1.7 Problems

Problem 1

Problem 1. Consider rolling a six-sided die. Let A be the set of outcomes where the roll is an even number. Let B be the set of outcomes where the roll is greater than 3. Calculate and compare the sets on both sides of De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c. \qquad (A \cap B)^c = A^c \cup B^c.$$

$A := \{2,4,6\}$ and $B := \{4,5,6\}$.
Hence, $A^c\{1,3,5\}$ and $B^c = 1,2,3\}$.
$A \cup B = \{2,4,5,6\}$ and $A \cap B = \{4,6\}$.
We see that
$(A \cup B)^c = \{2,4,5,6\}^c = \{1,3\}$
And $A^c \cap B^c = \{1,3\}$. Hence, the first equality holds.
Similarly,
$(A \cap B)^c = \{4,6\}^c = \{1,2,3,5\}$
$A^c \cap B^c = \{1,2,3,5\}$. Hence, the second equality holds.

Problem 2

Problem 2. Let A and B be two sets.

(a) Show that

$$A^c = (A^c \cap B) \cup (A^c \cap B^c), \qquad B^c = (A \cap B^c) \cup (A^c \cap B^c).$$

(b) Show that

$$(A \cap B)^c = (A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c).$$

(c) Consider rolling a fair six-sided die. Let A be the set of outcomes where the roll is an odd number. Let B be the set of outcomes where the roll is less than 4. Calculate the sets on both sides of the equality in part (b), and verify that the equality holds.

(a)
⇒

Let $x \in A^c$. Either $x \in B$ or $x \notin B$. If $x \in B$, then it follows that $x \in A^c \cap B$ and hence $x \in (A^c \cap B) \cup (A^c \cap B^c)$. If instead $x \notin B$, then it follows that $x \in A^c \cap B^c$ and hence $x \in (A^c \cap B) \cup (A^c \cap B^c)$. Either way the claim follows.

$\Leftarrow$

If $x \in (A^c \cap B) \cup (A^c \cap B^c)$, then we know either x\in A^c\cap Borx\in A^c\cap B^c $.Either way,$x\in A^c.

(b)

By applying De Morgan's law, we have $(A \cap B)^c = A^c \cup B^c$. Then, applying the results from (a), the claim follows.

(c)

$A = \{1, 2, 5\}$ and $B = \{1, 2, 3\}$.

$A \cap B = \{1, 3\}$ and $(A \cap B)^c = \{2, 4, 5, 6\}$.

$A^c = 2, 4, 6\}$ and $B^c = \{4, 5, 6\}$.

$A^c \cap B$=\{2\}$,$A^c\cap B^c=\{4,6\}$.$A\cap B^c=\{5\} $A \cap B^c = \{5\}$ cap B^c=\{5\} $capB^c = \{5\}$ p B^c=\{5\} $pB^c = \{5\}$ B^c=\{5\} $B^c = \{5\}$ c=\{5\} $c = \{5\}$ \{5\} $\{5\}$ 5\} $5\}$ }.

Finally,

$(A^c \cap B) \cup (A^c \cap B^c) \cup (A \cap B^c) = \{2, 4, 5, 6\}$

Problem 3

Problem 3.* Prove the identity

$$A \cup \left(\cap_{n=1}^{\infty} B_n \right) = \cap_{n=1}^{\infty} (A \cup B_n).$$

$\Rightarrow$

Let $x \in A \cup (\cap_{n=1}^{\infty} B_n$. Then either $x \in A$ or $x \in \cap_{n=1}^{\infty} B_n$. If $x \in A$, then obviously $x \in A \cup B_n$ for all n and consequently, $x \in \cap_{n=1}^{\infty}(A \cup B_n)$. If instead $x \in \cap_{n=1}^{\infty} B_n$, then $x \in B_n$ for all n and so $x \in A \cup B_n$ for all n. Hence, $x \in \cap_{n=1}^{\infty}(A \cup B_n)$. The result follows.

$\Leftarrow$

Let $x \in \cap_{n=1}^{\infty}(A \cup B_n)$. Either $x \in \cap_{n=1}^{\infty} B_n$ or $x \notin \cap_{n=1}^{\infty} B_n$. If $x \in \cap_{n=1}^{\infty} B_n$, then it immediately follows that $x \in A \cup (\cap_{n=1}^{\infty} B_n$. If instead $x \notin \cap_{n=1}^{\infty} B_n$, then we can write that $x \in A$ since there must be some n such that $x \notin B_n$, yet we know that $x \in A \cup B_n$. Since $x \in A$, it follows that $x \in A \cup (\cap_{n=1}^{\infty} B_n$.

Problem 4

Problem 5

Problem 5. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither a genius nor a chocolate lover.

$A = \{is\ \ genius\}$ and $B = \{loves\ \ chocolate\}$.

$P(A) = 0.6$, $P(B) = 0.7$ and $P(A \cap B) = 0.4$.

$P(A^c \cup B^c) = P(A \cap B)^c$

$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.7 - 0.4 = 0.9$

$P(A \cap B)^c = 1 - P(A \cap B) = 0.1$

Problem 6

Problem 6. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely, as are all odd faces. Construct a probabilistic model for a single roll of this die and find the probability that the outcome is less than 4.

Let $A = \{roll\ \ is\ \ even\}$ and $B = \{roll\ \ is\ \ odd\}$. If an even is twice as likely as an odd, while even faces are equally likely and odd faces are equally likely, then we can write

$P(A) = 2P(B)$ and $P(A) + P(B) = 1$.

Hence, $P(A) = 2/3$ and $P(B) = 1/3$, and since there are three faces in A and three in B, it follows that $P(i) = 2/9$ where i is an even face and $P(j) = 1/9$ where j is an odd face.

Hence, $P(\{roll < 4\}$=P(\{1,2,3\})=1/9+2/9+1/9=4/9.

Problem 7

Problem 7. A four-sided die is rolled repeatedly, until the first time (if ever) that an even number is obtained. What is the sample space for this experiment?

The sample space is the set of all possible outcomes. If an even is obtained, then the outcome will be of the form $(a_1, \cdots, a_n)$, where a_n is an element of $\{2, 4\}$ and each of the $a_1, \cdots, a_{n-1}$ is an element of $\{1, 3\}$. If an even is never obtained, then the outcome will be an infinite sequence where each element is a member of $\{1, 3\}$. The sample space is the collection of all these outcomes.

Problem 14

Problem 14. We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

(a) Find the probability that doubles are rolled.

(b) Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.

(c) Find the probability that at least one die roll is a 6.

(d) Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

(a) There are possible doubles, hence $P(A) = 6/36 = 1/6$.
(b) If $B\{i + j \leq 4\}$, then $P(B) = 1/6$, since there are six possibilities that sum to less than or equal to 4.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 1/18 \times 6/1 = 1/3$$.

(c) $P(A) = 11/36$, where $A = \{i = 6 \vee j = 6\}$.
(d) $P(B) = 30/36$, where $B = \{i \neq j\}$.
$P(A \cap B) = 10/36$
$P(A|B) = 10/36 \times 36/30 = 1/3$

Problem 15

Problem 15. A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is a head than if we know that at least one of the tosses is a head. Is she right? Does it make a difference if the coin is fair or unfair? How can we generalize Alice's reasoning?

Let $A = \{HH\}$.
$B = \{1st\ \ toss\ \ H\} = \{HT, HH\}$
$C = \{at\ \ least\ \ one\ \ toss\ \ H\} = \{HT, HH, TH\}$
It follows then that
$P(A) = 1/4$, $P(B) = 1/2$ and $P(C) = 3/4$.
$P(A \cap B) = 1/4$, and $P(A \cap C) = 1/4$.
Hence, $P(A|B) = 1/2$ and $P(A|C) = 1/3$.
Alice is right.

Given that $P(A \cap B) = P(A \cap C)$ irrespective of whether the coin is fair, we can assert that Alice will always be right since $P(C) \geq P(A)$ irrespective of whether the coin is fair. This is because $A \subset C$.

A generalisation of her reasoning is that given events A, B, and C such that $A \subset B \subset C$, then A is at least as likely to have occurred if we know B has occurred than if we know that C has occurred.

Problem 16

Problem 16. We are given three coins: one has heads in both faces, the second has tails in both faces, and the third has a head in one face and a tail in the other. We choose a coin at random, toss it, and the result is heads. What is the probability that the opposite face is tails?

Let A be the event that the fair coin is chosen and $B = \{H\}$. We know that $P(B|A) = 1/2$ and $P(A) = 1/3$, hence, $P(A \cap B) = 1/6$. Since $P(B) = 1/2$, it follows that $P(A|B) = 1/6 * 2/1 = 1/3$.

Problem 17

Problem 17. A batch of one hundred items is inspected by testing four randomly selected items. If one of the four is defective, the batch is rejected. What is the probability that the batch is accepted if it contains five defectives?

Let $A_i = \{ith\ \ draw\ \ not\ \ defective\}$.
Then $P(A_1 \cap A_2 \cap A_3 \cap A_4\}$ is the probability that the batch is accepted.
We can write

$$P(A_1 \cap A_2 \cap A_3 \cap A_4\} = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)P(A_4|A_1 \cap A_2 \cap A_3)$$
$$= 95/100 * 94/99 * 93/98 * 92/97 = 0.81187$$

Problem 18

Problem 18. Let A and B be events. Show that $\mathbf{P}(A \cap B \mid B) = \mathbf{P}(A \mid B)$, assuming that $\mathbf{P}(B) > 0$.

$$P(A \cap B|B) = \frac{P((A \cap B) \cap B}{P(B)} = \frac{P(A \cap B}{P(B)}$$

Problem 19

Problem 19. Alice searches for her term paper in her filing cabinet. which has several drawers. She knows that she left her term paper in drawer j with probability $p_j > 0$. The drawers are so messy that even if she correctly guesses that the term paper is in drawer i. the probability that she finds it is only d_i. Alice searches in a particular drawer. say drawer i. but the search is unsuccessful. Conditioned on this event, show that the probability that her paper is in drawer j, is given by

$$\frac{p_j}{1 - p_i d_i}, \qquad \text{if } j \neq i, \qquad\qquad \frac{p_i(1 - d_i)}{1 - p_i d_i}, \qquad \text{if } j = i.$$

Let A be the event that searching drawer i results in a failure. And B be that the paper is in drawer j.

We want to find the probability that the paper is in drawer j, given that searching drawer i results in a failure.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)P(A|B)}{P(A)}.$$

We already know that $P(B) = p_j$

We know that $P(A^c) = P(success|D_i) = p_i d_i$.

Hence, we can write $P(A) = 1 - p_i d_i$.

If $i = j$, then it follows easily that $P(A|B) = 1 - d_j$, and hence,

$$P(B|A) = \frac{p_j(1 - d_i)}{1 - p_i d_i}.$$

If $i \neq j$, then $P(A|B) = 1$ because if the paper is not in the drawer i then it is certain that searching the drawer i will result in failure. Hence,

$$P(B|A) = \frac{p_j}{1 - p_i d_i}.$$

Problem 20

Problem 20. How an inferior player with a superior strategy can gain an advantage. Boris is about to play a two-game chess match with an opponent, and wants to find the strategy that maximizes his winning chances. Each game ends with either a win by one of the players, or a draw. If the score is tied at the end of the two games, the match goes into sudden-death mode, and the players continue to play until the first time one of them wins a game (and the match). Boris has two playing styles. *timid* and *bold*, and he can choose one of the two at will in each game. no matter what style he chose in previous games. With timid play. he draws with probability $p_d > 0$, and he loses with probability $1 - p_d$. With bold play. he wins with probability p_w, and he loses with probability $1 - p_w$. Boris will always play bold during sudden death, but may switch style between games 1 and 2.

(a) Find the probability that Boris wins the match for each of the following strategies:
  (i) Play bold in both games 1 and 2.
  (ii) Play timid in both games 1 and 2.
  (iii) Play timid whenever he is ahead in the score. and play bold otherwise.

(b) Assume that $p_w < 1/2$, so Boris is the worse player, regardless of the playing style he adopts. Show that with the strategy in (iii) above. and depending on the values of p_w and p_d. Boris may have a better than a 50-50 chance to win the match. How do you explain this advantage?

(a)
(i)
$P(winning) = P(W_2 \cap W_1) + P(W_3 \cap L_2 \cap W_1) + P(W_3 \cap W_2 \cap L_1)$
$= P(W_1)P(W_2|W_1) + P(W_1)P(L_2|W_1)P(W_3|L_2 \cap W_1) + P(L_1)P(L_W|L_1)P(W_3|W_2 \cap L_1)$
$= p_w^2 + p_w(1-p_w)p_w + (1-p_w)p_w p_w$
$= p_w^2 + p_w^2(1-p_w)$

(ii)
$P(winning) = P(W_3 \cap D_3 \cap D_1) = P(D_1)P(D_2|D_1)P(W_3|D_2 \cap D_1)$
$= p_d^2 p_w$

(iii)
$P(winning) = P(D_2 \cap W_1) + P(W_3 \cap L_2 \cap W_1) + p(W_3 \cap W_2 \cap L_1)$
$= P(W_1)P(D_2|W_1) + P(W_1)P(L_2|W_1)P(W_3|L_2 \cap W_1) + P(L_1)P(W_2|L_1)P(W_3|W_2 \cap L_1)$
$= p_w p_d + p_w(1-p_d)p_w + (1-p_w)p_w^2.$

(b)
If $p_w < .5$, then it follows that $P(winning) < .25p_d + .375$. Hence, in the case where p_w is close to $.5$, p_d will need to be greater than $.5$. The closer p_d is to 1, the more allowance we can

have for a lower p_w. However, there isn't much allowance, as even if $p_w = 0.4$, p_d will need to be roughly $.9$.
The advantage is explained by the fact that Boris can adapt his strategy after a game play, while his opponent cannot.

Problem 21

Problem 21. Two players take turns removing a ball from a jar that initially contains m white and n black balls. The first player to remove a white ball wins. Develop a recursive formula that allows the convenient computation of the probability that the starting player wins.

Using the total probability theorem, we can write that the probability of drawing a white is
$\frac{m}{m+k} + \frac{m}{m+k-1}\frac{k}{m+k} + \frac{m}{m+k-2}\frac{k-1}{m+k-1}\frac{k}{m+k} + \cdots$.
From here we can conclude that a recursive formula would be
$p(m,k) = \frac{m}{m+k} + \frac{k}{m+k}p(m,k-1)$
Where $p(m,0) = 1$.

Problem 22

Problem 22. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$.

Using the total probability theorem, we can write
$p(w,k) = p(w,k-1)\frac{m+1}{m+n+1} + (1-p(w,k-1))\frac{m}{m+n+1}$
Then from induction on k, the result follows since
$p(w,k+1) = \frac{m}{m+n}\frac{m+1}{m+n+1} + \frac{n}{n+m}\frac{m}{m+n+1}$
$= \frac{m(m+1)+nm}{(m+n)(m+n+1)}$

Problem 23

Problem 23. We have two jars, each initially containing an equal number of balls. We perform four successive ball exchanges. In each exchange, we pick simultaneously and at random a ball from each jar and move it to the other jar. What is the probability that at the end of the four exchanges all the balls will be in the jar where they started?

Let $p_{n-i,i}(k)$ denote the probability that after k exchanges, for a jar with n balls, there will be i balls from the other jar, and $n-i$ balls from itself.

We want to calculate $p_{n,0}(4)$.
For this to be possible, we require that after three exchanges, there is only 1 ball from the other jar. In which case, we require exchanging this ball with the ball in the other jar that was originally in ours. This will result in all balls returning to their original jar. Since, there is only 1 ball from the other jar in ours, there is a $1/n$ chance that it is exchanged. Similarly, for receiving our ball from the other har. Hence, we can write
$p_{n,0}(4) = \frac{1}{n}\frac{1}{n}p_{n-1,1}(3)$.
To arrive at 1 foreign ball in our jar after three exchanges, there are three possibilities after two exchanges.

(1) All balls returned to their original jar after two exchanges, in which case, the only next possible event (probability is 1) is a foreign ball from each jar being exchanged, resulting in 1 foreign ball in our jar.
(2) One foreign ball in our jar after two exchanges. To get one foreign ball in our jar after the next exchange, we either
    (a) Give the current foreign ball in our jar and receive a different foreign ball from ther other jar; this occurs with probability $\frac{1}{n}\frac{n-1}{n}$, since there is only one foreign ball in our jar and there are $n-1$ foreign balls in the other jar.
    (b) Give an original from our jar and receive back the original in the other jar. This occurs with a probability of $\frac{n-1}{n}\frac{1}{n}$, since there are $n-1$ originals in our jar and one original (from our jar) in the other jar.
(3) Two foreign balls are in our jar after two exchanges. In which case, to arrive at one foreign ball after three exchanges, we require giving up one of the foreign balls in our jar, and receiving one of the originals in the other jar. This has a probability of $\frac{2}{n}\frac{2}{n}$.

Hence, we can write
$p_{n-1,1}(3) = p_{n,0}(2) + 2\frac{1}{n}\frac{n-1}{n}p_{n-1,1}(2) + \frac{2}{n}\frac{2}{n}p_{n-2,2}(2)$.
We next consider each possibility after one exchange.

(1) $p_{n,0}(2) = \frac{1}{n}\frac{1}{n}p_{n-1,1}(1)$ since this would require simultaneously returning the ball we received and receiving the ball we gave on the first exchange. Since there is only one ball that received and only one we gave, then the probabilities are $1/n$ and $1/n$.
(2) After one exchange, we have two options to arrive at $p_{n-1,1}(2)$.
    (a) Give back the foreign ball we just received and receive a new foreign ball, the probability would be $\frac{1}{n}\frac{n-1}{n}$.
    (b) Give a new ball originally from our jar and receive our original ball back. This would also occur with a likelihood of $\frac{1}{n}\frac{n-1}{n}$.

(3) To arrive at two foreign balls after two exchanges we would need to give a new ball originally from our jar and receive a new foreign ball. This would occur with a probability of $\frac{n-1}{n}\frac{n-1}{n}$.

Hence,

$p_{n,0}(2) = \frac{1}{n}\frac{1}{n}p_{n-1,1}(1)$

$p_{n-1,1}(2) = 2\frac{1}{n}\frac{n-1}{n}p_{n-1,1}(1)$

$p_{n-2,2}(2) = \frac{n-1}{n}\frac{n-1}{n}p_{n-1,1}(1)$

Since the first exchange will necessarily involve giving an original ball and receiving a foreign ball, we have $p_{n-1,1}(1) = 1$.

Putting all of this together it follows that

$p_{n,0}(4) = \frac{1}{n^4} + \frac{8(n-1)^2}{n^6}$

Problem 24

Problem 24. The prisoner's dilemma. The release of two out of three prisoners has been announced, but their identity is kept secret. One of the prisoners considers asking a friendly guard to tell him who is the prisoner other than himself that will be released, but hesitates based on the following rationale: at the prisoner's present state of knowledge, the probability of being released is 2/3, but after he knows the answer, the probability of being released will become 1/2, since there will be two prisoners (including himself) whose fate is unknown and exactly one of the two will be released. What is wrong with this line of reasoning?

Let $pris_i = \{\text{prisoner i is released}\}$ where $i \in [1,3] \subset \mathbb{N}$

and $A_i = \{\text{guard says prisoner i is released}\}$ where $i \in [1,2] \subset \mathbb{N}$.

Then we have four possible outcomes:

(1) $pris_1 \cap pris_2 \cap A_2$

(2) $pris_1 \cap pris_3 \cap A_3$

(3) $pris_2 \cap pris_3 \cap A_2$

(4) $pris_2 \cap pris_3 \cap A_3$

Note,

$P(pris_1 \cap pris_2 \cap A_2) = P(pris_1)P(pris_2|pris_1)P(A_2|pris_1 \cap pris_2)$

$= 2/3 \times 1/2 \times 1 = 1/3$

$P(pris_1 \cap pris_3 \cap A_3) = P(pris_1)P(pris_3|pris_1)P(A_3|pris_1 \cap pris_3)$

$= 2/3 \times 1/2 \times 1 = 1/3$

$P(pris_2 \cap pris_3 \cap A_2) = P(pris_2)P(pris_3|pris_2)P(A_2|pris_2 \cap pris_3)$

$P(pris_2 \cap pris_3 \cap A_3) = P(pris_2)P(pris_3|pris_2)P(A_3|pris_3 \cap pris_3)$

The probability of the last two possible outcomes cannot be calculated without specifying the complements $P(A_2|pris_2 \cap pris_3)$ and $P(A_3|pris_3 \cap pris_3)$. Where the guard is equally likely to reveal prisoner 2 or prisoner 3, we would have
$P(A_2|pris_2 \cap pris_3) = P(A_3|pris_3 \cap pris_3) = 1/2$.

We know from the total probability theorem that
$P(pris_1) = P(pris_1|A_2)P(A_2) + P(pris_1|A_3)P(A_3)$
We also know from the question that $P(pris_1) = 2/3$.
The prisoner believes that $P(pris_1|A_i) = 1/2$ where $i \in [1, 2] \subset \mathbb{N}$. However, it is not possible to know this without specifying $P(A_2|pris_2 \cap pris_3)$ and $P(A_3|pris_3 \cap pris_3)$.
This is because

$$P(pris_1|A_i) = \frac{P(pris_1 \cap A_i)}{P(A_i)}$$

It follows easily that $P(pris_1 \cap A_i) = P(pris_1 \cap pris_i \cap A_i)$ since we have assumed that the guard cannot lie.
Now,
$P(A_i) = P(pris_1 \cap pris_i \cap A_i) + P(pris_i \cap pris_j \cap A_i)$ and hence depends on the specification of $P(A_2|pris_2 \cap pris_3)$ and $P(A_3|pris_3 \cap pris_3)$ since
$P(pris_2 \cap pris_3 \cap A_2) = P(pris_2)P(pris_3|pris_2)P(A_2|pris_2 \cap pris_3)$
$P(pris_2 \cap pris_3 \cap A_3) = P(pris_2)P(pris_3|pris_2)P(A_3|pris_3 \cap pris_3)$

If we assume $P(A_2|pris_2 \cap pris_3) = P(A_3|pris_3 \cap pris_3) = 1/2$, then it can be shown that $P(pris_1|A_2) = P(pris_1|A_3) = 2/3$, in which case the prisoner would be wrong anyway.

Problem 25

Problem 25. A two-envelopes puzzle. You are handed two envelopes. and you know that each contains a positive integer dollar amount and that the two amounts are different. The values of these two amounts are modeled as constants that are unknown. Without knowing what the amounts are, you select at random one of the two envelopes, and after looking at the amount inside, you may switch envelopes if you wish. A friend claims that the following strategy will increase above 1/2 your probability of ending up with the envelope with the larger amount: toss a coin repeatedly. let X be equal to 1/2 plus the number of tosses required to obtain heads for the first time, and switch if the amount in the envelope you selected is less than the value of X. Is your friend correct?

Let $\underline{m}$ and $\overline{m}$ be the lower and higher amounts contained in the envelopes, respectively.

Consider the three events:
$A = \{X < \underline{m}\}$
$B = \{\underline{m} < X < \overline{m}\}$
$C = \{\overline{m} < X\}$
Note that $P(A) + P(B) + P(C) = 1$.
Define the respective events
$\overline{A}, \overline{B}, \overline{C} = \{A|B|C \text{ larger envelope selected first}\}$
$\underline{A}, \underline{B}, \underline{C} = \{A|B|C \text{ smaller envelope selected first}\}$
Finally, define the event
$W = \{$\overline{m}}

We want to determine whether $P(W) > 1/2$
Using the total probability theorem
$P(W|A) = 1/2P(W|\overline{A}) + 1/2P(W|\underline{A}) = 1/2(1) + 1/2(0) = 1/2$
$P(W|B) = 1/2P(W|\overline{B}) + 1/2P(W|\underline{B}) = 1/2(1) + 1/2(1) = 1$
$P(W|C) = 1/2P(W|\overline{C}) + 1/2P(W|\underline{C}) = 1/2(0) + 1/2(1) = 1/2$

Hence, using the total probability theorem again, we can write
$P(W) = P(A)P(W|A) + P(B)P(W|B) + P(C)P(W|C)$
$= 1/2P(A) + P(B) + 1/2P(C)$
$= 1/2P(A) + 1/2P(B) + 1/2P(C) + 1/2P(B)$
$= 1/2(P(A) + P(B) + P(C)) + 1/2P(B)$
$= 1/2 + 1/2P(B)$
Since $P(B) > 0$ it follows that $P(W) > 1/2$

Problem 26

Problem 26. The paradox of induction. Consider a statement whose truth is unknown. If we see many examples that are compatible with it, we are tempted to view the statement as more probable. Such reasoning is often referred to as *inductive inference* (in a philosophical, rather than mathematical sense). Consider now the statement that "all cows are white." An equivalent statement is that "everything that is not white is not a cow." We then observe several black crows. Our observations are clearly compatible with the statement. but do they make the hypothesis "all cows are white" more likely?

To analyze such a situation, we consider a probabilistic model. Let us assume that there are two possible states of the world, which we model as complementary events:

$$A: \text{all cows are white,}$$

$$A^c: 50\% \text{ of all cows are white.}$$

Let p be the prior probability $\mathbf{P}(A)$ that all cows are white. We make an observation of a cow or a crow, with probability q and $1-q$, respectively, independent of whether event A occurs or not. Assume that $0 < p < 1$, $0 < q < 1$, and that all crows are black.

(a) Given the event $B = \{\text{a black crow was observed}\}$, what is $\mathbf{P}(A \mid B)$?

(b) Given the event $C = \{\text{a white cow was observed}\}$, what is $\mathbf{P}(A \mid C)$?

We know

$P(A) = p$ and $P(B) = 1 - q$.

(a)

$P(B|A) = 1 - q$, since observing a black crow is not affected by where all cows are white or not.

Hence,

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{p(1-q)}{1-q} = p$$

(b)

$P(C|A) = q$, since if all cows are white, then the probability of observing a white cow is just the probability of observing a cow.

$P(C|A^c) = 0.5q$ which we can assert using the multiplication rule, since

$P(C) = P(\{\text{observe cow and cow is white}\}$

$= P(\text{observe cow})P(\text{observed cow is white — observe cow})$

$= q * 0.5$

Hence,

$$P(A|C) = \frac{P(A)P(C|A)}{P(C)} = \frac{P(A)P(C|A)}{P(A)P(C|A) + P(A^c)P(C|A^c)}$$

$$= \frac{pq}{pq + (1-p)0.5q} = \frac{2p}{p+1}$$

Note, $P(A|C) > p$, hence, observing a white cow makes A more likely to be true.

Problem 30

Problem 30. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He knows that each dog. independent of the other. will choose the correct path with probability p. The hunter decides to let each dog choose a path, and if they agree, take that one, and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path?

Consider the following events

$A_i = \{\text{dog}_\text{i} \text{ chooses correct path}\}$ for $i = 1, 2$

$W = \{\text{hunter chooses correct path}\}$

$B = \{\text{dogs choose same path}\}$

We know $P(A_i) = p$.

If the hunter only lets one dog decide, then $P(W) = p$. Hence, we want to find out if, under the current strategy, $P(W) > p$.

From the total probability theorem, we can write

$P(W) = P(B)P(W|B) + P(B^c)P(W|B^c)$

$P(B) = P(A_1 \cap A_2) + P(A_1^c \cap A_2^c)$.

Since, the A_i are independent, we know that

$P(B) = p^2 + (1-p)^2$.

Then, $P(B^c) = 2p(1-p)$

We also know that $P(W|B^c) = 0.5$ (hunter chooses randomly).

$$P(W|B) = \frac{P(W \cap B)}{P(B)} = \frac{p^2}{p^2 + (1-p)^2}$$

Where we have used the fact that

$P(W \cap B) = P(\text{both dogs choose correct path})$

Hence,

$$P(W) = \frac{(p^2 + (1-p)^2 p^2}{p^2 + (1-p)^2} + 0.5 * 2p(1-p)$$

$= p^2 + p(1-p) = p$.

Either strategy $P(W) = p$, so the hunters strategy is not better or worse.

Problem 31

Problem 31. Communication through a noisy channel. A source transmits a message (a string of symbols) through a noisy communication channel. Each symbol is 0 or 1 with probability p and $1-p$, respectively, and is received incorrectly with probability ϵ_0 and ϵ_1, respectively (see Fig. 1.18). Errors in different symbol transmissions are independent.

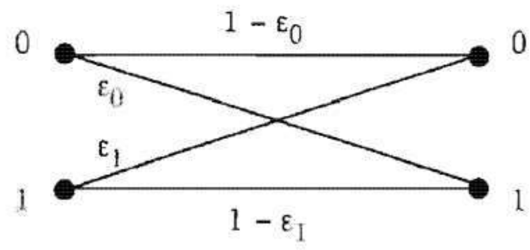

Figure 1.18: Error probabilities in a binary communication channel.

(a) What is the probability that the kth symbol is received correctly?

(b) What is the probability that the string of symbols 1011 is received correctly?

(c) In an effort to improve reliability, each symbol is transmitted three times and the received string is decoded by majority rule. In other words, a 0 (or 1) is transmitted as 000 (or 111, respectively), and it is decoded at the receiver as a 0 (or 1) if and only if the received three-symbol string contains at least two 0s (or 1s, respectively). What is the probability that a 0 is correctly decoded?

(d) For what values of ϵ_0 is there an improvement in the probability of correct decoding of a 0 when the scheme of part (c) is used?

(e) Suppose that the scheme of part (c) is used. What is the probability that a symbol was 0 given that the received string is 101?

(a)
$C_k = \{\text{kth symbol received shortly}\}$
We know that $P(0) = p$ and $P(1) = 1 - p$.
We also know that $P(C_k|0) = 1 - \epsilon_0$ and $P(C_k|1) = 1 - \epsilon_1$.
Using the total probability theorem, we have
$P(C_k) = P(0)P(C_k|0) + P(1)P(C_k|1)$
$P(C_k) = p(1 - \epsilon_0) + (1 - p)(1 - \epsilon_1)$

(b)
$S = 1011$

$SC = \{\text{S received correctly}\}$
$P(SC) = P(C_1 \cap C_2 \cap C_3 \cap C_4)$
$= P(C_1)P(C_2)P(C_3)P(C_4)$ since they are independent.
$= (1-\epsilon_0)(1-\epsilon_1)^3$

(c)
If $S = 000$, then the possible decoded string can be 100, 010, 001, or 000 for the string to be decoded correctly.
Hence, since these events are disjoint
$P(SC) = (1-\epsilon_0)^3 + 3\epsilon_0(1-\epsilon_0)^2$.

(d)
We need $(1-\epsilon_0)^3 + 3\epsilon_0(1-\epsilon_0)^2 > 1-\epsilon_0$, which follows if $\epsilon_2 < 1/2$.

(e)
We can find $P(0|101)$ using Bayes theorem.

$$P(0|101) = \frac{P(0)P(0|101)}{P(0)P(0|101) + P(1)P(1|101)}$$

Since $P(0|101) = \epsilon_0^2(1-\epsilon_0)$
And $P(1|101) = \epsilon_1(1-\epsilon_1)^2$

$$P(0|101) = \frac{p\epsilon_0^2(1-\epsilon_0)}{p\epsilon_0^2(1-\epsilon_0) + (1-p)\epsilon_1(1-\epsilon_1)^2}$$

Problem 32

Problem 32. The king's sibling. The king has only one sibling. What is the probability that the sibling is male? Assume that every birth results in a boy with probability 1/2, independent of other births. Be careful to state any additional assumptions you have to make in order to arrive at an answer.

The additional assumptions relate to the decisions made by the king's parents. They could have decided to keep having children until they had exactly two, in which case there are four possible outcomes:

- Boy boy
- Girl girl
- Boy girl
- Girl boy

The second one is eliminated automatically because we know that the king is a male. Hence, the likelihood that he king's sibling is male is ⅓
On the other hand, the parents could have decided to keep having children until one of them is male, in which case we know with certainty that the sibling is female.

Problem 33

Problem 33. Using a biased coin to make an unbiased decision. Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately. the only available coin is biased (though the bias is not known exactly). How can they use the biased coin to make a decision so that either option (opera or the movies) is equally likely to be chosen?

Flip the coin twice. If it's HT go to the opera. If it's TH, go to the movies. Repeat until a decision is made.

Problem 34

Problem 34. An electrical system consists of identical components. each of which is operational with probability p, independent of other components. The components are connected in three subsystems, as shown in Fig. 1.19. The system is operational if there is a path that starts at point A. ends at point B. and consists of operational components. What is the probability of this happening?

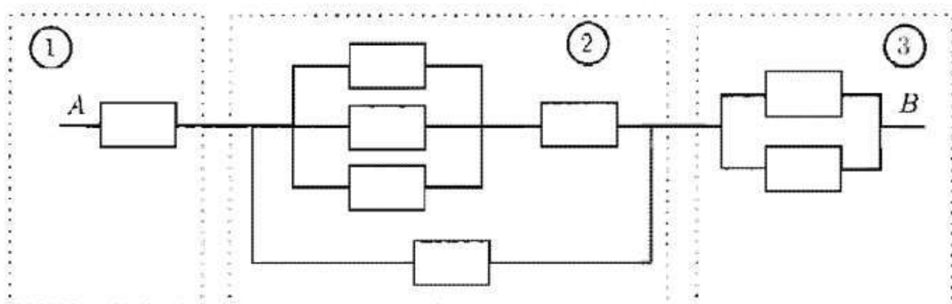

Figure 1.19: A system of identical components that consists of the three subsystems 1. 2, and 3. The system is operational if there is a path that starts at point A, ends at point B, and consists of operational components.

$P(S) = P(SS_1 \cap SS_2 \cap SS_2) = P(SS_1)P(SS_2)P(SS_3)$
$P(SS_1) = p$

$P(SS_2) = P(SS_2A \cup SS_2B) = P(SS_2A) + P(SS_2B) - P(SS_2A)P(SS_2B)$
$P(SS_2B) = p$
$P(SS_2A) = P(SS_2A_1)P(SS_2A_2)$
$P(SS_2A_1) = 1 - (1-p)^3$
$P(SS_2A_2) = p$
$P(SS_2A) = (1-(1-p)^3)p$
$P(SS_2) = (1-(1-p)^3)p + p - (1-(1-p)^3)p^2$

$$P(SS_3) = P(SS_3A \cup SS_3B) = P(SS_3A) + P(SS_3B) - P(SS_3A)P(SS_3B = 2p - p^2$$

Problem 35

Problem 35. Reliability of a k-out-of-n system. A system consists of n identical components, each of which is operational with probability p, independent of other components. The system is operational if at least k out of the n components are operational. What is the probability that the system is operational?

Let $A_i = \{\text{i units are working}\}$. Hence, the system is working if A_i for $i \geq k$.
Since each A_i is disjoint, we can write

$$P(working) = \sum_{i=k}^{n} P(A_i)$$.

Then, for some A_i, we can use the binomial coefficient to get the number of ways i out of n components are working in the system. Hence,

$$P(A_i) = \binom{n}{i} p^i (1-p)^{n-i}$$.

So it follows that

$$P(working) = \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$.

Problem 36

Problem 36. A power utility can supply electricity to a city from n different power plants. Power plant i fails with probability p_i, independent of the others.

(a) Suppose that any one plant can produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

(b) Suppose that two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

(a)

$$P(blackout) = \prod_{i=1}^{n} p_i$$

(b)
There is a black if n or $n-1$ plants fail. These events are disjoint, hence
$P(blackout) = P(\text{n fail}) + P(\text{n-1 fail})$

$$= \prod_{i=1}^{n} p_i + \sum_{i=i}^{n} (1 - p_i) \prod_{j=1, j\neq i}^{n} p_j$$

Problem 37

Problem 37. A cellular phone system services a population of n_1 "voice users" (those who occasionally need a voice connection) and n_2 "data users" (those who occasionally need a data connection). We estimate that at a given time, each user will need to be connected to the system with probability p_1 (for voice users) or p_2 (for data users), independent of other users. The data rate for a voice user is r_1 bits/sec and for a data user is r_2 bits/sec. The cellular system has a total capacity of c bits/sec. What is the probability that more users want to use the system than the system can accommodate?

The probability k_1 voice users want to access the phone network is

$$p_1(k_1) = \binom{n_1}{k_1} p_{11}^{k}(1-p_1)^{n_1-k_1}$$

If this happens, then the total rate required would be $k_1 r_1$.

Similarly, for k_2 data users on the data network, we have

$$p_2(k_2) = \binom{n_2}{k_2} p_{22}^{k}(1-p_2)^{n_2-k_2}$$

If this happens, then the total rate required would be $k_2 r_2$.

For given k_1 and k_2, the total rate required on the network would be $k_1 r_1 + k_2 r_2$. The network reaches capacity when $k_1 r_1 + k_2 r_2 > c$.

Hence, the probability that more capacity is required than possible is

$$\sum_{(k_1,k_2):k_1 r_1 + k_2 r_2 > c} p_1(k_1) p_2(k_2)$$

Problem 38

Problem 38. The problem of points. Telis and Wendy play a round of golf (18 holes) for a \$10 stake, and their probabilities of winning on any one hole are p and $1-p$, respectively, independent of their results in other holes. At the end of 10 holes, with the score 4 to 6 in favor of Wendy, Telis receives an urgent call and has to report back to work. They decide to split the stake in proportion to their probabilities of winning had they completed the round, as follows. If p_T and p_W are the conditional probabilities that Telis and Wendy, respectively, are ahead in the score after 18 holes given the 4-6 score after 10 holes, then Telis should get a fraction $p_T/(p_T+p_W)$ of the stake, and Wendy should get the remaining $p_W/(p_T+p_W)$. How much money should Telis get? *Note*: This is an example of the, so-called, problem of points, which played an important historical role in the development of probability theory. The problem was posed by Chevalier de Méré in the 17th century to Pascal, who introduced the idea that the stake of an interrupted game should be divided in proportion to the players' conditional probabilities of winning given the state of the game at the time of interruption. Pascal worked out some special cases and through a correspondence with Fermat, stimulated much thinking and several probability-related investigations.

Tellis wins the game if he gets at least 6 points in the next 10 rounds. Hence,

$$p_T = \sum_{i=6}^{8} \binom{8}{i} p^i (1-p)^{8-i}$$

$$p_W = \sum_{i=4}^{8} \binom{8}{i} (1-p)^i p^{8-i}$$

Problem 39

Problem 39. A particular class has had a history of low attendance. The annoyed professor decides that she will not lecture unless at least k of the n students enrolled in the class are present. Each student will independently show up with probability p_g if the weather is good, and with probability p_b if the weather is bad. Given the probability of bad weather on a given day, obtain an expression for the probability that the professor will teach her class on that day.

$P(\text{bad weather}) = p_w$

$P(teaching) = P(\text{bad weather})P(teaching|\text{bad weather}) + P(\text{good weather})P(teaching|\text{good weather})$

$$P(teaching|\text{bad weather}) = \sum_{i=k}^{k} \binom{n}{i} p_b^k (1-p_b)^{n-k}$$

$$P(teaching|\text{good weather}) = \sum_{i=k}^{k} \binom{n}{i} p_g^k (1-p_g)^{n-k}$$

Hence,

$$P(teaching) = p_w \sum_{i=k}^{k} \binom{n}{i} p_b^k (1-p_b)^{n-k} + (1-p_w) \sum_{i=k}^{k} \binom{n}{i} p_g^k (1-p_g)^{n-k}$$

Problem 40

Problem 40. Consider a coin that comes up heads with probability p and tails with probability $1-p$. Let q_n be the probability that after n independent tosses, there have been an even number of heads. Derive a recursion that relates q_n to q_{n-1}, and solve this recursion to establish the formula

$$q_n = \left(1 + (1-2p)^n\right)/2.$$

Let $A = \{\text{n-1 tosses has even heads}\}$

And $E = \{\text{nth toss is heads}\}$

Then

$q_n = P(A)P(E^c) + P(A^c)P(E) = q_{n-1}(1-p) + (1-q_{n+1})p$.

Where $q_0 = 1$.

Induction:

$n = 0$, then $(1 + (1 - 2p)^0)/2 = 1$.

Assume $q_{n-1} =$.
Hence,
$q_n = frac1 + (1 - 2p)^{n-1}2(1 - p) + (1 - frac1 + (1 - 2p)^{n-1}2)p$
$= frac1 - p + (1 - p)(1 - 2p)^{n-1} + 2p - p + p(1 - 2p)^{n-1}2$
$= frac1 + (1 - p)(1 - 2p)^{n-1} + p(1 - 2p)^{n-1}2$
$= frac1 + (1 - 2p)(1 - 2p)^{n-1}2$
$= frac1 + (1 - 2p)^n2$

Problem 49

Problem 49. De Méré's puzzle. A six-sided die is rolled three times independently. Which is more likely: a sum of 11 or a sum of 12? (This question was posed by the French nobleman de Méré to his friend Pascal in the 17th century.)

For a sum to 11, we have the following possible combinations:
(5,5,1),(5,3,3), (4,3,4), (3,2,6), (6,4,1), (5,4,2)
The first three have 3 permutations and the last three have 6 permutations. Hence, the total number of permutations is $3 \times 3 + 3 \times 6 = 27$.
For a sum to 12, we have the following possible combinations:
(4,4,4), (3,3,6), (5,2,5), (5,3,4), (6,4,2), (5,6,1)
The first has only 1 permutation, the next two have 3 permutations, while the last three have 6 permutations. Hence, the total number of permutations is 25.
Hence, a sum to 11 is more likely.

Problem 50

Problem 50. The birthday problem. Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year. independent of everyone else, and ignore the additional complication presented by leap years (i.e., assume that nobody is born on February 29). What is the probability that each person has a distinct birthday?

We can treat this problem according to the counting principle. Where each stage represents allocating a person to their birthday. The principle tells use that there are 365^n possible results.
Let $A = \{\text{each person has a distinct birthday}\}$
Now, if $n > 365$, then $P(A) = 0$, since we would have to allocate more than one person to a given day. Hence, let $n \leq 365$.
The problem boils down to counting the number of ways we can pick n birthdays out of 365 days of the year and arrange them in a sequence. If we could allocate more than one person to a day, then it wouldn't be a sequence. This allows us to apply permutations. In which case,

$$P(A) = \frac{365!}{(365-n)!}/365^n$$

Problem 51

Problem 51. An urn contains m red and n white balls.

(a) We draw two balls randomly and simultaneously. Describe the sample space and calculate the probability that the selected balls are of different color, by using two approaches: a counting approach based on the discrete uniform law, and a sequential approach based on the multiplication rule.

(b) We roll a fair 3-sided die whose faces are labeled 1,2,3, and if k comes up. we remove k balls from the urn at random and put them aside. Describe the sample space and calculate the probability that all of the balls drawn are red. using a divide-and-conquer approach and the total probability theorem.

Although the two balls are drawn simultaneously, we still need to consider them as separate draws as the order matters. That is, a white-red draw is distinct from a red-white draw.

(a)

Considering the problem using the counting approach, the sample space is (i, j) for $1 \leq i \leq m$ and $1 \leq i \leq n$. We assert that on the first draw there are $n+m$ possible outcomes. Then on the second draw, there are $n+m-1$ possible outcomes. Hence, the total possible outcomes for a simultaneous draw is $(n+m)(n+m-1)$. There are nm possible outcomes for the draws to be distinct if either a red or white ball is drawn first.

Hence,

$$P(distinct) = \frac{nm}{(n+m-1)(mn)} + \frac{nm}{(n+m-1)(mn)} = \frac{2nm}{(n+m-1)(mn)}$$

From a multiplication perspective, we consider the sample space as $\{RR, RW, WR, WW\}$.

Then,

$P(distinct) = P(\{RW, WR\}) = P(RW) + P(WR)$, since RW and WR are disjoint.

Where

$$P(RW) = \frac{m}{n+m}\frac{n}{n+m-1} = \frac{nm}{(n+m-1)(mn)}$$

$$P(WR) = \frac{n}{n+m}\frac{m}{n+m-1} = \frac{nm}{(n+m-1)(mn)}$$

Hence,

$$P(distinct) = \frac{2nm}{(n+m-1)(mn)}$$

(b)

Let $B = \{\text{all balls drawn are red}\}$
And $A_k = \{\text{k comes up}\}$, where $k = 1, 2, 3$
Then,
$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$
Where $P(A_1) = P(A_2) = P(A_3) = 1/3$
And

$$P(B|A_1) = \frac{m}{m+n}$$

$$P(B|A_2) = \frac{m}{m+n}\frac{m-1}{m+n-1}$$

$$P(B|A_3) = \frac{m}{m+n}\frac{m-2}{m+n}\frac{m}{m+n-2}$$

Problem 52

Problem 52. We deal from a well-shuffled 52-card deck. Calculate the probability that the 13th card is the first king to be dealt.

The sample space is the total number of ways to arrange a sequence of 13 cards from the deck. Hence,

$$\frac{52!}{39!}$$

To count the total number of ways we can arrange the sequence with the 13th card being king, we first consider counting the number of ways to arrange 12 cards with none of them a king which is $\frac{48!}{36!}$. Then we consider the number of ways to arrange 4 kings in a sequence of 1, which is $\frac{4!}{3!} = 4$.
Hence,

$$P(A) = \frac{\frac{48!}{36!} \cdot 4}{\frac{52!}{39!}}$$

Problem 53

Problem 53. Ninety students, including Joe and Jane, are to be split into three classes of equal size, and this is to be done at random. What is the probability that Joe and Jane end up in the same class?

We can think about this question simply. Assign Joe to some class. Then for Jane, there are 89 possible outcomes, but only 29 of which put her in the same class as Joe. Hence, the answer is 29/89.

Problem 54

Problem 54. Twenty distinct cars park in the same parking lot every day. Ten of these cars are US-made. while the other ten are foreign-made. The parking lot has exactly twenty spaces. all in a row. so the cars park side by side. However. the drivers have varying schedules. so the position any car might take on a certain day is random.

(a) In how many different ways can the cars line up?

(b) What is the probability that on a given day, the cars will park in such a way that they alternate (no two US-made are adjacent and no two foreign-made are adjacent)?

(a)
20!

(b) $\dfrac{20 \cdot 10! \cdot 9!}{20!}$

Problem 55

Problem 55. Eight rooks are placed in distinct squares of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another, i.e.. that there is no row or column with more than one rook.

The total number of possible ways to arrange the rooks (without restrictions) is $\frac{64!}{56!}$, which represents the number 8 permutations of 64 distinct places. With the above restriction, consider: when placing the first rook, we have 8×8 places. When placing the second rook, we have 7×7 places to choose from, and so forth. Hence, the probability is

$$\frac{(8!)^2}{\frac{64!}{56!}}$$

Problem 56

Problem 56. An academic department offers 8 lower level courses: $\{L_1, L_2, \ldots, L_8\}$ and 10 higher level courses: $\{H_1, H_2, \ldots, H_{10}\}$. A valid curriculum consists of 4 lower level courses. and 3 higher level courses.

(a) How many different curricula are possible?

(b) Suppose that $\{H_1, \ldots, H_5\}$ have L_1 as a prerequisite, and $\{H_6, \ldots H_{10}\}$ have L_2 and L_3 as prerequisites. i.e.. any curricula which involve, say, one of $\{H_1, \ldots, H_5\}$ must also include L_1. How many different curricula are there?

(a)
We can use combinations in this problem because the order doesn't matter.
Hence, we have

$$\binom{8}{4}\binom{10}{3}$$

(b)
We have to consider 4 individual cases. Note, the first part of the curriculum must contain L_1 or L_2 and L_3, otherwise the second part will take place.
(i)
L_1 is in the first part of the curriculum and not L_2 or L_3. Then there are 3 remaining places to be filled with 5 courses. In the second part of the curriculum, there are 3 places to be filled by 5 courses. Hence,

$$\binom{5}{3}\binom{5}{3}$$

(ii)
L_1, L_2 and L_3. In which case, there is one place to be filled by 5 courses. For the second part of the course, there are 3 places to be filled by 10 courses. Hence,

$$\binom{5}{1}\binom{10}{3}$$

(iii)
L_1 and L_2 are in the first part and L_1 isn't. Then, there are two places to be filled by 5 courses in the first part, and 3 places to be filled by 5 courses in the second. Hence,

$$\binom{5}{2}\binom{5}{3}$$

(iv)
L_1 is in the first part and either L_2 or L_3 as well. In which case, assuming one or the other, there are 2 places to be filled by 5 places (since we know if L_2 then not L_3 and vice versa). The second part has 3 places to be filled by 5 courses. We must times this by 2 to account for the two possibilities of either L_2 or L_3 in the first part.

$$2\binom{5}{2}\binom{5}{3}$$

Problem 57

Problem 57. How many 6-word sentences can be made using each of the 26 letters of the alphabet exactly once? A word is defined as a nonempty (possibly jibberish) sequence of letters.

We break this problem into two parts. Firstly, we consider the order of the letters across the whole sentence, which is just $26!$. Then, we consider how the sentence is broken up into words, we note that this is essentially asking how to place 5 blanks within 25 positions, where the order doesn't matter because a blank is a blank. Hence, the answer is

$$26!\binom{25}{5}$$

Problem 58

Problem 58. We draw the top 7 cards from a well-shuffled standard 52-card deck. Find the probability that:

(a) The 7 cards include exactly 3 aces.

(b) The 7 cards include exactly 2 kings.

(c) The probability that the 7 cards include exactly 3 aces. or exactly 2 kings, or both.

(a)
The sample space is all the ways of drawing out 7 cards from the 52 card deck. The order of the 7 cards does not matter, so we can use combinations.
$\binom{52}{7}$.
We can then select 3 out of the 4 aces and 4 out of the 48 remaining cards.
Hence,

$$P(7 \text{ cards include exactly 3 aces}) = \frac{\binom{4}{3}\binom{48}{4}}{\binom{52}{7}}$$

(b)
This is calculated in a similar way

$$P(7 \text{ cards include exactly 2 kings}) = \frac{\binom{4}{2}\binom{48}{5}}{\binom{52}{7}}$$

(c)
We want to calculate $P(A \cup B)$ where $A = \{7 \text{ cards include exactly 3 aces}\}$ and $B = \{7 \text{ cards include exactly 2 kings}\}$.
Since the events are not disjoint (they can happen at the same time), we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where

$$P(A \cap B) = \frac{\binom{4}{3}\binom{4}{2}\binom{44}{2}}{\binom{52}{7}}$$

Problem 59

Problem 59. A parking lot contains 100 cars, k of which happen to be lemons. We select m of these cars at random and take them for a test drive. Find the probability that n of the cars tested turn out to be lemons.

The sample space is $\binom{100}{m}$. Then we choose n from k available lemons, and $n-k$ good cars from $100-k$ available good cars.
Hence,

$$P(A) = \frac{\binom{k}{n}\binom{100-k}{m-n}}{\binom{100}{m}}$$

Problem 60

Problem 60. A well-shuffled 52-card deck is dealt to 4 players. Find the probability that each of the players gets an ace.

This is a partition problem.

The sample space is

$$\frac{52!}{13!13!13!13!}$$

We first consider the number of ways we can divide the 4 aces into 4 groups.

$$\frac{4!}{1!1!1!1!}$$

Then we consider how to divide the remaining 48 cards to 4 groups of 12.

$$\frac{48!}{12!12!12!12!}$$

Hence, the answer is

$$\frac{\frac{4!}{1!1!1!1!}\frac{48!}{12!12!12!12!}}{\frac{52!}{13!13!13!13!}}$$