2.8 Problems

Problem 1

Problem 1. The MIT soccer team has 2 games scheduled for one weekend. It has a 0.4 probability of not losing the first game. and a 0.7 probability of not losing the second game, independent of the first. If it does not lose a particular game, the team is equally likely to win or tie. independent of what happens in the other game. The MIT team will receive 2 points for a win, 1 for a tie. and 0 for a loss. Find the PMF of the number of points that the team earns over the weekend.

$P(X=0)=0.6\times 0.3=0.18$

$P(X=1)=0.6\times 0.35+0.2\times 0.3=0.27$

$P(X=2)=0.2\times 0.35+0.2\times 0.3+0.6\times 0.35=0.34$

$P(X=3)=0.2\times 0.35+0.2\times 0.3=0.14$

$P(X=4)=0.2\times 0.35=0.07$

Note that $P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)=1$ and hence it is a legitimate PMF. .

Problem 2

Problem 2. You go to a party with 500 guests. What is the probability that exactly one other guest has the same birthday as you? Calculate this exactly and also approximately by using the Poisson PMF. (For simplicity. exclude birthdays on February 29.)

Using the binomial to calculate exactly, we can write

$$\binom{499}{1}(1/365)(364/365)^{498}=0.3486986$$

Or approximately using the Poisson PMF

$$e^{-499/365}(499/365)=0.3483963$$

Problem 3

Problem 3. Fischer and Spassky play a chess match in which the first player to win a game wins the match. After 10 successive draws. the match is declared drawn. Each game is won by Fischer with probability 0.4. is won by Spassky with probability 0.3. and is a draw with probability 0.3. independent of previous games.

(a) What is the probability that Fischer wins the match?

(b) What is the PMF of the duration of the match?

(a)

$$P(\text{Fischer wins}) = \sum_{l=1}^{10} 0.4(0.3)^{l-1} = 0.5714$$

(b)
We use something similar to a geometric random variable, except that we do not have infinite matches, as we are essentially how many games until someone wins or until 10 games, $p = 0.7$. Hence,

$$p_X(k) = \begin{cases} (0.3)^{k-1}0.7, & k = 1 \cdots 9 \\ 0.3^k, & k = 9 \\ 0, & \text{otherwise} \end{cases}$$

Problem 4

Problem 4. An internet service provider uses 50 modems to serve the needs of 1000 customers. It is estimated that at a given time. each customer will need a connection with probability 0.01, independent of the other customers.

(a) What is the PMF of the number of modems in use at the given time?

(b) Repeat part (a) by approximating the PMF of the number of customers that need a connection with a Poisson PMF.

(c) What is the probability that there are more customers needing a connection than there are modems? Provide an exact. as well as an approximate formula based on the Poisson approximation of part (b).

(a)
We can use a binomial PMF here. If $k < 50$, then we require the probability that there are exactly k customers who require a connection. If $k = 50$, then we require the probability that more than 50 customers require a connection. Hence,

$$p_X(k) = \begin{cases} \binom{1000}{k} 0.01^k 0.99^{1000-k}, & k = 0 \cdots 49 \\ \sum_{k=50}^{1000} \binom{1000}{k} 0.01^k 0.99^{1000-k}, & k = 50 \end{cases}$$

(b)
Since $= np = 0.01 \times 1000 = 10$, then

$$p_X(k) = \begin{cases} e^{-10}\frac{10^k}{k!}, & k = 0 \cdots 49 \\ \sum_{k=50}^{1000} e^{-10}\frac{10^k}{k!}, & k = 50 \end{cases}$$

(c)
Using the binomial PMF we can calculate exactly that

$$p(X > 50) = \sum_{k=51}^{1} 0000.01^k 0.99^{1000-k} = 1.5557 \times 10^-20$$

Or approximately using the Poisson PMF

$$p(X > 50) = \sum_{k=51}^{1} 000e^{-10}\frac{10^k}{k!} = 3.6 \times 10^-20$$

Problem 5

Problem 5. A packet communication system consists of a buffer that stores packets from some source, and a communication line that retrieves packets from the buffer and transmits them to a receiver. The system operates in time-slot pairs. In the first slot, the system stores a number of packets that are generated by the source according to a Poisson PMF with parameter λ; however, the maximum number of packets that can be stored is a given integer b, and packets arriving to a full buffer are discarded. In the second slot, the system transmits either all the stored packets or c packets (whichever is less). Here, c is a given integer with $0 < c < b$.

(a) Assuming that at the beginning of the first slot the buffer is empty, find the PMF of the number of packets stored at the end of the first slot and at the end of the second slot.

(b) What is the probability that some packets get discarded during the first slot?

(a)
If $k < b$, then the number of packets after the first slot is simply the Poisson PMF

$$p(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

If $k = b$, then

$$p(X = b) = \sum_{k=b}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = 1 - \sum_{k=0}^{b-1} e^{-\lambda}\frac{\lambda^k}{k!}$$

Let Y be the number of packets after the second slot, and let $min(X, c)$ be the number of packets transmitted. Then,

$$P(Y = 0) = P(X \leq c) = \sum_{k=0}^{c} e^{-\lambda}\frac{\lambda^k}{k!}$$

The maximum number of packets that are left after transmission is $b - c$.
Hence,

$$P(Y = k) = e^{-\lambda}\frac{\lambda^{c+k}}{(c+k)!}$$ for $0 < k < b - c$

And,

$$P(Y = b - c) = P(X = b) = 1 - \sum_{k=0}^{b-1} e^{-\lambda}\frac{\lambda^k}{k!}$$

(b)
This is simply $P(X = b)$, which has been shown already.

Problem 6

Problem 6. The Celtics and the Lakers are set to play a playoff series of n basketball games, where n is odd. The Celtics have a probability p of winning any one game, independent of other games.

(a) Find the values of p for which $n = 5$ is better for the Celtics than $n = 3$.

(b) Generalize part (a), i.e., for any $k > 0$, find the values for p for which $n = 2k + 1$ is better for the Celtics than $n = 2k - 1$.

(b)
Let N be the number of wins after $2k-1$ games. Let A be the event such that the Celtics win where $n = 2k+1$, and B the event such that the Celtics win where $n = 2k-1$.
Hence,

$P(A) = P(N \geq k+1) + P(N = k)[p(1-p) + (1-p)p + p^2] + P(N = k-1)p^2$
$= P(N \geq k+1) + P(N = k)(2p - p^2) + P(N = k-1)p^2$
$P(B) = P(N \geq k) = P(N \geq k+1) + P(N \leq k)$

We want to find the values of p such that $P(A) > P(B)$.
Hence,

$P(N \geq k+1) + P(N = k)(2p - p^2) + P(N = k-1)p^2 > P(N \geq k+1) + P(N \leq k)$

$P(N = k-1)p^2 > P(N = k)(1-p)^2$

$\binom{2k-1}{k-1} p^{k-1}(1-p)^k p^2 > \binom{2k-1}{k} p^k (1-p)^{k-1}(1-p)^2$

Since $\binom{2k-1}{k-1} = \binom{2k-1}{k}$, it follows that $p > 1/2$.

Problem 7

Problem 7. You just rented a large house and the realtor gave you 5 keys, one for each of the 5 doors of the house. Unfortunately, all keys look identical. so to open the front door, you try them at random.

(a) Find the PMF of the number of trials you will need to open the door, under the following alternative assumptions: (1) after an unsuccessful trial. you mark the corresponding key. so that you never try it again. and (2) at each trial you are equally likely to choose any key.

(b) Repeat part (a) for the case where the realtor gave you an extra duplicate key for each of the 5 doors.

(a)
(1)
For every $1 \leq n \leq 5$, we have $P(X = n) = 1/5$.
For example, $P(X = 5) = 4/5 * 3/4 * 2/3 * 1/2 * 1 = 1/5$
(2)

We can use the geometric PMF, where $p = 1/5$. Note, you may never choose the right key.
$P(X = n) = (4/5)^{n-1} 1/5$

(b)
(1)
$$P(X = n) = \frac{8!(10-n)2}{10!}$$
(2)
Same as (a)(2).

Problem 8

Problem 8. Recursive computation of the binomial PMF. Let X be a binomial random variable with parameters n and p. Show that its PMF can be computed by starting with $p_X(0) = (1-p)^n$. and then using the recursive formula

$$p_X(k+1) = \frac{p}{1-p} \cdot \frac{n-k}{k+1} \cdot p_X(k), \qquad k = 0, 1, \ldots, n-1.$$

$$\frac{p}{1-p}\frac{n-k}{k+1}p_X(k) = \frac{p}{1-p}\frac{n-k}{k+1}\binom{n}{k}p^k(1-p)^{n-k}$$
$$= \binom{n}{k+1}p^{k+1}(1-p)^{n-(k+1)} = p_X(k+1)$$

Problem 9

Problem 9. Form of the binomial PMF. Consider a binomial random variable X with parameters n and p. Let k^* be the largest integer that is less than or equal to $(n+1)p$. Show that the PMF $p_X(k)$ is monotonically nondecreasing with k in the range from 0 to k^*. and is monotonically decreasing with k for $k \geq k^*$.

Using information from Problem 8, we can write
$$\frac{p_X(k)}{p_X(k-1)} = \frac{p}{1-p}\frac{n-(k-1)}{k}$$
$$= \frac{(n+1)p - kp)}{k - kp}$$

Since $k \leq (n+1)p$ it follows that $k - kp \leq (n+1)p - kp$ and hence $1 \leq \frac{p_X(k)}{p_X(k-1)}$, which implies the PMF is monotonically nondecreasing.

If $k > k*$, then $1 > \frac{p_X(k)}{p_X(k-1)}$ and the PMF is monotonically decreasing.

Problem 10

Problem 10. Form of the Poisson PMF. Let X be a Poisson random variable with parameter λ. Show that the PMF $p_X(k)$ increases monotonically with k up to the point where k reaches the largest integer not exceeding λ, and after that point decreases monotonically with k.

It can be easily shown that $p_X(k) = \frac{\lambda}{k} p_X(k-1)$, hence, the PMF is monotonically increasing if $\lambda > k$ and monotonically decreasing otherwise.

Problem 13

Problem 13. A family has 5 natural children and has adopted 2 girls. Each natural child has equal probability of being a girl or a boy, independent of the other children. Find the PMF of the number of girls out of the 7 children.

Let X be the number of girls out of the natural children. Then,

$$p(X=x) = \begin{cases} \binom{5}{x}(1/2)^5, & 0 \le x \le 5 \\ 0, & otherwise \end{cases}$$

Let Y be the number of girls out of the 7 children, then obviously $Y = X + 2$. Hence, it is a function of X, and we can write

$$p_Y(y) = \sum_{\{x|x+2=y\}} p_X(x) = p_Y(y-2)$$

Hence,

$$p_Y(y) = \begin{cases} \binom{5}{y-2}(1/2)^5, & 2 \le y \le 7 \\ 0, & otherwise \end{cases}$$

Problem 14

Problem 14. Let X be a random variable that takes values from 0 to 9 with equal probability 1/10.

(a) Find the PMF of the random variable $Y = X \bmod(3)$.

(b) Find the PMF of the random variable $Y = 5 \bmod(X+1)$.

(a)

$$p_Y(0) = p_X(0) + p_X(3) + p_X(6) + p_X(9) = 4/10,$$
$$p_Y(1) = p_X(1) + p_X(4) + p_X(7) = 3/10,$$
$$p_Y(2) = p_X(2) + p_X(5) + p_X(8) = 3/10,$$
$$p_Y(y) = 0, \qquad \text{if } y \notin \{0, 1, 2\}.$$

(b)

$$p_Y(y) = \begin{cases} 2/10, & \text{if } y = 0, \\ 2/10, & \text{if } y = 1, \\ 1/10, & \text{if } y = 2, \\ 5/10, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 15

Problem 15. Let K be a random variable that takes, with equal probability $1/(2n+1)$, the integer values in the interval $[-n, n]$. Find the PMF of the random variable $Y = \ln X$. where $X = a^{|K|}$. and a is a positive number.

We have

$p_K(k) = \frac{1}{2n+1}$ for $-n \leq k \leq n$

Then,

$$p_X(x) = \sum_{\{k|a^{|k|}=x\}} p_K(k) = p_K(log_a x)$$

Note that for some k, $x = a^k$ and $x = a^{-k}$. Hence,

$$p_X(x) = \begin{cases} 1/(2n+1), & 0 \\ 2/(2n+1), & a \leq x \leq a^n \end{cases}$$

Similarly,

$$p_Y(y) = \sum_{\{x|lnx=y\}} p_X(x) = p_X(e^y)$$

Hence,

$$p_Y(y) = \begin{cases} 1/(2n+1), & ln1 \\ 2/(2n+1), & lna \leq y \leq nlna \end{cases}$$

Problem 16

Problem 16. Let X be a random variable with PMF

$$p_X(x) = \begin{cases} x^2/a, & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a and $\mathbf{E}[X]$.

(b) What is the PMF of the random variable $Z = \left(X - \mathbf{E}[X]\right)^2$?

(c) Using the result from part (b), find the variance of X.

(d) Find the variance of X using the formula $\text{var}(X) = \sum_x \left(x - \mathbf{E}[X]\right)^2 p_X(x)$.

(a)

The PMF has to sum to 1 and so $a = 28$. $E[X] = 0$ because the PMF is symmetric.

(b)
Since $E[X] = 0$,
$p_Z(z) = p_X(\sqrt{z})$. Note that $z = x^2 = (-x)^2$, hence
$p_Z(z) = \frac{z}{14}$ for $z = 1, 4, 9$

(c)
The $Var(X) = 7$

(d)
NA

Problem 17

Problem 17. A city's temperature is modeled as a random variable with mean and standard deviation both equal to 10 degrees Celsius. A day is described as "normal" if the temperature during that day ranges within one standard deviation from the mean. What would be the temperature range for a normal day if temperature were expressed in degrees Fahrenheit?

The conversion formula is given by

$$Y = \frac{9}{5}X + 32$$

Hence,

$$E[Y] = \frac{9}{5} \times E[X] = 50$$

And

$$var(Y) = \frac{9}{5}^2 var(X) = 324$$ and $\sigma_X = 18$

Problem 18

Problem 18. Let a and b be positive integers with $a \leq b$, and let X be a random variable that takes as values, with equal probability, the powers of 2 in the interval $[2^a, 2^b]$. Find the expected value and the variance of X.

$p_X(x) = \frac{1}{b-a+1}$ for $x = 2^k$ where $a \leq k \leq b$

Then

$$E[X] = \frac{1}{b-a+1}\sum_{k=a}^{b} 2^k$$

Also, if $Y = X^2$, then

$p_Y(y) = p_X(\sqrt{y}) = \dfrac{1}{b-a+1}$ for $y = 2^{2^k}$ where $a \le k \le b$

Hence,

$$E[X^2] = \frac{1}{b-a+1}\sum_{k=a}^{b} 2^{2^k}$$

And

$$var(X) = E[X^2] - (E[X])^2$$

Problem 19

Problem 19. A prize is randomly placed in one of ten boxes, numbered from 1 to 10. You search for the prize by asking yes-no questions. Find the expected number of questions until you are sure about the location of the prize, under each of the following strategies.

(a) An enumeration strategy: you ask questions of the form "is it in box k?".

(b) A bisection strategy: you eliminate as close to half of the remaining boxes as possible by asking questions of the form "is it in a box numbered less than or equal to k?".

(a)

Let X be the number of questions asked until you are sure about the prize's location under the first strategy. Then,

$$p_X(x) = \begin{cases} 1/10, & 1 \le x \le 8 \\ 2/10, & x = 9 \end{cases}$$

And hence,

$$E[X] = \frac{1}{10}\sum_{x=1}^{8} x + \frac{2}{10} \times 9 = 5.4$$

(b)

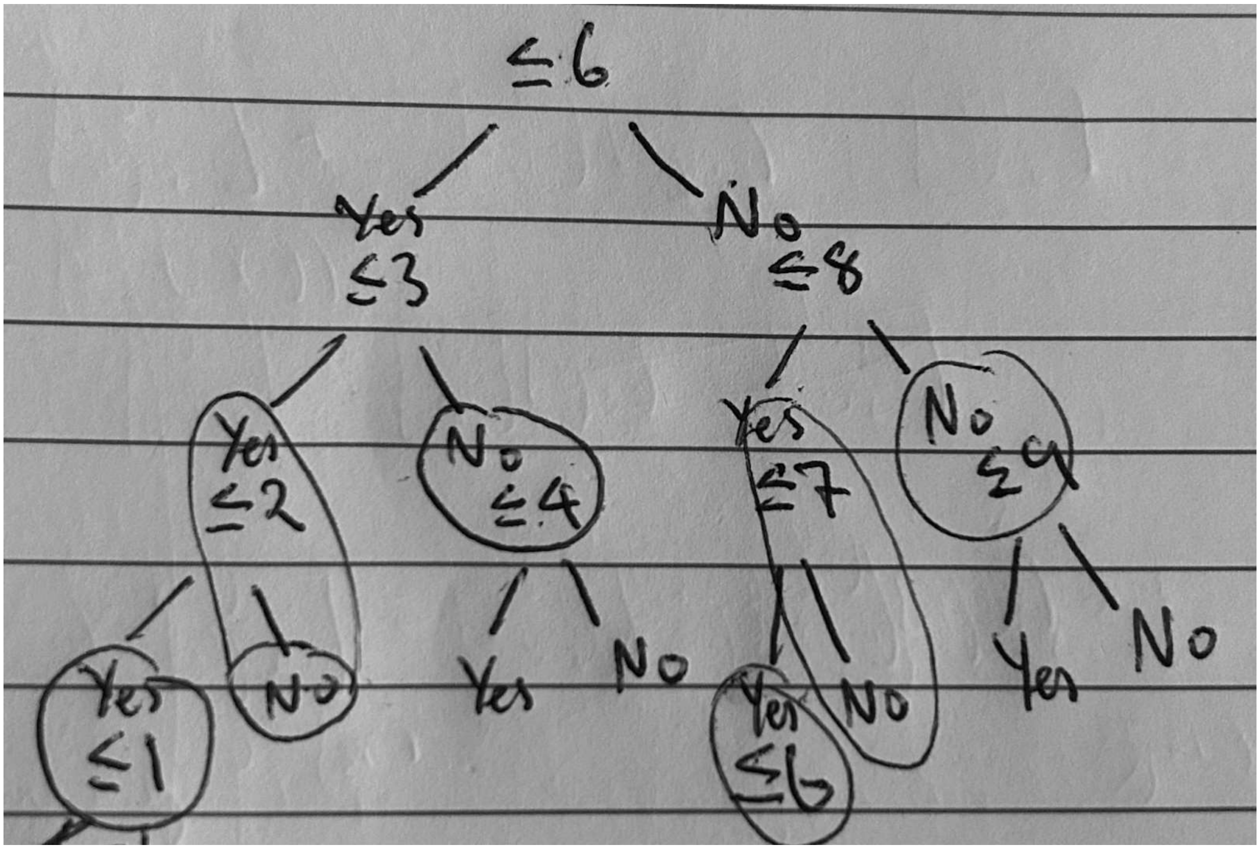
See the below graph that shows how the questions are asked. A question that leads you to know where the prize is located is circled. For example, after asking if the location is less than or equal to 6, and if it's less than or equal to 3, asking if it is less than or equal to 4 will give certainty irrespective of the response. If the answer is yes to the last question, then we know it is in 4; if the answer is no, we know it is in 5. If the location is 4 or 5, three questions are required to find the location.

Hence, letting Y be the number of questions asked, we have

$$p_Y(y) = \begin{cases} 6/10, & x = 3 \\ 4/10, & x = 4 \end{cases}$$

And,

$$E[X] = 3.4$$



Problem 20

Problem 20. As an advertising campaign, a chocolate factory places golden tickets in some of its candy bars, with the promise that a golden ticket is worth a trip through the chocolate factory, and all the chocolate you can eat for life. If the probability of finding a golden ticket is p, find the mean and the variance of the number of candy bars you need to eat to find a ticket.

On the first trial we either succeed with probability p, or we fail with probability $1-p$. If we fail the remaining mean number of trials until a success is identical to the original mean, since all trials are independent. Hence,

$E[X] = p + (1-p)(1+E[X])$

Which gives

$E[X] = 1/p$

Since $E[X^2] = 1/p$, we have

$var(X) = E[X^2] - (E[X]^2 = \frac{p-1}{p^2}$

Problem 21

Problem 21. St. Petersburg paradox. You toss independently a fair coin and you count the number of tosses until the first tail appears. If this number is n, you receive 2^n dollars. What is the expected amount that you will receive? How much would you be willing to pay to play this game?

$$E[N] = \sum_n np_N(n) = \sum_n 2^n 0.5^n = \sum_n 1 = \infty$$

Problem 22

Problem 22. Two coins are simultaneously tossed until one of them comes up a head and the other a tail. The first coin comes up a head with probability p and the second with probability q. All tosses are assumed independent.

(a) Find the PMF, the expected value, and the variance of the number of tosses.

(b) What is the probability that the last toss of the first coin is a head?

(a)
This is effectively a geometric random variable with probability of success being
$r = p(1-q) + (1-p)q$
In which case,
$E[X] = 1/r$ and $var(x) = \frac{r-1}{r^2}$
(b)
$$P(\text{first coin heads}|\text{last toss}) = \frac{p(1-q)}{r}$$

Problem 23

Problem 23.

(a) A fair coin is tossed repeatedly and independently until two consecutive heads or two consecutive tails appear. Find the PMF, the expected value, and the variance of the number of tosses.

(b) Assume now that the coin is tossed until we obtain a tail that is immediately preceded by a head. Find the PMF and the expected value of the number of tosses.

(a)
Let X be the number of required tosses. Then
$p_X(x) = 0.5^{x-2}0.5 =$ for $2 \leq x \leq \infty$.
This is not quite a geometric random variable. However, consider $Y = X - 1$.
Then we have,
$p_Y(y) = p_X(y+1) = 0.5^{y-1}0.5$ for $1 \leq y \leq \infty$.
Y is a geometric random variable and so we have
$E[Y] = 1/0.5 = 2$ and $var(Y) = (1-0.5)/(0.5^2) = 2$

Then,
$E[X] = E[Y] + 1 = 3$ and $var(X) = var(Y) = 2$
(b)
Let X be the number of required tosses.
Then,
$p_X(x) = 0.5^x(x-1)$ for $x \geq 2$.
Hence,
$$E[X] = \sum_{x=2}^{\infty} x(x-1)0.5^x = \sum_{x=1}^{\infty} x(x-1)0.5^x$$
$$= \sum_{x=1}^{\infty} x^2 0.5^x - \sum_{x=1}^{\infty} x0.5^x$$
Let Y be a geometric random variable with $p = 0.5$.
Then,
$$E[Y] = \sum_{y=1}^{\infty} y0.5^y$$
And
$$E[Y^2] = \sum_{y=1}^{\infty} y^2 0.5^y$$
Hence,
$E[X] = E[Y^2] - E[Y]$
Now,
$E[Y^2] = var(Y) + (E[Y])^2 = 2 + 4 = 6$
And
$E[X] = 6 - 2 = 4$

Problem 24

Problem 24. A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period, and assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying

$$-2 \leq x \leq 4, \qquad -1 \leq y - x \leq 1.$$

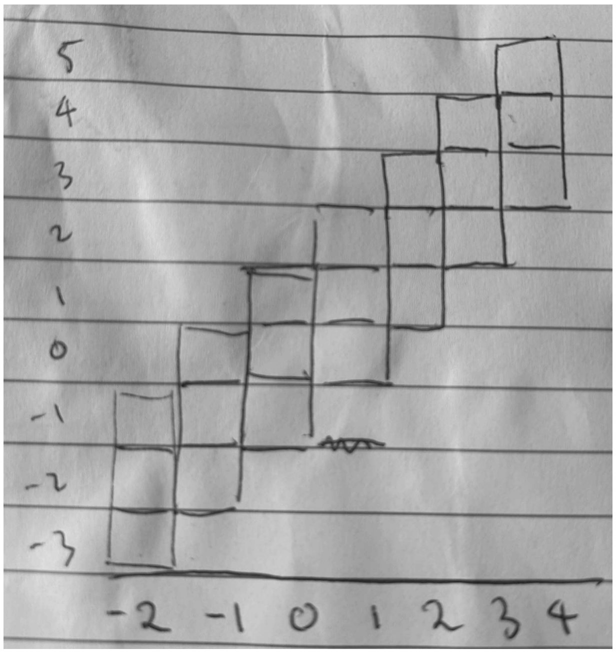
(a) Find the marginal PMFs and the means of X and Y.

(b) Find the mean of the trader's profit.

(a)
We have,
$R = \{(x, y) | -2 \leq x \leq 4, -1 \leq y - x \leq 1\}$

And since there are 7 possible values in x and for each x there are 3 possible values of y, we have
$|R| = 21$
It follows then that
$p_X(x) = 3/21$ for $-2 \leq x \leq 4$ since the PMF is uniform.
It follows that $E[X] = 1$

To determine the marginal PMF of y, we can use the tabular method:



Hence,

$$
p_Y(y) = \begin{cases} 1/21, & \text{if } y = -3, \\ 2/21, & \text{if } y = -2, \\ 3/21, & \text{if } y = -1, 0, 1, 2, 3, \\ 2/21, & \text{if } y = 4, \\ 1/21, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}
$$

And it follows that $E[Y] = 1$

(b)
The profit is given by
$P = 100X + 200Y$
And hence
$E[P] = 100E[X] + 200E[Y] = 300$

Problem 25

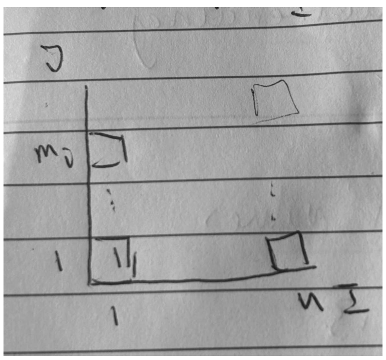
Problem 25. A class of n students takes a test consisting of m questions. Suppose that student i submitted answers to the first m_i questions.

(a) The grader randomly picks one answer, call it (I, J), where I is the student ID number (taking values $1, \ldots, n$) and J is the question number (taking values $1, \ldots, m$). Assume that all answers are equally likely to be picked. Calculate the joint and the marginal PMFs of I and J.

(b) Assume that an answer to question j, if submitted by student i, is correct with probability p_{ij}. Each answer gets a points if it is correct and gets b points otherwise. Calculate the expected value of the score of student i.

(a)

Using the tabular method,



We can easily see that

$p_{I,J}(i,j) = \frac{1}{\sum_{i=1}^{n} m_i}$

And

$p_I(i) = \frac{m_i}{\sum_{k=1}^{n} m_k}$

And

$p_J(j) = \frac{l_j}{\sum_{k=1}^{n} m_k}$ where l_j is the number of students such that $j \leq m_i$.

(b)

Let Y be the student's score on the whole test and let Z_j be the student's score on question j. Then,

$Y = \sum_{j=1}^{m_i} Z_j$

And

$E[Z_j] = ap_{ij} + b(1 - p_{ij})$

Hence,

$$E[Y] = \sum_{j=1}^{m_i} E[Z_j] = \sum_{j=1}^{m_i} ap_{ij} + b(1 - p_{ij})$$

Problem 26

Problem 26. PMF of the minimum of several random variables. On a given day. your golf score takes values from the range 101 to 110. with probability 0.1, independent of other days. Determined to improve your score, you decide to play on three different days and declare as your score the minimum X of the scores X_1, X_2, and X_3 on the different days.

(a) Calculate the PMF of X.

(b) By how much has your expected score improved as a result of playing on three days?

(a)
We can define the PMF of X as
$p_X(k) = p_X(x > k-1) - p_X(x > k)$ for $101 \le k \le 110$
Also,
$p_X(x > k) = p_{X_1}(x_1 > k)p_{X_2}(x_2 > k)p_{X_3}(x_3 > k)$
$= \dfrac{(110-k)^3}{10^3}$
Hence,

$$p_X(k) = \begin{cases} \dfrac{(111-k)^3 - (110-k)^3}{10^3}, & \text{if } k = 101, \ldots 110, \\ 0, & \text{otherwise.} \end{cases}$$

(b)
$E[X_i] = 105.5$ and $E[X] = 103.025$

Problem 31

Problem 31. Consider four independent rolls of a 6-sided die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y?

The joint PMF is given by
$p_{X,Y}(x,y) = p_{X|Y}(x|y)p_Y(y)$
Where we have
$p_Y(y) = \binom{4}{y}(\frac{5}{6})^{4-x}(\frac{1}{6})^y$

For $p_{X|Y}$ consider that if we know, for example, $Y = 2$, i.e, of the four rolls two 2s came up, then we know at most only two 1s can be obtained in the remaining two rolls. We also know that the probability of a 1 appearing is $1/5$ since it is certain that a 2 will not appear in the remaining two rolls.

Hence,
$p_{X|Y} = \binom{4-y}{x}(\frac{4}{5})^{4-y-x}(\frac{1}{5})^{y}$ for $0 \le x+y \le 4$

$p_{X,Y}(x,y)$ follows easily.

Problem 32

Problem 32. D. Bernoulli's problem of joint lives. Consider $2m$ persons forming m couples who live together at a given time. Suppose that at some later time, the probability of each person being alive is p, independent of other persons. At that later time. let A be the number of persons that are alive and let S be the number of couples in which both partners are alive. For any survivor number a, find $\mathbf{E}[S \mid A = a]$.

Don't understand the solution.

Problem 33

Problem 33.* A coin that has probability of heads equal to p is tossed successively and independently until a head comes twice in a row or a tail comes twice in a row. Find the expected value of the number of tosses.

Let H_k and T_k be the events that a head or a tail occurs at the kth toss. Then, H_1 and T_1 partition the sample space. Hence,
$E[X] = pE[X|H_1] + qE[X|H_2]$

We can use the total expectation theorem like so
$E[X|H_1] = P(H_2|H_1)E[X|H_2 \cap H_1] + P(T_2|H_1)E[X|T_2 \cap H_1]$
$= pE[X|H_2 \cap H_1] + qE[X|T_2 \cap H_1]$

Now,
$E[X|H_2 \cap H_1] = 2$
And
$E[X|T_2 \cap H_1] = 1 + E[X|T_1]$
Since we can model the expected number of tosses given H_1 and T_2 as the number of tosses from T_1 plus one toss.
It follows then that
$\mathbf{E}[X \mid T_1] = 2q + p\big(1 + \mathbf{E}[X \mid H_1]\big)$
Using a similar argument we can write
$E[X|H_1] = 2p + q(1 + E[X|H_1])$
Hence,
$\mathbf{E}[X \mid T_1] = \frac{2+p^2}{1-pq}$

and

$$\mathbf{E}[X \mid H_1] = \frac{2+q^2}{1-pq}.$$

So

$$\mathbf{E}[X] = \frac{2+pq}{1-pq}$$

Where we have used the fact that $p+q=1$

Problem 38

Problem 38. Alice passes through four traffic lights on her way to work, and each light is equally likely to be green or red. independent of the others.

(a) What is the PMF, the mean, and the variance of the number of red lights that Alice encounters?

(b) Suppose that each red light delays Alice by exactly two minutes. What is the variance of Alice's commuting time?

(a)
The traffic lights follow a binomial distribution and hence
$E[X] = np = 2$
And
$var[X] = np(1-p) = 1$

(b)
We have to calculate the variance of $2X$ which is $4var(X) = 4$.

Problem 39

Problem 39. Each morning, Hungry Harry eats some eggs. On any given morning. the number of eggs he eats is equally likely to be 1. 2, 3. 4, 5, or 6, independent of what he has done in the past. Let X be the number of eggs that Harry eats in 10 days. Find the mean and variance of X.

Let Y be the number of eggs Harry eats in a day. Then,
E[Y]=3.5andvar(Y)=35/12 $var(Y) = 35/12$ r(Y)=35/12 $r(Y) = 35/12$ Y)=35/12 $Y) = 35/12$ =35/12 $= 35/12$ 5/12 $5/12$ 12 12 $$
Hence,
$E[X] = 35$ and $var(X) = 350/12$

Problem 40

Problem 40. A particular professor is known for his arbitrary grading policies. Each paper receives a grade from the set $\{A, A-, B+, B, B-, C+\}$, with equal probability, independent of other papers. How many papers do you expect to hand in before you receive each possible grade at least once?

Let a success be the receipt of a new grade. Let X_i be the number of papers between the last success and the ith success (except the first paper received).
Then,

$$E[X] = 1 + \sum_{i=1}^{5} E[X_i]$$

Notice that X_i is a geometric variable with $p = (6-i)/6$. Then,
$E[X_i] = 6/(6-i)$
Hence,
$E[X] = 14.7$

Problem 41

Problem 41. You drive to work 5 days a week for a full year (50 weeks), and with probability $p = 0.02$ you get a traffic ticket on any given day, independent of other days. Let X be the total number of tickets you get in the year.

(a) What is the probability that the number of tickets you get is exactly equal to the expected value of X?

(b) Calculate approximately the probability in (a) using a Poisson approximation.

(c) Any one of the tickets is \$10 or \$20 or \$50 with respective probabilities 0.5, 0.3, and 0.2, and independent of other tickets. Find the mean and the variance of the amount of money you pay in traffic tickets during the year.

(d) Suppose you don't know the probability p of getting a ticket. but you got 5 tickets during the year, and you estimate p by the sample mean

$$\hat{p} = \frac{5}{250} = 0.02.$$

What is the range of possible values of p assuming that the difference between p and the sample mean $\hat{p}$ is within 5 times the standard deviation of the sample mean?

(a)
We have a binomial random variable. Hence,
$E[X] = 250\dot{0}.02 = 5$
Then,

$$p_X(x) = \binom{250}{5} 0.98^{245} 0.02^5 = 0.177$$

(b)

$\lambda = 5$

$$p_X(x) = e^{-5} \frac{5^5}{5!}$$

(c)

Let Y be the dollar value of tickets you receive in a year. Let Y_i be the dollar value of the ticket on a given day. Then,

$$\mathbf{P}(Y_i = y) = \begin{cases} 0.98, & \text{if } y = 0, \\ 0.01, & \text{if } y = 10, \\ 0.006, & \text{if } y = 20, \\ 0.004, & \text{if } y = 50. \end{cases}$$

Hence,

$E[Y_i] = 0.42$

$var(Y_i) = E[Y_i^2 - E[Y_i]^2 = 13.22$

So,

$E[Y] = 250\dot{E}[Y_i] = 105$

$var[Y] = 250\dot{v}ar[Y_i] = 3305$

(d)

Since X is a binomial random variable and

Problem 42

Problem 42. **Computational problem.** Here is a probabilistic method for computing the area of a given subset S of the unit square. The method uses a sequence of independent random selections of points in the unit square $[0, 1] \times [0, 1]$, according to a uniform probability law. If the ith point belongs to the subset S the value of a random variable X_i is set to 1, and otherwise it is set to 0. Let $X_1, X_2, \ldots$ be the sequence of random variables thus defined, and for any n, let

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

(a) Show that $\mathbf{E}[S_n]$ is equal to the area of the subset S. and that $\text{var}(S_n)$ diminishes to 0 as n increases.

(b) Show that to calculate S_n. it is sufficient to know S_{n-1} and X_n, so the past values of X_k, $k = 1, \ldots, n-1$, do not need to be remembered. Give a formula.

(c) Write a computer program to generate S_n for $n = 1, 2, \ldots, 10000$, using the computer's random number generator, for the case where the subset S is the circle inscribed within the unit square. How can you use your program to measure experimentally the value of π?

(d) Use a similar computer program to calculate approximately the area of the set of all (x, y) that lie within the unit square and satisfy $0 \leq \cos \pi x + \sin \pi y \leq 1$.