf{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbf{R}$. Thus a function $f(x, y)$ which is jointly continuous in (x, y) is also continuous in each variable x, y separately.

Since f is continuous, we know that for every sequence $(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ that converges to (x_o, y_o), it follows that $(f(x^{(n)}, y^{(n)}))_{n=1}^{\infty}$ converges to $f(x_o, y_o)$.
Let $x_o \in \mathbb{R}$ and define $g : y \to f(x_o, y)$. Let $(y^{(n)})_{n=1}^{\infty}$ be a sequence that converges to y_o. It is obvious that (x_o) converges to x_o. Then from Proposition 1.1.18 (AII), it follows that $(x_o, y^{(n)})_{n=1}^{\infty}$ converges to (x_o, y_o). Then, using the continuity of f, it follows that $(f(x_o, y^{(n)}))_{n=1}^{\infty}$ converges to $f(x_o, y_o)$. We can then write that $(g(y^{(n)}))_{n=1}^{\infty}$ converges to $f(x_o, y_o)$, as desired, since have defined g in this way.
A similar argument shows that $g : x \to f(x, y)$ is continuous for every $y \in \mathbb{R}$.

Exercise 2.2.11

Exercise 2.2.11. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the function defined by $f(x,y) := \frac{xy}{x^2+y^2}$ when $(x,y) \neq (0,0)$, and $f(x,y) = 0$ otherwise. Show that for each fixed $x \in \mathbf{R}$, the function $y \mapsto f(x,y)$ is continuous on $\mathbf{R}$, and that for each fixed $y \in \mathbf{R}$, the function $x \mapsto f(x,y)$ is continuous on $\mathbf{R}$, but that the function $f : \mathbf{R}^2 \to \mathbf{R}$ is not continuous on $\mathbf{R}^2$. This shows that the converse to Exercise 2.2.10 fails; it is possible to be continuous in each variable separately without being jointly continuous.

Let $x' \in \mathbb{B}$. Define $g(y) = f(x', y)$. We know that $h(y) = y$ is a continuous function. If $x' = 0$, then $g(y) = 0$ for all y, and hence is continuous. If $x'0$, then we know that $g(y)$ is continuous from Corollary 2.2.3, since $(x')^2 + y^2$ is never equal to 0.
A similar argument shows that $x \to f(x, y')$ is continuous for every $y' \in \mathbb{R}$.

To show that $f(x,y)$ is not continuous, we need to find a convergent sequence $(x^{(n)}, y^{(n)})_{n=1}^{\infty}$ such that $f(x^{(n)}, y^{(n)})$ does not converge to $f(x_o, y_o)$.
Define $(x^{(n)})_{n=1}^{\infty}$ as the sequence $0.1, 0.01, 0.001, \ldots$, and define $(y^{(n)})_{n=1}^{\infty}$ as the same sequence. Both sequences obviously converge to 0. Now, $f(x^{(n)}, y^{(n)}) = 0.5$ for every n, since $x^{(n)} = y^{(n)}$ for every n. Hence, $f(x^{(n)}, y^{(n)})$ converges to 0.5, while $f(0,0) = 0$. Hence, $f(x,y)$ is not continuous.

Section 3

Theorem 2.3.5

Theorem 2.3.5. *Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that (X, d_X) is compact. If $f : X \to Y$ is function, then f is continuous if and only if it is uniformly continuous.*

If f is uniformly continuous, continuity follows easily.

Suppose f is continuous. Let $\epsilon > 0$. Then we can write,
$\forall x_o [d_X(x, x_o) < \delta(x_o) \to d_Y(f(x), f(x_o)) < \epsilon/2]$, where $\delta(x_o)$ depends on x_o.

If $d_X(x, x_o) < \delta(x_o)/2$ and $d_X(x', x) < \delta(x_o)/2$, then we can write $d_X(x_o, x') \leq d(x_o, x) + d(x', x) < \delta(x_o)/2 + \delta(x_o)/2 = \delta(x_o)$. Since $d_X(x, x_o) < \delta(x_o) < \delta$, we can write $d_Y(f(x), f(x')) \leq d_Y(f(x), f(x_o)) + d_Y(f(x'), f(x_o)) < \epsilon/2 + \epsilon/2 = \epsilon$. This will come in handy later in the proof.
Now, consider the possible infinite set of balls:
$\{B_{(X,d_X)}(x_o, \delta(x_o)/2 : x_o \in X\}$
Since X is compact, the balls are open and the union of them covers X, we can apply Theorem 1.5.8 to assert that there are a finite number of points $x_1, \cdots, x_n$ such that

$$X \subseteq \bigcup_{j=1}^{n} B_{(X,d_X)}(x_j, \delta(x_j)/2)$$

Define $\delta := min_{j=1}^{n}\delta(x_j)/2$. Since each $\delta(x_j)/2$ is positive, and $n < \infty$, we know that $\delta < 0$. Let $x, x' \in X$ such that $d_X(x, x') < \delta$. Since the X is covered by the $B_{(X,d_X)}(x_j, \delta(x_j)/2)$, it follows that there is some j such that $x \in B_{(X,d_X)}(x_j, \delta(x_j)/2)$, and, consequently, $d_X(x, x_j) < \delta(x_j)/2$. Since $d_X(x, x') < \delta$ and δ is the minimum, we have $d_X(x, x') < \delta(x_j)/2$. From the previous discussion, it follows that $d_Y(f(x), f(x')) < \epsilon$. Since x, x' are arbitrary and δ is not, we have shown uniform continuity.

The proof sets out to find the required δ rather than working by contradiction. Because the function is continuous, we can always find some zone of stability for every $x_o \in X$. We take the collection of balls whose centres are all the x_o and whose distances provide the individual zone of stability. Obviously, this collection covers X.
The crux of the proof is moving from the possibly infinite collection to a finite collection that covers X using Theorem 1.5.8. By doing so, we guarantee that there is some non-zero minimum distance, below which the function has uniformly continuous behaviour. If the collection remained infinite, then there would be no guarantee that the minimum (infimum) is non-zero, because it could be zero instead. The compactness of X provides Theorem 1.5.8, and the strucutre of metric spaces (the triangle inequality) ensures that $d_Y(f(x), f(x')) < \epsilon$ at the end of the proof.

Exercise 2.3.1

Exercise 2.3.1. Prove Theorem 2.3.1.

Theorem 2.3.1 (Continuous maps preserve compactness). *Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := \{f(x) : x \in K\}$ of K is also compact.*

Let $(f(x^{(n)}))_{n=m}^{\infty}$ be a sequence in $f(K)$. We then take the sequence $(x^{n)})_{n=m}^{\infty}$ in K. Since K is compact, we know that there is some convergent subsequence $(x^{n_j)})_{j=1}^{\infty}$ in K. Since f is continuous, it follows that $(f(x^{n_j}))_{j=1}^{\infty}$ is a convergent sequence in $f(K)$, which is also a subsequence of $(f(x^{(n)}))_{n=m}^{\infty}$. It follows that $f(K)$ is compact.

Exercise 2.3.2

Exercise 2.3.2. Prove Proposition 2.3.2. (Hint: modify the proof of Proposition 9.6.7.)

Proposition 2.3.2 (Maximum principle). *Let (X, d) be a compact metric space, and let $f : X \to \mathbf{R}$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

Since X is compact and f is continuous, we know from Theorem 2.3.1 that $f(X)$ is compact. From Theorem 1.5.7, and the fact that $f(X) \subset \mathbb{R}$, it follows that $f(X)$ is bounded and closed.

Exercise 2.3.3

Exercise 2.3.3. Show that every uniformly continuous function is continuous, but give an example that shows that not every continuous function is uniformly continuous.

Continuity follows easily from uniform continuity.

$f(x) = 1/x$ is continuous but not uniformly so.

Exercise 2.3.4

Exercise 2.3.4. Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Show that $g \circ f : X \to Z$ is also uniformly continuous.

The argument follows in a similar fashion to Exercise 2.1.3.

Exercise 2.3.5

Exercise 2.3.5. Let (X, d_X) be a metric space, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be uniformly continuous functions. Show that the direct sum $f \oplus g : X \to \mathbf{R}^2$ defined by $f \oplus g(x) := (f(x), g(x))$ is uniformly continuous.

Using the fact that $d_{l^\infty} \leq d_{l^2} \leq d_{l^1}$, and

$$d_{l^1}(f(x), f(x')) + d_{l^1}(g(x), g(x')) = |f(x) - f(x')| + |g(x) - g(x')| = d_{l^1}((f(x), g(x)), (f(x'), g(x'))) \leq \epsilon/2 + \epsilon/2 = \epsilon$$,

the conclusion follows.

Exercise 2.3.6

Exercise 2.3.6. Show that the addition function $(x, y) \mapsto x + y$ and the subtraction function $(x, y) \mapsto x - y$ are uniformly continuous from $\mathbf{R}^2$ to $\mathbf{R}$, but the multiplication function $(x, y) \mapsto xy$ is not. Conclude that if $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ are uniformly continuous functions on a metric space (X, d), then $f + g : X \to \mathbf{R}$ and $f - g : X \to \mathbf{R}$ are also uniformly continuous. Give an example to show that $fg : X \to \mathbf{R}$ need not be uniformly continuous. What is the situation for $\max(f, g)$, $\min(f, g)$, f/g, and cf for a real number c?

$f(x) = x$ and $g(x) = x$ are uniformly continuous. We can also show $d_{l^1}(x+y, x'+y') = |x+y-x'+y'| = |x-x'+y-y'| \leq |x-x'| + |y-y'| < \epsilon$ from $d_{l^1}((x, y), (x', y')) < \delta$. The conclusion follows that the addition function is uniformly continuous.

We can show that the subtraction function is uniformly continuous for d_{l^1} by using the fact that cf is uniformly continuous.

For every δ there is some n such that $1/n < \delta$. Hence, it follows that $d_{l^1}((n,n),(n+1/n,n)) = |n-n-1| + |n-n| = 1 < \delta$. We can then write $d_{l^1}(nn, n(n+1)) = 1$. It follows that what for whatever δ we choose we cannot assert $d_{l^1}((n,n),(n+1/n,n)) <\!\!\rightarrow d_{l^1}(nn, n(n+1))$ for every ϵ, for every $(x,y),(x',y') \in \mathbb{R}^2$, and as such the multiplication function is not uniformly continuous.

Since f and g are uniformly continuous, we know that $f \oplus g$ is uniformly continuous from Exercise 2.3.5. From Exercise 2.3.4, we know that the composition of $f \oplus g$ and the addition function is uniformly continuous. This composition is equal to $f+g$.
In a similar manner, we can show that $f-g$ is uniformly continuous.

x^2 is not uniformly continuous, which is an example of fg not being so.

max, mix, and cf are uniformly continuous. f/g is not uniformly continuous since $1/x$ is not a continuous function.

Section 4

Theorem 2.4.5

Theorem 2.4.5. *Let X be a subset of the real line $\mathbf{R}$. Then the following statements are equivalent.*

(a) *X is connected.*

(b) *Whenever $x, y \in X$ and $x < y$, the interval $[x,y]$ is also contained in X.*

(c) *X is an interval (in the sense of Definition 9.1.1).*

(a) ⇒ (b)
Let X be connected. Assume, by contradiction, that there is some $x, y \in X$ such that $x < y$ and $[x,y] \nsubseteq X$. There is some $z \notin X$ such that $x < z < y$. Take $(-\infty, z)$ and (z, ∞). They are non-empty, open relative to X and also cover X. Hence, X is disconnected. A contradiction.

(b) ⇒ (a)

Let X be some set such that $\forall x, y \in X[x < y \to [x,y]X]$. Assume, by contradiction, that X is disconnected. Then, we know there is some V and W that are non-empty, disjoint, and open relative to X, such that $V \cup W = X$. Since V and W are disjoint and non-empty, we can choose some $x \in V$ and $y \in W$ such that xy. We will assume $x < y$ w.l.o.g.
Since $x < y$, by assumption this implies $[x,y] \subseteq X$. Consider the set $[x,y] \cap V$. It is bounded and non-empty, which implies that there is some supremum, $z := sup([x,y] \cap V)$. Since $z \in [x,y]$, it follows that $z \in X$, and hence $z \in V$ or $z \in W$.
Assume $z \in V$. Since $y \in W$, and V and W are disjoint, we know that zy. V is open relative to X, which means we can construct a ball centred at z that is contained in V, whose ambient space is X, namely, $B_{(X,d)}(z,r) \subseteq V$. Since $[x,y] \subset X$ and $z \in [x,y]$, we can restrict the ambient space to $[x,y]$ and the ball would still be contained in V. However, this guarantees that there is some z_o contained in the ball and that is greater than z. This is a contradiction since z is the supremum of $[x,y] \cap V$.
Now assume $z \in W$. Then we know zx. Since W is open relative to X, we can construct a ball $B_{([x,y],d)}(z,r) \subseteq W$. However, this would imply there is some $z_o < z$ that is an upper bound of $[x,y] \cap V$, a contradiction of z being the supremum.
Hence, X is connected.

This argument relies on the contradiction that arises from having a disconnected set, X, that also contains any closed interval where the bounds are contained in the set. From a high level, we can choose our x and y such that one of each are coming from the disjoint sets, V and W, which make up X. Since they are disjoint, there will be a gap in values, but these values are in the closed interval, which is contained in X. This is a contradiction.
To achieve the contradiction more precisely, we take the supremum, z, of the disjunct between the closed interval and V. z is an element of V or W.
If V, then, because of the disjoint sets relative openness, we can construct a ball that is contained in V and whose ambient space is $[x,y]$. This ensures that we can find some z_o that is greater than z and still contained in the disjunct of the closed interval and V. This is a contradiction since z would no longer be an upper bound.
If z is contained in W, which is relatively open, we can construct a ball that is contained in W and whose ambient space is $[x,y]$. This ensures that we can find some z_o that is less than z but also an upper bound for the disjunct of $[x,y]$ and V. A contradiction since z is the least upper bound.

Exercise 2.4.1

Exercise 2.4.1. Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.

We know there is some x and y in E. Define $V := \{x\}$, and define $W := E \setminus V$. Obviously, they are disjoint and their union equals E. If we let $r = 0.5$, then the ball centred at at x, $B_{E,d_{disc}}(x, r)$ is contained in V, and hence V is open relative to E. Similarly, for any $z \in W$, the ball $B_{E,d_{disc}}(x, r)$ is contained in W, and hence W is open relative to E. This shows that E is disconnected.

Exercise 2.4.2

Exercise 2.4.2. Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. (Hint: use Exercise 2.4.1.)

If f is constant, then f is necessarily continuous.

Let f be continuous. Assume, by contradiction, that f is non-constant. Then $f(X) \subseteq Y$ has more than two elements. Since $f(X)$ is discrete, Exercise 2.4.1 implies that $f(X)$ is disconnected. However, from Theorem 2.4.6, we can assert that $f(X)$ is connected, since X is connected and f is continuous. This is a contradiction, hence, f is constant.

Exercise 2.4.3

Exercise 2.4.3. Prove the equivalence of statements (b) and (c) in Theorem 2.4.5.

(*b*) *Whenever* $x, y \in X$ *and* $x < y$, *the interval* $[x, y]$ *is also contained in* X.

(*c*) X *is an interval (in the sense of Definition 9.1.1).*

Definition 9.1.1 (Intervals). Let $a, b \in \mathbf{R}^*$ be extended real numbers. We define the *closed interval* $[a, b]$ by

$$[a, b] := \{x \in \mathbf{R}^* : a \leq x \leq b\},$$

the *half-open intervals* $[a, b)$ and $(a, b]$ by

$$[a, b) := \{x \in \mathbf{R}^* : a \leq x < b\}; \quad (a, b] := \{x \in \mathbf{R}^* : a < x \leq b\},$$

and the *open intervals* (a, b) by

$$(a, b) := \{x \in \mathbf{R}^* : a < x < b\}.$$

(b) ⇒ (c)
Since $X \subseteq \mathbb{R}$, we know that X is a subset of $\{a, b\}$, where $a, b \in \mathbb{R}*$ are the infimum and supremum, respectively. Suppose X is not an interval, then there is some $z \notin X$ such that $a < z < b$. We know there are some $c, d \in X$ such that $a \leq c < z < d \geq b$, otherwise X would not be a subset of $\{a, b\}$. Using (b), we can assert that the interval $[c, d] \subseteq X$. Since $z \in [c, d]$, it follows that $z \in X$, which is a contradiction. Hence, X is an interval.

(c) ⇒ (b)
Let X be an interval. Let $x, y \in X$ such that $x < y$. Suppose $[x, y]$ is not contained in X, then there is some $z \notin X$ such that $x < z < y$. However, this contradicts that fact that X is an interval since $a < z < b$.

Exercise 2.4.4

Exercise 2.4.4. Prove Theorem 2.4.6. (Hint: the formulation of continuity in Theorem 2.1.5(c) is the most convenient to use.)

Theorem 2.4.6 (Continuity preserves connectedness). *Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected subset of X. Then $f(E)$ is also connected.*

From Theorem 2.1.5(c):

(c) *Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.*

Assume, by contradiction, that $f(E)$ is disconnected. Then, we know there is some V and some W that are disjoint, non-empty, and open, such that $V \cup W = f(E)$. Since f is continuous, and the sets are open, we can apply Theorem 2.1.5(c). Hence, we have $f^{-1}(V)$ and $f^{-1}(W)$ open in E. Obviously, they are non-empty, and we can write $f^{-1}(V) \cap f^{-1}(W) = E$ since $V \cup W = f(E)$. Now, assume by contradiction that there is some x in $f^{-1}(V)$ and in $f^{-1}(W)$, in which case they are not disjoint. Then we can write $f(x) \in V$ and $f(x) \in W$, which is a contradiction. Hence, $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint. This implies that E is disconnected, an obvious contradiction. Hence, $f(E)$ is connected.

This is an argument by contradiction. By assuming that $f(E)$ is disconnected, we can assert that we have two disjoint open sets that make up $f(E)$. Using the continuity of f we can then assert that the reverse images of these disjoint sets are also open and disjoint. Since they also make up E, it follows that E is disconnected, which is a contradiction.

Exercise 2.4.5

Exercise 2.4.5. Use Theorem 2.4.6 to prove Corollary 2.4.7.

Corollary 2.4.7 (Intermediate value theorem). *Let $f : X \to \mathbf{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.*

Let $a, b \in E$. Let y be between $f(a)$ and $f(b)$. Suppose, w.l.o.g, that $f(a) \leq y \leq f(b)$. Since X is connected, and f is continuous, we know that $f(E)$ is connected. It follows then that $[f(a), f(b)] \subseteq f(E)$. Hence, $y \in f(E)$, which implies that there is some $c \in E$ such that $f(c) = y$.

Exercise 2.4.6

Exercise 2.4.6. Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha \in I}$ be a collection of connected sets in X. Suppose also that $\bigcap_{\alpha \in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha \in I} E_\alpha$ is connected.

Asumme, by contradiction, that $\cup_{\alpha \in I} E_\alpha$ is disconnected. Then, there is some V and some W that are open, disjoint, and non-empty, such that $V \cup W = \cup_{\alpha \in I} E_\alpha$. Applying Theorem 1.3.4, we can assert that $V \cap E_\beta$ is open for every β, since V is open and $E_\beta \subseteq \cup_{\alpha \in I} E_\alpha$. Assume

$V \cap E_\beta$ is empty for some β. Then, since $E_\beta \subseteq \cup_{\alpha \in I} E_\alpha$ is non-empty, and $V \cup W = \cup_{\alpha \in I} E_\alpha$, we know that $E_\beta \subseteq W$. This implies that there is some E_α in V that is disjoint from E_β; however, since $\cap_{\alpha \in I} E_\alpha$ is non-empty, this is a contradiction. Hence, $V \cap E_\beta$ is non-empty. Similarly, we can show that $W \cap E_\beta$ is open and non-empty for every β. Let $x \in E_\beta$, then either $x \in W$ or $x \in V$. Hence, either $x \in V \cap E_\beta$ or $W \cap E_\beta$. Since x was arbitrary, it follows that $V \cap E_\beta \cap E_\beta$, since $V \cup W$ makes up the collection union. Hence, we have shown that E_β is disconnected, which is a contradiction. Hence, $\cup_{\alpha \in I} E_\alpha$ is connected.

Exercise 2.4.7

Exercise 2.4.7. Let (X, d) be a metric space, and let E be a subset of X. We say that E is *path-connected* iff, for every $x, y \in E$, there exists a continuous function $\gamma : [0,1] \to E$ from the unit interval $[0,1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every path-connected set is connected. (The converse is false, but is a bit tricky to show and will not be detailed here.)

Assume E is disconnected. There is some V and some W that are disjoint, open, and non-empty, such that $V \cup W = E$. Let $x \in V$ and $x \in W$. Then, since E is path connected, we know there is some continuous γ such that $F := \gamma([0,1]) \subseteq E$ and $x, y \in F$. Since γ is continuous and $[0,1]$ is connected, applying Theorem 2.4.6, we know that F is connected. Using Theorem 1.3.4, we know that $V \cap F$ and $W \cap F$ are open relative to F. With similar reasoning to Exercise 2.4.6, we can show that $(V \cap F) \cup (W \cap F) = F$. Since these intersections are obviously non-empty, it follows that F is disconnected. This is a contradiction. Hence, E is connected.

Exercise 2.4.8

Exercise 2.4.8. Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?

Assume $\bar{E}$ is disconnected. So we have some V and some W that are disjoint, open, and non-empty, such that $V \cup W = E$. Consider $V \cap E$ and $W \cap E$. From Theorem 1.3.4, we know they are open. They are also disjoint from each other. Take some $x \in V$. Then either $x \in E$ or $x \in \bar{E}$. If $x \in E$, then obviously $V \cap E$ is non-empty. If $x \in \bar{E}$, then since V is open, we know then is some ball that is contained in V. Since x is a boundary point, and hence an adherent point, we know this ball must contain elements of E. Hence, $V \cap E$ is non-empty. By way of a similar argument, we can show that $W \cap E$ is non-empty. Since obviously $(V \cap E) \cup (V \cap E$, we have shown that E is disconnected. This is a contradiction, hence, $\bar{E}$ is connected.

The converse is not true since $[0,1) \cup (1,2]$ is disconnected, but $[0,2]$ is connected.

Exercise 2.4.9

Exercise 2.4.9. Let (X, d) be a metric space. Let us define a relation $x \sim y$ on X by declaring $x \sim y$ iff there exists a connected subset of X which contains both x and y. Show that this is an equivalence relation (i.e., it obeys the reflexive, symmetric, and transitive axioms). Also, show that the equivalence classes of this relation (i.e., the sets of the form $\{y \in X : y \sim x\}$ for some $x \in X$) are all closed and connected. (Hint: use Exercise 2.4.6 and Exercise 2.4.8.) These sets are known as the *connected components* of X.

Reflexive:

Let $x \in X$. We know that $\{x\}$ is connected and $x \in \{x\}$. Hence, we have found some connected subset of X that contains x, and $x \sim x$.

Symmetry is obvious.

Transitive:

Let $x \sim y$ and $y \sim z$. Then we know there is some $Y \subseteq X$ that is connected and contains x, y. We also know there is some $Z \subseteq X$ that is connected and contains y, z. Since y is in both sets, we know that $Z \cap Y$ is non-empty. Hence, using Exercise 2.4.6, we can assert that $Z \cup Y$ is connected. Since also $Z \cup Y$ contains x, z, we have shown that $x \sim z$.

Let $x \in X$, and define $E := \{y \in X : y \sim x\}$. Hence, for every $y \in Y$, we know there is some $Y \subseteq X$ that is connected and contains x, y. Choosing Y_y for each y, we consider the intersection $\cap_{y \in E} Y_y$, which is non-empty since each Y_y contains x. Hence, using Exercise 2.4.6, we can assert that the union $\cup_{y \in E} Y_y$ is connected.

Let $y \in E$. We know that $y \in Y_y$, and so $y \in \cup_{y \in E} Y_y$. Hence, $E \subseteq \cup_{y \in E} Y_y$. Let $y' \in \cup_{y \in E} Y_y$. Then we know there is some y such that $y' \in Y_y$. Since Y_y is connected and contains y', x, it follows that $y' \in E$. Hence, $E = \cup_{y \in E} Y_y$ and E is connected.

Let y be a boundary point of E. We know that $\bar{E}$ is connected from Exercise 2.4.8. Hence, $\bar{E}$ contains y, and it obviously also contains x. Hence $y \in E$, and E contains all its boundary points, so is closed.

Exercise 2.4.10

Exercise 2.4.10. Combine Proposition 2.3.2 and Corollary 2.4.7 to deduce a theorem for continuous functions on a compact connected domain which generalizes Corollary 9.7.4.

Proposition 2.3.2 (Maximum principle). *Let (X, d) be a compact metric space, and let $f : X \to \mathbf{R}$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

Corollary 2.4.7 (Intermediate value theorem). *Let $f : X \to \mathbf{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.*

Corollary 9.7.4 (Images of continuous functions). *Let $a < b$, and let $f : [a, b] \to \mathbf{R}$ be a continuous function on $[a, b]$. Let $M := \sup_{x\in[a,b]} f(x)$ be the maximum value of f, and let $m := \inf_{x\in[a,b]} f(x)$ be the minimum value. Let y be a real number between m and M (i.e., $m \leq y \leq M$). Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.*

Generalisation of 9.7.4:
Let (X, d) be a compact metric space, and let $f : X \to \mathbb{R}$ be a continuous function. Let $M := sup_{x\in X} f(x)$ and $m := inf_{x\in X} f(x)$ be the maximum and minimum of f, respectively. Let $y \in \mathbb{R}$ such that $m \leq y \leq M$. Then, there exists some $x \in X$ such that $f(x) = y$. Furthermore, $f(X) = [m, M]$.

Proof:
Since X is compact and f is continuous, we know from Proposition 2.3.2 that there are x_{min} and x_{max} in X such that $f(x_{min}) = m$ and $f(x_{max}) = M$. Let $y \in [m, M]$, then from Corollary 2.4.7, we know that there is some $x \in X$ such that $f(x) = y$.

Section 5

Exercise 2.5.1

Exercise 2.5.1. Let X be an arbitrary set, and let $\mathcal{F} := \{\emptyset, X\}$. Show that $(X, \mathcal{F})$ is a topology (called the *trivial topology* on X). If X contains more than one element, show that the trivial topology cannot be obtained from by placing a metric d on X. Show that this topological space is both compact and connected.

$\varnothing$ and X are both obviously in $\mathbb{F}$. The only possible intersection is $\varnothing \cap X = X$ is in $\mathbb{F}$, and the only union $\varnothing \cup X = X$ is in $\mathbb{F}$ as well.

The only open cover of X is itself, and it is obviously a finite cover. Hence, X is compact. Suppose X is disconnected, in which case we have V and W which are disjoint, open, and non-empty such that $V \cup W = X$. Since the topological space X only contains the empty set and X, V and W must be these two sets. However, the empty set cannot be non-empty, so we have a contradiction. X is connected.

Exercise 2.5.2

Exercise 2.5.2. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of convergence of sequences in Definition 1.1.14 and Definition 2.5.4 coincide.

Definition 1.1.14 (Convergence of sequences in metric spaces). Let m be an integer, (X, d) be a metric space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X (i.e., for every natural number $n \geq m$, we assume that $x^{(n)}$ is an element of X). Let x be a point in X. We say that $(x^{(n)})_{n=m}^{\infty}$ *converges to x with respect to the metric d*, if and only if the limit $\lim_{n\to\infty} d(x^{(n)}, x)$ exists and is equal to 0. In other words, $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d if and only if for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(n)}, x) \leq \varepsilon$ for all $n \geq N$. (Why are these two definitions equivalent?)

Definition 2.5.4 (Topological convergence)**.** Let m be an integer, $(X, \mathcal{F})$ be a topological space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X. Let x be a point in X. We say that $(x^{(n)})_{n=m}^{\infty}$ *converges to* x if and only if, for every neighbourhood V of x, there exists an $N \geq m$ such that $x^{(n)} \in V$ for all $n \geq N$.

Assume that $(x^{(n)})_{n=m}^{\infty}$ converges in terms of Definition 1.1.14. Then we know that for every ϵ there is some $N \geq m$ such that for every $n \geq N$, we have $d(x^{(n)}, x) \leq \epsilon$. Let V be a neighbourhood of x. Since V is open, we know that there is some $r > 0$ such that $B(x,r) \subseteq V$. In particular, we can write that there is some $N \geq m$ such that for every $n \geq N$, we have $d(x^{(n)}, x) \leq r$. It follows then that $x^{(n)} \in B(x,r)$ and consequently, $x^{(n)} \in V$. Hence, for every neighbourhood V of x there is some $N \geq m$ such that for every $n \geq N$, we have $x^{(n)} \in V$.

Now assume $(x^{(n)})_{n=m}^{\infty}$ converges in terms of Definition 2.5.4. Then we know that for every neighbourhood V of x there is some $N \geq m$ such that for every $n \geq N$, we have $x^{(n)} \in V$. Let $\epsilon > 0$. We know that $B(x,\epsilon)$ is open and that $x \in B(x,\epsilon)$. Obviously, $B(x,\epsilon)$ is contained in X. Hence, this ball is a neighbourhood of x, and we can write that there is some $N \geq m$ such that for every $n \geq N$, we have $x^{(n)} \in B(x,\epsilon)$. It then immediately follows that there is some $N \geq m$ such that for every $n \geq N$, we have $d(x^{(n)}, x) < \epsilon$. Since ϵ was arbitrary, we have shown that the sequence converges in term of Definition 1.1.14.

Exercise 2.5.3

Exercise 2.5.3. Let (X, d) be a metric space (and hence a topological space). Show that the two notions of interior, exterior, and boundary in Definition 1.2.5 and Definition 2.5.5 coincide.

Definition 1.2.5 (Interior, exterior, boundary)**.** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

Definition 2.5.5 (Interior, exterior, boundary). Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a neighbourhood V of x_0 such that $V \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a neighbourhood V of x_0 such that $V \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

Interior point:
Suppose x_o is an interior point in terms of Definition 1.2.5. Then we know there is some $r > 0$ such that $B(x_o, r) \subseteq E$. Since $B(x_o, r)$ is a neighbourhood itself, it follows that x_o is an interior point in terms of Definition 2.5.5.
Suppose x_o is an interior point in terms of Definition 2.5.5. Then we know there is some neighbourhood V such that $V \subseteq E$. Since V is open, we know there is some r such that $B(x_o, r) \subseteq V$. It follows then that $B(x_o, r) \subseteq E$, and Definition 1.2.5 holds.

Exterior point:
Suppose x_o is an exterior point in terms of Definition 1.2.5. Then we know there is some $r > 0$ such that $B(x_o, r) \cap E = \emptyset$. Since $B(x_o, r)$ is a neighbourhood itself, it follows that x_o is an exterior point in terms of Definition 2.5.5.
Suppose x_o is an exterior point in terms of Definition 2.5.5. Then we know there is some neighbourhood V such that $V \cap E = \emptyset$. Since V is open, we know there is some r such that $B(x_o, r) \subseteq V$. It follows then that $B(x_o, r) \cap E = \emptyset$, and Definition 1.2.5 holds.

Boundary point:
This follows easily from above.

Exercise 2.5.4

Exercise 2.5.4. A topological space $(X, \mathcal{F})$ is said to be *Hausdorff* if given any two distinct points $x, y \in X$, there exists a neighbourhood V of x and a neighbourhood W of y such that $V \cap W = \emptyset$. Show that any topological space coming from a metric space is Hausdorff, and show that the trivial topology is not Hausdorff. Show that the analogue of Proposition 1.1.20 holds for Hausdorff topological spaces, but give an example of a non-Hausdorff topological space in which Proposition 1.1.20 fails. (In practice, most topological spaces one works with are Hausdorff; non-Hausdorff topological spaces tend to be so pathological that it is not very profitable to work with them.)

Proposition 1.1.20 (Uniqueness of limits). *Let (X, d) be a metric space, and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence in X. Suppose that there are two points $x, x' \in X$ such that $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d, and $(x^{(n)})_{n=m}^{\infty}$ also converges to x' with respect to d. Then we have $x = x'$.*

Let (X, d) is a metric space considered as a topological space. Let $x, y \in X$ be distinct. Then $d(x, y) = r > 0$. The balls $B(x, r/2)$ and $B(y, r/2)$ are neighbourhoods of x and y, respectively, and are disjoint. Hence, the space is Hausdorff.

The trivial topology is not Hausdorff since for every distinct $x, y \in X$, the only neighbourhood is X itself, and hence they cannot have distinct, disjoint neighbourhoods.

Proposition 1.1.20 Analogue:
Let $(X, \mathbb{F})$ be a Hausdorff topological space, and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence that converges in X. If $(x^{(n)})_{n=m}^{\infty}$ converges to x and also converges to x', then $x = x'$.

Proof:
Suppose x and x' are distinct. Since X is Hausdorff, then we know there is some neighbour of x, V, and some neighbourhood of x', W such that $V \cap W = \emptyset$. Since the sequence converges to x, we can write $\forall n \geq N(x^{(n)} \in V)$ for some N. Since the sequence converges to x', we can write $\forall n \geq N'(x^{(n)} \in W)$ for some N'. Taking $M := max(N, N')$, we can write that for every $n \geq M$, $x^{(n)} \in W$ and $x^{(n)} \in V$. However, this is a contradiction since V and W are disjoint.

Example:
$(X, \mathbb{F})$, where $X := [0, 2]$ and $\mathbb{F} = \{\emptyset, [0, 1), (1, 2], [0, 2]\}$. Consider the sequence $(x^{(n)})_{n=m}^{\infty} = 1.9, 1.99, 1.999, 1.9999, \cdots$. The sequence converges to 2 and also 1.9, since every neighbourhood that contains 2 and 1.9 also contains all $x^{(n)}$.

Exercise 2.5.5

Exercise 2.5.5. Given any totally ordered set X with order relation $\leq$, declare a set $V \subset X$ to be *open* if for every $x \in V$ there exists a set I which is an interval $\{y \in X : a < y < b\}$ for some $a, b \in X$, a ray $\{y \in X : a < y\}$ for some $a \in X$, the ray $\{y \in X : y < b\}$ for some $b \in X$, or the whole space X, which contains x and is contained in V. Let $\mathcal{F}$ be the set of all open subsets of X. Show that $(X, \mathcal{F})$ is a topology (this is the *order topology* on the totally ordered set $(X, \leq)$) which is Hausdorff in the sense of Exercise 2.5.4. Show that on the real line $\mathbf{R}$ (with the standard ordering $\leq$), the order topology matches the standard topology (i.e., the topology arising from the standard metric). If instead one applies this to the extended real line $\mathbf{R}^*$, show that $\mathbf{R}$ is an open set with boundary $\{-\infty, +\infty\}$. If $(x_n)_{n=1}^{\infty}$ is a sequence of numbers in $\mathbf{R}$ (and hence in $\mathbf{R}^*$), show that x_n converges to $+\infty$ if and only if $\liminf_{n\to\infty} x_n = +\infty$, and x_n converges to $-\infty$ if and only if $\limsup_{n\to\infty} x_n = -\infty$.

The empty set and X are vacuously open sets of the topological space.
Let $V_1 \cdots V_n \in \mathbb{F}$. Then for every $1 \leq i \leq n$, there is some I_i of the form $\{y \in X a_i < y < b_i\}$, or $\{y : a_i < y\}$, or $\{y : y < b_i\}$, or is X itself.Take $A := max_{1 \leq i \leq n} a_i$ and $B := min_{1 \leq i \leq n} b_i$, and define $I := \{y \in X : A < y < B\}$. Let $y \in I$, then obviously $a_i < y < b_i$ for every i, and so y is an element of the intersection of the V_i. Hence, I is contained in the intersection, which implies that the intersection is contained in $\mathbb{F}$.
Suppose $(V_\alpha)_{\alpha \in I}$ is a family of sets in $\mathbb{F}$. Then for any V_α, we have some I_α contained in V_α. Since V_α is contained in the union of the family, it follows that I_α is contained in the union of the family, and the union is contained in $\mathbb{F}$.

Let $x, y \in X$. Then either there is some $z \in X$ such that $x < z < y$, or there is no such z. If there is such a z, then we can take $I_x := \{x' \in X : x' < z\}$ and $I_y := \{x' \in X : z < x'\}$. These intervals are neighbourhoods of x and y, respectively, and are disjoint, showing that the topology is Hausdorff. If there is no such z, then we can take $I_x\{x' \in X : x' < y\}$ and $I_y := \{x' \in X : x < x'\}$, which are disjoint neighbourhoods as well.

Let V be an open set in the standard topology. Then for every $x \in V$, we have a ball $B(x, r) \subseteq V$. So if we take the interval $I := \{y \in X : x - r < y < x + r\}$, which is contained in V, we have shown that V is open in the ordered topology.
Let V be an open set in the ordered topology. Then for every $x \in V$, we know that $a < x < b$. If we take $r := min(x - a, b - x)$, then the ball $B(x, r)$ is contained in V, and x is an interior point. Hence, V is open in the standard topology.

Let $x \in \mathbb{R}*$. Then the interval $I := \{y \in \mathbb{R}* : x-1 < x < x+1\}$ is contained in $\mathbb{R}*$. Hence, $\mathbb{R}$ is open in $\mathbb{R}*$. Obviously the bounds of $\mathbb{R}$ are $\{-\infty, \infty\}$.

If $(x_n)_{n=m}^{\infty}$ is a sequence in $\mathbb{R}$ that converges to ∞, then by a similar argument to Proposition 6.4.12 (AI), we can show that the limit inferior is equal to ∞. The converse is also true. A similar argument applies for when the sequence converges to $-\infty$.

Exercise 2.5.6

Exercise 2.5.6. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-finite (which means that $X \backslash E$ is finite). Show that $(X, \mathcal{F})$ is a topology (this is called the *cofinite topology* on X) which is not Hausdorff in the sense of Exercise 2.5.4, and is compact and connected. Also, show that if $x \in X$ $(V_n)_{n=1}^{\infty}$ is any countable collection of open sets containing x, then $\bigcap_{n=1}^{\infty} V_n \neq \{x\}$. Use this to show that the cofinite topology cannot be obtained by placing a metric d on X. (Hint: what is the set $\bigcap_{n=1}^{\infty} B(x, 1/n)$ equal to in a metric space?)

The empty set is obviously open in the topology. Since $X = \emptyset$, which is finite, X is also open in the topology.

Let $V_1 \cdots V_n \in \mathbb{F}$. Then $X \setminus (V_1 \cap \cdots \cap V_n) = X \setminus V_1 \cup \cdots \cup X \setminus V_n$, from Proposition 3.1.28(h) (AI). Since each of $X \setminus X_i$ is finite and n is finite, it follows that $X \setminus (V_1 \cap \cdots \cap V_n)$ is finite, and $V_1 \cap \cdots \cap V_n$ is open in the topology.

Let $(V_\alpha)_{\alpha \in I} \in \mathbb{F}$. Then using De Morgan's Law again, we have $X \setminus \cup_{\alpha \in I} V_\alpha = \cap_{\alpha \in I} X \setminus V_\alpha \subseteq V_\alpha$. Hence, $X \setminus \cup_{\alpha \in I} V_\alpha$ is finite and the union is contained in $\mathbb{F}$

.

Suppose the topology is Hausdorff. Then for some distinct x, y, we have some neighbourhoods V and W that are non-empty and disjoint. It follows that $X \setminus V$ and $X \setminus W$ are finite. Since V and W are disjoint, we know that $V \subseteq X \setminus W$ is finite. But then it would follow that $V \cup X \setminus V$ is finite, which is a contradiction, since X is uncountable. Hence, the topology is not Hausdorff.

Suppose (V_α) is a collection of open sets that covers X. Hence, for any $x \in X$, we have $x \in V_\alpha$ for some α. We also know that $X \setminus V_\alpha$ is finite. Hence, for every $y \in X \setminus V_\alpha$, we can choose some V_β. Combining these with V_α, we have a finite subcover. Hence, X is compact.

If X was disconnected, then we would have V, W open, non-empty, and disjoint. It is easy to show that $V = X \setminus W$ and $W = X \setminus V$. Hence, V and W are finite and so is$V \cup W$. It follows then that X is finite, which is a contradiction. Hence, X is connected.

Suppose $(V_\alpha)_{n=m}^\infty$ is a countable collection of open sets containing x. Now suppose that $\cap_{n=m}^\infty V_n = \{x\}$. Since $X \setminus \{x\}$ is uncountable, and $X \setminus \{x\} = X \setminus \cap_{n=m}^\infty V_n = \cup_{n=1}^\infty X \setminus V_n$, it follows that $\cup_{n=1}^\infty X \setminus V_n$ is uncountable. However, since $X \setminus V_n$ is finite for every n and we have a countable collection, it follows that $\cup_{n=1}^\infty X \setminus V_n$ is at most uncountable. This is a contradiction, and hence we have $\cap_{n=m}^\infty V_n \{x\}$.

Suppose we can obtain the topology by placing a metric on X. Now assume that $\cap_{n=1}^\infty B(x, 1/n)\{x\}$. Then there is some $y \in \cap_{n=1}^\infty B(x, 1/n)$ where yx. Then we have $d(x, y) = r > 0$. We know there is some m such that $r > 1/m$. Hence, $y \notin B(x, 1/m)$, and $y \notin \cap_{n=1}^\infty B(x, 1/n)$. This is a contradiction, so $\cap_{n=1}^\infty B(x, 1/n) = \{x\}$. However, this is a contradiction, since we have already shown that the intersection of a countable collection of open sets in the topology cannot equal $\{x\}$.

Exercise 2.5.7

Exercise 2.5.7. Let X be an uncountable set, and let $\mathcal{F}$ be the collection of all subsets E in X which are either empty or co-countable (which means that $X \backslash E$ is at most countable). Show that $(X, \mathcal{F})$ is a topology (this is called the *cocountable topology* on X) which is not Hausdorff in the sense of Exercise 2.5.4, and connected, but cannot arise from a metric space and is not compact.

The empty set is obviously open in the topology. Since $X = \emptyset$, which is at most countable, X is also open in the topology.

Let $V_1 \cdots V_n \in \mathbb{F}$. Then $X \setminus (V_1 \cap \cdots \cap V_n) = X \setminus V_1 \cup \cdots \cup X \setminus V_n$, from Proposition 3.1.28(h) (AI). Since each of $X \setminus X_i$ is at most countable and n is finite, it follows that $X \setminus (V_1 \cap \cdots \cap V_n)$ is at most countable, and $V_1 \cap \cdots \cap V_n$ is open in the topology.

Let $(V_\alpha)_{\alpha \in I} \in \mathbb{F}$. Then using De Morgan's Law again, we have $X \setminus \cup_{\alpha \in I} V_\alpha = \cap_{\alpha \in I} X \setminus V_\alpha \subseteq V_\alpha$. Hence, $X \setminus \cup_{\alpha \in I} V_\alpha$ is at most countable and the union is contained in $\mathbb{F}$.

Suppose the topology is Hausdorff. Then for some distinct x, y, we have some neighbourhoods V and W that are open, non-empty and disjoint. It follows that $X \setminus V$ and $X \setminus W$ are at most countable. Since V and W are disjoint, we know that $V \subseteq X \setminus W$ is at most countable. But then it would follow that $V \cup X \setminus V$ is at most countable, which is a contradiction, since X is uncountable. Hence, the topology is not Hausdorff.

If X was disconnected, then we would have V, W open, non-empty, and disjoint. It is easy to show that $V = X \setminus W$ and $W = X \setminus V$. Hence, V and W are at most countable and so is

$V \cup W$. It follows then that X is at most countable, which is a contradiction. Hence, X is connected.

Suppose $(V_\alpha)_{n=m}^\infty$ is a countable collection of open sets containing x. Now suppose that $\cap_{n=m}^\infty V_n = \{x\}$. Since $X \setminus \{x\}$ is uncountable, and $X \setminus \{x\} = X \setminus \cap_{n=m}^\infty V_n = \cup_{n=1}^\infty X \setminus V_n$, it follows that $\cup_{n=1}^\infty X \setminus V_n$ is uncountable. However, since $X \setminus V_n$ is at most countable for every n and we have a countable collection, it follows that $\cup_{n=1}^\infty X \setminus V_n$ is at most uncountable. This is a contradiction, and hence we have $\cap_{n=m}^\infty V_n \{x\}$.
Suppose we can obtain the topology by placing a metric on X. Now assume that $\cap_{n=1}^\infty B(x, 1/n)\{x\}$. Then there is some $y \in \cap_{n=1}^\infty B(x, 1/n)$ where yx. Then we have $d(x, y) = r > 0$. We know there is some m such that $r > 1/m$. Hence, $y \notin B(x, 1/m)$, and $y \notin \cap_{n=1}^\infty B(x, 1/n)$. This is a contradiction, so $\cap_{n=1}^\infty B(x, 1/n) = \{x\}$. However, this is a contradiction, since we have already shown that the intersection of a countable collection of open sets in the topology cannot equal $\{x\}$.

Choose some $E \subset X$ that is countable, in which case, we can write $E = \{x_1, x_2, \cdots\}$. Define $E_i = (X \setminus E) \cup \{x_i\}$. Obviously, $E_i \subseteq X$. We can write
$X \setminus E_i = X \setminus [(X \setminus E) \cup \{x_i\}] = [X \setminus (X \setminus E)] \cap (X \setminus \{x_i\})$
$= E \cap (X \setminus \{x_i\} = E \setminus \{x_i\}$.
Since E is countable, we know $E \setminus \{x_i\}$ is at most countable, which implies that E_i is open in the topology. Let $x \in X$. Then $x \in E \vee x \notin E$. If $x \in E$, then $x \in E_i$ for some i. If $x \notin E$, then $x \in X \setminus E$, and so $x \in (X \setminus E) \cup \{x_i\} = E_i$ for every i. Either way $x \in \cup_{i=1}^\infty E_i$. Since x was arbitrary, we have shown that $X \subset \cup_{i=1}^\infty E_i$, and this union is an open subcover. However, if we take some finite subset, $\cup_{i=1}^n E_i$, it follows that $x_i \notin \cup_{i=1}^n E_i$ for some $m \in \mathbb{N}$. Hence, there is no finite subcover, and the topology is not compact.

Exercise 2.5.9

Exercise 2.5.9. Let $(X, \mathcal{F})$ be a compact topological space. Assume that this space is *first countable*, which means that for every $x \in X$ there exists a countable collection $V_1, V_2, \ldots$ of neighbourhoods of x, such that every neighbourhood of x contains one of the V_n. Show that every sequence in X has a convergent subsequence, by modifying Exercise 1.5.11. Explain why this does not contradict Exercise 2.5.8.

Exercise 1.5.11. Let (X, d) have the property that every open cover of X has a finite subcover. Show that X is compact. (Hint: if X is not compact, then by Exercise 1.5.9, there is a sequence $(x^{(n)})_{n=1}^\infty$ with no limit points. Then for every $x \in X$ there exists a ball $B(x, \varepsilon)$ containing x which contains at most finitely many elements of this sequence. Now use the hypothesis.)

Let $(x^{(n)})_{n=1}^\infty$ be a sequence in X. Suppose that for every $x \in X$, there is some neighbourhood of x that only contains a finite number of $x^{(n)}$ in the sequence. Then, choosing the neighbourhoods V_x, we know that $X \subseteq \cup_{x \in X} V_x$. Since X is compact, we know there is some finite Y such that $X \subseteq \cup_{x \in Y} V_x$. However, this would imply that one of the V_x contains an infinite number of $x^{(n)}$, a contradiction. Hence, there is some $c \in X$ such that every neighbourhood of c contains an infinite number of $x^{(n)}$. Also, for this c let $V_1, V_2, \cdots$ be the countable collection of neighbourhoods.

Define $U_1 := V_1$, and $U_n = \cap_{i=1}^n V_i$. Also define $A_k \{x^{(n)} : n > k\}$. We know that U_1 is a neighbourhood of c, and so there is some $y_1 \in A_0$ such that $y_1 \in U_1$. Similarly, we know U_2 is a neighbourhood of c, since it is a finite intersection of neighbourhoods, so there is some $y_2 \in A_1$ such that $y_2 \in U_2$. We can continue doing this, since otherwise U_N would only contain a finite number of $x^{(n)}$, which is a contradiction, since every U_n is a neighbourhood. Hence, we can construct a sequence $(y^{(n)})_{n=1}^\infty$, or equivalently, $(x_{(n_k)})_{k=1}^\infty$, that is a subsequence of $(x^{(n)})_{n=1}^\infty$. To show that this subsequence converges to c, let V be a neighbourhood of c. Then we know there is some $V_N \subseteq V$. We know also that for every $n \geq N$, $V_n \subseteq V_N$, hence, $y^{(n)} \in V_n \subseteq V_N$. And we have shown that for every neighbourhood V of c, there is an N such that for every $n \geq N$, $x^{(n_k)} \in V$.

This proof constructs a subsequence by repeatedly taking a finite intersection of neighbourhoods, where subsequent intersections are subsets of previous ones. Due to the topology being first countable, we could ensure that $A_{k-1} \cap U_k$ was non-empty for each k, and hence the subsequence could be constructed.

This doesn't contradict exercise 2.5.8 because $w_1 + 1$ is not first countable.

Exercise 2.5.10

Exercise 2.5.10. Prove the following partial analogue of Proposition 1.2.10 for topological spaces: (c) implies both (a) and (b), which are equivalent to each other. Show that in the co-countable topology in Exercise 2.5.7, it is possible for (a) and (b) to hold without (c) holding.

Proposition 1.2.10. *Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are logically equivalent.*

(a) x_0 is an adherent point of E.

(b) x_0 is either an interior point or a boundary point of E.

(c) There exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d.

(c) ⇒ (a)
Let V be a neighbourhood of x_o. Then, since the sequence converges to x_o, we know there is some N such that for every $n \geq N$ we have $x^{(n)} \in V$. Since the sequence is in E, we know that $V \cap E \emptyset$. It follows that x_o is an adherent point of x_o.

(a) ⇒ (b)
Since x_o is an adherent point, we know that it is not an exterior point. We know that is either an interior point or not an interior point. Either way, the claim follows.

(b) ⇒ (a)
Since x_o is an interior point or a boundary point, we know that it is not an exterior point. Hence, it is not the case that there is some neighbourhood V of x_o such that $V \cap E \emptyset$. It follows that x_o is an adherent point.

Let X be a co-countable topology. Take $E \subseteq X$ as countable, and let $x \in E$. Define $Y := X \subseteq E$, then Y is open in X. Suppose V is a neighbourhood of x. Suppose $V \cap Y = \emptyset$. Then $V \subseteq E$, since E is the complement of Y. Since Y is uncountable, it follows that $X \subseteq V$ is uncountable. However, this is a contradiction since V is a neighbourhood. Hence, $V \cap Y \emptyset$. It follows that x is an adherent point of Y.

Suppose $(x^{(n)})_{n=m}^{\infty}$ is some sequence that converges to x. Then we know that for every neighbourhood V of x, there is some $N \geq m$ such that for every $n \geq N$, $x^{(n)} \in V$. Consider $V := (Y \cup \{x\}) \setminus \{x^{(n)} : n \in \mathbb{N}\}$. Obviously, $x \in V$, and since $X \setminus V = (E \setminus \{x\}) \cup \{x^{(n)} : n \in \mathbb{N}\}$, which is countable, then it follows that V is a neighbourhood that does not contain any $x^{(n)}$, which is a contradiction. Hence, the sequence does not converge, and (c) does not hold.

Exercise 2.5.11

Exercise 2.5.11. Let E be a subset of a topological space $(X, \mathcal{F})$. Show that E is open if and only if every element of E is an interior point, and show that E is closed if and only if E contains all of its adherent points. Prove analogues of Proposition 1.2.15(e)-(h) (some of these are automatic by definition). If we assume in addition that X is Hausdorff, prove an analogue of Proposition 1.2.15(d) also, but give an example to show that (d) can fail when X is not Hausdorff.

Let E be open, and let $x \in E$. Then E is a neighbourhood of x, and since $E \subset E$, it follows that x is an interior point.
Now suppose that E only contains interior points. Then we know for every $x \in E$, there is some neighbourhood V_x such that $V_x \subseteq E$. We know that $\cup_{x \in E} V_x$ is open. It follows easily that $E = \cup_{x \in E} V_x$, and hence E is open.

We define a set K in a topological space (X, F) to be closed iff its complement $X \setminus K$ is open.

Let E be closed. Then we know $X \setminus E$ is open. Let x be an adherent point of E. Then it is an interior point or a boundary point. If it is an interior point, then obviously $x \in E$. If it is a boundary point, then know for every neighbourhood V, $V \cap E = \emptyset$. Since it is not an interior point, we know that V for every V. Hence, $V \cap (X \setminus E) = \emptyset$ and $V \nsubseteq (X \setminus E)$ for every V. It follows that x is a boundary point of $X \setminus E$. We know $x \notin X \setminus E$ since it is open, hence $x \in E$.
Let E contain all its adherent points. Let x be a boundary point of $X \setminus E$. Since $V \cap (X \setminus E) \neq \emptyset$ and $V \nsubseteq (X \setminus E)$, for every V, we know that x is a boundary point of E. Then $x \in E$, since it contains all adherent points, and $x \notin X \setminus E$. Hence $X \setminus E$ is closed and E is open.

(e) If E is a subset of X, then E is open if and only if the complement $X \backslash E := \{x \in X : x \notin E\}$ is closed.

This follows from the definition.

(f) If $E_1, \ldots, E_n$ are a finite collection of open sets in X, then $E_1 \cap E_2 \cap \ldots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is also closed.

First half of this follows from the definition of open sets in a topology.

If $F_1 \cdots F_n$ are closed, then $X \setminus F_1 \cdots X \setminus F_n$ are closed. Then $X \setminus F_1 \cap \cdots \setminus F_n$ is open, and $X \setminus (F_1 \cup \cdots_n)$ is open, hence, $F_1 \cup \cdots_n$ is closed.

(g) If $\{E_\alpha\}_{\alpha \in I}$ is a collection of open sets in X (where the index set I could be finite, countable, or uncountable), then the union $\bigcup_{\alpha \in I} E_\alpha := \{x \in X : x \in E_\alpha \text{ for some } \alpha \in I\}$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in I} F_\alpha := \{x \in X : x \in F_\alpha \text{ for all } \alpha \in I\}$ is also closed.

The first part follows from the definition.

$\{F_\alpha\}_{\alpha \in I}$ are closed. So $\{X \setminus F_\alpha\}_{\alpha \in I}$ are open. Then $\cup_{\alpha \in I} X \setminus F_\alpha$ is open. Hence, $X \setminus \cap_{\alpha \in I} F_\alpha$ is open and hence $\cap_{\alpha \in I} F_\alpha$ is closed.

(h) If E is any subset of X, then $\mathrm{int}(E)$ is the largest open set which is contained in E; in other words, $\mathrm{int}(E)$ is open, and given any other open set $V \subseteq E$, we have $V \subseteq \mathrm{int}(E)$. Similarly $\overline{E}$ is the smallest closed set which contains E; in other words, $\overline{E}$ is closed, and given any other closed set $K \supset E$, $K \supset \overline{E}$.

Suppose x is a boundary point of $int(E)$, then $V \cap int(E) \neq \emptyset$ and $V \nsubseteq int(E)$ for every neighbourhood V. Obviously, $V \cap E \neq \emptyset$ for every neighbourhood V. Suppose $V \subseteq E$ for one of these neighbourhoods. Then since $V \nsubseteq int(E)$, it follows that there is some boundary point y of E such that $y \in V$. However, since $V \subset E$ it follows that y is an interior point, which is a contradiction. Hence, $V \nsubseteq E$, and x is also a boundary point of E. Since $int(E)$ does not contain boundary points of E, it follows that $x \notin int(E)$, and $int(E)$ is open.
Let $V \subseteq E$ be open. Let $x \in V$. Then, obviously, V is a neighbourhood of x and, since $V \subseteq E$, we know x is not a boundary point. Hence, $x \in int(E)$ and $V \subseteq int(E)$.

Let x be a boundary point of $\bar{E}$. Then since $V \nsubseteq \bar{E}$ for every V, it follows easily that $V \nsubseteq E$. Since $\bar{E}$ contains only adherent points of E, we know that x is an adherent point of E. Hence, x is a boundary point of E, and $x \in \bar{E}$. Hence, $\bar{E}$ is closed.
Let $E \subset K$ be closed. Let x be a boundary point of E. We can assert that either $V \nsubseteq K$ for every V, or there is some V such that $V \subset K$. Since x is a boundary point of E, we know that $V \cap E \neq \emptyset$ for every V. It follows that $V \cap K \neq \emptyset$ for every V since $E \subset K$. Hence, if we assume $V \nsubseteq K$ for every V, it follows that x is a boundary point of K, and an element of K since K is closed. If we assume instead that there is some V such that $V \subset K$, then it follows

immediately that x is an element of K. Either way, all boundary points of E are elements of K. Hence, $\bar{E}$ is a subset of K.

(d) Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

Suppose X is Hausdorff. Let x be a boundary point of $\{x_o\}$. Then we know that for every neighbourhood V of x, $V \cap \{x_o\} \neq \emptyset$, which implies that $x_o \in V$ for every V. Suppose $x \neq x_o$. Then, since X is Hausdorf, we have some open V and W such that $x \in V$ and $x_o \in W$, and $V \cap W = \emptyset$. However, V is a neighbourhood, we know that $x_o \in V$, since V and W are disjoint. Hence, $x = x_o$. Then $x \in \{x_o\}$ and $\{x_o\}$ is closed.

Exercise 2.5.12

Exercise 2.5.12. Show that the pair $(Y, \mathcal{F}_Y)$ defined in Definition 2.5.7 is indeed a topological space.

Since $\emptyset \cap Y = \emptyset$ and $Y \cap X = Y$, then we know $\emptyset, Y \in \mathbb{F}_Y$.
Suppose $W_1 \cdots W_n \in \mathbb{F}_Y$. Then
$W_1 \cap \cdots \cap W_n = (V_1 \cap Y) \cap \cdots \cap (V_n \cap Y) = (V_1 \cap \cdots \cap_n) \cap Y$. Since $V_1 \cap \cdots \cap_n \in \mathbb{F}$, it follows that $(V_1 \cap \cdots \cap_n) \cap Y \in \mathbb{F}_Y$, and consequently, $W_1 \cap \cdots \cap W_n \in \mathbb{F}_Y$.
Suppose $(W_\alpha)_{\alpha \in I} \in \mathbb{F}_Y$. Then $\cup_{\alpha \in I}(W_\alpha) = \cup_{\alpha \in I}(V_\alpha \cap Y) = (\cup_{\alpha \in I} V_\alpha) \cap Y$. Since $\cup_{\alpha \in I} V_\alpha \in \mathbb{F}$, we know $\cup_{\alpha \in I}(W_\alpha) \in \mathbb{F}_Y$.

Exercise 2.5.13

Exercise 2.5.13. Generalize Corollary 1.5.9 to compact sets in a topological space.

Corollary 1.5.9. *Let (X, d) be a metric space, and let $K_1, K_2, K_3, \ldots$ be a sequence of non-empty compact subsets of X such that*

$$K_1 \supset K_2 \supset K_3 \supset \ldots.$$

Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

The topology is assumed to be Hausdorff (errata).

Let K_n, for some $n \geq 1$. Let x_o be an adherent point of K_n. Then for every neighbourhood V of x_o, we have $V \cap K_n \neq \emptyset$. Assume $x_o \notin K_n$. Then for every $x \in K_n$, using the Hausdorff property, we have some neighbourhood F_x of x and some neighbourhood G_{x_o} of x_o. We can choose these F_x's are construct a subcover $K_n \subset \cup_{x \in K_n} F_x$. Since K_n is compact, we know there is some finite $\bar{K} \subset K_n$ such that $K_n \subset \cup_{x \in \bar{K}} F_x$. If we take $\cap_{x \in \bar{K}} G_x$, we know that this is open, since all the G_x are open. Obviously, $x_o \in \cap_{x \in \bar{K}} G_x$ and $\cap_{x \in \bar{K}} G_x \cap K_n = \emptyset$, which is a contradiction, since x_o is an adherent point.
Since the K_n are closed, it follows that the $X \setminus K_n$ are open.

In a similar argument to Exercise 1.5.6, the claim follows.

Exercise 2.5.14

Exercise 2.5.14. Generalize Theorem 1.5.10 to compact sets in a topological space.

Theorem 1.5.10. *Let (X, d) be a metric space.*

(a) If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact if and only if Z is closed.

(b) If $Y_1, \ldots, Y_n$ are a finite collection of compact subsets of X, then their union $Y_1 \cup \ldots \cup Y_n$ is also compact.

(c) Every finite subset of X (including the empty set) is compact.

X is assumed to be Hausdorff.

(a)
If Z is compact, then from a similar argument to Exercise 2.5.13, it is closed.
Suppose Z is closed. Then $X \setminus Z$ is open. Hence, for any cover subcover of Z, adding $X \setminus Z$ will make it a subcover of X. Since X is compact, we know there is a finite subcover of X. Removing $X \setminus Z$, we have a finite subcover that covers Z.

(b)
Suppose we have some subcover of $\cup_{i=1}^{n} Y_i$. Then it is a subcover for each Y_n, and it follows that we have finite subcover for each Y_n. Taking the union of each finite subcover, the union will still be finite, and it will also cover $\cup_{i=1}^{n} Y_i$. Hence, $\cup_{i=1}^{n} Y_i$ is compact.

(c)
Since the singleton set is closed, and hence compact from (a), it follows easily that any finite set is compact.

Exercise 2.5.15

Exercise 2.5.15. Let (X, d_X) and (Y, d_Y) be metric spaces (and hence a topological space). Show that the two notions continuity (both at a point, and on the whole domain) of a function $f : X \to Y$ in Definition 2.1.1 and Definition 2.5.8 coincide.

Definition 2.1.1 (Continuous functions). Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

Definition 2.5.8 (Continuous functions). Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is *continuous at* x_0 iff for every neighbourhood V of $f(x_0)$, there exists a neighbourhood U of x_0 such that $f(U) \subseteq V$. We say that f is *continuous* iff it is continuous at every point $x \in X$.

Suppose f is continuous in terms of Definition 2.1.1. Then let V be a neighbourhood of $f(x_o)$. Since V is open, we know that $f(x_o)$ is an interior point, and, hence, there is some r such that $B(f(x_o), r) \subseteq V$. We can then assert that there is some δ such that $d_X(x, x_o) < \delta \to d_Y(f(x), f(x_o)) < r$. Defining $U := B(x_o, \delta)$, it follows that if $f(x) \in f(U)$, then $x \in U$ and $d_X(x, x_o) < \delta$, so $d_Y(f(x), f(x_o) < r$ and $f(x) \in V$, since \$B(f(x_o),r)\subseteq V$.Hence,$f(U)\subseteq V $seteqV$ teq V.efinition 2.5.8 follows from the result.

Suppose f is continuous in terms of Definition 2.5.8. Let $\epsilon > 0$. Then $B(f(x_o), \epsilon)$ is a neighbourhood of $f(x_o)$. We can assert that there is some neighbourhood U of x_o such that $f(U) \subseteq B(f(x_o), \epsilon)$. Since x_o is an interior point of U, we know that there is some r such that $B(x_o, r) \subseteq U$. Hence, if $d_X(x, x_o) <$, then $x \in U$ and $f(x) \in f(U)$, hence, $f(x) \in B(f(x_o), \epsilon)$ and $d_Y(f(x), f(x_o)) < \epsilon$. Definition 2.1.1 follows from the result.

Exercise 2.5.16

Exercise 2.5.16. Show that when Theorem 2.1.4 is extended to topological spaces, that (a) implies (b). (The converse is false, but constructing an example is difficult.) Show that when Theorem 2.1.5 is extended to topological spaces, that (a), (c), (d) are all equivalent to each other, and imply (b). (Again, the converse implications are false, but difficult to prove.)

Theorem 2.1.4 (Continuity preserves convergence). *Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following three statements are logically equivalent:*

(a) f is continuous at x_0.

(b) Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.

(c) For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$.

Suppose $(x^{(n)})_{n=1}^{\infty}$ converges to x_o in X. Suppose V is a neighbourhood of $f(x_o)$. Then we have some neighbourhood U of x_o such that $f(U) \subseteq V$, since f is continuous at x_o. We know that there is some $N \geq m$ such that $x^{(n)} \in V$ for every $n \geq N$, since the sequence converges to x_o U is a neighbour of x_o. For any $x^{(n)} \in U$, it follows that $f(x^{(n)}) \in U$, so $f(x^{(n)}) \in V$. The claims follows.

Theorem 2.1.5. *Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following four statements are equivalent:*

(a) *f is continuous.*

(b) *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*

(c) *Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.*

(d) *Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.*

(a) ⇒ (c)

Suppose V is open in Y. Let $x \in f^{-1}(V)$. Since $f(x) \in V$ and V is open, it follows that $f(U) \subseteq V$ for some U. Hence, we can choose U_x for every $x \in f^{-1}(V)$. Consider $\cup_{x \in f^{-1}(V)} U_x$. Obviously, $f^{-1}(V) \subseteq \cup_{x \in f^{-1}(V)} U_x$. Now suppose $x_o \in \cup_{x \in f^{-1}(V)} U_x$. Then $x_o \in U_x$ for some U_x. Since $f(U_x) \subseteq V$, we have $f(x_o) \in V$ and $x_o \in f^{-1}(V)$ Hence, we have shown that $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. Since the U_x are open, we know that $\cup_{x \in f^{-1}(V)} U_x$ is open. The claim follows.

(c) ⇒ (a)

Suppose $x_o \in X$. Suppose V is a neighbourhood of $f(x_o)$. Then V is open in Y and $x_o \in f^{-1}(V)$. Consider $f(f^{-1}(V))$. It is obviously a subset of Y. Hence, we have shown the result.

(a) ⇒ (d)

Since F is closed in Y, we have $Y \setminus F$ is open in Y. Hence, using (c), which is implied by (a), we can assert that $f^{-1}(Y \setminus F)$ is open in X. So $X \setminus f^{-1}(Y \setminus F)$ is closed in X. It is easy enough to show that $f^{-1}(F) = X \setminus f^{-1}(Y \setminus F)$ and hence $f^{-1}(F)$ is closed in X.

(d) ⇒ (c)

Suppose V is open in Y. Then $Y \setminus V$ is closed in Y, and, using (d), $f^{-1}(Y \setminus V)$ is closed in X. Hence, $X \setminus f^{-1}(Y \setminus V)$ is open in X. It follows easily enough that $X \setminus f^{-1}(Y \setminus V) = f^{-1}(V)$, and so the claim follows.

(a) ⇒ (b)
The claim follows easily from our generalisation of Theorem 2.1.4.

Exercise 2.5.17

Exercise 2.5.17. Generalize both Theorem 2.3.1 and Proposition 2.3.2 to compact sets in a topological space.

Theorem 2.3.1 (Continuous maps preserve compactness). *Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := \{f(x) : x \in K\}$ of K is also compact.*

Proposition 2.3.2 (Maximum principle). *Let (X, d) be a compact metric space, and let $f : X \to \mathbf{R}$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

Let $f(K) \subseteq \cup_{\alpha \in I} W_\alpha$ be a subcover. We know that for each W_α, $f^{-1}(W_\alpha)$ is open in X, since each W_α is open in Y, and f is continuous. It follows easily that $K \subseteq \cup_{\alpha \in I} f^{-1}(W_\alpha)$. Since K is compact, we can find a finite J such that $K \subseteq \cup_{\alpha \in J} f^{-1}(W_\alpha)$. It follows easily that $f(K) \subseteq \cup_{\alpha \in J} W_\alpha$. The claim follows.

The second argument follows in a similar manner to Exercise 2.3.2.