Class Questions

Part 1

1.1 Exercise 1

Let A, B and C be three events. Express the following events versus A, B and C:

- Only A occurs.
- At most two events occur among the 3 events.
- Only two events occur.
- At least 2 events occur.

1.
$A \cap \bar{B} \cap \bar{C}$

2.
$A \cap \bar{B} \cap C = \bar{A} \cap \bar{B} \cap \bar{C}$

3.
$(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

4.
$(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C) \cup (A \cap B \cap C)$

1.
$A_i = \{\text{roll 1 gives i}\}$, $B_i = \{\text{roll 2 gives i}\}$, $C_i = \{\text{roll 3 gives i}\}$
$S_i = \{\text{3 rolls gives same number i}\}$
$S = \{\text{3 rolls gives same number}\}$
$p(S_i) = p(A_i \cap B_i \cap C_i) = p(A_i)p(B_i)p(C_i) = 1/6 \times 1/6 \times 1/6$
$p(S) = p(\cup_{i=1}^{6} S_i) = \sum_{i=1}^{6} p(S_i) = 6p(S_i) = 1/36$

2.
$p(A) = 3 \times 6 \times (1/6 \times 1/6 \times 4/6) = 5/12$

3.
$p(A) = 6 \times (1/6 \times 5/6 \times 4/6 \times) = 5/9$

1.3 Exercise 3 : Bridge player

How many different hands (13 cards) can have a bridge player (deck having 52 cards)? Find the following probabilities:

- Probability to get exactly one ace ?
- Probability to get at least one ace ?
- Probability to get at least one ace and at least one king ?
- Probability to get at least one ace or at least one king ?
- Probability to get only cards from three or less different colors ?

The sample space is given by $\Omega = \binom{52}{13}$.

1.

$N = 4 \times \binom{48}{12}$.

2.

$A = \{\text{at least one ace}\}$

$p(A) = 1 - p(\bar{A})$

$\bar{A} = \{\text{no aces}\}$

$card(\bar{A}) = \binom{48}{13}$

$p(A) = 1 - \frac{card(\bar{A})}{\Omega}$

3.

$A = \{\text{at least one ace}\}$ $B = \{\text{at least one kind}\}$

$p(A \cap B) = 1 - p(A \bar{\cap} B)$

$p(A \bar{\cap} B) = p(\bar{A}) + p(\bar{B}) - p(\bar{A} \cup \bar{B})$

$p(\bar{A}) = p(\bar{B}) = \frac{\binom{48}{13}}{\Omega}$

$p(\bar{A} \cup \bar{B}) = \frac{\binom{44}{13}}{\Omega}$

Hence,

$p(A \cap B) = 1 - 2\frac{\binom{48}{13}}{\Omega} + \frac{\binom{44}{13}}{\Omega}$

4.

$A_1 = \{\text{at least one diamond}\}$

$A_2 = \{\text{at least one heart}\}$

$A_3 = \{\text{at least one club}\}$

$A_4 = \{\text{at least one pique}\}$

$E = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4$

$$p(E) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{\text{all groups G of k events}} P(\cap_{A_i \in G} A_i)$$

Since for each group G of k events, the probability will be the same, we have

$$p(E) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P(G_k)$$

$$= \binom{4}{1} \frac{\binom{39}{13}}{\Omega} - \binom{4}{2} \frac{\binom{26}{13}}{\Omega} + \binom{4}{3} \frac{\binom{13}{13}}{\Omega}$$

Where we have used the fact that $P(\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3 \cup \bar{A}_4) = 0$

1.4 Exercise 4 : Mad postman

In a real, a postman has to distribute N letters in p boxes. r_i letters are for the box i. Thus we have $r_1 + r_2 + \ldots + r_p = N$. He does his job randomly. Find the following probabilities :

- Probability that the box 1 is correctly filled?
- Probability that the box 1 contains no letter of other neighbours ?
- Probability that box 1 has k letters ?
- Probability that no box is empty ?

1.

$\Omega = p^N$

$card(A) = (p-1)^{N-r_1}$

$$p(A) = \frac{(p-1)^{N-r_1}}{p^N}$$

2.

$card(A) = p^{r_1}(p-1)^{N-r_1}$

$$p(A) = \frac{p^{r_1}(p-1)^{N-r_1}}{p^N}$$

3.

$$card(A) = \binom{N}{k}(p-1)^{N-k}$$

$$p(A) = \frac{\binom{N}{k}(p-1)^{N-k}}{\Omega}$$

4.

$A_i = \text{box not empty}$

$E = \cap_{i=1}^{p} A_i$

$p(E) = 1 - p(\bar{E})$

$$p(\bar{E}) = p(\cup_{i=1}^{p} \bar{A}_i) = \sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}p(G_k)$$

Where

$G_k = \cap_{i=1}^{k} \bar{A}_i$

Hence, $p(G_k) = \dfrac{(p-k)^N}{p^N}$

Hence,

$$p(E) = \sum_{k=1}^{p}(-1)^{k+1}\binom{p}{k}\frac{(p-k)^N}{p^N}$$

Part 2

2.1 Exercise 1 : Cards player

A and B play cards. B gets r cards from a well-shuffled 52 cards deck. If B has the spade king, he wins.

- Find the probability that B wins ?
- A tricks and keeps k cards before B gets his cards. What is the probability that B wins ?

$W = \{\text{B wins}\}\}$

1.

$$p(W) = \frac{\binom{51}{r-1}}{\binom{52}{r}} = \frac{r}{52}$$

2.

$T = \{\text{A has king spade in k cards}\}$

If T occurs, then $p(W) = 0$. Hence,

$p(W) = p(W \cap \bar{T}) = p(W|\bar{T})p(\bar{T})$

$$p(\bar{T}) = 1 - p(T) = \frac{52-k}{52}$$

$$p(W|\bar{T}) = \frac{r}{52-k}$$

Hence,

$$p(W) = \frac{r}{52}$$

2.2 Exercise 2

A table has 9 drawers. A letter has a probability of 1/2 to be in this table. If the letter is in this table, each drawer has the same probability to contain the letter.

- What is the probability that the letter is in the drawer 9 ?
- The 8 first drawers are opened and are empty. What is the probability that the letter is in the last drawer ?

$L = \{\text{letter is in table}\}$

$D_i = \{\text{letter is in drawer i}\}$

$p(D_i|L) = 1/9$

1.

Since $p(D_i|\bar{L}) = 0$, we have

$p(D_i) = p(D_i|L)p(L) = 1/18$

2.

$$p(D_9| \cap_{i=1}^{8} \bar{D}_i) = \frac{p(D_9 \cap \cap_{i=1}^{8} \bar{D}_i)}{p(\cap_{i=1}^{8} \bar{D}_i)}$$

$p(D_9 \cap \cap_{i=1}^{8} \bar{D}_i) = p(D_9) = 1/18$

$p(\cap_{i=1}^{8} \bar{D}_i) = p(D_9 \cup \bar{L}) = 1/18 + 1/2 = 10/18$

Hence,

$p(D_9| \cap_{i=1}^{8} \bar{D}_i) = 1/10$

Part 3

3.1 Exercise 1

Let us consider a waiting queue. The number of clients in the queue follows a Poisson law with parameter λ. The probability that the client gets the service is equal to p. Define the law of the random variable that counts the number of satisfied clients.

Let X be the number of clients in the queue. Let Y be a Bernoulli random variable of whether a given client is satisfied or not. Let N be the number of satisfied clients in the queue.

Hence,

$p_X(x) = e^{-\lambda}\frac{\lambda^x}{x!}$

$p_Y(y) = p$ if the client is satisfied and $p_Y(y) = 1 - p$ otherwise.

Given $X = x$ we can write

$$p(N = k|X = x) = \binom{x}{k} p^k (1-p)^{x-k}$$

To find $p(N = k)$, we use the total probability theorem.

$$\begin{aligned} p(N = k) &= \sum_{x=k}^{\infty} p(N = k \cap X = x) \\ &= \sum_{x=k}^{\infty} p(N = k|X = x)p(X = x) \\ &= \sum_{x=k}^{\infty} \binom{x}{k} p^k (1-p)^{x-k} e^{-\lambda}\frac{\lambda^x}{x!} \end{aligned}$$

Note that sum starts at $x = k$ because the $x \geq k$.

Manipulating this and using the power series expansion of e^x, we can write

$$p(N = k) = e^{-\lambda p}\frac{(\lambda p)^k}{k!}$$

Which is a Poisson random variable with parameter λp.

3.2 Exercise 2

Let X be a random discrete variable that takes its value in $\mathbb{N}^*$ and such that $P[X = k] = \frac{\alpha}{k!}$.

- Calculate α.
- Calculate $E[X]$ and $E[X(X-1)]$. Deduce the variance of X.

We can use the normalisation property to determine α.

$$\sum_{k=1}^{\infty} P[X = k] = 1$$

Hence,

$$\sum_{k=1}^{\infty} \frac{\alpha}{k!}$$

$$= \alpha \sum_{k=1}^{\infty} \frac{1}{k!}$$

$$= \alpha(\sum_{k=0}^{\infty} \frac{1}{k!} - 1)$$

$$= \alpha(e - 1)$$

Hence,

$$\alpha = \frac{1}{e-1}$$

$$E[X] = \sum_{k=1}^{\infty} k \frac{\alpha}{k!}$$
$$= \alpha \sum_{k=1}^{\infty} \frac{k}{k!}$$
$$= \alpha \sum_{k=1}^{\infty} \frac{1}{(k-1)!}$$
$$= \alpha \sum_{k=0}^{\infty} \frac{1}{k!}$$
$$= \alpha e$$

$$E[X(X-1)] = \sum_{k=1}^{\infty} k(k-1) \frac{\alpha}{k!}$$
$$= \sum_{k=2}^{\infty} k(k-1) \frac{\alpha}{k!}$$ since when $k = 1$, the term equals 0
$$= \sum_{k=2}^{\infty} \frac{\alpha}{(k-2)!}$$
$$= \alpha \sum_{k=0}^{\infty} \frac{1}{k!}$$
$$= \alpha e$$

Also
$E[X(X-1)] = E[X^2] - E[X] = \alpha e$
Hence,
$E[X^2] = 2\alpha e$
And
$var(X) = E[X^2] - (E[X]^2) = 2\alpha e - (\alpha e)^2$

3.3 Exercise 3

We have an urn filled with N balls numerated from 1 to N. We do k pullings without putting back the balls. Let's X be the smallest number obtained.

- Find the distribution of X. Verify the normalisation condition.
- Show that

$$E[X] = \sum_{i=1}^{N-k+1} P[X \geq i].$$

Let $A_i = \{\text{i is the smallest number drawn}\}$.
Then,

$$\Omega = \binom{N}{k}$$

$card(A_1)$ will just equal the probability of drawing a 1 because if we draw 1, then it will be the smallest number.
Hence,

$$card(A_1) = \binom{N-1}{k-1}$$

For $card(A_2)$, it equals the probability of drawing a 2 and not drawing a 1, since otherwise, 2 will not be the smallest number. Hence,

$$card(A_2) = \binom{N-2}{k-1}$$

In general,

$$card(A_n) = \binom{N-n}{k-1}$$

Hence,

$$p_X(x) = \frac{\binom{N-x}{k-1}}{\binom{N}{k}}$$

To verify the normalisation property we write

$$\sum_{x=1}^{N-k+1} \frac{\binom{N-x}{k-1}}{\binom{N}{k}}$$

First, note that using Pascal's formula

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Hence,

$$\binom{N}{k} = \binom{N-1}{k-1} + \binom{N-1}{k}$$

$$= \binom{N-1}{k-1} + \binom{N-2}{k-1} + \binom{N-2}{k}$$
$$= \binom{N-1}{k-1} + \binom{N-2}{k-1} + \binom{N-3}{k-1} + \binom{N-3}{k}$$
$$\cdots$$
$$= \binom{N-1}{k-1} + \cdots + \binom{k}{k-1} + \binom{k}{k}$$
$$= \binom{N-1}{k-1} + \cdots + \binom{k}{k-1} + \binom{k-1}{k-1}$$
$$\sum_{x=1}^{N-k+1} \binom{N-x}{k-1}$$

Where we have used the fact that $\binom{k}{k} = \binom{k-1}{k-1}$

Hence,

$$\sum_{x=1}^{N-k+1} \frac{\binom{N-x}{k-1}}{\binom{N}{k}}$$
$$= \frac{\binom{N}{k}}{\binom{N}{k}}$$
$$= 1$$

Let $M = N - k + 1$.

Note that

$$P[X \geq i] = \sum_{j=i}^{M} P[X = j]$$

Hence,

$$\sum_{i=1}^{M} P[X \geq i]$$
$$= \sum_{i=1}^{M} \sum_{j=i}^{M} P[X = j]$$

Now, let

$$\mathbf{1}_{j \geq i} = \begin{cases} 1 & j \geq i \\ 0 & \text{otherwise} \end{cases}$$

Then, we can continue

$$= \sum_{i=1}^{M} \sum_{j=1}^{M} \mathbf{1}_{j \geq i} P[X = j]$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \mathbf{1}_{j \geq i} P[X = j]$$
$$= \sum_{j=1}^{M} P[X = j] \sum_{i=1}^{M} \mathbf{1}_{j \geq i}$$
$$= \sum_{j=1}^{M} P[X = j] j$$
$$= \mathbb{E}[X]$$

Part 4

4.1 Exercise 1

Let us define X a random variable taking its value in $[-1, 1]$. The probability density f_X of X forms a isosceles triangle (the hypotenuse is on the x-axis).

- Find the expression of $f_x(x)$ and the expression of the cumulative distribution function $F_x(x)$.
- Calculate the mathematical expectation of X and its variance.
- Calculate $P[X \leq \frac{1}{4}, X^2 \leq \frac{1}{4}]$.
- Let us define another random variable by $Y = X^2$. Calculate the cumulative distribution function of Y. Find its mean.

We know that the area of the triangle is equal to 1 from the normalisation property. We also know that the base of the triangle is 2.
Hence,
$A = 0.5 \times 2 \times h = 1$
$h = 1$

$$f_X = \begin{cases} x + 1 & x \in [-1, 0] \\ 1 - x & x \in [0, 1] \end{cases}$$

Then, since $F_X = \int_{-\infty}^{x} f_X dx$, we have
For $x \in [-1, 0]$
$$F_X = \int_{-1}^{x} u + 1 du$$
$$= \frac{(x+1)^2}{2}$$

For $x \in [0, 1]$

$$F_X = \int_{-1}^{x} f_X du$$
$$= 0.5 + \int_{0}^{x} 1 - x du$$
$$= \frac{1 + 2x - x^2}{2}$$

Next,

$$E[X] = \int_{-\infty}^{\infty} x f_X dx$$
$$= \int_{-1}^{0} x(x+1)dx + \int_{0}^{1} x(1-x)dx$$
$$= 0$$

And

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X dx$$
$$= \int_{-1}^{0} x^2(x+1)dx + \int_{0}^{1} x^2(1-x)dx$$
$$= 1/6$$

Hence,

$$var(X) = 1/6$$

$$= P[X \leq 1/4 \cap (-1/2 \leq X \leq 1/2)]$$
$$= P[-1/2 \leq X \leq 1/4]$$
$$= \int_{-1/2}^{1/4} f_X dx$$
$$= \int_{-1/2}^{0} x + 1 dx + \int_{0}^{1/4} 1 - x dx$$
$$= [\frac{x^2}{2} + x]-1/2^0 + [x - \frac{x^2}{2}]_0^{1/4}$$
$$= \frac{19}{32}$$

$$F_Y(y) = P[Y \leq y]$$
$$= P[X^2 \leq y]$$
$$= P[-\sqrt{y} \leq X \leq \sqrt{y}]$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$
$$= \frac{1 + 2\sqrt{y} - \sqrt{y}^2}{2} - \frac{(-\sqrt{y}+1)^2}{2}$$
$$= 2\sqrt{y} - y$$

4.2 Exercise 2

Let X be an absolutely continuous random variable. Its density probability is given by :

$$f_X(x) = Ke^{-\lambda x} \text{ for } x \geq 0$$

and $f_X(x) = 0$ for $x < 0$.

- Under wath condition on λ, $f_X(x)$ is a probability density? Express K versus λ.
- Let x and x_0 be two reals greater than zero. Calculate $P[X > x|X > x_0]$. Comment your result.

For f_X to be a PDF, then it must obey the normalisation property.
Hence,

$$\int_{\infty}^{\infty} f_X dx$$
$$= \int_0^{\infty} Ke^{-\lambda x} dx$$
$$= \frac{-K}{\lambda}[e^{-\lambda x}]_0^{\infty}$$
$$= \frac{-K}{\lambda}[\lim_{t\to\infty} e^{-\lambda t} - 1]$$

If $\lambda < 0$, then the integral diverges, and the f_X is not a PDF.
If $\lambda = 0$, then the integral is equal to 0 and f_X is not a PDF.
If $\lambda > 0$, then the integral converges to K/λ. In which case, f_X is a PDF if also $\lambda = K$.

Let $x > x_o$ if $x_o \geq x$ then $P[X > x|X > x_o] = 1$.

$$P[X > x|X > x_o] = \frac{P[X > x \cap X > x_o]}{P[X > x_o]} = \frac{P[X > x]}{P[X > x_o]}$$

Note that

$$P[X > x] = 1 - P[X \leq x] = 1 - \int_{-\infty}^{x} f_X du$$
$$= 1 - \int_0^x \lambda e^{-\lambda u} du$$
$$= 1 - [-e^{-\lambda u}]_0^{\infty}$$
$$= e^{\lambda x}$$

Hence,

$$P[X > x | X > x_0] = \frac{e^{\lambda x}}{e^{\lambda x}} = \frac{e^{\lambda x}}{e^{\lambda x_o}}$$
$$= e^{-\lambda(x - x_0)}$$
$$= P[X > x - x_o]$$

This shows that, assuming $x > x_0$, if we know that $X > x_0$, then the probability that $X > x$ is simply the probability that X is greater than the difference between x and x_0.
Letting $t = x - x_0$, then if we know that $X > x_0$, then the probability for any additional t, i.e. $P[X > x_0 + t]$, does not depend on the initial start time. This is the memorylessness property.

4.3 Exercise 3

Let X be a random variable absolutely continuous that takes its value in $\mathbb{B}$. And Y a random variable given by $X = ln(Y)$.

- Find the set of values taken by Y.
- Express the relation between F_Y and F_x and between f_Y and f_x.

We assume that X is a Gaussian random variable of mean μ and variance σ^2.

- Calculate the mean of Y.
- Let's define the median value m of X as

$$P[X \geq m] \geq \frac{1}{2} \text{ and } P[X \leq m] \geq \frac{1}{2}.$$

  For an absolutely continuous random variable, find the characterisation of m using F_X. Deduce the median value of X and of Y.

- Let us define $Z = \frac{1}{\sigma e^{\mu}}(Y - e^{\mu})$. Calculate the probability density of Z and its limit when σ tends to zero.

*$\mathbb{B}$ is supposed to be $\mathbb{R}$.

$$Y = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+$$

$$F_Y = P[Y \leq y]$$
$$= P[e^X \leq y]$$
$$= P[ln(y) \leq X]$$
$$= F_X(ln(y)$$

$$f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y} F_Y$$

$$= \frac{\mathrm{d}}{\mathrm{d}y} F_X(ln(y)$$
$$= \frac{1}{y} f_X(ln(y))$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy$$
$$= \int_{-\infty}^{\infty} y \frac{1}{y} f_X(ln(y)) dy$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(ln(y)-\mu)^2}{2\sigma^2}} dy$$

From here, we can change the variables. Let $t = ln(y)$.
Then,
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} e^t dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{t-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

We develop the exponential further.
$$t - \frac{(t-\mu)^2}{2\sigma^2} = \frac{1}{2\sigma^2}[2\sigma^2 t - (t-\mu)^2]$$
$$= \frac{-1}{2\sigma^2}[t^2 - 2(\mu+\sigma^2)t + \mu^2]$$
$$= \frac{-1}{2\sigma^2}[t^2 - 2(\mu+\sigma^2)t + (\mu+\sigma^2)^2 - (\mu+\sigma^2)^2 + \mu^2]$$
$$= \frac{-1}{2\sigma^2}(t - (\mu+\sigma^2))^2 - (\mu+\sigma^2)^2 + \mu^2]$$
$$= \frac{-1}{2\sigma^2}(t - (\mu+\sigma^2))^2 - \sigma^2(2\mu+\sigma^2)]$$

Hence, we can write the integral as
$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(t-(\mu+\sigma^2))^2}{2\sigma^2}} e^{\frac{2\mu+\sigma^2}{2}} dt$$
$$= e^{\frac{2\mu+\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(t-(\mu+\sigma^2))^2}{2\sigma^2}} dt$$
$$= e^{\frac{2\mu+\sigma^2}{2}}$$

Where we have used the fact that
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(t-(\mu+\sigma^2))^2}{2\sigma^2}} dt = 1$$

Since this is just a normal distribution over t with mean $\mu + \sigma^2$ and variance σ^2.

We can write the median as

$P[X \leq m_x] = 1/2$

And then write

$F_X(m_x) = 1/2$

Since X is normal we immediately recognise that $m_x = \mu$, since it is symmetric.

For Y,

$P[Y \leq m_y]$

$= P[e^X \leq m_y]$

$= P[X \leq ln(m_y)$

$= F_X(ln(m_y))$=1/2 $= 1/2$ /2 $/2$ $$

Hence, this is satisfied when

$m_x = ln(m_y)$

Or, equivalently,

$m_y = e^{m_x}$

For Z,

$P[Z \leq z] = P[\frac{1}{\sigma e^\mu}(Y - e^\mu) \leq z]$

$= P[Y \leq e^\mu(\sigma z + 1)$

$= P[X \leq ln(e^\mu(\sigma z + 1))]$

$= F_X(ln(e^\mu(\sigma z + 1)))$

Then, using

$\frac{d}{dx}F(x) = f_X(x)$

We have

$\frac{d}{dz}F(ln(e^\mu(\sigma z + 1))) = \frac{\sigma e^\mu}{e^\mu(\sigma z + 1)} f_X(ln(e^\mu(\sigma z + 1)))$

$= \frac{1}{\sqrt{2\pi}(\sigma z + 1)} e^{-\frac{(\mu + ln(\sigma z + 1) - \mu)^2}{2\sigma^2}}$

$= \frac{1}{\sqrt{2\pi}(\sigma z + 1)} e^{-\frac{ln(\sigma z + 1)^2}{2\sigma^2}}$

Note that in general

$ln(x + 1) \to x$ and $x \to 0$

Hence,

$f_Z(z) = \frac{1}{\sqrt{2\pi}(\sigma z + 1)} e^{-\frac{z^2}{2}}$

As $z \to 0$

Part 5

5.1 Exercise 1

Let (X, Y) be a couple of independent random variables. We assume that X is chosen according to an uniform probability law in [0,1] and Y follows an exponential law with parameter λ.

Let us define another random variable couple (U, V) as : $U = X + Y$ and $V = X - Y$.

- Find the joint density probability of (U, V).
- Deduce the marginal law of U and V.
- Calculate the covariance matrices of $[XY]^T$ and $[UV]^T$.

5.2 Exercise 2

Let X and Y be two real independent random variables. X and Y follow the same exponential law with parameter 1. Let us define $Z = X + Y$.

- find the conditional law of Z knowing X noted $F_{Z|X=x}(z)$.
- Deduce the law of Z.

$$\begin{aligned}
F_{Z|X=x}(z) &= P[Z \leq z|X = x] \\
&= P[X + Y|X = x \leq z] \\
&= P[x + Y \leq z] \\
&= P[Y \leq z = x] \\
&= F_Y(z - x) \\
&= 1 - e^{x-z} \text{ if } z \geq z, \text{ 0 otherwise.}
\end{aligned}$$

Using the total probability theorem, we can write

$$\begin{aligned}
F_Z(z) &= P[Z \leq z] \\
&= \int_{-\infty}^{\infty} P[Z \leq z|X = x] f_X(x) dx \\
&= \int_{-\infty}^{z} F_{Z|X=x}(z) f_X(x) dx \\
&= \int_{0}^{z} (1 - e^{x-z}) e^{-x} dx \\
&= \int_{0}^{z} e^{-x} - e^{-z}) dx \\
&= 1 - e^{-z}(1 + z)
\end{aligned}$$

5.3 Exercise 3

Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed random variables. X_i follows a normal law centered with variance equal to one. Let us define two random variables as follows:

$$\begin{aligned} M_n &= \frac{1}{n}\sum_{i=1}^{n} X_i \\ V_n &= \frac{1}{n}\sum_{i=1}^{n}(X_i - M_n)^2 \end{aligned}$$

Let A be a square matrix of size $n \times n$, orthogonal and such that the terms in its first line are all equal to $\frac{1}{\sqrt{n}}$. Let us define $Y = AX$ where $X = [X_1, \ldots, X_n]^T$.

- Find the law of Y.
- Express M_n and V_n versus the components of Y.
- Are the variables M_n and V_n independent ?
- Find the law of M_n.
- Show that if X follows a normal law centered and with a variance equal to one then X^2 follows a gamma law noted $\gamma(\frac{1}{2}, \frac{1}{2})$.
- Let Z_1 and Z_2 be two independent random variables. Their law is respectively $\gamma(\theta_1, \lambda)$ and $\gamma(\theta_2, \lambda)$. What is the law of $Z = Z_1 + Z_2$?
- Deduce from the previous questions the law of V_n.

Part 6

6.2 Exercice 2

A Manufacture of Woven has a new machine. to evaluate the performance of this machine, we manufacture pieces of 10 cm and count the number of flaws on each piece. After 126 observations, we have the following distribution:

Nb flaws k	Nb pieces n_k
0	44
1	49
2	24
3	7
4	2

1. Compute $\hat{m}$ and $\hat{\sigma}^2$.
2. Assume that the number of flaws follows a Poisson law of parameter λ, find the Maximum Likelihood estimator of λ.

1.

$$\hat{m} = \frac{44(0) + 49(1) + 24(2) + 7(3) + 2(4)}{126} = 1$$
$$\hat{\sigma^2} = \frac{44(-1)^2 + 49(0)^2 + 24(1)^2 + 7(2)^2 + 2(3)^3}{126}$$

2.

$$p(x;\lambda) = \prod_{x=1}^{N} P[X = x]$$
$$= \prod_{x=1}^{N} \frac{\lambda^x}{x!} e^{-\lambda}$$
$$= (\prod_{x=1}^{N} \frac{1}{x!}) \lambda^{\sum_{i=1}^{N} x} e^{-N\lambda}$$

Then take the log

$$L(x;\lambda) = ln(\prod_{x=1}^{N} \frac{1}{x!}) + ln(\lambda^{\sum_{i=1}^{N} x}) + ln(e^{-N\lambda})$$
$$= ln(\prod_{x=1}^{N} \frac{1}{x!}) + \sum_{i=1}^{N} x ln(\lambda) - N\lambda$$

Then

$$\frac{dL}{d\lambda} = \frac{1}{\lambda} \sum_{i=1}^{N} x - N = 0$$

Hence,

$$\lambda = \frac{\sum_{i=1}^{N} x}{N}$$

6.3 Exercice 3

Let's X be a random variable following a Bernouilli law: $P[X = u] = p^{1-u}(1-p)^u$ with $u = 0, 1$.

1. Using the moment method, find a estimator of p.
2. Find the Maximum Likelihood estimator of p.

1.

$$E[X] = p \times 0 + (1-p) \times 1$$
$$= 1 - p$$

Hence,

$$p = 1 - \hat{m}.$$

2.

$$l(p; x_1, \cdots, x_n) = \prod_{i=1}^{n} p^{1-x_i}(1-p)^{x_i}$$
$$= p^{\sum_{i=1}^{n} 1-x_i}(1-p)^{\sum_{i=1}^{n} x_i}$$
Hence,
$$L(p; x_1, \cdots, x_n) = lnp^{\sum_{i=1}^{n} 1-x_i} + ln(1-p)^{\sum_{i=1}^{n} x_i}$$
$$= (\sum_{i=1}^{n} 1 - x_i)lnp + (\sum_{i=1}^{n} x_i)ln(1-p)$$
And,
$$\frac{dL}{dp} = \frac{\sum_{i=1}^{n} 1 - x_i}{p} - \frac{\sum_{i=1}^{n} x_i}{1-p}$$
From here, we can write
$$p = 1 - \frac{\sum_{i=1}^{n} x_i}{n} = 1 - \hat{m}$$

6.4 Exercice 4

Mr Aimeté-ésse wants to know how many Carps he has in his pond. He asks to his nephew Luke, who managed with brillant success his study at the famous Télécom Bretagne university. Luke could use the force to answer the question, however he prefers to carry on a non invasive fishing action and manage to mark 50 carps. These marked fishies are then releasd in the pond. After a few days, another fishing action is done, 100 carps are catched. We observed that 26 are marked.

Let's us defined by N the number of carps in the pond.

1. Luke propose to use the bernouilli law to estimate N. What should he define as random variable? Give the different parameters of this random variable.
2. Find the ML estimator of the proportion of marked carps in the pond. Deduce an estimator of N.
3. Gives the intervalle of confidence $(1 - \alpha = 0.95)$ on p and on N.

Let $X =$ whether the fish is marked.
Then
$$E[X] = p$$
And
$$p = \frac{50}{N}$$

2.
$$l(p; x_1 \cdots x_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$
It follows easily by logging both sides that
$$p = \frac{26}{100} = 0.26$$
And

$N = 50/p = 192$

3.

We use the fact $\dfrac{\hat{p} - p}{\sigma/\sqrt{n}} \to N(0,1)$ as $n \to \infty$.

For the 95% confidence interval, we have

$$-1.96 \leq \frac{\hat{p} - p}{\sigma/\sqrt{n}} \leq 1.96$$

And

$$\hat{p} - \frac{1.96\sigma}{\sqrt{n}} \leq p \leq \hat{p} + \frac{1.96\sigma}{\sqrt{n}}$$

We need to use an estimator for σ.

We can use either

$\sigma^2 = \hat{p}(1 - \hat{p}) = 0.1924$ and so $\sigma = 0.4386$

Or

$$\sigma^2 = \frac{1}{n-1} \sum_{i=1}^{1} 00(x_i - \hat{m})^2 = 0.1943$$

and so $\sigma = 0.44$

Using the first estimator, we have the interval as

$[0.174, 0.346]$

Part 7

7.1 Using the inversion of the cumulative distribution function

The easiest way to generate a random variable Y having a cumulative distribution function $F_Y(y) = P[Y \leq y]$ consists in using a good uniform generator in $[0,1]$ then we generate a $U \sim \mathcal{U}[0,1]$ and we use the transformation $Y = F_Y^{-1}(U)$.

1. Let us define $U \sim \mathcal{U}[0,1]$ and let G be a bijective growing function. Find the expression of $P[G(U) \leq y]$ and deduce that $Y = G(U)$ has F_Y as cumulative distribution function if and only if $G = F_Y^{-1}$.

2. Generate from this method a random variable Y having an exponential law (noted $\epsilon(\lambda)$) of mean $1/\lambda$ (for example you can take $\lambda = 2$). The exponential law of mean $1/\lambda$ has for complementary cumulative distribution function :

$$F_Y^c(y) = P[Y \geq y] = 1 - F_Y(y) = e^{-\lambda y}\mathbb{I}_{\mathbb{R}^+}(y) + \mathbb{I}_{\mathbb{R}^{*-}}(y).$$

3. Generate $T = 10000$ independent realisations of a random variable Y of law $\epsilon(\lambda)$. Calculate analytically the expression of the mean and variance of Y. Calculate the empirical mean and variance of the generated samples. Comment.

4. Verify the adequation of the generated samples with the theoretical complementary cumulative distribution function.

1. $P[G(U) \leq y]$
$= P[U \leq G^{-1}(y)]$
$= G^{-1}(y)$
If $G = F_Y^{-1}$, then $F_y = G^{-1}$
Hence,
$P[G(U) \leq y] = F_Y(y)$
$F_{G(U)}(y) = F_y(y)$

2.
$F_Y = 1 - e^{-\lambda x}$
And hence,
$$Y = -\frac{log(1-x)}{\lambda}$$

3.
```
N = 100000
lambda_ = 2

U = rand(N)
E = -log(1-U) / lambda_

# Analytical expressions
true_mean = 1/lambda_
true_var = 1/lambda_**2

sample_mean = E.mean()
sample_var = E.var()
```

7.2 Box and Muller method

A very useful law is the Gaussian law (due to the central limit theorem). We may have a generator for this law (*randn* in matlab) but it is not always the case. It is then useful to be able to generate random variables that follow this law. In this section, we present a classical method due to Box and Muller (1958). This method allows us to generate 2 independent realisations of a Gaussian law $\mathcal{N}(0,1)$.

1. Let X_1 and X_2 be two Gaussian independent random variables following $\mathcal{N}(0,1)$. Let us define $Z = X_1^2 + X_2^2$. Show that Z follows an exponential law of mean $\frac{1}{\lambda}$ with $\lambda = 1/2$.

   First of all find the expression of $P[Z \geq z]$. Express $P[Z \geq z]$ as the double integral of the density function of X_1 and X_2. Resolve this integral (hits: you can try this transformation: $x_1 = rcos\theta$ and $x_2 = rsin\theta$). You should find that : $P[Z \geq z] = e^{-z/2}\mathbb{I}_{\mathbb{R}^+}(z) + \mathbb{I}_{\mathbb{R}^{+-}}(z)$.

Let X_1, X_2 be independent normal random variables with mean μ and variance σ^2.
Let $Z = X_1^2 + X_2^2$.
Then,

$$P[Z > z] = P[X_1^2 + X_2^2 > z]$$
$$= \int\int_{x_1^2+x_2^2>z} f_{X_1,X_2}(x_1, x_2)dx_1dx_2$$
$$= \int\int_{x_1^2+x_2^2>z} f_{X_1}(x_1)f_{X_2}(x_2)dx_1dx_2 \quad \text{(independence)}$$
$$= \int\int_{x_1^2+x_2^2>z} \frac{1}{2\pi}e^{-\frac{x_1^2+x_2^2}{2}}dx_1dx_2$$

Then, we change variables with $x_1 = rcos\theta$ and $r = rsin\theta$, in which case $x_1^2 + x_2^2 = r^2$.
And
$|J| = r$.
Hence, we can continue with

$$= \int_0^{2\pi}\int_{r^2>z}^{\infty} \frac{1}{2\pi}re^{-\frac{r^2}{2}}drd\theta$$
$$= \int_0^{2\pi}\int_{r>\sqrt{z}}^{\infty} \frac{1}{2\pi}re^{-\frac{r^2}{2}}drd\theta$$
$$= \int_0^{2\pi} \frac{1}{2\pi}[-e^{-\frac{r^2}{2}}]_{\sqrt{2}}^{\infty}d\theta$$
$$= \int_0^{2\pi} \frac{1}{2\pi}e^{-\frac{z}{2}}d\theta$$
$$= e^{-\frac{z}{2}}$$

Since $P[Z > z] = e^{-z/2}$, we have $F_Z(z) = 1 - e^{-z/2}$, and consequently $f_Z(z) = \frac{1}{2}e^{-z/2}$, which shows that Z follows an exponential law with $\lambda = \frac{1}{2}$.

2. Let X_1 and X_2 be two Gaussian independent random variables following $\mathcal{N}(0,1)$ and let (r,θ) be the polar coordinates of the point having (X_1, X_2) as Cartesian coordinates. Show, using the previous question, that r^2 follows an exponential law of parameter $1/2$. Justify that θ follows an uniform law that takes its value in $[0, 2\pi]$.

Since $z = r^2$, the first result follows immediately.
From the working out in the previous question we know that the marginal distribution of θ is given by

$$f_\theta(\theta) = \int_0^{\infty} \frac{1}{2\pi}re^{-\frac{r^2}{2}}dr = \frac{1}{2\pi}$$

Since the density is a constant, we know that it's a uniform distribution. Since also the denominator is 2π, the interval of $[0, 2\pi]$ follows.

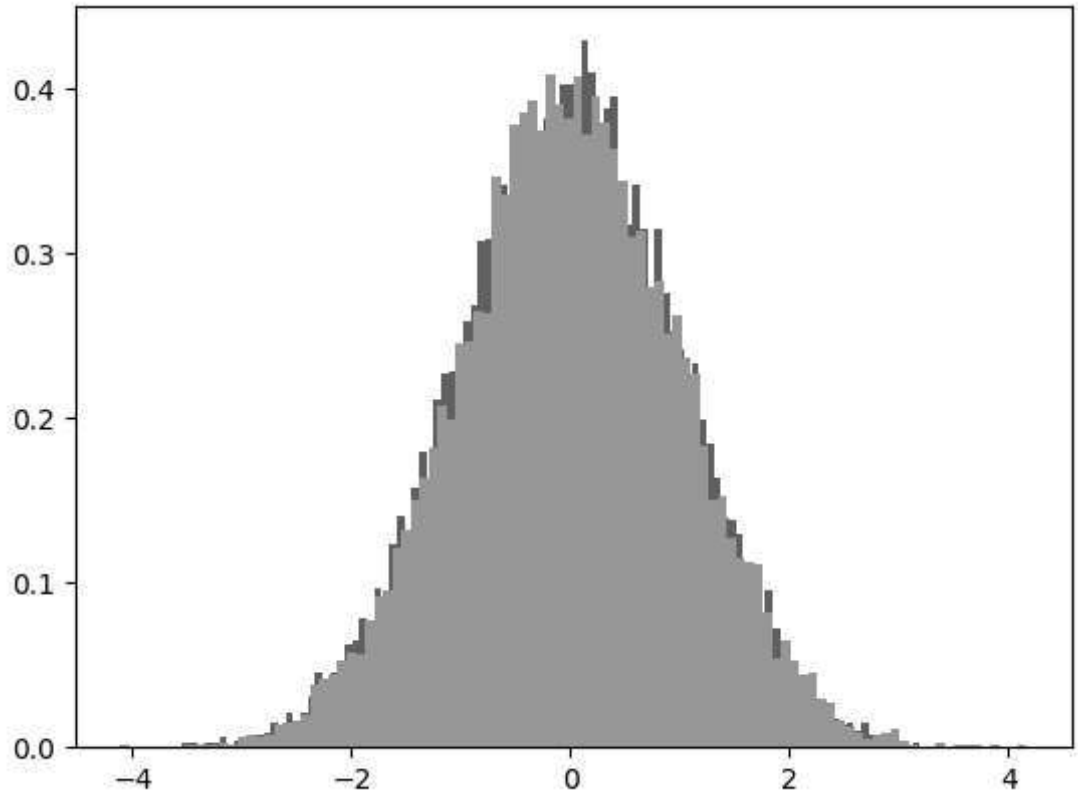
3. Deduce a method to generate two independent random variables of law $\mathcal{N}(0, 1)$.

1. Generate
2. Generate $\theta = U[0, 2 * \pi]$
3. Set $x1 = sqrtZcos\theta$ and $x2 = sqrtZsin\theta$

4. Recall the transformation to be done to a random variable $X \sim \mathcal{N}(0, 1)$ to obtain a random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$. Generate $T = 10000$ realisations of a random variable $Y \sim \mathcal{N}(5, 4)$.

5. Verify that the mean and variance of the generated samples are in concordance with the assigned value. Plot the histogram of the samples. Comment.

```
# Part 7 Exercise 2
from numpy.random import exponential, uniform
from numpy import cos, sin
from math import pi
import matplotlib.pyplot as plt

# 3
lambda_ = 2
n = 10000
Z = exponential(lambda_, n)
Theta = uniform(0, 2*pi, n)

X1 = Z**0.5*cos(Theta)
X2 = Z**0.5*sin(Theta)

plt.hist(X1, bins=100, density=True);
plt.hist(X2, bins=100, density=True);

# 4
T1 = X1*2+5
T2 = X2*2+5

# 5
T1.mean() # 4.95
T1.var()  # 3.94

T2.mean() # 4.99
T2.var()  # 4.12
```



Part 8

8 Random variables Generation: Acceptance-Rejection

The inversion method of the cumulative distribution function cannot always be used to generate a random variable. Indeed F_Y is not always bijective (for example random variable that takes discrete values) or F_Y is bijective but we don't have the analytic form of F_Y^{-1}. Therefore, another method has been developed: the Acceptance-Rejection method.

We would like to generate a density law f using a generator of a density law g. To perform the Acceptance/Rejection method, we need to find a constant M such that$f(X) \leq Mg(X)$ for all X.

The method is then the following:

1. Generate $X \sim g$ et $U \sim \mathcal{U}([0,1])$.
2. if $U \leq f(X)/Mg(X)$, we accept $Y = X$ as a realisation of the density law f else if we repeat the operation.

Questions:

1. Show that we necessary have $M \geq 1$.

Since $f(X) \leq Mg(X)$, we can write

$$\int_{-\infty}^{\infty} f(X)dx \leq \int_{-\infty}^{\infty} Mg(X)dx$$

And hence, from the normalisation property
$1 \leq M$.

2. Can we generate a Cauchy law $\mathcal{C}(0,1)$ $f(x) = \frac{1}{\pi(1+x^2)}$ from a normal law $\mathcal{N}(0,1)$? Why?

To generate f from g we need an M such that

$$\frac{1}{\pi(1+x^2)} \leq \frac{M}{\sqrt{2\pi}} e^{-x^2/2}$$

Which implies that we need an M such that

$$\frac{\sqrt{2\pi} e^{x^2/2}}{\pi(1+x^2)} \leq M$$ for every x

However, the LHS is an unbounded function since the numerator tends to infinity much faster than the denominator tends to 0 as x increases.
Hence, no such M exists and we cannot generate a Cauchy variable using a normal variable.
Note however that we could generate a normal from a cauchy.

3. We want to show that the acceptance/rejection method gives effectively a random variable following the density law f.

   (a) Justify the following equality:

$$P[Y \leq y] = \sum_{i=0}^{+\infty} P\left[X \leq y, U \leq \frac{f(X)}{Mg(X)}\right] \left(P\left[U \geq \frac{f(X)}{Mg(X)}\right]\right)^i.$$

   (b) Show that :

$$P\left[X \leq y, U \leq \frac{f(X)}{Mg(X)}\right] = \frac{1}{M} F_f(y)$$

   where $F_f(y) = \int_{-\infty}^{y} f(t)dt$.

   (c) Deduce that $P[Y \leq y] = F_f(y)$.

(a)
Let N be the number of iterations needed for an acceptance. Then N is a geometric random variable and

$$P[N = i] = P[Acceptance]P[Rejection]^{i-1}$$

Since N partitions the sample space, we can write

$$PY \leq y] = \sum_{i=1}^{\infty} P[Y \leq y \cap N = i]$$

$$= \sum_{i=1}^{\infty} P[Y \leq y \cap \text{Acceptance on i} \cap \text{i-1 Rejections}]$$

Since the rejections are independent from acceptance, and consequently also from $Y \leq y$, we can write

$$= \sum_{i=1}^{\infty} P[Y \leq y \cap \text{Acceptance on i}] P[\text{i-1 Rejections}]$$

$$= \sum_{i=1}^{\infty} P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}]P[U > \frac{f(X)}{Mg(X)}]^{i-1}$$
$$= \sum_{i=0}^{\infty} P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}]P[U > \frac{f(X)}{Mg(X)}]^{i}$$

Where we have used

$$P[\text{Acceptance}] = P[U \leq \frac{f(X)}{Mg(X)}]$$
$$P[\text{Rejection}] = P[U < \frac{f(X)}{Mg(X)}]$$

(b)

$$P[X \leq y, U \leq \frac{f(X)}{Mg(X)}] = \int_{-\infty}^{y} \int_{0}^{\frac{f(x)}{Mg(x)}} f_X(x) f_U(u) du dx$$
$$= \int_{-\infty}^{y} f_X(x) \int_{0}^{\frac{f(x)}{Mg(x)}} du dx$$
$$= \int_{-\infty}^{y} f_X(x) \frac{f(x)}{Mg(x)} dx$$
$$= \int_{-\infty}^{y} f_X(x) \frac{f(x)}{Mg(x)} dx$$
$$= \frac{1}{M} \int_{-\infty}^{y} f(x) dx$$
$$= \frac{1}{M} F_f(y)$$

Where we have used the fact that $f_X(x) = g(x)$

(c)

$$P[Y \leq y] = \sum_{i=0}^{\infty} P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}]P[U > \frac{f(X)}{Mg(X)}]^{i}$$
$$= P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}] \sum_{i=0}^{\infty} P[U > \frac{f(X)}{Mg(X)}]^{i}$$
$$= P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}] \frac{1}{1 - P[U > \frac{f(X)}{Mg(X)}]}$$
$$= P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}] \frac{1}{P[U \leq \frac{f(X)}{Mg(X)}]}$$

Where we have noted that $P[U > \frac{f(X)}{Mg(X)}]$ is always less than or equal to 1, and we recognise the geometric series.
Hence,

$$P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}] = P[U \leq \frac{f(X)}{Mg(X)}]P[Y \leq y]$$

Now,

$$P[U \leq \frac{f(X)}{Mg(X)}] = \int_{-\infty}^{\infty} \int_{0}^{\frac{f(x)}{Mg(x)}} f_X(x) f_U(u) du dx$$
$$= \frac{1}{M}$$

Where we have used a similar working to that in (b).
Hence,

$$P[Y \leq y, U \leq \frac{f(X)}{Mg(X)}] = \frac{1}{M} P[Y \leq y] = \frac{1}{M} F_Y(y)$$

From the result in (b), it follows that $F_Y(y) = F_f(y)$

4. Show that the acceptance probability after one iteration of the algorithm is $\frac{1}{M}$ and that the mean of iterations needed to accept a value for Y is equal to M. What does that mean concerning the choice of the density law f?

We have already shown $P[U \leq \frac{f(X)}{Mg(X)}] = \frac{1}{M}$, which is the acceptance probability.
Since the number of iterations is a geometric random variable, we know that $E[N] = 1/p = M$, where p is the acceptance probability.

5. Application: Generation of a truncated Gaussian law. The goal of this question is to study and compare 2 different methods of generation of a Gaussian random variable left truncated to a threshold s.

   (a) **First method**

   We consider the following algorithm:

   i. Generate $X \sim \mathcal{N}(0, 1)$
   ii. If $X > s$ then we accept $Y = X$ else if return to i.

   The previous algorithm allows to generate a random variable Y having a density function $f(y) = \alpha \mathbb{I}_{y>s} e^{-y^2/2}$.

   i. Show that the normalisation constant α is equal to:

$$\alpha = \sqrt{\frac{2}{\pi}} \frac{1}{\text{erfc}(\frac{s}{\sqrt{2}})}$$

   with $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du$.

i.

Since $f(y)$ is a density function, we know that

$$\int_{y>s}^{\infty} \alpha e^{-\frac{y^2}{2}} dy = 1$$

Where we have used the fact that $f(y)$ is only non-zero for $y > s$.

By changing the variables with $u = y/\sqrt{2}$, we have

$$\int_{u>s/\sqrt{2}}^{\infty} \alpha\sqrt{2} e^{-u^2} du = 1$$

And

$$\sqrt{\frac{\pi}{2}} \int_{u>s/\sqrt{2}}^{\infty} \alpha \frac{2}{\sqrt{\pi}} e^{-u^2} du = 1$$

Which easily leads to the result.

ii.

ii. We set $M_1^{-1} = \frac{1}{2}\text{erfc}(\frac{s}{\sqrt{2}})$. Show that the acceptance probability after one iteration is equal to M_1^{-1} and that the mean of needed iterations to accept a value for Y is equal to M_1.

Acceptance probability is given by

$$P[X > s] = \int_{s}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Here, we can use the same change in variables as in i.

$$= \int_{s/\sqrt{2}}^{\infty} \frac{\sqrt{2}}{\sqrt{2\pi}} e^{-y^2} du$$

$$= \int_{s/\sqrt{2}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-y^2} du$$

$$= \frac{1}{2} \int_{s/\sqrt{2}}^{\infty} \frac{2}{\sqrt{\pi}} e^{-y^2} du$$

$$= \frac{1}{2} erfc(s/\sqrt{2})$$

Since the number of iterations for acceptance is a geometric random variable, we have immediately that the mean is equal to M_1.

(b) **Acceptance - Rejection method**

We implement the acceptance/rejection method with the exponential law $g(x) = \mathbb{I}_{x \geq s} \lambda e^{-\lambda(x-s)}$ as instrumental law.

i. Generate an exponential random variable of law $g(x) = \mathbb{I}_{x \geq s} \lambda e^{-\lambda(x-s)}$. Let us define $U \sim \mathcal{U}([,])$ and $X = s - \frac{1}{\lambda} lnU$. Show that X has $g(x)$ as density function.

$$P[X \leq x] = P[s - \frac{1}{\lambda} lnU \leq x]$$
$$= P[-\frac{1}{\lambda} lnU \leq x - s]$$
$$= 1 - P[U \leq e^{-\lambda(x-s)}]$$

Then, we have

$F_X(x) = 1 - F_U(e^{-\lambda(x-s)})$

Then,

$f_X(x) = \lambda e^{-\lambda(x-s)}$

ii. Show that $\frac{f(x)}{g(x)}$ is over-valued by $M_2 = \frac{\alpha}{\lambda} e^{\frac{\lambda^2}{2} - \lambda s}$.

We need to find an M_2 such that $\frac{f(x)}{g(x)} \leq M_2$.

Note,

$$\frac{f(x)}{g(x)} = \frac{\alpha e^{\frac{-x^2}{2}}}{\lambda e^{-\lambda(x-s)}}$$
$$= \frac{\alpha}{\lambda} e^{\frac{-x^2}{2} + \lambda x - s\lambda}$$

Note that this expression is bounded. To find the maximum, take the derivative of the log.

Let y be the logged expression

$$y = ln(\frac{\alpha}{\lambda}) - \frac{x^2}{2} + \lambda x - s\lambda$$

Then,

$y' = -x + \lambda$.

Hence, the maximum is achieved when $x = \lambda$.

So, we can set

$$M_2 = \frac{\alpha}{\lambda} e^{\frac{\lambda^2}{2} - \lambda s}$$

iii. Deduce that the value of λ that minimizes the number of iterations in the acceptance-rejection algorithm is equal to :

$$\lambda = \frac{s + \sqrt{s^2 + 4}}{2}$$

and find the mean of iterations needed to accept one value of Y for that choice of λ.

To minimise the number of iterations it's necessary that we minimise M_2. Hence, we take the derivative of M_2 with respect to λ. First, we take the log

$$L = ln(\alpha) + ln(\frac{1}{\lambda}) + \frac{\lambda^2}{2} - \lambda s$$

Then,

$$L' = -\frac{1}{\lambda} + \lambda - s = 0$$

This gives

$$\lambda = \frac{s \pm \sqrt{s^2+4}}{2}$$

Since $\sqrt{s^2+4} > s$, and $\lambda > 0$, the result follows.

The mean number of iterations is M_2.

(c) **Comparison of the two methods**

i. Compare M_1 and M_2. Show that for large value of s the second method is better than the first one in terms of the mean of needed iterations. Estimate numerically the value of s for which the acceptance-rejection method is better than the first one.

Part 10

10 Statistics

10.1 Rayleigh distribution

The Rayleigh distribution is often used in the reverberation problems related to radar and sonar. Its probability density function (pdf) is given by:

$$\begin{aligned} f_X(x) &= \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \text{ if } x \geq 0 \\ &= 0 \text{ elsewhere} \end{aligned} \tag{3}$$

1. Show that if U and V follows Gaussian distribution $\mathcal{N}(0, \sigma^2)$, then $X = \sqrt{U^2+V^2}$ follow a Rayleigh distribution.

   (hint: compute the cumulative distribution function $F_X(x) = P[X \leq x]$ using polar coordinates).

2. Using the previous result, generate 1000 samples following a Rayleigh distribution of parameter $\sigma^2 = 0.5$.

3. Plot the histogram of the generated variables. Normalize the histogram and verify that the samples follow a Rayleigh distribution of parameter $\sigma^2 = 0.5$.

4. Calculate the mean and variance of the generated variables. Compare your answers to the theoretical expressions : $m_x = \sigma\sqrt{\pi/2}$ and $Var(X) = \frac{4-\pi}{2}\sigma^2$.

5. Do the previous questions again but with 10000 samples. Comment your results.

6. Show that if Y follows an exponential distribution of parameter $\lambda = 1/2\sigma^2$, then $X = \sqrt{Y}$ follows a Rayleigh distribution.

7. Using the previous result, generate 1000 samples following a Rayleigh distribution of parameter $\sigma^2 = 0.5$. Plot the histogram and verify that theses samples follow a Rayleigh distribution.

1.

$$F_X(x) = P[X \leq x] = P[\sqrt{U^2+V^2} \leq x]$$

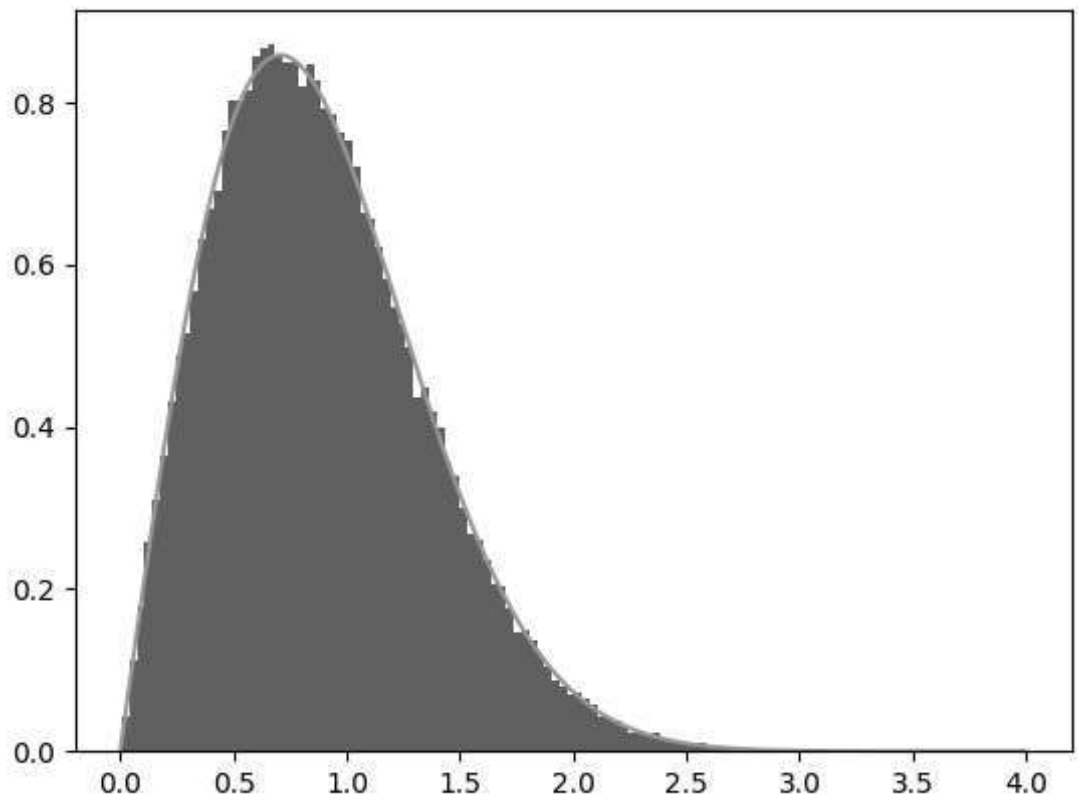
$$
\begin{aligned}
&= \int\int_{u^2+v^2\leq x^2} f_{U,V}(u,v)dudv \\
&= \int\int_{u^2+v^2\leq x^2} f_U(u)f_V(v)dudv \\
&= \int\int_{u^2+v^2\leq x^2} \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{u^2}{2\sigma^2}}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{v^2}{2\sigma^2}}dudv \\
&= \frac{1}{2\pi\sigma^2}\int\int_{u^2+v^2\leq x^2} e^{-\frac{u^2+v^2}{2\sigma^2}}dudv \\
&= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_{r\leq x} e^{-\frac{r^2}{2\sigma^2}}rdrd\theta \\
&= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}[-\sigma^2 e^{-\frac{r^2}{2\sigma^2}}]_0^x d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} 1-e^{-\frac{x^2}{2\sigma^2}}d\theta \\
&= 1-e^{-\frac{x^2}{2\sigma^2}}
\end{aligned}
$$

Then

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$$

2./3.

```
from numpy import random, sqrt, pi, exp, linspace, log
import matplotlib.pyplot as plt

sigma = sqrt(0.5)

n = 100000
U = random.normal(0, sigma, n)
V = random.normal(0,sigma, n)

X = sqrt(U**2 + V**2)

n_vec = linspace(0,4,200)
true_X = (n_vec / sigma**2)*exp(-n_vec**2/(2*sigma**2))

plt.hist(X, bins=100, density=True)
plt.plot(n_vec, true_X)
```



4.

```
true_mean = sigma*sqrt(pi/2)        # 0.8862269
true_var = sigma**2 * (4 - pi) / 2  # 0.2146018
X.mean()                            # 0.8862094
X.var()                             # 0.2143287
```

5.

Estimation is better with larger N.

6.

$$
\begin{aligned}
F_X(x) &= P[X \le x] = P[\sqrt{y} \le x] \\
&= P[Y \le x^2] \\
&= \int_0^{x^2} \frac{1}{2\sigma^2} e^{-\frac{y}{2\sigma^2}} dy
\end{aligned}
$$

Then, it follows easily that

$$f_X(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$$

Since

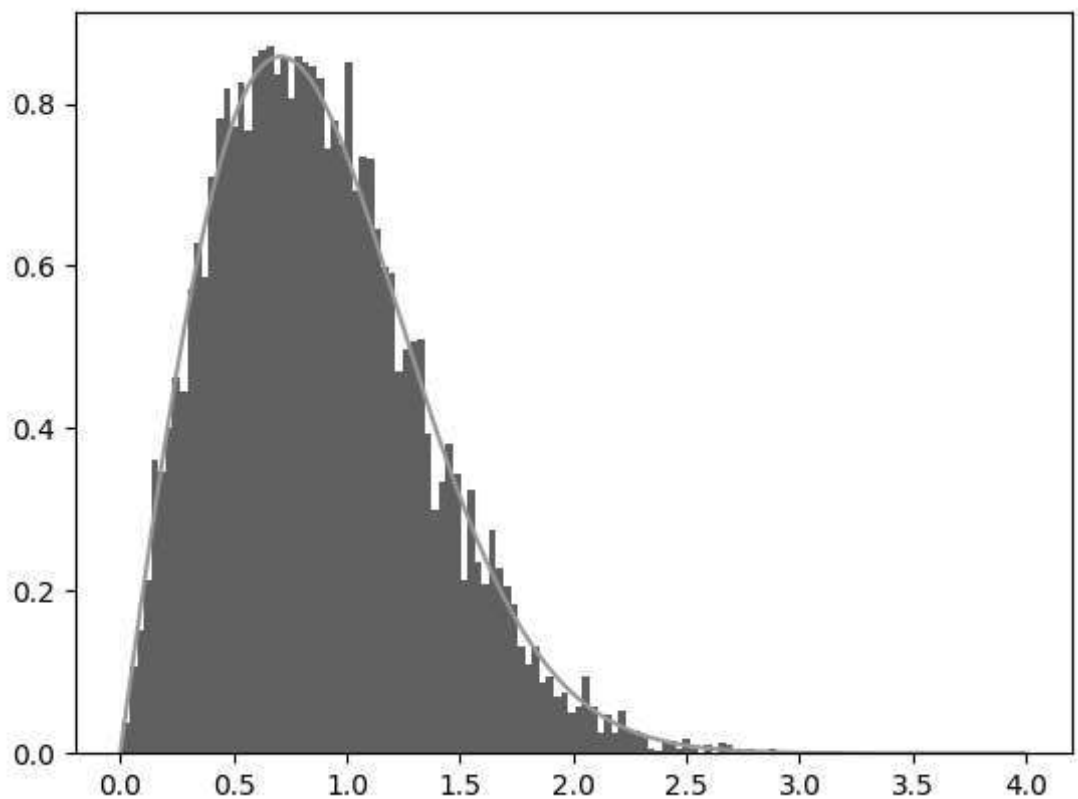
$$F_X(x) = F_Y(x^2)$$

So

$$
\begin{aligned}
\frac{d}{dx}F_X(x) &= \frac{d}{dx}F_Y(x^2) \\
&= \frac{d}{dx}x^2\frac{d}{dy}F_Y(x^2) \\
&= 2xf_Y(x^2) \\
&= \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}
\end{aligned}
$$

7.

```
n = 10000
sigma = sqrt(0.5)
lambda_ = 1/(2*sigma**2)

U = random.random(n)
Y = -log(U)/lambda_
X = sqrt(Y)

plt.hist(X, bins=100, density=True)
plt.plot(n_vec, true_X)
```



10.2 Weibull distribution

The exponential distribution and Rayleigh distributions are particular cases of the Weibull distribution. The Weibull distribution is genereally used to estimate the average life of materials. Its probability density function is given by:

$$\begin{aligned} f_X(x) &= \frac{\beta}{\eta}\left(\frac{x}{\eta}\right)^{\beta-1} e^{-\left(\frac{x}{\eta}\right)^{\beta}} \text{ if } x \geq 0 \\ &= 0 \text{ elsewhere} \end{aligned} \tag{4}$$

1. Verify that the corresponding cumulative distribution function is equal to : $F_X(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^{\beta}}$.
2. What type of distribution do we get if we set the parameters of the Weibull distribution to $\beta = 1, \eta = 1/\lambda$?
3. Show that if $X = \eta(-log(U))^{1/\beta}$ with U a uniform random variable on the intervalle $[0, 1]$, then X follows a Weibull distribution of parameters (β, η).
4. Using the previous results, generate 1000 samples following a Weibull distribution of parameters $(\beta = 1.4, \eta = 76)$. Plot the histogram of the generated samples and verify that theses samples follow a Weibull distribution.

1.

$$\begin{aligned} F_X(x) &= \int_{-\infty}^{x} f_X(t)dt \\ &= \int_{-\infty}^{x} \frac{\beta}{\eta}(\frac{t}{\eta})^{\beta-1} e^{-(\frac{x}{\eta})^{\beta}} \\ &= [-e^{-(\frac{t}{\eta})^{\beta}}]_{-\infty}^{x} \\ &= 1 - e^{-(\frac{x}{\eta})^{\beta}} \end{aligned}$$

2.

$F_X(x) = 1 - e^{-\lambda x}$, an exponential distribution.

3.

$$\begin{aligned} F_X(x) &= P[X \leq x] \\ &= P[\eta(-\ln(U))^{\frac{1}{\beta}} \leq x] \\ &= P[U \geq e^{-(\frac{x}{\eta})^{\beta}}] \\ &= 1 - P[U \leq e^{-(\frac{x}{\eta})^{\beta}}] \\ &= 1 - e^{-(\frac{x}{\eta})^{\beta}} \end{aligned}$$

And so

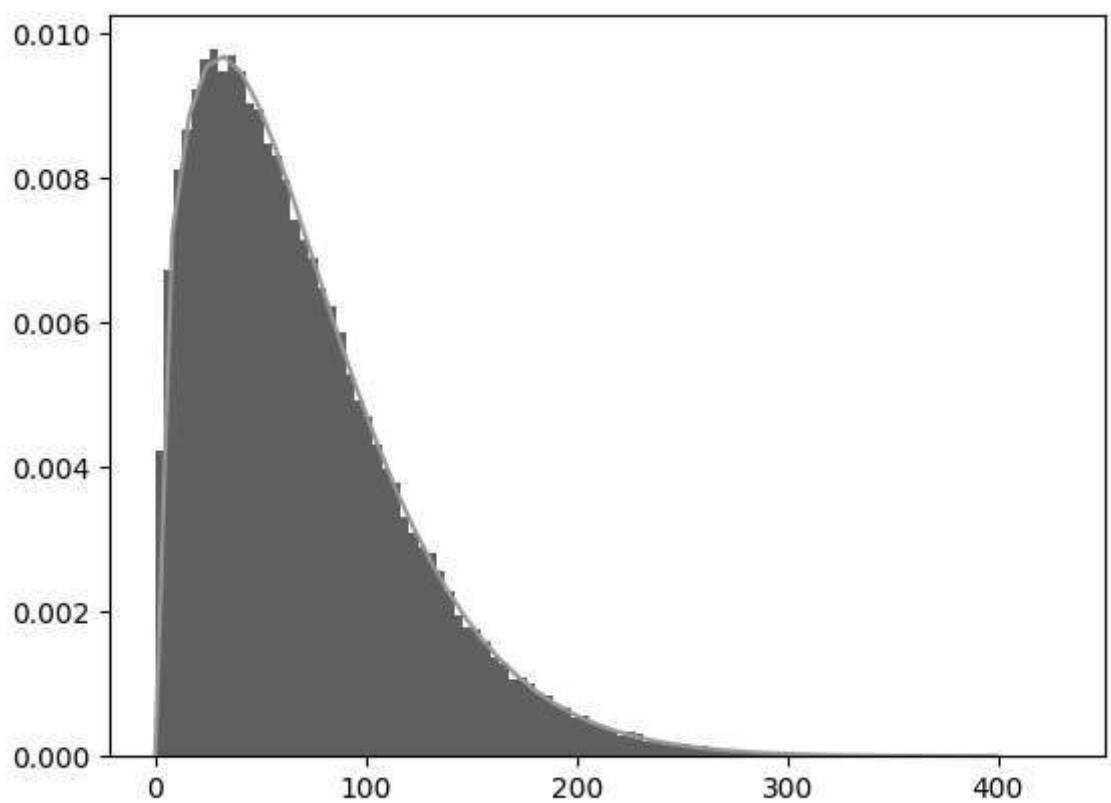
$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{\beta}{\eta}(\frac{x}{\eta})^{\beta-1} e^{-(\frac{x}{\eta})^{\beta}}$$

Hence,

$X \sim \text{Weibull}(\beta, \eta)$

4.

```
beta = 1.4
eta = 76

n = 100000

U = random.random(n)
X = eta*((-log(U))**(1/beta))

n_vec = linspace(0, 400)
X_true = (beta/eta)*((n_vec / eta)**(beta-1))*exp(-((n_vec/eta)**beta))

plt.hist(X, density=True, bins=100)
plt.plot(n_vec, X_true)
```



10.3 Estimation of parameters

1. Find the mean of a Rayleigh distribution of parameter σ. Let us consider the estimator of the mean: $\hat{m} = \frac{1}{N}\sum_{i=1}^{N} x_i$. Express σ with respect to $\hat{m}$ to obtained an expression of the estimator of σ.

2. Find an estimator of σ using the Maximum Likelihood Estimator.

3. Let N take the values (2 5 10 100 1000 10000). For each value of N, run a Monte Carlo test of 1000 trials and calculate the two previous estimators for a Rayleigh distribution of parameter $\sigma^2 = 0.5$. Plot for each estimator the biais on the estimation of σ and the associated variance. Comment your results.

1.

2.

$$l(\sigma; x_1, \cdots, x_n) = \prod_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i^2}{2\sigma^2}}$$

$$= \frac{\prod_{i=1}^{n} x_i}{\sigma^{2n}} e^{-\frac{\sum_{i=1}^{n} x^2}{2\sigma^2}}$$

$$L = -\ln\sigma^{2n} - \frac{\sum_{i=1}^{n} x_i^2}{2\sigma^2}$$

$$\frac{dL}{dx} = \frac{-2n}{\sigma} + \frac{\sum_{i=1}^{n} x_i^2}{\sigma^3} = 0$$

Hence,

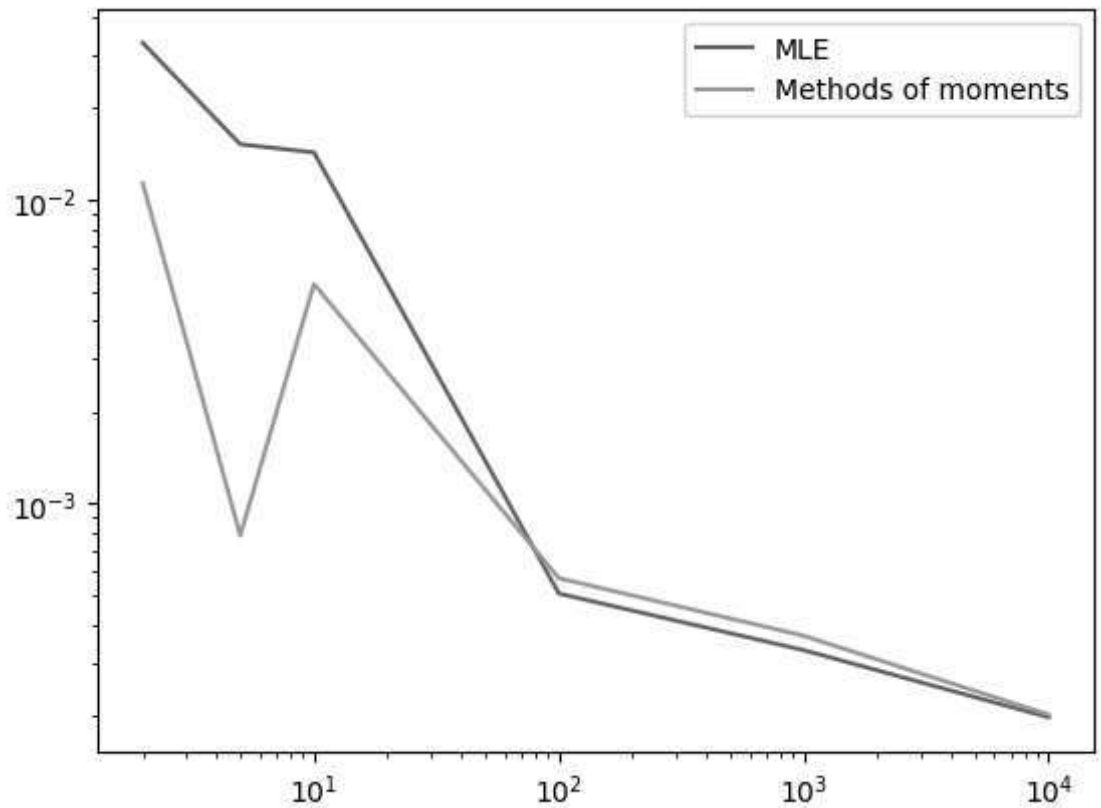
$$\sigma^2 = \frac{\sum_{i=1}^{n} x_i^2}{2n}$$

3.

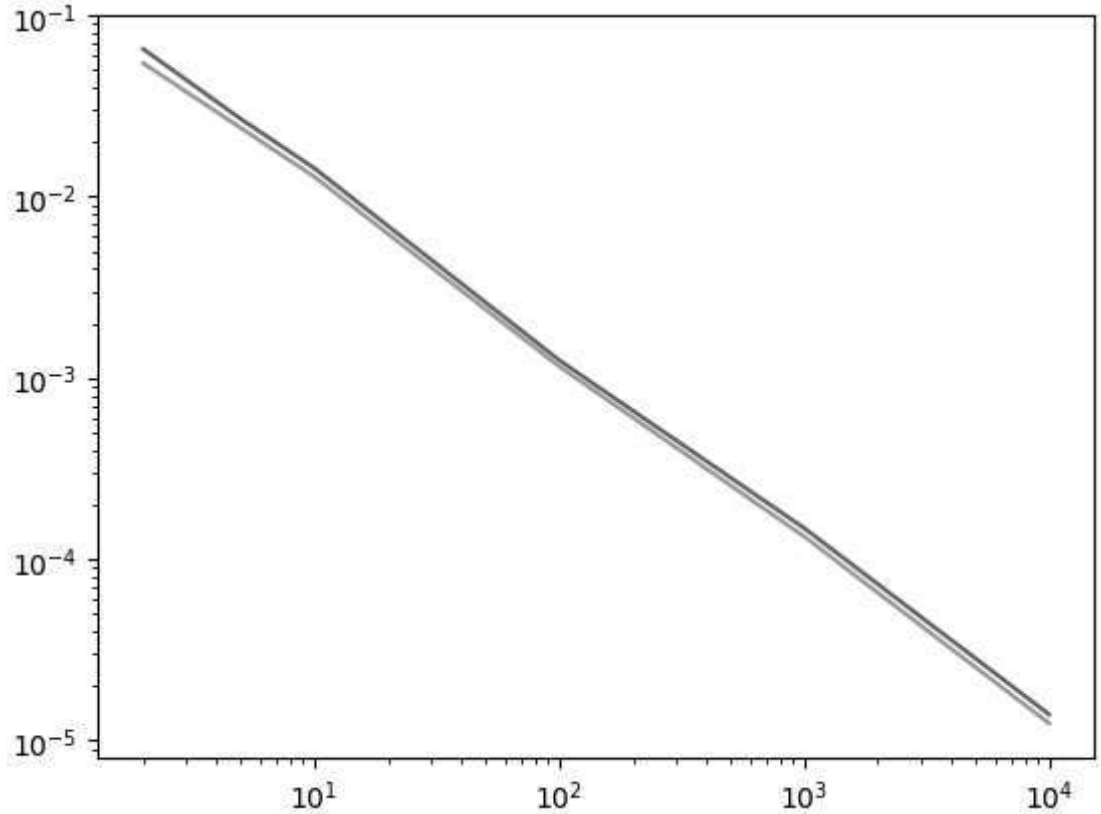
```
from numpy import sqrt, log, exp, linspace, pi, array
from numpy.random import random
from matplotlib import pyplot as plt



def rayleigh_pdf(x_array, sigma):
    return (x_array / sigma ** 2) * exp(- (x_array ** 2) / (2 * (sigma **
2)))



sigma = sqrt(0.5)
nb_experiments = 1000
```

```
def methods_of_moments(R_array):
    return sqrt(2 / pi) * R_array.mean()

def MLE(R_array):
    return sqrt((R_array ** 2).mean() / 2)


def get_rayleigh_estimators(N):
    sigma_hat1s = []
    sigma_hat2s = []
    for _ in range(nb_experiments):
        U = random(N)
        Y_exp = -2 * (sigma ** 2) * log(U)
        R1 = sqrt(Y_exp)

        sigma_hat1 = methods_of_moments(R1)
        sigma_hat2 =  MLE(R1)

        sigma_hat1s.append(sigma_hat1)
        sigma_hat2s.append(sigma_hat2)

    return array(sigma_hat1s), array(sigma_hat2s)


def biases(N):
    sigma_hat1s, sigma_hat2s = get_rayleigh_estimators(N)
    return (
        (sigma_hat1s.mean() - sigma),
        (sigma_hat2s.mean() - sigma),
        sigma_hat1s.var(),
        sigma_hat2s.var()
    )

N = [2, 5, 10, 100, 1000, 10000]
sigma1_biases = []
sigma2_biases = []
sigma1_vars = []
sigma2_vars = []
for n in N:
    bias1, bias2, var1, var2 = biases(n)
```

```
    sigma1_biases.append(abs(bias1))
    sigma2_biases.append(abs(bias2))
    sigma1_vars.append(var1)
    sigma2_vars.append(var2)

plt.yscale('log')
plt.xscale('log')
plt.plot(N, sigma2_biases, label='MLE')
plt.plot(N, sigma1_biases, label='Methods of moments')
plt.legend()
```




```
plt.yscale('log')
plt.xscale('log')
plt.plot(N, sigma1_vars)
plt.plot(N, sigma2_vars)
```



Other Proofs

Expectation of sums is sum of expectations

$$
\begin{aligned}
E[X+Y] &= \int_{\mathbb{R}} \int_{\mathbb{R}} (x+y) f_{X,Y}(x,y) dx dy \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} x f_{X,Y}(x,y) + y f_{X,Y}(x,y) dx dy \\
&= \int_{\mathbb{R}} x \int_{\mathbb{R}} f_{X,Y}(x,y) dy dx + \int_{\mathbb{R}} y \int_{\mathbb{R}} f_{X,Y}(x,y) dx dy \\
&= \int_{\mathbb{R}} x f_X(x) dx + \int_{\mathbb{R}} y f_Y(y) dy \\
&= E[X] + E[Y]
\end{aligned}
$$

Where we have used the definition of marginal distribution and expectation. Note that this result does not require X and Y to be independent.

Sum of Variances

Assume X and Y are independent.

$$
\begin{aligned}
var(X+Y) &= E[(X+Y)^2] - E[X+Y]^2 \\
&= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 \\
&= E[X^2] + 2E[XY] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2 \\
&= E[X^2] + 2E[X]E[Y] + E[Y^2] - E[X]^2 - 2E[X]E[Y] - E[Y]^2 \\
&= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 \\
&= var(X) + var(Y)
\end{aligned}
$$

Variance estimator of i.i.d random variables

Note that $E[\hat{m}] = m$

$$
\begin{aligned}
E[\hat{\sigma}^2] &= \frac{1}{n}\sum_{i=1}^{n} E[(X_i - \hat{m})^2] \\
&= \frac{1}{n}\sum_{i=1}^{n}(E[X_i^2] - 2E[X_i\hat{m}] + E[\hat{m}^2]) \\
&= \frac{1}{n}\sum_{i=1}^{n}(E[X_i^2] - \frac{2}{n}E[X_i\sum_{j=1}^{n} X_j] + \frac{1}{n^2}E[(\sum_{j=1}^{n} X_j)^2]) \\
&= \frac{1}{n}\sum_{i=1}^{n}(E[X_i^2] - \frac{2}{n}(E[X_i^2] + (n-1)m^2) + \frac{1}{n^2}(\sum_{j=1}^{n} E[X_j]^2 + n(n-1)m^2)
\end{aligned}
$$

Call Types

Let $X \sim exp(\lambda_1)$ and $Y \sim exp(\lambda_2)$ be independent.

We want to calculate the probability for the time until the next call. Then,

$Z = min(X, Y)$.

And

$$
\begin{aligned}
P[Z > z] &= P[min(X,Y) > z] = P[X > z \cap Y > z] \\
&= \int_z^\infty \int_z^\infty f_{X,Y}(x,y)dxdy \\
&= \int_z^\infty \int_z^\infty f_X(x)f_Y(y)dxdy \quad \text{(independence)} \\
&= \int_z^\infty f_Y(y) \int_z^\infty \lambda_1 e^{-\lambda_1 x}dxdy \\
&= \int_z^\infty f_Y(y)e^{-\lambda_1 z}dy \\
&= e^{-(\lambda_1+\lambda_2)x}
\end{aligned}
$$

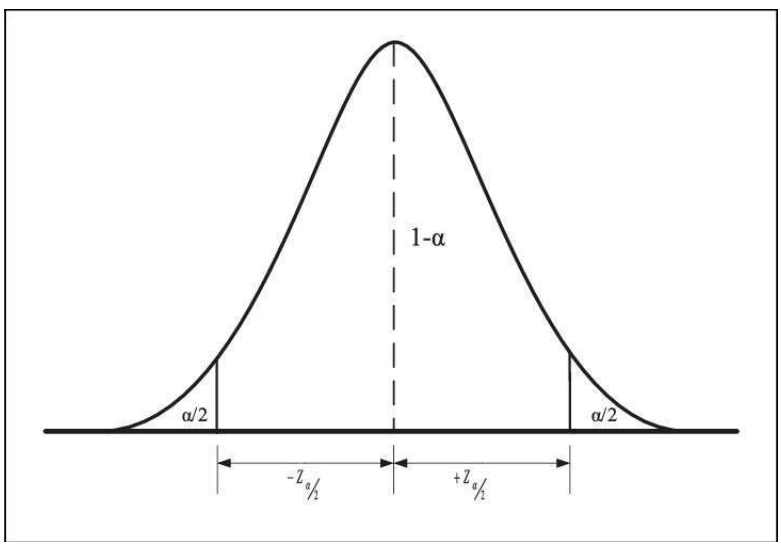
Hence,

$$min(X, Y) \sim exp(\lambda_1 + \lambda_2)$$

If we want to calculate the probability that the next call is of type X, we have

$$\begin{aligned} P[X < Y] &= \int_{-\infty}^{\infty} \int_{x}^{\infty} f_{X,Y}(x, y) dy dx \\ &= \int_{-\infty}^{\infty} f_X(x) \int_{x}^{\infty} \lambda_2 e^{-\lambda_2 x} dy dx \\ &= \int_{-\infty}^{\infty} \lambda_1 e^{-(\lambda_1+\lambda_2)x} dy \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

Confidence Intervals with erfc



Consider the above graph of a normal distribution $N(0, 1)$. Given some value of α, we want to find the z. Hence, we want to solve the following equation

$$\int_0^z f_X(x) dx = \frac{1-\alpha}{2}$$

$$\int_0^z \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx = \frac{1-\alpha}{2}$$

To express this in terms of the error function, we change the variables with .

$$\int_0^{z/\sqrt{2}} \frac{1}{\sqrt{\pi}} e^{-t^2} dt = \frac{1-\alpha}{2}$$

Hence,

$$z = \sqrt{2} erc^{-1}(1-\alpha)$$

The sum of independent exponential random variables gives the gamma distribution

The gamma distribution with shape parameter n and rate parameter λ is given by

$$\begin{aligned} f(x) &= \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \\ &= \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x} \end{aligned}$$

Since $\Gamma(n) = (n-1)!$ for any natural number n.

We will prove by induction.

The base case:

Let $Z = Y_1 + Y_2$, which we evaluate using convolution.

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{Y_1}(y_1) f_{Y_2}(z - y_1) dy_1 \\ &= \int_{-\infty}^{\infty} \lambda e^{-\lambda y_1} \lambda^{-\lambda(z-y_1)} dy_1 \\ &= \lambda^2 \int_0^z e^{-\lambda z} dy_1 \\ &= \lambda^2 e^{-\lambda z} \int_0^z dy_1 \\ &= \lambda^2 e^{-\lambda z} [y_1] 0^z \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

The bounds are determined as such because the exponential distribution is 0 for $x < 0$ and since we plugging $z - y_1$ into f_{Y_2}, we know that $y_2 = z - y_1 > 0$ and so $z > y_1$.

Induction assumption:

Let

$$Z_n = \sum_{i=1}^{n+1} Y_i$$

So

$$f_{Z_n}(z_n) = \frac{\lambda^n}{(n-1)!} z_n^{n-1} e^{-\lambda z_n}$$

Then

$$\begin{aligned} f_{Z_{n+1}}(z_{n+1}) &= \int_{-\infty}^{\infty} f_{Z_n}(z_n) f_{Y_{n+2}}(z_{n+1} - z_n) dz_n \\ &= \int_0^{z_{n+1}} \frac{\lambda^n}{(n-1)!} z_n^{n-1} e^{-\lambda z_n} \lambda e^{-\lambda(z_{n+1} - z_n)} dz_n \\ &= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda z_{n+1}} \int_0^{z_{n+1}} z_n^{n-1} dz_n \end{aligned}$$

$$= \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda z_{n+1}} [\frac{1}{n} z_n^n]_0^{z_{n+1}}$$
$$= \frac{\lambda^{n+1}}{n!} z_{n+1}^n e^{-\lambda z_{n+1}}$$