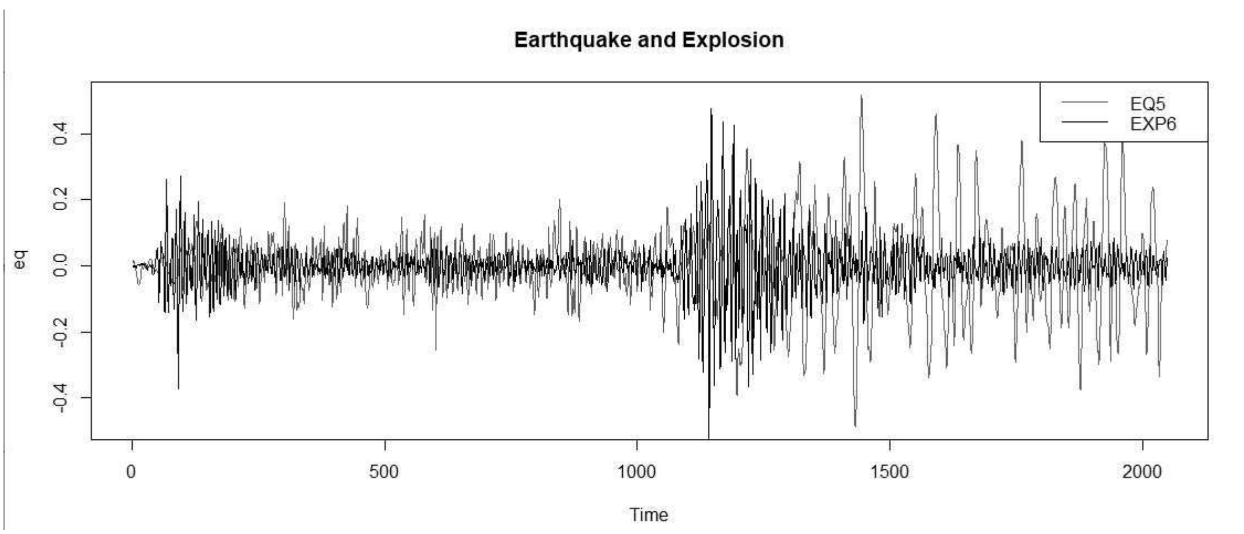
Chapter 1

1.1

1.1 To compare the earthquake and explosion signals, plot the data displayed in Fig. 1.7 on the same graph using different colors or different line types and comment on the results. (The R code in Example 1.11 may be of help on how to add lines to existing plots.)

```
par(mfrow=c(2,1))
eq = EQ5; exp = EXP6
plot(eq, main="Earthquake and Explosion", col=4)
lines(exp, col=1)
legend(x="topright", legend=c("EQ5", "EXP6"), lty=c(1,1), col = c(4,1))
```



Both waveforms show two distinct phases in their amplitudes. While both experience relatively low levels of amplitude in the first phase, being seemingly indistinguishable, they are more distinct in the second phase. The explosion shows a large spike in amplitude, which quickly reduces to levels in the first phase. The earthquake, on the other hand, experiences a maintained increase in amplitude, although, the frequency of the waves for the earthquake are markedly lower than that of the explosion. The ratio of the amplitudes between phases for the explosion is about 1, given that the spike in amplitude is temporary, while the same ratio for the earthquake is 0.5, as the increase is maintained.

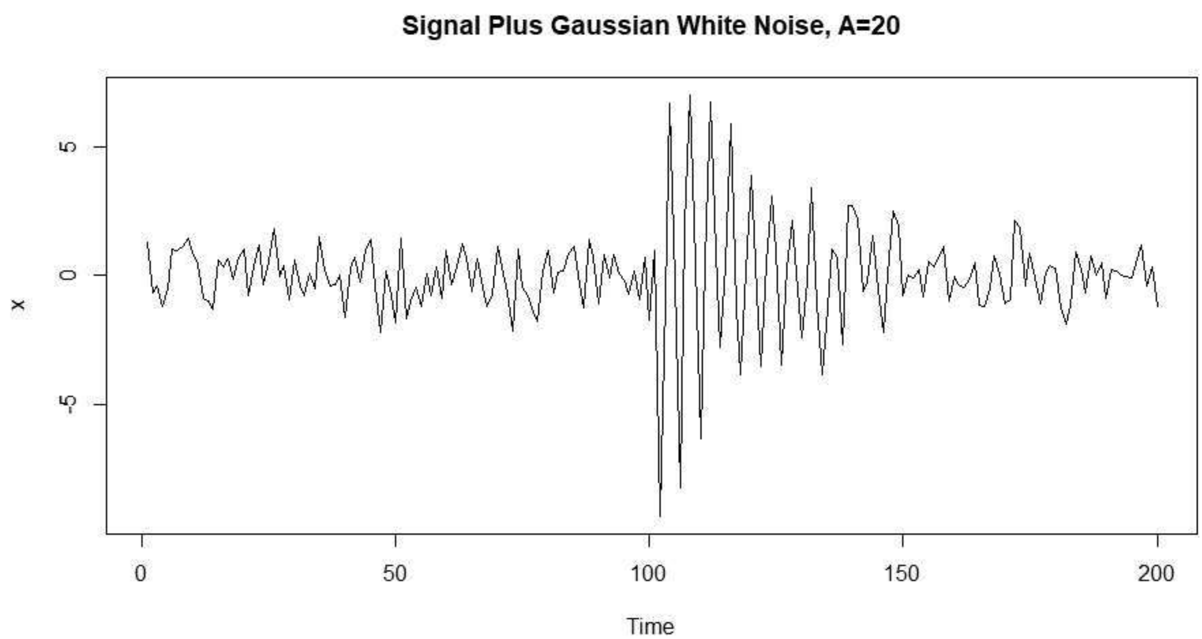
1.2

1.2 Consider a signal-plus-noise model of the general form $x_t = s_t + w_t$, where w_t is Gaussian white noise with $\sigma_w^2 = 1$. Simulate and plot $n = 200$ observations from each of the following two models.

(a) $x_t = s_t + w_t$, for $t = 1, \dots, 200$, where

$$s_t = \begin{cases} 0, & t = 1, \dots, 100 \\ 10 \exp\{-\frac{(t-100)}{20}\} \cos(2\pi t/4), & t = 101, \dots, 200. \end{cases}$$

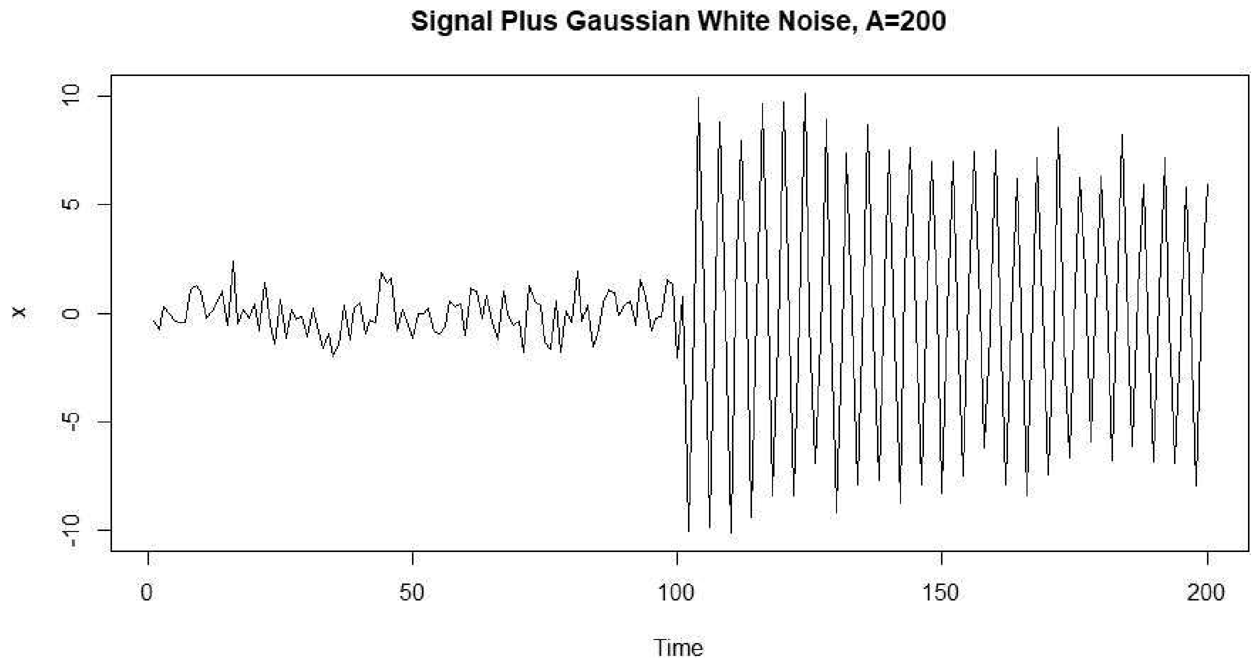
```
s1 = rep(0,100)
s2 = 10*exp(-(1:100)/20)*cos(2*pi*101:200/4)
s = c(s1, s2)
w = rnorm(200)
x = s + w
plot.ts(x, main="Signal Plus Gaussian White Noise")
```



(b) $x_t = s_t + w_t$, for $t = 1, \dots, 200$, where

$$s_t = \begin{cases} 0, & t = 1, \dots, 100 \\ 10 \exp\{-\frac{(t-100)}{200}\} \cos(2\pi t/4), & t = 101, \dots, 200. \end{cases}$$

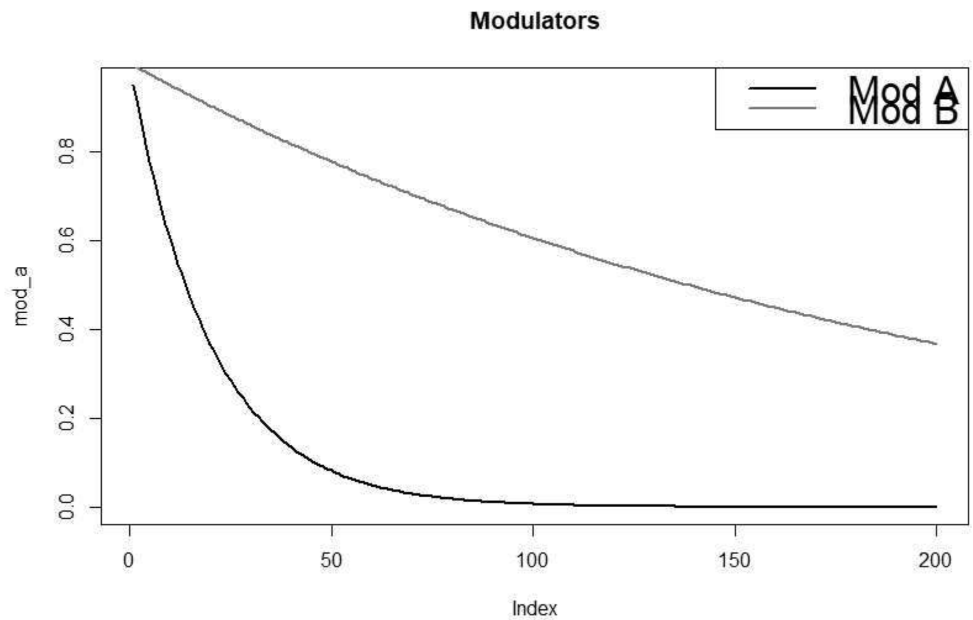
```
s1 = rep(0,100)
s2 = 10*exp(-(1:100)/200)*cos(2*pi*101:200/4)
s = c(s1, s2)
w = rnorm(200)
x = s + w
plot.ts(x, main="Signal Plus Gaussian White Noise")
```



(c) Compare the general appearance of the series (a) and (b) with the earthquake series and the explosion series shown in Fig. 1.7. In addition, plot (or sketch) and compare the signal modulators (a) $\exp\{-t/20\}$ and (b) $\exp\{-t/200\}$, for $t = 1, 2, \ldots, 100$.

Both series can be partitioned into two distinct phases, in a similar way to the explosion and earthquake waveforms. However, series A more resembles the explosion, while series B more resembles the earthquake. This is because series A experiences a decline in amplitude relatively quickly after the spike, while series B maintains the increase.

```
mod_a = exp(-(1:200)/20)
mod_b = exp(-(1:200)/200)
plot(mod_a, type="l", lwd=2, main="Modulators")
lines(mod_b, type="l", col=2, lwd=2)
legend(x="topright", legend=c("Mod A", "Mod B"), lty=1, col=1:2, lwd=2, cex=2)
```



1.3

1.3 (a) Generate $n = 100$ observations from the autoregression

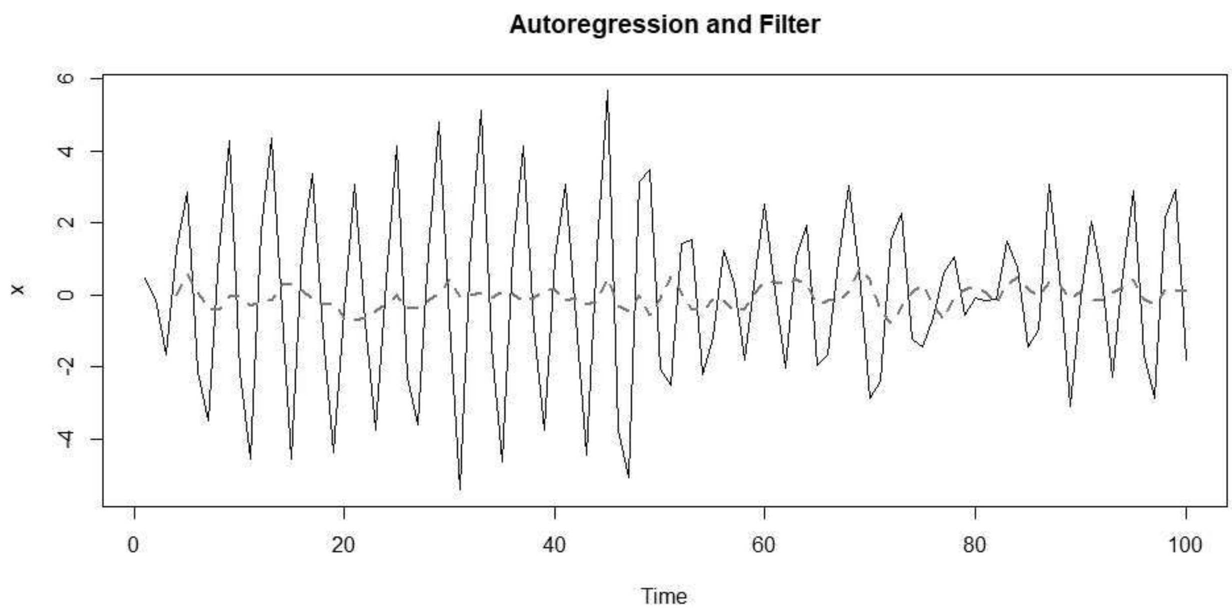
$$x_t = -.9x_{t-2} + w_t$$

with $\sigma_w = 1$, using the method described in Example 1.10. Next, apply the moving average filter

$$v_t = (x_t + x_{t-1} + x_{t-2} + x_{t-3})/4$$

to x_t, the data you generated. Now plot x_t as a line and superimpose v_t as a dashed line. Comment on the behavior of x_t and how applying the moving average filter changes that behavior. [*Hints:* Use `v = filter(x, rep(1/4, 4), sides = 1)` for the filter and note that the R code in Example 1.11 may be of help on how to add lines to existing plots.]

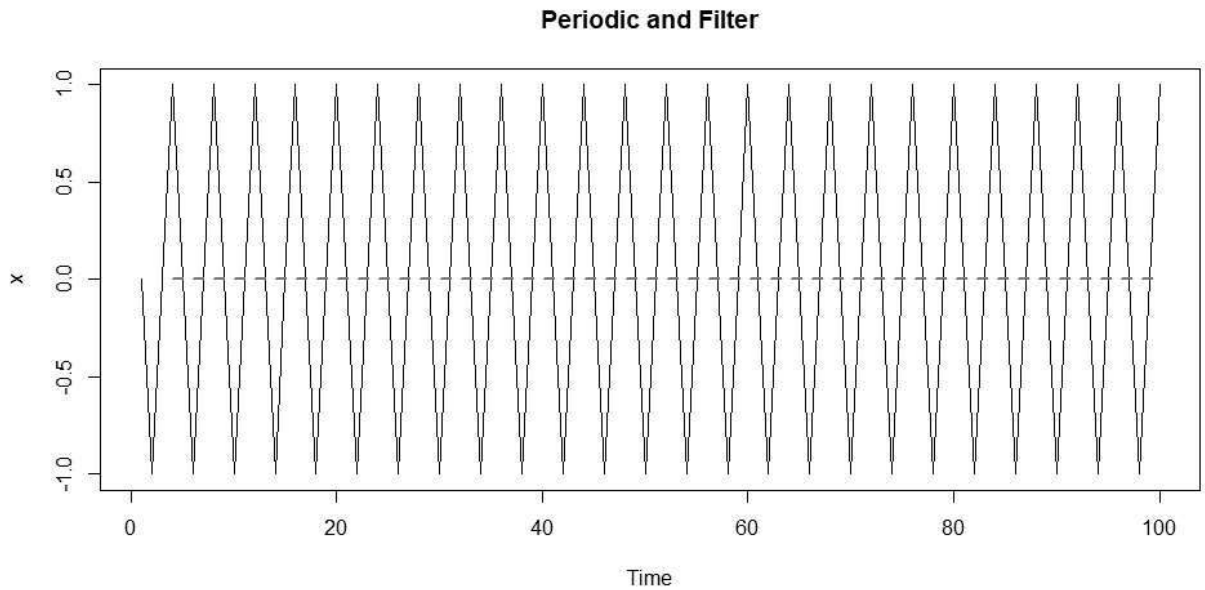
```
n = 150
w = rnorm(n,0,1)
x = filter(w, filter=c(0,-.9),method="recursive")[-(1:50)]
plot.ts(x, main='Autoregression and Filter', lwd=1)
v = filter(x, filter=rep(1/4, 4), sides=1)
lines(v, lty=2, col=2, lwd=2)
```



(b) Repeat (a) but with

$$x_t = \cos(2\pi t/4).$$

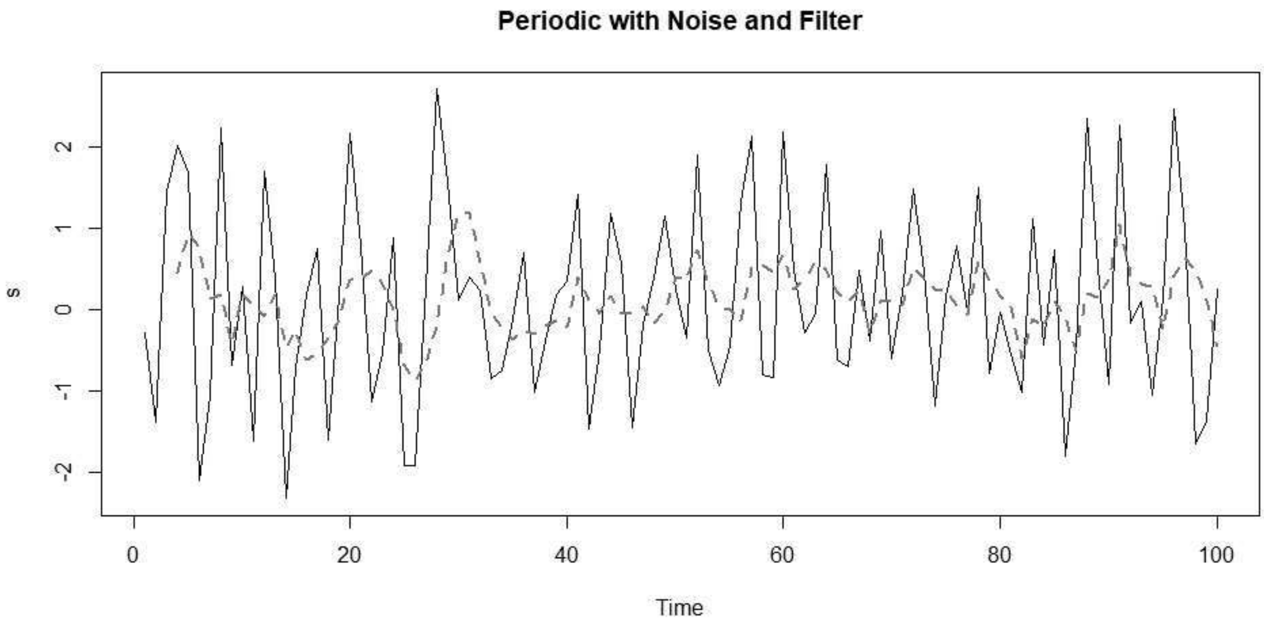
```
x = cos(2*pi*1:100/4)
plot.ts(x, main="Periodic and Filter")
v = filter(x, filter=rep(1/4, 4), sides=1)
lines(v, lty=2, col=2, lwd=2)
```



(c) Repeat (b) but with added N(0, 1) noise,

$$x_t = \cos(2\pi t/4) + w_t.$$

```
w = rnorm(100); x = cos(2*pi*1:100/4)
s = x + w
plot.ts(s, main="Periodic with Noise and Filter")
v = filter(s, filter=rep(1/4, 4), sides=1)
lines(v, lty=2, col=2, lwd=2)
```



(d) Compare and contrast (a)–(c); i.e., how does the moving average change each series.

The moving averages for all series are smoothed out versions of each series. We observe that the amplitude of the MAs is significantly less and they stay much closer to the mean, which is 0 for each series. The MA of series B appears as a flat line, when it is in fact a series of very small, yet non-zero values. This is because the underlying series is sinusoidal with no noise, and for this series in particular, every second value is 1 or -1 repeating; these cancel out leaving a small, non-zero value behind. Once noise is introduced, as in series C, the filter is no longer flat and behaves more like the filter of series A. This is because the white noise is deviating from the mean in some unpredictable way. Instead of 1 / -1 repeating, like in series B, a random difference is given, resulting in a slightly more random filter.

1.4

1.4 Show that the autocovariance function can be written as

$$\gamma(s,t) = \mathrm{E}[(x_s - \mu_s)(x_t - \mu_t)] = \mathrm{E}(x_s x_t) - \mu_s \mu_t,$$

where $\mathrm{E}[x_t] = \mu_t$.

$$
\begin{aligned}
&\mathbb{E}[(x_s-\mu_s)(x_t-\mu_t)] = \mathbb{E}[x_s x_t - x_s\mu_t - \mu_s x_t + \mu_s\mu_t] \\
&= \mathbb{E}[x_s x_t] - \mathbb{E}[x_s\mu_t] - \mathbb{E}[\mu_s x_t] + \mathbb{E}[\mu_s\mu_t] \\
&= \mathbb{E}[x_s x_t] - \mu_s\mu_t - \mu_s\mu_t + \mu_s\mu_t \\
&= \mathbb{E}[x_s x_t] - \mu_s\mu_t
\end{aligned}
$$

1.5

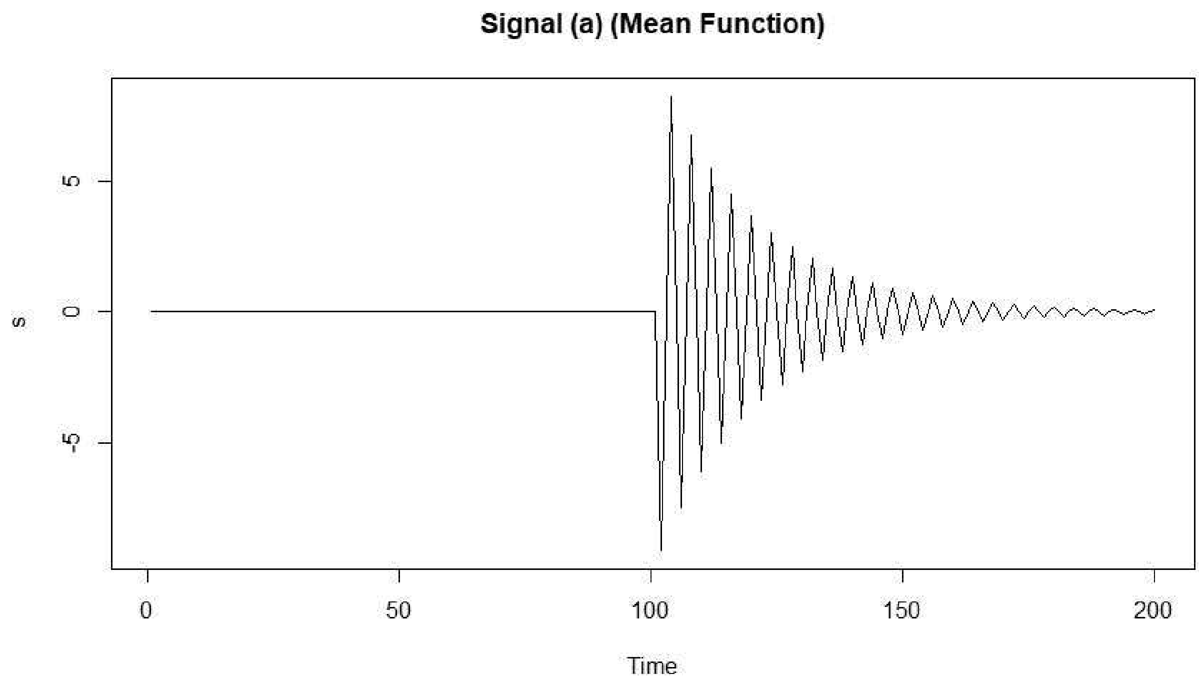
1.5 For the two series, x_t, in Problem 1.2 (a) and (b):

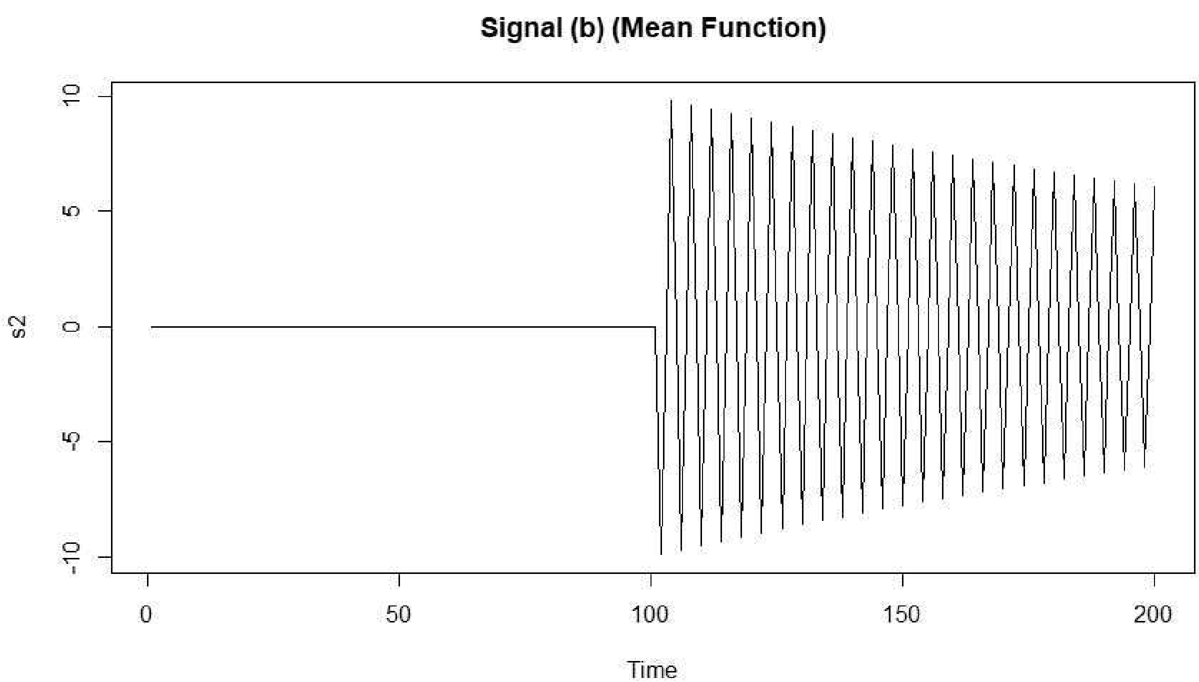
(a) Compute and plot the mean functions $\mu_x(t)$, for $t = 1, \ldots, 200$.

Both mean functions are calculated as follows:

$$
\begin{aligned}
&x_t = s_t + w_t \\
&\mathbb{E}[x_t] = \mathbb{E}[s_t + w_t] \\
&= \mathbb{E}[s_t] + \mathbb{E}[w_t] = s_t
\end{aligned}
$$

Where the mean function is just the underlying signal.





(b) Calculate the autocovariance functions, $\gamma_x(s,t)$, for $s, t = 1, \ldots, 200$.

$$
\begin{aligned}
x_t &= s_t + w_t \\
x_s &= s_s + w_s \\
\mathbb{E}[x_t] &= s_t \\
\mathbb{E}[x_s] &= s_s \\
\mathbb{E}[(x_s - \mu_s)(x_t - \mu_t)] &= \mathbb{E}[(s_s + w_s - s_s)(s_t + w_t - s_t)] \\
&= \mathbb{E}[w_s w_t]
\end{aligned}
$$

Above is 1 when $s = t$ and 0 otherwise.

1.6

1.6 Consider the time series

$$x_t = \beta_1 + \beta_2 t + w_t,$$

where β_1 and β_2 are known constants and w_t is a white noise process with variance σ_w^2.

(a) Determine whether x_t is stationary.

The mean of x_t is given by:

$$\mathbb{E}[x_t] = \beta_1 + \beta_2 t$$

Which clearly depends on time and, hence, the series is not stationary. Note, the autocovariance does not depend on time.

$$
\begin{aligned}
\mathbb{E}[x_s] &= \beta_1 + \beta_2 s \\
\mathbb{E}[(x_t - \mu_t)(x_s - \mu_s)] &= \mathbb{E}[(\beta_1 + \beta_2 t + w_t - \beta_1 + \beta_2 t)(\beta_1 + \beta_2 s + w_t - \beta_1 + \beta_2 s)] \\
&= \mathbb{E}[w_t w_s]
\end{aligned}
$$

As the above is 1 when $t = s$ and 0 otherwise. The series could be considered trend stationary.

(b) Show that the process $y_t = x_t - x_{t-1}$ is stationary.

$$
\begin{aligned}
x_t &= \beta_1 + \beta_2 t + w_t \\
x_{t-1} &= \beta_{t-1} + \beta_2 t - 1 + w_{t-1} \\
Y_t &= x_t - x_{t-1} = \beta_1 + \beta_2 t + w_t - \beta_{t-1} - \beta_2 t - 1 - w_{t-1} \\
&= \beta_2 + w_t - w_{t-1}
\end{aligned}
$$

Hence, the mean is given by

$\mathbb{E}[y_t] = \beta_2$

And the autocovariance

$\mathbb{E}[(y_t - \mu_t)(y_s - \mu_s)] = \mathbb{E}[(w_t - w_{t-1})(w_s - w_{s-1})]$

Which clearly do not depend on time.

(c) Show that the mean of the moving average

$$v_t = \frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}$$

is $\beta_1 + \beta_2 t$, and give a simplified expression for the autocovariance function.

$$\mathbb{E}[v_t] = \mathbb{E}[\frac{1}{2q+1} \sum_{j=-q}^{q} x_{t-j}]$$
$$= \frac{1}{2q+1} \sum_{j=-q}^{q} \mathbb{E}[x_{t-j}]$$
$$= \frac{1}{2q+1} \sum_{j=-q}^{q} \beta_1 + \beta_2(t-j)$$
$$= \frac{1}{2q+1} (\sum_{j=-q}^{q} \beta_1 + \beta_2 t - \sum_{j=-q}^{q} \beta_2 j)$$
$$= \frac{2q+1}{2q+1} (\beta_1 + \beta_2 t) = \beta_1 + \beta_2 t$$

1.7

1.7 For a moving average process of the form

$$x_t = w_{t-1} + 2w_t + w_{t+1},$$

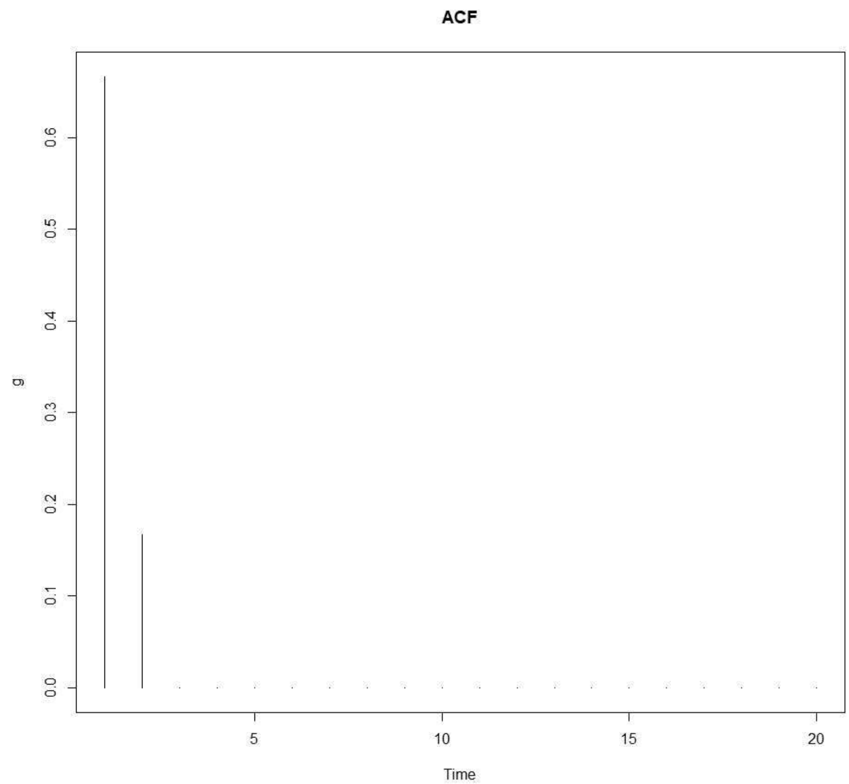
where w_t are independent with zero means and variance σ_w^2, determine the autocovariance and autocorrelation functions as a function of lag $h = s - t$ and plot the ACF as a function of h.

$$\mathbb{E}[x_t] = 0$$
$$\gamma(h) = \mathbb{E}[x_t x_{t+h}] = \mathbb{E}[(w_{t-1} + 2w_t + w_{t+h})(w_{t-1} + 2w_t + w_{t+h})]$$

Multiplying this out, we will see:

$$\gamma(h) = \begin{cases} 6\sigma_w^2 & h = 0 \\ 4\sigma_w^2 & \text{h=}\pm 1 \\ \sigma_w^2 & \text{h=}\pm 2 \\ 0 & |h|\text{>}2. \end{cases}$$

```
acf <- function(h) {
  if (h == 0) {
    return(1)
  } else if (h == 1 || h == -1) {
    return(4/6)
  } else if (h == 2 || h == -2) {
    return(1/6)
  } else {
    return(0)
  }
}
x = 1:20
g = lapply(x, acf)
plot.ts(g, type="h", main="ACF")
```



1.8

1.8 Consider the random walk with drift model

$$x_t = \delta + x_{t-1} + w_t,$$

for $t = 1, 2, \ldots$, with $x_0 = 0$, where w_t is white noise with variance σ_w^2.

(a) Show that the model can be written as $x_t = \delta t + \sum_{k=1}^{t} w_k$.

Prove by induction, where the base case is $t = 1$.

$$x_1 = \delta + x_0 + w_1 = \delta + w_1 = \delta t + \sum_{k=1}^{t} w_k$$

Then, by assumption,

$$x_{t+1} = \delta + x_t + w_{t+1} = \delta + \delta t + \sum_{k=1}^{t} w_k + w_t$$

$$= \delta(t+1) + \sum_{k=1}^{t+1} w_k$$

Which completes the proof.

(b) Find the mean function and the autocovariance function of x_t.

$$\mathbb{E}[x_t] = \mathbb{E}[\delta t + \sum_{k=1}^{t} w_k] = \delta t$$

$$\gamma(h) = \mathbb{E}[(\delta t + \sum_{k=1}^{t} w_k - \delta t)(\delta(t+h) + \sum_{k=1}^{t+h} w_k - \delta(t+h))]$$

$$\mathbb{E}[\sum_{j=1}^{t}\sum_{k=1}^{t+h} w_j w_k], \quad k \neq j \;\; w_k w_j = 0$$

$$= \sum_{k=1}^{t} \mathbb{E}[w_k^2] = t\sigma_w^2$$

(c) Argue that x_t is not stationary.

The mean function is not constant, as it depends on time. The autocovariance function also depends on time and not through the difference, h.

(d) Show $\rho_x(t-1,t) = \sqrt{\frac{t-1}{t}} \to 1$ as $t \to \infty$. What is the implication of this result?

$$\gamma(t-1,t) = (t-1)\sigma_w^2$$
$$\gamma(t-1,t-1) = (t-1)\sigma_w^2$$
$$\gamma(t,t) = t\sigma_w^2$$
$$\rho(t-1,t) = \frac{\gamma(t-1,t)}{\sqrt{\gamma(t-1,t-1)\gamma(t,t)}} = \frac{(t-1)\sigma_w^2}{\sqrt{(t-1)\sigma_w^2 t\sigma_w^2}} = \sqrt{\frac{t-1}{t}}$$

The implication is that as t increases, the correlation between adjacent lags increases. This means that the series is changing slowly.

(e) Suggest a transformation to make the series stationary, and prove that the transformed series is stationary. (Hint: See Problem 1.6b.)

$$y_t = x_t - x_{t-1}$$
$$\mathbb{E}[y_t] = \delta + w_t$$
$$\gamma(h) = \mathbb{E}[(\delta + w_t - \delta)(\delta + w_{t+h} - \delta)] = \mathbb{E}[w_t w_{t+h}]$$

Differencing leads to a stationary series.

1.9

1.9 A time series with a periodic component can be constructed from

$$x_t = U_1 \sin(2\pi\omega_0 t) + U_2 \cos(2\pi\omega_0 t),$$

where U_1 and U_2 are independent random variables with zero means and $E(U_1^2) = E(U_2^2) = \sigma^2$. The constant ω_0 determines the period or time it takes the process to make one complete cycle. Show that this series is weakly stationary with autocovariance function

$$\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h).$$

$$\gamma = \mathbb{E}[(U_1 sin(2\pi w_o t) + U_2 cos(2\pi w_o t))(U_1 sin(2\pi w_o (t+h)) + U_2 cos(2\pi w_o (t+h))]$$

$$\begin{aligned}
&= \mathbb{E}[U_1^2 sin(2\pi w_o t) sin(2\pi w_o (t+h)) + U_1 U_2 sin(2\pi w_o t) cos(2\pi w_o (t+h)) \\
&+ U_2 U_1 cos(2\pi w_o t) sin(2\pi w_o (t+h)) + U_2^2 cos(2\pi w_o t) cos(2\pi w_o (t+h))] \\
&= \mathbb{E}[U_1^2 \frac{cos(\theta - \phi) - cos(\theta + \phi)}{2} + U_1 U_2 \frac{sin(\theta + \phi) + sin(\theta + \phi)}{2} \\
&+ U_2 U_1 \frac{sin(\theta + \phi) - sin(\theta + \phi)}{2} + U_2^2 \frac{cos(\theta + \phi) + cos(\theta + \phi)}{2}] \\
&= \sigma^2 cos(\theta - \phi) + \mathbb{E}[U_2 U_1] sin(\theta + \phi) = \sigma^2 cos(2\pi w_o(-h)) = \sigma^2 cos(2\pi w_o h)
\end{aligned}$$

Where we have defined $\theta = 2\pi w_o t$ and $\phi = 2\pi w_o (t+h)$ and used the trig identities:

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

1.10

1.10 Suppose we would like to predict a single stationary series x_t with zero mean and autocorrelation function $\gamma(h)$ at some time in the future, say, $t+\ell$, for $\ell > 0$.

(a) If we predict using only x_t and some scale multiplier A, show that the mean-square prediction error

$$MSE(A) = \mathrm{E}[(x_{t+\ell} - Ax_t)^2]$$

is minimized by the value

$$A = \rho(\ell).$$

$$MSE(A) = \mathbb{E}[(x_{t+l} - Ax_t)^2] = \mathbb{E}[x_{t+l}^2 - 2Ax_t x_{t+l} + A^2 x_t^2]$$
$$= A^2\mathbb{E}[x_{t+l}^2] - 2A\mathbb{E}[x_t x_{t+l}] + -\mathbb{E}[x_{t+l}^2]$$
$$= A^2\gamma(0) - 2A\gamma(l) + -\gamma(0)$$

Taking the derivative with respect to A and setting it to 0 gives

$$A = \frac{\gamma(l)}{\gamma(0)} = \rho(l)$$

(b) Show that the minimum mean-square prediction error is

$$MSE(A) = \gamma(0)[1 - \rho^2(\ell)].$$

$$MSE(A) = \rho(l)^{22}\gamma(0) - 2\rho(l)\gamma(l) + \gamma(0) = (\rho(l)^2 - \frac{2\rho(l)\gamma(l)}{\gamma(0)} + 1)\gamma(0)$$
$$= (\rho(l)(\rho(l) - \frac{2\rho(l)}{\rho(0)}) + 1)\gamma(0) = (1 - \rho^2(l))\gamma(0)$$

(c) Show that if $x_{t+\ell} = Ax_t$, then $\rho(\ell) = 1$ if $A > 0$, and $\rho(\ell) = -1$ if $A < 0$.

$$MSE(A) = \mathbb{E}[(Ax_t - Ax_t)^2] = 0 = \gamma(0)[1 - \rho^2(l)]$$
$$\rho^2(l) = 1, \ \ \rho(l) = +/-1$$

Since $A = \rho(l)$, the claim follows.

1.11

1.11 Consider the linear process defined in (1.31).

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{1.31}$$

$$\gamma_x(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j \tag{1.32}$$

(a) Verify that the autocovariance function of the process is given by (1.32). Use the result to verify your answer to Problem 1.7. *Hint:* For $h \geq 0$, $\text{cov}(x_{t+h}, x_t) = \text{cov}(\sum_k \psi_k w_{t+h-k}, \sum_j \psi_j w_{t-j})$. For each $j \in \mathbb{Z}$, the only "survivor" will be when $k = h + j$.

$$\gamma(h) = \mathbb{E}[\sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \sum_{k=-\infty}^{\infty} \psi_k w_{t-k}] = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j$$

Since $w_{t-j} = w_{t+h-k}$ when $k = h + j$

For Problem 1.7, we set $\psi_{-1} = 1, \ \ \psi_0 = 2 \ \ \psi_1 = 1,$ and $\phi_j = 0$ for all other j. This will produce the desired result.

(b) Show that x_t exists as a limit in mean square (see Appendix A).

Similar proof can be found in Example A.2.

1.12

1.12 For two weakly stationary series x_t and y_t, verify (1.30).

$$\rho_{xy}(h) = \rho_{yx}(-h), \tag{1.30}$$

$$\gamma_{xy}(h) = \text{E}[(x_{t+h} - \mu_x)(y_t - \mu_y)] = \text{E}[(y_t - \mu_y)(x_{t+h} - \mu_x)] = \gamma_{yx}(-h)$$

1.13

1.13 Consider the two series

$$x_t = w_t$$

$$y_t = w_t - \theta w_{t-1} + u_t,$$

where w_t and u_t are independent white noise series with variances σ_w^2 and σ_u^2, respectively, and θ is an unspecified constant.

(a) Express the ACF, $\rho_y(h)$, for $h = 0, \pm1, \pm2, \dots$ of the series y_t as a function of σ_w^2, σ_u^2, and θ.

$\gamma_y(h) = \mathbb{E}[y_t y_{t+h}] = \mathbb{E}[(w_t - \theta w_{t-1} + u_t)(w_{t+h} - \theta w_{t+h-1} + u_{t+h})]$

Multiplying this out, we see that:

$$\gamma_y(h) = \begin{cases} \sigma_w^2(1+\theta^2) + \sigma_u^2 & h = 0 \\ -\theta\sigma_w^2 & h = \pm 1 \\ 0 & |h| > 1. \end{cases}$$

And the $\rho(h)$ are calculated accordingly.

(b) Determine the CCF, $\rho_{xy}(h)$ relating x_t and y_t.

$\gamma_{xy}(h) = \mathbb{E}[x_{t+h} y_t] = \mathbb{E}[w_t w_{t+h} - \theta w_{t-1} w_{t+h} + u_t w_{t+h}]$

Multiplying this out, we see:

$$\gamma_{xy}(h) = \begin{cases} \sigma_w^2 & h = 0 \\ -\theta\sigma_w^2 & h = -1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\gamma_x(h) = \begin{cases} \sigma_w^2 & h = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0)\gamma_y(0)}}.$$

(c) Show that x_t and y_t are jointly stationary.

Both x_t and y_t are stationary, and the cross-covariance function is a function only of h.

1.14

1.14 Let x_t be a stationary normal process with mean μ_x and autocovariance function $\gamma(h)$. Define the nonlinear time series

$$y_t = \exp\{x_t\}.$$

(a) Express the mean function $\mathrm{E}(y_t)$ in terms of μ_x and $\gamma(0)$. The moment generating function of a normal random variable x with mean μ and variance σ^2 is

$$M_x(\lambda) = \mathrm{E}[\exp\{\lambda x\}] = \exp\left\{\mu\lambda + \frac{1}{2}\sigma^2\lambda^2\right\}.$$

$$\mathrm{E}(y_t) = \mathrm{E}(\exp\{x_t\}) = \exp\left\{\mu_x + \frac{1}{2}\gamma_x(0)\right\}$$

(b) Determine the autocovariance function of y_t. The sum of the two normal random variables $x_{t+h} + x_t$ is still a normal random variable.

$$x_t' = x_t + x_{t+h}$$
$$\mathbb{E}[x_t'] = 2\mu_x$$
$$\gamma_{x_t'}(k) = \mathbb{E}[(x_t + x_{t+h} - 2\mu_x)(x_{t+k} + x_{t+h+k} - 2\mu_x)]$$
$$\gamma_{x_t'}(0) = 2(\gamma_x(0) + \gamma_x(h))$$

$$\begin{aligned}\gamma_y(h) &= \mathrm{E}(y_{t+h}y_t) - \mathrm{E}(y_{t+h})\mathrm{E}(y_t)\\ &= \exp\{2\mu_x + \gamma_x(0) + \gamma_x(h)\} - \left(\exp\{\mu_x + \frac{1}{2}\gamma_x(0)\}\right)^2\\ &= \exp\{2\mu_x + \gamma_x(0)\}\left(\exp\{\gamma_x(h)\} - 1\right).\end{aligned}$$

1.15

1.15 Let w_t, for $t = 0, \pm1, \pm2, \ldots$ be a normal white noise process, and consider the series

$$x_t = w_t w_{t-1}.$$

Determine the mean and autocovariance function of x_t, and state whether it is stationary.

$$\mu_{x,t} = \mathrm{E}(x_t) = \mathrm{E}(w_t w_{t-1}) = \mathrm{E}(w_t)\mathrm{E}(w_{t-1}) = 0;$$

$$\gamma_x(0) = \mathrm{E}(w_t w_{t-1} w_t w_{t-1}) = \mathrm{E}(w_t^2)\mathrm{E}(w_{t-1}^2) = \sigma_w^2\sigma_w^2 = \sigma_w^4$$

Similarly, $\gamma_x(h) = 0$ for $|h| \geq 1$.
This series is white noise and hence stationary.

1.16

1.16 Consider the series

$$x_t = \sin(2\pi Ut),$$

$t = 1, 2, \ldots$, where U has a uniform distribution on the interval $(0, 1)$.

(a) Prove x_t is weakly stationary.

$$\mathrm{E}(x_t) = \int_0^1 \sin(2\pi ut)du = -\frac{1}{2\pi t}\cos(2\pi ut)\Big|_0^1 = -\frac{1}{2\pi t}\left[\cos(2\pi t) - 1\right]$$

For every integer t, this function equals 0.
…

(b) Prove x_t is not strictly stationary.
See solutions manual.

1.17

1.17 Suppose we have the linear process x_t generated by

$$x_t = w_t - \theta w_{t-1},$$

$t = 0, 1, 2, \ldots$, where $\{w_t\}$ is independent and identically distributed with characteristic function $\phi_w(\cdot)$, and θ is a fixed constant. [Replace "characteristic function" with "moment generating function" if instructed to do so.]

(a) Express the joint characteristic function of $x_1, x_2, \ldots, x_n$, say,

$$\phi_{x_1, x_2, \ldots, x_n}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

in terms of $\phi_w(\cdot)$.

See solutions manual.

(b) Deduce from (a) that x_t is strictly stationary.

See solutions manual.

1.18

1.18 Suppose that x_t is a linear process of the form (1.31). Prove

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty.$$

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty. \tag{1.31}$$

Substituting the below from Problem 1.11 into the required sum:

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j$$

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| = \sigma_w^2 \sum_{h=-\infty}^{\infty} \Big| \sum_{j=-\infty}^{\infty} \psi_{j+h}\psi_j \Big| \le \sigma_w^2 \sum_{h=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |\psi_{j+h}||\psi_j|$$

$$= \sigma_w^2 \sum_{k=-\infty}^{\infty} |\psi_k| \sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

1.19

1.19 Suppose $x_t = \mu + w_t + \theta w_{t-1}$, where $w_t \sim wn(0, \sigma_w^2)$.

(a) Show that mean function is $\mathrm{E}(x_t) = \mu$.

$$\mathrm{E}(x_t) = \mathrm{E}(\mu + w_t + \theta w_{t-1}) = \mu.$$

(b) Show that the autocovariance function of x_t is given by $\gamma_x(0) = \sigma_w^2(1 + \theta^2)$, $\gamma_x(\pm 1) = \sigma_w^2 \theta$, and $\gamma_x(h) = 0$ otherwise.

$\gamma(h) = \mathbb{E}[(w_t + \theta w_{t-1})(w_{t+h} + \theta w_{t+h-1})]$

Multiplying this out, we see:

$\gamma_x(0) = (1 + \theta^2)\sigma_w^2$, $\gamma_x(\pm 1) = \theta\sigma_w^2$, and 0 otherwise.

(c) Show that x_t is stationary for all values of $\theta \in \mathbb{R}$.

From (a) and (b) we see that, for any θ, both the mean function and autocovariance function are independent of time.

(d) Use (1.35) to calculate $\mathrm{var}(\bar{x})$ for estimating μ when (i) $\theta = 1$, (ii) $\theta = 0$, and (iii) $\theta = -1$

$\gamma_x(h)$ is 0 for all values except $h = 0, +/-1$. Hence, (1.35) becomes:

$$\text{var}(\bar{x}) = \frac{1}{n}\left[\gamma_x(0) + \frac{2(n-1)}{n}\gamma_x(1)\right]$$

$var(\bar{x}) = \frac{\sigma_w^2}{n}[4 - \frac{2}{n}]$ when $\theta = 1$.

$\text{var}(\bar{x}) = \sigma_w^2/n$. when $\theta = 0$. The variance is four times smaller when uncorrelated than in the above case.

$var(\bar{x}) = \frac{2\sigma_w^2}{n^2}$ when $\theta = -1$. The variance is smaller than in the uncorrelated case.

(e) In time series, the sample size n is typically large, so that $\frac{(n-1)}{n} \approx 1$. With this as a consideration, comment on the results of part (d); in particular, how does the accuracy in the estimate of the mean μ change for the three different cases?

Conclusions regarding variance stay similar.

1.20

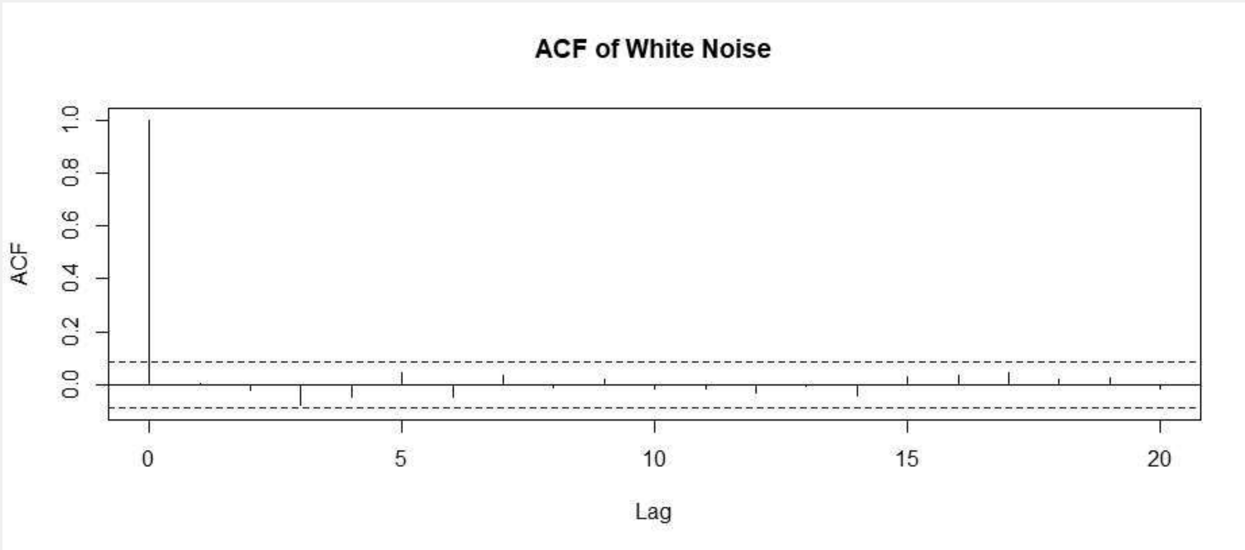
1.20 (a) Simulate a series of $n = 500$ Gaussian white noise observations as in Example 1.8 and compute the sample ACF, $\hat{\rho}(h)$, to lag 20. Compare the sample ACF you obtain to the actual ACF, $\rho(h)$. [Recall Example 1.19.]

The theoretical ACF is given by dividing below values by σ_w^2:

$$\gamma_w(h) = \text{cov}(w_{t+h}, w_t) = \begin{cases} \sigma_w^2 & h = 0, \\ 0 & h \neq 0. \end{cases}$$

```
n = 500
w = rnorm(n)
acf(w, main="ACF of White Noise", type="correlation", lag.max=20)
```

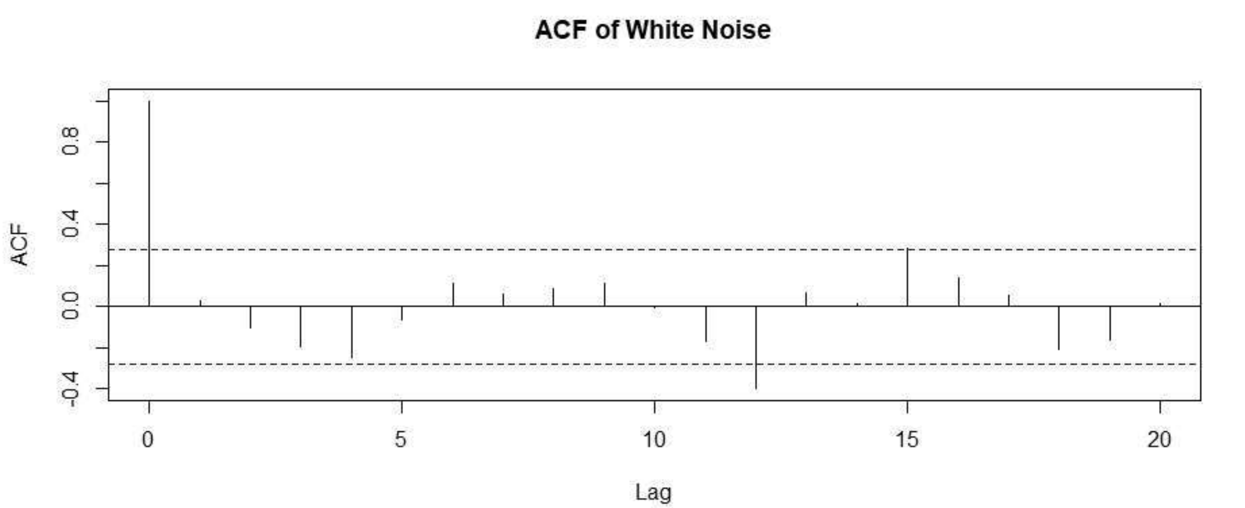
And the actual ACF of this realisation is given by:



We can see that after the first lag, there are non-zero ACF values. However, they remain below 2 standard deviations and hence are insignificant.

(b) Repeat part (a) using only $n = 50$. How does changing n affect the results?

```
n = 50
w = rnorm(n)
acf(w, main="ACF of White Noise", type="correlation", lag.max=20)
```



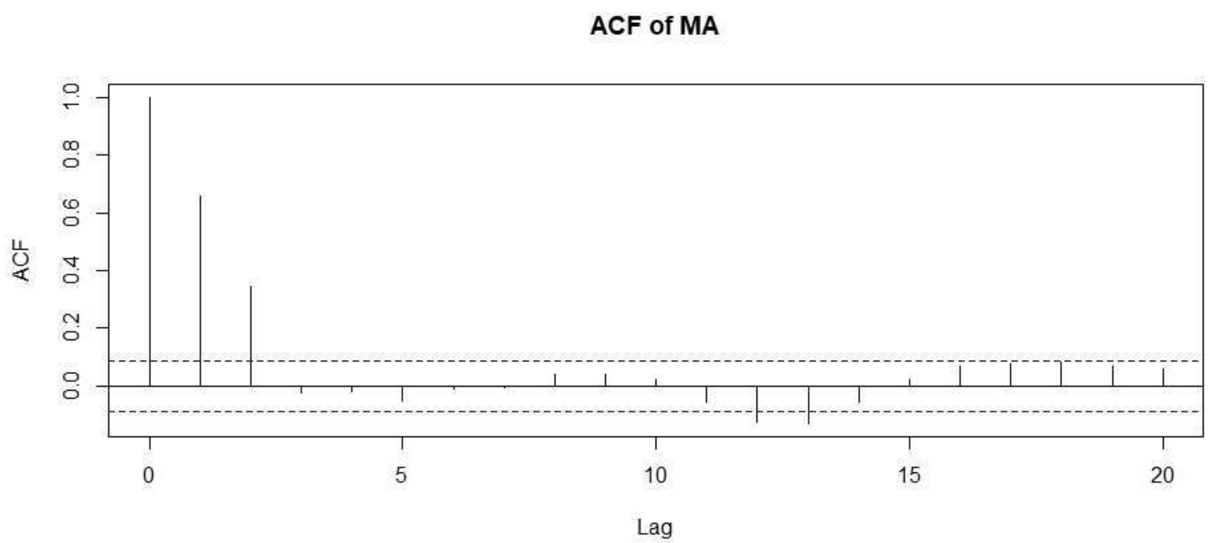
The bounds are larger and we see that some values penetrate the significance level.

1.21

1.21 (a) Simulate a series of $n = 500$ moving average observations as in Example 1.9 and compute the sample ACF, $\hat{\rho}(h)$, to lag 20. Compare the sample ACF you obtain to the actual ACF, $\rho(h)$. [Recall Example 1.20.]

$$\rho_v(h) = \begin{cases} 1 & h = 0, \\ \frac{2}{3} & h = \pm 1, \\ \frac{1}{3} & h = \pm 2, \\ 0 & |h| > 2. \end{cases}$$

```
n = 500
w = rnorm(n)
x = filter(w, sides=2, filter=rep(1/3,3))
acf(x, main="ACF of MA", type="correlation", lag.max=20, na.action = na.pass)
```

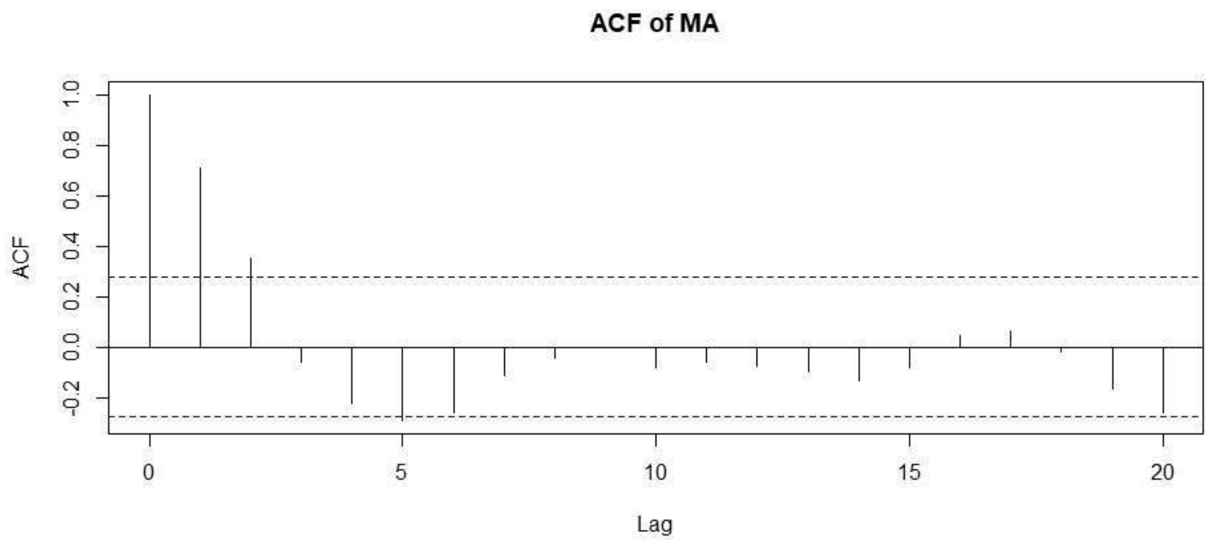


We see that the sample ACF closely reflects the theoretical ACF, as lags 2 and 3 are approximately ⅔ and ⅓, respectively. We do see non-zero values for lags greater than 2, which is contradictory to the actual ACF, however, these are mostly insignificant. One or two values penetrate the significant level; a larger sample size is probably required to avoid this.

(b) Repeat part (a) using only $n = 50$. How does changing n affect the results?

```
n = 50
w = rnorm(n)
x = filter(w, sides=2, filter=rep(1/3,3))
```

```
acf(x, main="ACF of MA", type="correlation", lag.max=20, na.action =
na.pass)
```



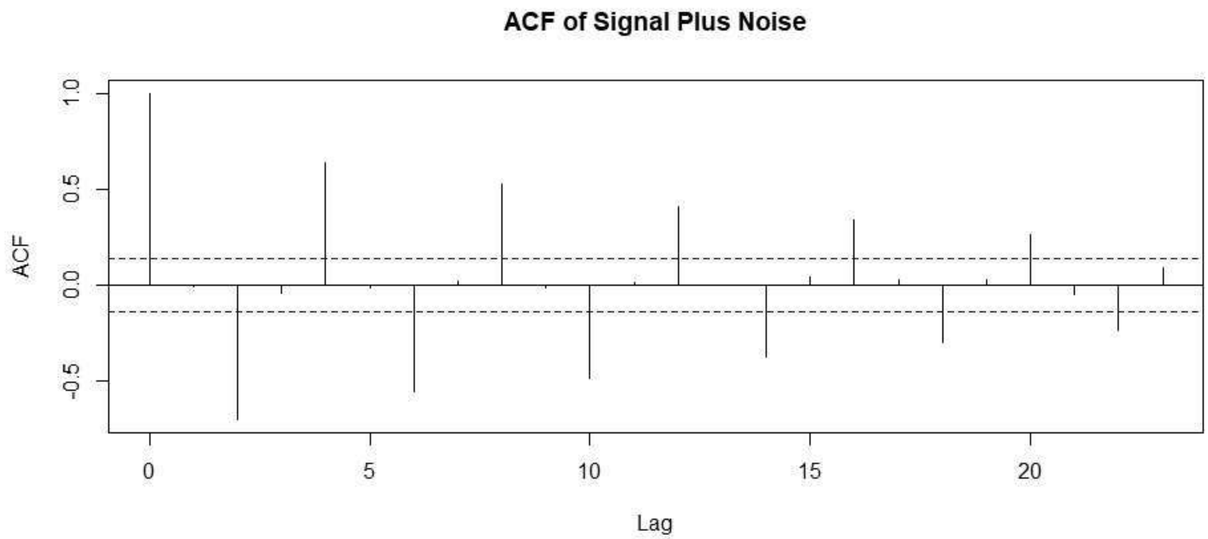
We observe similar results, however, lags greater than 2 have larger non-zero values.

1.22

1.22 Although the model in Problem 1.2(a) is not stationary (Why?), the sample ACF can be informative. For the data you generated in that problem, calculate and plot the sample ACF, and then comment.

The model is not stationary because the mean is not constant, instead it depends on time.

```
#Same code from Problem 1.2(a)
acf(x, main="ACF of Signal Plus Noise")
```

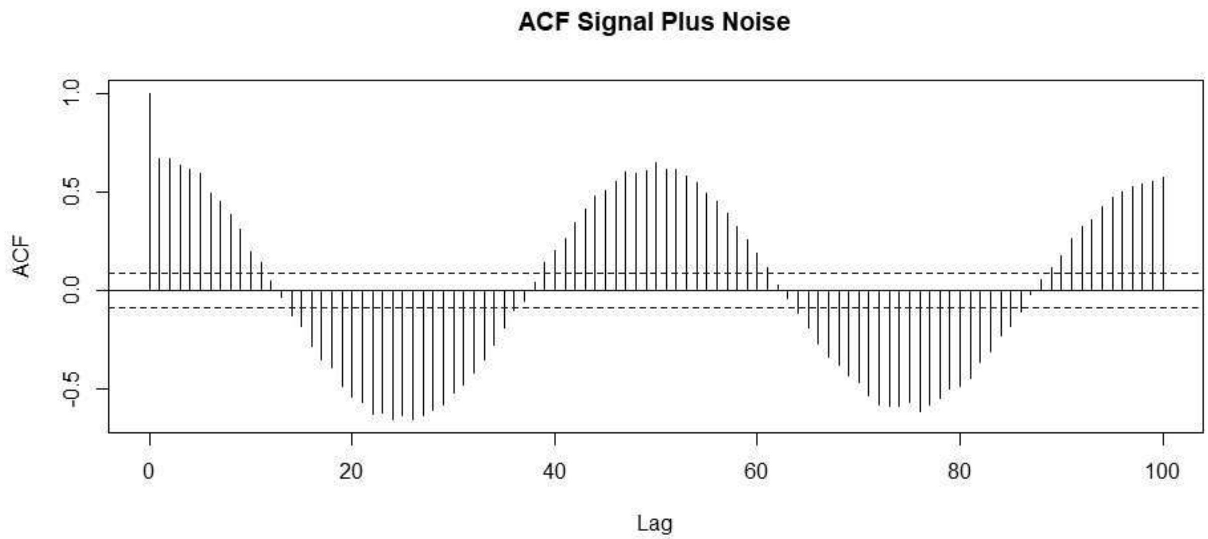


The ACF has statistically significant lags up to beyond lag 20. The height of each is slowly declining as the lags increase. In previous ACFs, where the underlying model is stationary, we observe rather that any correlation stops after the first few lags. Here, the correlation is decreasing at a steady rate. We can also see that significance is only occurring every second lag and that every second significant lag itself is negative. So, there must be some cyclic behaviour going on in the underlying series.

1.23

1.23 Simulate a series of $n = 500$ observations from the signal-plus-noise model presented in Example 1.12 with $\sigma_w^2 = 1$. Compute the sample ACF to lag 100 of the data you generated and comment.

```
n = 500; w = rnorm(n)
s = 2*cos(2*pi*(1:n + 15)/50)
x = s + w
acf(x, main="ACF Signal Plus Noise", lag.max=100)
```



The ACF is showing cyclic behaviour, reaching a full cycle around lag 50, which makes sense given that the frequency of the model is 50.

1.24

1.24 For the time series y_t described in Example 1.26, verify the stated result that $\rho_y(1) = -.47$ and $\rho_y(h) = 0$ for $h > 1$.

Use results from Problem 1.19, given that x_t has zero mean and the variables are independent.

$\gamma_x(0) = (1+\theta^2)\sigma_w^2$, $\gamma_x(\pm 1) = \theta\sigma_w^2$, and 0 otherwise.

1.25

1.25 A real-valued function $g(t)$, defined on the integers, is non-negative definite if and only if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i g(t_i - t_j) a_j \geq 0$$

for all positive integers n and for all vectors $a = (a_1, a_2, \ldots, a_n)'$ and $t = (t_1, t_2, \ldots, t_n)'$. For the matrix $G = \{g(t_i - t_j);\ i, j = 1, 2, \ldots, n\}$, this implies that $a'Ga \geq 0$ for all vectors a. It is called positive definite if we can replace '$\geq$' with '$>$' for all $a \neq 0$, the zero vector.

(a) Prove that $\gamma(h)$, the autocovariance function of a stationary process, is a non-negative definite function.

The variance is always non-negative. Hence,

$$var(\sum_{s=1}^{n} a_s x_s) = cov(\sum_{s=1}^{n} a_s x_s, \sum_{t=1}^{n} a_t x_t)$$

$$= \mathbb{E}[(\sum_{s=1}^{n} a_s x_s - \mu_x \sum_{s=1}^{n} a_s)(\sum_{t=1}^{n} a_t x_t - \mu_x \sum_{t=1}^{n} a_t)] = \mathbb{E}[\sum_{s=1}^{n}\sum_{t=1}^{n}(a_s a_t x_s x_t - a_s a_t \mu_x^2)]$$

$$= \sum_{s=1}^{n}\sum_{t=1}^{n}(a_s \gamma(s-t) a_t) \geq 0$$

(b) Verify that the sample autocovariance $\hat{\gamma}(h)$ is a non-negative definite function.

Let $Y_t = x_t - \bar{x}$ for $t = 1...n\, t = 1...n$

$$D = \begin{pmatrix} 0 & 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_n \\ 0 & \cdots & 0 & Y_1 & Y_2 & \cdots & Y_n & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & Y_1 & Y_2 & \cdots & Y_n & 0 & \cdots & 0 \end{pmatrix}$$

$$\hat{\Gamma}_n = \{\hat{\gamma}(s-t)\}_{s,t=1}^{n} = \frac{1}{n} DD'.$$

$$\boldsymbol{a}'\hat{\Gamma}_n\boldsymbol{a} = \frac{1}{n}\boldsymbol{a}'DD'\boldsymbol{a} = \frac{1}{n}\boldsymbol{c}'\boldsymbol{c} = \sum_{i=1}^{n} c_i^2 \geq 0$$

Since $\hat{\Gamma}_n$ is symmetric, it will be positive definite if its eigenvalues, $\hat{\gamma}(0)$, are positive, which requires a non-zero sample variance.

1.26

1.26 Consider a collection of time series $x_{1t}, x_{2t}, \ldots, x_{Nt}$ that are observing some common signal μ_t observed in noise processes $e_{1t}, e_{2t}, \ldots, e_{Nt}$, with a model for the j-th observed series given by

$$x_{jt} = \mu_t + e_{jt}.$$

Suppose the noise series have zero means and are uncorrelated for different j. The common autocovariance functions of all series are given by $\gamma_e(s,t)$. Define the sample mean

$$\bar{x}_t = \frac{1}{N}\sum_{j=1}^{N} x_{jt}.$$

(a) Show that $E[\bar{x}_t] = \mu_t$.

$$\mathrm{E}\bar{x}_t = \frac{1}{N}\sum_{j=1}^{N} x_{jt} = \frac{1}{N}\sum_{j=1}^{n} \mu_t = \frac{N\mu_t}{N} = \mu_t$$

(b) Show that $\mathrm{E}[(\bar{x}_t - \mu)^2)] = N^{-1}\gamma_e(t,t)$.

$$\mathbb{E}[(\bar{x}_t - \mu_t)^2] = \mathbb{E}[\bar{x}_t\bar{x}_t] - \mu_t^2 = \mathbb{E}[\frac{1}{N}\sum_{j=1}^{N} x_{jt}\sum_{k=1}^{N} x_{kt}] - \mu_t^2$$
$$= \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}\mathbb{E}[x_{jt}x_{kt}] - (\frac{1}{N}\sum_{j=1}^{N}\mu_t)^2 = \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}(\mathbb{E}[x_{jt}x_{kt} - \mu_t^2])$$
$$= \frac{1}{N^2}\sum_{j=1}^{N}\sum_{k=1}^{N}(\mathbb{E}[e_{jt}e_{kt}]) = N^{-1}\gamma_e(t,t)$$

(c) How can we use the results in estimating the common signal?

As the number of series with the same underlying signal increases, the variance decreases.

1.27

1.27 A concept used in *geostatistics*, see Journel and Huijbregts [109] or Cressie [45], is that of the *variogram*, defined for a spatial process x_s, $s = (s_1, s_2)$, for $s_1, s_2 = 0, \pm1, \pm2, \ldots$, as

$$V_x(h) = \frac{1}{2}\mathrm{E}[(x_{s+h} - x_s)^2],$$

where $h = (h_1, h_2)$, for $h_1, h_2 = 0, \pm1, \pm2, \ldots$ Show that, for a stationary process, the variogram and autocovariance functions can be related through

$$V_x(h) = \gamma(0) - \gamma(h),$$

where $\gamma(h)$ is the usual lag h covariance function and $0 = (0, 0)$. Note the easy extension to any spatial dimension.

$$V_x(h) = \frac{1}{2}\mathbb{E}[(x_{s+h} - x_s)^2] = \frac{1}{2}\mathbb{E}[x_{s+h}^2 - x_{s+h}x_s - x_{s+h}x_s + x_s^2]$$
$$= \frac{1}{2}(\gamma(0) + \mu_x^2 - \gamma(h) - \mu_x^2 - \gamma(h) - \mu_x^2 + \gamma(0) + \mu_x^2) = \gamma(0) - \gamma(h)$$