$$\mathbf{P}(N = n \,|\, Y = y) = \frac{p_N(n) f_{Y|N}(y \,|\, n)}{\sum_i p_N(i) f_{Y|N}(y \,|\, i)}$$

Inference Based on Discrete Observations
We can rearrange the above formula to write

$$f_{Y|A}(y) = \frac{f_Y(y)\,\mathbf{P}(A \,|\, Y = y)}{\mathbf{P}(A)}$$

Which then gives

$$f_{Y|A}(y) = \frac{f_Y(y)\,\mathbf{P}(A \,|\, Y = y)}{\int_{-\infty}^{\infty} f_Y(t)\,\mathbf{P}(A \,|\, Y = t)\,dt}$$

This can be used to make inference about the random variable Y when an event A is observed.

3.7 Problems

Problem 1

Problem 1. Let X be uniformly distributed in the unit interval $[0, 1]$. Consider the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} 1, & \text{if } x \leq 1/3, \\ 2, & \text{if } x > 1/3. \end{cases}$$

Find the expected value of Y by first deriving its PMF. Verify the result using the expected value rule.

The PMF of Y is given by

$$p_Y(1) = P(X \leq 1/3) = 1/3$$
$$p_Y(2) = P(X > 1/3) = 2/3$$

Hence,

$$\mathbf{E}[Y] = \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot 2 = \frac{5}{3}.$$

To verify using the expected value rule,

$$\mathbf{E}[Y] = \int_0^1 g(x) f_X(x)\,dx = \int_0^{1/3} dx + \int_{1/3}^1 2\,dx = \frac{5}{3}.$$

Problem 2

Problem 2. Laplace random variable. Let X have the PDF

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

where λ is a positive scalar. Verify that f_X satisfies the normalization condition, and evaluate the mean and variance of X.

We have

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|}\, dx = 2 \cdot \frac{1}{2} \int_0^{\infty} \lambda e^{-\lambda x}\, dx = 2 \cdot \frac{1}{2} = 1,$$

Where we can break the integral using $|x|$ into 2 times the integral from 0 to ∞, given the symmetry.

Also given the symmetry of the PDF, we have

$$E[X] = 0$$

Then, to calculate the variance, first we have

$$E[X^2] = \int_{-\infty}^{\infty} x^2 \frac{\lambda}{2} e^{-\lambda|x|}$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}$$

Then, using integration by parts we have

$$= (-x^2 e^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} 2xe^{-\lambda x}\, dx$$

$$= 0 + \frac{2}{\lambda}\mathbf{E}[X]$$

$$= \frac{2}{\lambda^2}.$$

Hence,

$$\mathrm{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 2/\lambda^2.$$

Problem 5

Problem 5. Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X.

To calculate the CDF, we can take, A_x, the area of the triangle, whose base runs parallel through the point x.

Then $h_x = h - x$ and $b_x = b(h-x)/h$, where b_x is calculated using the fact that the two triangles are similar.

Then,

$$A_x = \frac{b(h-x)^2}{2h}$$

And

$$P(X > x) = A/A_x.$$

Hence,

$$F_X(x) = 1 - P(X > x) = 1 - (\frac{h-x}{h})^2$$

The PDF is calculated as the derivative of $F_X(x)$. Hence,

$$f_X(x) = \frac{2(h-x)}{h^2}$$

Problem 6

Problem 6. Calamity Jane goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of her. The service time of the customer ahead, if present, is exponentially distributed with parameter λ. What is the CDF of Jane's waiting time?

W: waiting time
C: customer present

$P[W \leq w] = P[W \leq w|C]P[C] + P[W \leq w|\overline{C}]P[\overline{C}]$
$= 0.5(1 - e^{-\lambda x}) + 0.5$

Since

$P[W \leq w|C] = 1 - e^{-\lambda x}$

And

$P[W \leq w|\overline{C}] = 1$

Problem 7

Problem 7. Alvin throws darts at a circular target of radius r and is equally likely to hit any point in the target. Let X be the distance of Alvin's hit from the center.

(a) Find the PDF, the mean. and the variance of X.

(b) The target has an inner circle of radius t. If $X \leq t$, Alvin gets a score of $S = 1/X$. Otherwise his score is $S = 0$. Find the CDF of S. Is S a continuous random variable?

(a)

To determine the PDF, we can consider the probability of the dart landing in a ring with radius x, where the ring has width dx.

$$\text{area of ring} = 2\pi x dx$$

Where 2π is the circumference.
The probability is then

$$\frac{2\pi x dx}{\pi r^2}$$

And so

$$f_X(x) = \frac{2x}{r^2}$$

It's easy to verify that

$$\int_0^r \frac{2x}{r^2} = 1$$

$$E[X] = \int_0^r x \cdot \frac{2x}{r^2} dx = \frac{2}{3} r$$

$$E[X^2] = \int_0^r x^2 \cdot \frac{2x}{r^2} dx = \frac{1}{2} r$$

$$var(X) = E[X^2] - E[X]^2 = \frac{1}{18} r^2$$

(b)

If $x \leq t$, then $\frac{1}{s} \leq t$ so that $\frac{1}{t} \leq s$. In which case, we also have

$$P[S \leq s] = P[\frac{1}{X} \leq s]$$

$$= P[\frac{1}{s} \leq X]$$

$$= 1 - P[X \leq \frac{1}{s}]$$

$$= 1 - \frac{1}{s^2 r^2}$$

Hence,

$$P[S \leq s] = \begin{cases} 0 & s < \frac{1}{t} \\ 1 - \frac{1}{s^2 r^2} & \frac{1}{t} \leq s \end{cases}$$

S is neither continuous nor discrete. It is mixed.

Problem 8

Problem 8. Consider two continuous random variables Y and Z, and a random variable X that is equal to Y with probability p and to Z with probability $1-p$.

(a) Show that the PDF of X is given by

$$f_X(x) = pf_Y(x) + (1-p)f_Z(x).$$

(b) Calculate the CDF of the two-sided exponential random variable that has PDF given by

$$f_X(x) = \begin{cases} p\lambda e^{\lambda x}, & \text{if } x < 0, \\ (1-p)\lambda e^{-\lambda x}, & \text{if } x \geq 0, \end{cases}$$

where $\lambda > 0$ and $0 < p < 1$.

(a)

$P(X \leq x) = P(X \leq x|X = Y)P(X = Y) + P(X \leq x|X = Z)P(X = Z)$

$= pP(Y \leq x) + (1-p)P(Z \leq x)$

$= pF_Y(x) + (1-p)F_Z(x)$

Differentiating easily leads to the result.

(b)

For $x > 0$

$F_X(x) = \int_{-\infty}^{x} p\lambda e^{\lambda t}\mathbf{1}_{t<0} + (1-p)\lambda e^{-\lambda t}\mathbf{1}_{t\leq 0}dt$

$= \int_{-\infty}^{0} p\lambda e^{\lambda t}dt + \int_{0}^{x} (1-p)\lambda e^{-\lambda t}dt$

$= pe^{\lambda t}\Big|_{-\infty}^{0} - (1-p)e^{-\lambda t}\Big|_{0}^{x}$

$= 1 - (1-p)e^{-\lambda x}$

For $x < 0$

$F_X(x) = \int_{-\infty}^{x} p\lambda e^{\lambda t}dt$

$= pe^{\lambda x}$

Problem 11

Problem 11. Let X and Y be normal random variables with means 0 and 1, respectively, and variances 1 and 4, respectively.

(a) Find $\mathbf{P}(X \leq 1.5)$ and $\mathbf{P}(X \leq -1)$.

(b) Find the PDF of $(Y-1)/2$.

(c) Find $\mathbf{P}(-1 \leq Y \leq 1)$.

(a)

Use the standard normal table

$P[X \leq 1.5] = 0.93319$

$[X \leq -1] = P[X > 1] = 1 - P[X < 1] = 1 - 0.84134 = 0.15866$

(b)

$E[\frac{Y-1}{2}] = \frac{1}{2}E[Y-1] = \frac{1}{2}E[Y] - \frac{1}{2} = 0$

$var(\frac{Y-1}{2}) = \frac{1}{4}(var(Y) - var(1)) = 1$

Hence

$\frac{Y-1}{2} \sim N(0,1)$

(c)

$$\begin{aligned}
\mathbf{P}(-1 \leq Y \leq 1) &= \mathbf{P}\big(-1 \leq (Y-1)/2 \leq 0\big) \\
&= \mathbf{P}(-1 \leq Z \leq 0) \\
&= \mathbf{P}(0 \leq Z \leq 1) \\
&= \Phi(1) - \Phi(0) \\
&= 0.8413 - 0.5 \\
&= 0.3413.
\end{aligned}$$

Problem 12

Problem 12. Let X be a normal random variable with zero mean and standard deviation σ. Use the normal tables to compute the probabilities of the events $\{X \geq k\sigma\}$ and $\{|X| \leq k\sigma\}$ for $k = 1, 2, 3$.

$X/\sigma \sim N(0,1)$

$P[X \geq k\sigma] = 1 - P[\frac{X}{\sigma} \leq k] = 0.15866$ for $k = 1$

$$\begin{aligned}
P[|X| \leq k\sigma] &= P[-k\sigma \leq X \leq k\sigma] \\
&= P[X \leq k\sigma] - P[X \leq -k\sigma]
\end{aligned}$$

$= P[X \leq k\sigma] - 1 + P[X \leq k\sigma]$
$= 2P[X \leq k\sigma] - 1$
$= 0.68268$ for $k = 1$

Problem 13

Problem 13. A city's temperature is modeled as a normal random variable with mean and standard deviation both equal to 10 degrees Celsius. What is the probability that the temperature at a randomly chosen time will be less than or equal to 59 degrees Fahrenheit?

59 degrees Fahrenheit is 15 degrees Celsius.
We know that $(T - 10)/10$ is standard normal.
Hence,
$P[T \leq 15] = P[(T - 10)/10 \leq 0.5] = 0.69146$

Problem 14

Problem 14.* Show that the normal PDF satisfies the normalization property. *Hint:* The integral $\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$ is equal to the square root of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-y^2/2}\,dx\,dy,$$

and the latter integral can be evaluated by transforming to polar coordinates.

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2/2}e^{-y^2/2}dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}dxdy \\ &= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2/2}rdrd\theta \\ &= \int_{0}^{2\pi} -e^{r^2/2}\Big|_0^{\infty} d\theta \\ &= \int_{0}^{2\pi} d\theta \\ &= 2\pi\end{aligned}$$

Then
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\sqrt{2\pi} = 1$$

If the normal distribution is non standard, then a change of variables can be applied to yield the result.

Problem 15

Problem 15. A point is chosen at random (according to a uniform PDF) within a semicircle of the form $\{(x, y) \mid x^2 + y^2 \leq r^2,\ y \geq 0\}$, for some given $r > 0$.

(a) Find the joint PDF of the coordinates X and Y of the chosen point.

(b) Find the marginal PDF of Y and use it to find $\mathbf{E}[Y]$.

(c) Check your answer in (b) by computing $\mathbf{E}[Y]$ directly without using the marginal PDF of Y.

(a)

$$f_{X,Y}(x,y) = \frac{2}{\pi r_0^2}$$

The normalisation property can be verified by using polar coordinates.

$$\int_0^{\pi}\int_0^{r_0}\frac{2}{\pi r_0^2} r dr d\theta = \frac{2}{\pi r_0^2}\int_0^{\pi}\frac{r^2}{2}\bigg|_0^{r_0}$$

$$= \int_0^{\pi}\frac{1}{\pi}d\theta = 1$$

(b)

$$f_Y(y) = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\frac{2}{\pi r^2}dx$$

$$= \frac{2}{\pi r^2}\sqrt{r^2-y^2}$$ for $0 \leq y \leq r$

Then

$$E[Y] = \int_0^r \frac{2}{\pi r^2}\sqrt{r^2-y^2}dy = \frac{4r}{3\pi}$$

(c)

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)dxdy$$

Apply polar coordinates

$$E[Y] = \int_0^{\pi}\int_0^r s\sin\theta\frac{2}{\pi r^2}sdsd\theta$$

$$= \frac{4r}{3\pi}$$

Problem 16. Consider the following variant of Buffon's needle problem (Example 3.11), which was investigated by Laplace. A needle of length l is dropped on a plane surface that is partitioned in rectangles by horizontal lines that are a apart and vertical lines that are b apart. Suppose that the needle's length l satisfies $l < a$ and $l < b$. What is the expected number of rectangle sides crossed by the needle? What is the probability that the needle will cross at least one side of some rectangle?

Problem 17.* Estimating an expected value by simulation using samples of another random variable. Let $Y_1, \ldots, Y_n$ be independent random variables drawn from a common and known PDF f_Y. Let S be the set of all possible values of Y_i, $S = \{y \mid f_Y(y) > 0\}$. Let X be a random variable with known PDF f_X, such that $f_X(y) = 0$, for all $y \notin S$. Consider the random variable

$$Z = \frac{1}{n} \sum_{i=1}^{n} Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}.$$

Show that

$$\mathbf{E}[Z] = \mathbf{E}[X].$$

$$\mathbf{E}\left[Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}\right] = \int_S y \frac{f_X(y)}{f_Y(y)} f_Y(y)\, dy = \int_S y f_X(y)\, dy = \mathbf{E}[X]$$

$$\mathbf{E}[Z] = \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}\left[Y_i \frac{f_X(Y_i)}{f_Y(Y_i)}\right] = \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[X] = \mathbf{E}[X]$$

Problem 18

Problem 18. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 2\}$.

(a) Find $\mathbf{E}[X]$, $\mathbf{P}(A)$, $f_{X|A}(x)$, and $\mathbf{E}[X \mid A]$.

(b) Let $Y = X^2$. Find $\mathbf{E}[Y]$ and $\mathrm{var}(Y)$.

(a)

$$\mathrm{E}[X] = \int_1^3 \frac{x^2}{4}\, dx = \left.\frac{x^3}{12}\right|_1^3 = \frac{27}{12} - \frac{1}{12} = \frac{26}{12} = \frac{13}{6}$$

$$\mathbf{P}(A) = \int_2^3 \frac{x}{4}\,dx = \frac{x^2}{8}\Big|_2^3 = \frac{9}{8} - \frac{4}{8} = \frac{5}{8}$$

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(A)}, & \text{if } x \in A, \\ 0, & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \dfrac{2x}{5}, & \text{if } 2 \le x \le 3, \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{E}[X \mid A] = \int_2^3 x \cdot \frac{2x}{5}\,dx = \frac{2x^3}{15}\Big|_2^3 = \frac{54}{15} - \frac{16}{15} = \frac{38}{15}$$

(b)

$$\mathbf{E}[Y] = \mathbf{E}[X^2] = \int_1^3 \frac{x^3}{4}\,dx = 5$$

$$\mathbf{E}[Y^2] = \mathbf{E}[X^4] = \int_1^3 \frac{x^5}{4}\,dx = \frac{91}{3}$$

$$\mathrm{var}(Y) = \mathbf{E}[Y^2] - \big(\mathbf{E}[Y]\big)^2 = \frac{91}{3} - 5^2 = \frac{16}{3}$$

Problem 20

Problem 20. An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean thirty minutes. The first student arrives on time, but the second student arrives five minutes late. What is the expected value of the time between the arrival of the first student and the departure of the second student?