Heavy Tails

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We can *invert* cumulatives;

$$F^{-1}(\alpha) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}: \ F(x) > \alpha\}$$

Let's now look at a *QQ Plot*

`https://en.wikipedia.org/wiki/Q-Q_plot`

Assume that we have two cumulatives F_1 and F_2, which are integrals of two positive density functions;

$$F_i(t) \stackrel{\text{def}}{=} \int_{s=-\infty}^{t} f_i(s)ds. \qquad i \in \{1,2\}, t \in \mathbb{R}$$

Let's plot $(F_1^{-1}(\alpha), F_2^{-1}(\alpha))$ for $\alpha \in (0,1)$; this is a *parametric plot*.

Suppose that this plot is a line;

$$F_2^{-1}(\alpha) = mF_1^{-1}(\alpha) + b \qquad \alpha \in (0,1) \tag{1}$$

Recalling how to take derivatives of inverse functions, we then have that

$$\frac{1}{f_2(F_2^{-1}(\alpha))} = \frac{m}{f_1(F_1^{-1}(\alpha))} \qquad \alpha \in (0,1)$$

or rather, using (1),

$$\frac{1}{f_2(mF_1^{-1}(\alpha) + b)} = \frac{m}{f_1(F_1^{-1}(\alpha))}$$

which implies (you might take $x' = F_1^{-1}(\alpha)$) that

$$f_1(x') = mf_2(mx' + b). \qquad x' \in \mathbb{R}$$

If X_1 and X_2 are random variables with cumulatives, respectively, F_1 and F_2, this in turn implies that, in law,

$$X_2 = mX_1 + b.$$

One often plots a reference line through the points corresponding to $\alpha = \frac{1}{4}$ and $\alpha = \frac{3}{4}$.

Let's see what could happen with a *heavy-tailed* distribution. Assume that at the center of the QQ plot, $F_2^{-2}(\alpha) \approx F_1^{-1}(\alpha)$, so that near the center of the distribution F_1 and F_2 agree with each other. Let's assume, however, that $F_2^{-1}(\alpha) \ll F_1^{-1}(\alpha)$ when $\alpha \approx 0$. Setting

$$x_i \stackrel{\text{def}}{=} F_i^{-1}(\alpha)$$

for $i \in \{1, 2\}$, the inequality $x_2 \ll x_1$ means that F_2 accumulates mass α further to the left than F_1. The left tail of F_2 is thus *fatter* than the left tail of F_1.
Similarly, let's assume that $F_2^{-1}(\alpha) \gg F_1^{-1}(\alpha)$ for $\alpha \approx 1$. Again using (**??**), the inequality $x_2 \gg x_1$ means that F_2 accumulates mass α further to the right than F_1, meaning that F_2 accumulates mass in the right tail at a more extreme value than F_1; the right tail of F_2 is thus *fatter* than the right- tail of F_1.