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Comparison of Integration Methods with Analytical Solution

Introduction

This report presents a comparative study between three numerical integration methods—Explicit Euler, Implicit Euler, and Runge-Kutta 4 (RK4)—and the analytical solution of a second-order linear differential equation. The equation models a damped, driven mechanical system and is expressed as:

- $9.09\,\ddot{x} + 8.92\,\dot{x} + 0.89\,x = 9.87$

The goal is to analyze the behavior of these numerical schemes for varying time steps (h) and to evaluate their accuracy and stability by comparing them with the analytical solution.

Function which I changed according to my variant :-

```
import numpy as np
import matplotlib.pyplot as plt

# Coefficients
a, b, c, d = 9.09, 8.92, 0.89, 9.87

def lin2nd_order_dynamics(x):

    x1, x2 = x[0], x[1]
    x1dot = x2
    x2dot = (d - b * x2 - c * x1) / a
    return np.array([x1dot, x2dot])
```

Analytical Solution

Dividing the given equation by the leading coefficient a = 9.09 gives the standard form:

- $\ddot{x} + 0.9813\,\dot{x} + 0.0979\,x = 1.0858$

The constant forcing term on the right-hand side leads to a steady-state (particular) solution:

- $x_p = d / c = 9.87 / 0.89 = 11.0899$

The homogeneous solution corresponds to the characteristic equation $r^2 + 0.9813r + 0.0979 = 0$, with roots:

- $r_1 = -0.8686$, $r_2 = -0.1127$

Thus, the general analytical solution is given by:

- $x(t) = C_1\, e^{(-0.8686t)} + C_2\, e^{(-0.1127t)} + 11.0899$

Using the initial conditions $x(0) = 0.1$ and $\dot{x}(0) = 0$, the constants are calculated as:

- $C_1 = 1.639$, $C_2 = -12.629$

The complete analytical solution becomes:

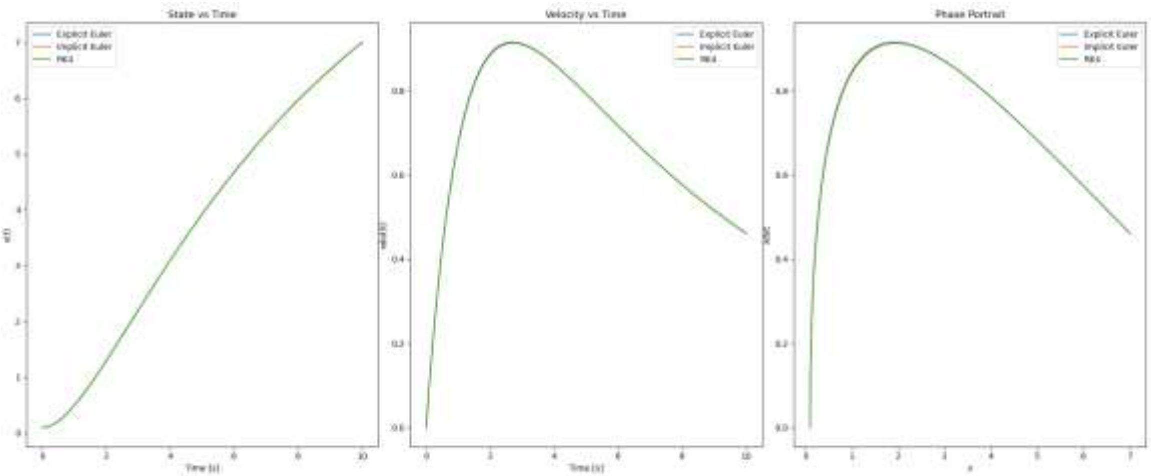
- $x(t) = 1.639\, e^{(-0.8686t)} - 12.629\, e^{(-0.1127t)} + 11.0899$

Graphical Comparison

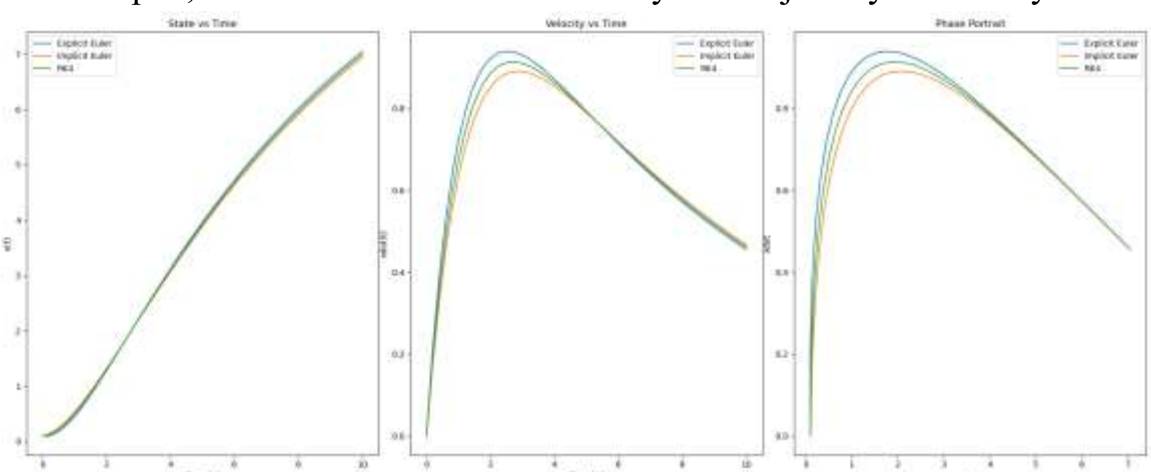
The system was solved using three integration methods—Explicit Euler, Implicit Euler, and Runge-Kutta 4—with a time step of h = 0.01 s and 0.3 s and total time Tf = 10 s. All three methods produced results that closely followed the analytical curve.

Key observations:

With h = 0.01, all methods overlap almost perfectly with the analytical solution. The explicit and implicit Euler methods show minor deviations, while RK4 remains indistinguishable from the analytical curve.



• When the time step is increased to h = 0.2, differences become visible: the explicit Euler method overshoots, the implicit Euler method becomes overdamped, and RK4 still follows the analytical trajectory accurately.



Discussion

The analytical and numerical comparisons highlight the trade-offs inherent to each integration method:

1. Explicit Euler: Simple and fast, but accuracy decreases with larger step sizes. It tends to overestimate the response and can become unstable if h exceeds the stability limit.

2. Implicit Euler: Unconditionally stable for linear systems, but introduces additional numerical damping. At larger step sizes, it underestimates the system response and reaches steady state prematurely.

3. Runge-Kutta 4 (RK4): Provides excellent accuracy and stability, even for larger time steps. It tracks the analytical solution precisely and is the most reliable for smooth systems like this one.

The phase portrait clearly shows the difference in behavior: explicit Euler produces an outward-leaning trajectory, implicit Euler curves inward due to overdamping, and RK4 lies between them, matching the true analytical path.

Effect of Step Size (h) and Final Time (Tf)

Increasing the step size (h) affects the accuracy and stability of the methods:

- Explicit Euler: Overpredicts position and velocity; may become unstable for large h.
- Implicit Euler: Always stable but overly damped; accuracy reduces with large h.
- RK4: Maintains high accuracy even for relatively large h.

Increasing the final time (Tf) simply extends the simulation duration, allowing all curves to settle to the steady-state value $x_s = d / c = 11.09$. It does not influence numerical accuracy, but it shows the full convergence behavior.

Conclusion

The comparative analysis demonstrates that all three numerical methods effectively approximate the analytical solution for small time steps. Explicit and implicit Euler methods exhibit predictable deviations—overestimation and overdamping respectively—when the step size increases, while RK4 maintains excellent agreement with the analytical solution even for coarser discretization. For this non-stiff, overdamped system, all methods are stable, but RK4 is preferred for high-accuracy applications.